Elliptic Threefolds
with
Trivial Canonical Bundles

by

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Abstract

We classify elliptic 3-folds $\pi : X \to S$ with $K_X \cong \mathcal{O}_X$ by classifying the base surface S. An approach for constructing examples of such elliptic 3-folds with $q(X) = 0$ will be presented.

Introduction

By an elliptic 3-fold we shall mean a fibration $\pi : X \longrightarrow S$ of a smooth projective 3-fold X over a smooth projective surface S such that general fibers are smooth elliptic curves. Here by a fibration we mean a proper surjective holomorphic map with connected fibers. Throughout this article we do not assume that π admits a section.

Elliptic 3-folds are higher-dimensional analogues of elliptic surfaces. In this article we shall consider fibrations $\pi : X \longrightarrow S$ of a smooth projective 3-fold X with $K_X \cong \mathcal{O}_X$ over a smooth projective surface S. Note that by the adjunction formula, general fibers of π are smooth elliptic curves and therefore $\pi : X \longrightarrow S$ is an elliptic 3-fold. We shall classify such elliptic 3-folds by classifying the base surface S. The main results are stated in Theorems 2.2.17, 3.1.3 and 3.2.1. Our method of proof will be completely elementary.

The contents of this article are organized as follows: in § 1 we will establish the basic formulas and prove that the anticanonical bundle of S is nef, § 2 and § 3 will be devoted to the cases $q(X) = 0$ and $q(X) \geq 1$ respectively, § 4 deals with construction of examples. Unfortunately non-trivial examples for the case $q(X) \geq 1$ are much harder to come by. Therefore we will restrict ourselves to the case $q(X) = 0$ only. We will discuss a unified construction (Theorem 4.5) which yields examples for the majority of cases predicted by our classification.

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Contents

NOTATIONS

K_M: the canonical line bundle of a complex manifold M,

$\kappa(M)$: Kodaira dimension of a complex manifold M,

Ω^i_M: sheaf of germs of holomorphic sections of i-forms on a complex manifold M,

$q(M)$: the complex dimension of $H^1(M, \mathcal{O}_M)$,

ω_M: sheaf of germs of holomorphic sections of n-forms on a complex manifold M of dimension n,

$R^i\pi_*\mathcal{F}$: the i-th higher direct image sheaf of a coherent sheaf $\mathcal{F}$ on M under π,

$\Gamma(M, L)$: the space of sections of a holomorphic line bundle L on a complex manifold M,

$e(M)$: the topological Euler number of a complex manifold M,

$\kappa^{-1}(M)$: the anti-Kodaira dimension of a complex manifold M,

$\diamondsuit$: end of proof of an assertion.

All varieties are defined over the field of complex numbers.

§1 Preliminaries

In this section we will derive an inequality relating invariants of X and S. We will also prove an intersection formula by a spectral sequence computation. A Kähler-Einstein metric on X will then be used to conclude that the anticanonical bundle of S is numerically effective.

§1.1 AN INEQUALITY AND AN INTERSECTION FORMULA

We start with a simple observation.

Proposition 1.1.1

Let $\pi : X \to S$ be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$. Then we have

$$R^i \pi_* \omega_X = \begin{cases} \mathcal{O}_S, & i = 0 \\ \omega_S, & i = 1 \\ 0, & i \geq 2. \end{cases}$$

Proof

Since π is proper and has connected fibers, $\pi_* \omega_X \cong \pi_* \mathcal{O}_X \cong \mathcal{O}_S$. The rest follows directly from Kollár ([11], Theorem 2.1 and Proposition 7.6).◇

Proposition 1.1.2

Let $\pi : X \to S$ be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$. Then we have

$$q(S) \leq q(X) \leq q(S) + p_g(S).$$

Proof

We have an exact sequence

$$0 \to H^1(S, \pi_* \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(S, R^1 \pi_* \mathcal{O}_X) \to \cdots.$$

Using Proposition 1.1.1 we immediately arrive at the inequalities. ◇

Proposition 1.1.3

Let $\pi : X \to S$ be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$. For any divisor C on S, we have

$$-C \cdot K_S = \frac{1}{12}\pi^*(c_1[C]) \cdot c_2(X),$$

where $[C]$ is the holomorphic line bundle on S associated to the divisor C.

Proof

By Hirzebruch-Riemann-Roch on X,

$$\mathcal{X}(X, \pi^*[C]) = \{\mathrm{ch}(\pi^*[C]) \cdot \mathrm{Td}(X)\}_3,$$

where $\{*\}_3$ denotes evaluation of the degree 3 term of $*$ on the fundamental cycle $[X]$. As $c_1^3(\pi^*[C]) = 0$ and $c_1(X) = 0$, the right hand side equals $\frac{1}{12}\pi^*(c_1[C]) \cdot c_2(X)$.

By definition, $\mathcal{X}(X, \pi^*[C]) = \sum_{i=0}^{3}(-1)^i h^i(X, \pi^*[C])$. To compute $h^i(X, \pi^*[C])$, we look at the Leray spectral sequence whose E_2 terms are given by

$$E_2^{p,q} = H^p(S, R^q\pi_*(\pi^*[C])) \Rightarrow H^{p+q}(X, \pi^*[C]).$$

Using Proposition 1.1.1 and the projection formula ([6], p.253), we have

$$R^q\pi_*(\pi^*[C]) = \begin{cases} [C], & q = 0 \\ [C] \otimes \omega_S, & q = 1 \\ 0, & q \geq 2 \end{cases}$$

Therefore $E_2^{p,q} = 0$ for all $q \geq 2$. Also, $E_2^{p,q} = 0$ for all $p \geq 3$ since $\dim S = 2$. Hence the spectral sequence degenerates at E_3 level, and therefore $H^i(X, \pi^*[C]) \cong \bigoplus_{i=p+q} E_3^{p,q}$.

A straight forward computation gives

$$\begin{aligned} H^0(X, \pi^*[C]) &\cong H^0(S, [C]), \\ H^1(X, \pi^*[C]) &\cong H^1(S, [C]) \oplus \mathrm{Ker}\ d_2, \\ H^2(X, \pi^*[C]) &\cong H^1(S, [C] \otimes \omega_S) \oplus \frac{H^2(S, [C])}{\mathrm{im}\ d_2}, \\ H^3(X, \pi^*[C]) &\cong H^2(S, [C] \otimes \omega_S), \end{aligned}$$

where $d_2 : H^0(S, [C] \otimes \omega_S) \to H^2(S, [C])$ is the differential on the E_2 level. By summing them up, we have

$$\begin{aligned} \mathcal{X}(X, \pi^*[C]) &= \mathcal{X}(S, [C]) - \mathcal{X}(S, [C] \otimes \omega_S) \\ &= -C \cdot K_S. \quad \text{(By Riemann} - \text{Roch on S)} \end{aligned}$$

Thus

$$-C \cdot K_S = \frac{1}{12}\pi^*(c_1[C]) \cdot c_2(X). \Diamond$$

§1.2 NUMERICAL EFFECTIVENESS OF $-K_S$

Let D be a divisor on a smooth projective manifold M. D is said to be nef if $D \cdot C \geq 0$ for all irreducible curve C on M. Here by a curve we shall always mean an effective divisor.

<u>**Proposition 1.2.1**</u>

Let $\pi : X \to S$ be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$. Then $-K_S$ is nef.

<u>**Proof**</u>

Let C be an irreducible curve on S. Since the line bundle $[\pi^*C]$ comes from the divisor $D = \pi^*C$, $c_1[\pi^*C]$ is represented by the Poincaré dual η_D of the divisor D ([5], p.141). D is effective since C is. Write $D = \Sigma_i a_i D_i$, where each D_i is an irreducible component of D and $a_i \geq 0$ for all i. We have $\eta_D = \Sigma_i a_i \eta_{D_i}$. By Proposition 1.1.3

$$\begin{aligned} -C \cdot K_S &= \frac{1}{12} c_1([\pi^*C]) \cdot c_2(X) \\ &= \frac{1}{12}\int_X \eta_D \wedge c_2(X) \qquad \text{(by definition of Poincaré dual),} \\ &= \frac{1}{12}\sum_i a_i \int_{D_i} j^* c_2(X) \end{aligned}$$

where $j : D_i \to X$ denotes the inclusion. We may assume that each D_i is a smooth complex submanifold of X without affecting the value of the integral.

By a theorem of Chern([3]), $c_2(X) = -\frac{1}{8\pi^2}(\Omega_j^j \wedge \Omega_k^k - \Omega_l^k \wedge \Omega_k^l)$, where $\Omega_l^k = R_{lkpq}\omega^p \wedge \bar{\omega}^q$ is the curvature given by a hermitian metric (g_{ij}) on X expressed in terms of a unitary coframe $(\omega^1, \omega^2, \omega^3)$.

As $c_1(X)$ vanishes, by the solution to the Calabi conjecture by Yau ([18]), we may choose a Kähler-Einstein metric (g_{ij}) on X with Ricci curvature $r_{pq} = R_{jjpq} = 0$ for all p and q. Thus

$$\begin{aligned}\Omega_j^j &= R_{jjpq}\omega^p \wedge \bar{\omega}^q \\ &= r_{pq}\omega^p \wedge \bar{\omega}^q = 0.\end{aligned}$$

Also, locally we may choose an adapted unitary coframe $(\omega^1, \omega^2, \omega^3)$ on X such that $(j^*\omega^1, j^*\omega^2)$ is a unitary coframe for the induced metric (j^*g_{ij}) on D_i and $j^*\omega^3 = 0$. The volume form of D_i is equal to $d\mu_{D_i} = -\frac{1}{4}j^*(\omega^1 \wedge \bar{\omega}^1 \wedge \omega^2 \wedge \bar{\omega}^2)$.

Using $j^*\omega^3 = 0$, the only terms survived in $j^*c_2(X)$ are $\omega^1 \wedge \bar{\omega}^1 \wedge \omega^2 \wedge \bar{\omega}^2$, $\omega^1 \wedge \bar{\omega}^2 \wedge \omega^2 \wedge \bar{\omega}^1$, $\omega^2 \wedge \bar{\omega}^1 \wedge \omega^1 \wedge \bar{\omega}^2$ and $\omega^2 \wedge \bar{\omega}^2 \wedge \omega^1 \wedge \bar{\omega}^1$. Therefore

$j^*c_2(X) = \frac{1}{8\pi^2}j^*(-2R_{lk12}R_{kl21})j^*(\omega^1 \wedge \bar{\omega}^1 \wedge \omega^2 \wedge \bar{\omega}^2)$.

Thus

$$\begin{aligned}\int_{D_i} c_2(X) &= \frac{1}{8\pi^2}\int_{D_i}(-2R_{lk12}R_{kl21})(-4d\mu_{D_i}) \\ &= \frac{1}{\pi^2}\int_{D_i}|R_{lk12}|^2 d\mu_{D_i} \\ &\geq 0.\end{aligned}$$

Hence $-K_S \cdot C \geq 0$ and $-K_S$ is nef.◇

We may now set off to classify S. Note that since the Kodaira dimension $\kappa(X)$ of X is zero, we have $q(X) \leq \dim X = 3$ ([8], Corollary 2). We will consider the situation for each value of $q(X)$ separately.

§2 The case $q(X) = 0$

Throughout this section X will denote a smooth projective 3-fold with $K_X \cong \mathcal{O}_X$ and $q(X) = 0$, i.e. a Calabi-Yau 3-fold. X automatically satisfies $h^0(X, \Omega_X^2) = 0$ by Serre duality. We record the following simple observation.

Claim

Let $\pi : X \to S$ be a fibration of a Calabi-Yau 3-fold X over a smooth compact complex surface S. Then S is projective.

Proof

Using $\pi_* \mathcal{O}_X \cong \mathcal{O}_S$ and the exact sequence

$$0 \to H^1(S, \pi_* \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(S, R^1 \pi_* \mathcal{O}_X) \to \cdots,$$

we have $h^{0,1}(S) = 0$. Also,$h^{0,2}(S) = h^{2,0}(S) = \dim\ H^0(S, \Omega_S^2) = 0$ because X does not have non-trivial holomorphic 2-forms. Therefore the first Chern class map $H^1(S, \mathcal{O}_S^*) \to H^2(S, Z)$ is an isomorphism.

If $b_1(S)$ were odd, we would have $1 + b_1(S) = 2h^{0,1}(S) = 0$, which is absurd. Thus $b_1(S)$ is even and $b^+(S) = 1 + 2h^{2,0}(S) = 1$. Hence there exists $\alpha \in H^2(S, Z)$ with $\alpha^2 > 0$. By the fact that the first Chern class map is an isomorphism, there exists a holomorphic line bundle L on S with $c_1(L) = \alpha$. Therefore $c_1^2(L) = \alpha^2 > 0$, which implies that S is projective.$\diamondsuit$

Thus for the case $K_X \cong \mathcal{O}_X$ and $q(X) = 0$, there is no loss in generality by letting the base surface S to be projective in our definition of elliptic 3-folds.

§2.1 RATIONALITY OF S

Before we prove that the base surface S is rational, we need some preliminaries which are well-known, but we include them for completeness.

Let M be a compact Kähler manifold of complex dimension n. A holomorphic tensor field of type (p, q) on M is defined to be a global holomorphic section of $\otimes_p T'_M \otimes \otimes_q \Omega_M^1$, where p and q are non-negative integers. We have the following result by a Bochner type argument.

Proposition 2.1.1

Let M be a compact Kähler manifold of complex dimension n with $c_1(M) = 0$. Then holomorphic tensor fields of type (p, q) on M are parallel.

Proof

By the solution to the Calabi conjecture by Yau ([18]), we can choose a Kähler-Einstein metric (g_{ij}) on M with Ricci curvature $r_{ij} = cg_{ij} = 0$. The metric (g_{ij})

induces a metric g^p_q on $\otimes_p T'_M \otimes \otimes_q \Omega^1_M$. Denote by $\| \sigma \|$ the length of a holomorphic tensor field σ of type (p,q) on M under the metric g^p_q. By a straight forward computation, we have

$$\begin{aligned}\triangle \| \sigma \|^2 &= \triangle g^p_q(\sigma \otimes \bar{\sigma}) \\ &= g^{kl} \frac{\partial^2}{\partial z^k \partial \bar{z}^l} g^p_q(\sigma \otimes \bar{\sigma}) \\ &= \| \nabla \sigma \|^2 + Q(\sigma),\end{aligned}$$

where $Q(\sigma) = c(q-p) \| \sigma \|^2 = 0$. Therefore $\triangle \| \sigma \|^2 = \| \nabla\sigma \|^2$. By Hopf's maximum principle ([7]), $\triangle \| \sigma \|^2$ is identically zero on M, so that $\nabla \sigma = 0$, i.e. σ is parallel.$\diamondsuit$

Again let M be a compact Kähler manifold of complex dimension n with $c_1(M) = 0$. By works of Bogomolov, the universal covering $\widetilde{M}$ of M is biholomorphic to a product

$$\mathcal{C}^k \times \prod_i U_i \times \prod_j V_j,$$

where

(i) $\mathcal{C}^k$ is the usual complex Euclidean space with the standard Kähler metric;

(ii) each U_i is a simply-connected compact Kähler manifold of odd complex dimension $u_i \geq 3$ with trivial canonical bundle and with irreducible holonomy group $SU(u_i)$;

(iii) each V_j is a simply-connected compact Kähler manifold of even complex dimension v_j with trivial canonical bundle and with irreducible holonomy group $Sp(\frac{v_j}{2})$.

Applying this to a Calabi-Yau 3-fold X, we have the following

Proposition 2.1.2

Let X be a Calabi-Yau 3-fold. Then $h^0(X, \otimes_m \Omega^1_X) = 0$ for all positive integers m.

Proof

If σ were a non-trivial global holomorphic section of $\otimes_m \Omega^1_X$, consider its lifting $\tilde{\sigma}$ to the universal cover $\widetilde{X}$ of X. Since $\pi_1(X)$ is finite ([1],§3, Proposition 2), $\widetilde{X}$ does not contain Euclidean factors. On individual factors U_i and V_j of $\widetilde{X}$, $\tilde{\sigma}$ is decomposed into holomorphic tensor fields of types $(0, m_i)$ and $(0, n_j)$ respectively, which are parallel by Proposition 2.1.1 and hence are identically zero by irreducible holonomy. Thus $\tilde{\sigma}$ is identically zero and so is σ.◇

Corollary 2.1.3

Let $\pi : X \to S$ be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$ and $q(X) = 0$. Then S is rational.

Proof

We have $q(S) = 0$ because $q(X) = 0$. We only need to prove that $h^0(S, K_S^n) = 0$ for all positive integers n.

If, on the contrary, that there were a non-trivial holomorphic section σ of $K_S^n = \otimes_n(\wedge^2\Omega_S^1)$ for some positive integer n, $\pi^*\sigma$ would then be a non-trivial global holomorphic section of $\otimes_n(\wedge^2\Omega_X^1)$. As $\otimes_n(\wedge^2\Omega_X^1)$ is a sub-bundle of $\otimes_{2n}(\Omega_X^1)$, $\pi^*\sigma$ would give a non-trivial global holomorphic section of $\otimes_{2n}(\Omega_X^1)$, which is impossible by Proposition 2.1.2.

Thus S is rational.◇

§2.2 DETERMINATION OF S

We need to determine all rational surfaces S with $-K_S$ nef. We start by noting a couple of elementary observations.

Proposition 2.2.1

Let S be a rational surface with $-K_S$ nef. Then $c_1^2(S) \geq 0$, $h^0(S, -K_S) \geq 1$ and $C^2 \geq -2$ for all smooth irreducible curves C on S.

Proof

Since $-K_S$ is nef, $c_1^2(S) \geq 0$ by Kleiman ([9]). Using Riemann-Roch and $h^0(S, K_S^2) = 0$, we have $h^0(S, -K_S) = 1 + c_1^2(S) + h^1(S, -K_S) \geq 1$. The last assertion follows from the genus formula.◇

Proposition 2.2.2

Let $b : \widetilde{S} \to S$ be a finite succession of blow-ups of a smooth compact complex surface S. If $-K_{\widetilde{S}}$ is nef, so is $-K_S$.

Proof

We can write

$$\widetilde{S} = S_m \xrightarrow{b_m} S_{m-1} \xrightarrow{b_{m-1}} \cdots \longrightarrow S_1 \xrightarrow{b_1} S_0 = S,$$

where $b = b_1 \circ \cdots \circ b_m$ and each b_i is a blow-up at a single point p_i of S_{i-1}. It suffices to show that $-K_{S_{i-1}}$ is nef under the assumption that $-K_{S_i}$ is nef. For simplicity we write p_i as p.

Let C be an irreducible curve on S_{i-1}. Then $b_i^*(C) = \widehat{C} + mE$, where $\widehat{C}$ is the proper transform of C, E is the exceptional curve of the blow-up b_i and $m = \mathrm{mult}_p(C) \geq 0$. Since $\widehat{C}$ is still an irreducible curve on S_i, we have

$$\begin{aligned} 0 \leq \widehat{C} \cdot (-K_{S_i}) &= (b_i^*(C) - mE)(b_i^*(-K_{S_{i-1}}) - E) \\ &= C \cdot (-K_{S_{i-1}}) - m. \quad \text{Thus} \\ C \cdot (-K_{S_{i-1}}) &\geq m \geq 0. \end{aligned}$$

Hence $-K_{S_{i-1}}$ is nef.$\diamondsuit$

Proposition 2.2.3

Let S be a minimal rational surface with $-K_S$ nef. Then S is either $\mathcal{CP}^2$, $\mathcal{CP}^1 \times \mathcal{CP}^1$ or the Hirzebruch surface Σ_2.

Proof

All minimal rational surfaces are among $\mathcal{CP}^2$ or Σ_n, $n = 0, 2, 3, \cdots$, where Σ_n is the n-th Hirzebruch surface.

$-K_{\mathcal{CP}^2} = 3H$ is ample and hence nef. For Σ_n's, we have

$$-K_{\Sigma_n} = 2E_0 + (2-n)F,\ E_0^2 = n,\ E_0 \cdot F = 1,\ E_\infty \sim E_0 - nF,$$

where E_0, E_∞ and F are the zero-section, ∞-section and a fiber of the projection $p : \Sigma_n \longrightarrow \mathcal{CP}^1$ respectively.

For $-K_{\Sigma_n}$ to be nef,

$$0 \leq (-K_{\Sigma_n}) \cdot E_0 = n + 2,$$
$$0 \leq (-K_{\Sigma_n}) \cdot F = 2, \text{ and}$$
$$0 \leq (-K_{\Sigma_n}) \cdot E_\infty = 2 - n.$$

Therefore $n = 0, 1$ or 2. But Σ_1 is not minimal because it is $\mathcal{CP}^2$ blown up at one point. We are left with $\Sigma_0 \cong \mathcal{CP}^1 \times \mathcal{CP}^1$ and Σ_2.◇

Since $c_1^2(\mathcal{CP}^2) = 9$ and $c_1^2(\mathcal{CP}^1 \times \mathcal{CP}^1) = c_1^2(\Sigma_2) = 8$, it follows that a rational surface S with $-K_S$ nef may be obtained by blowing up
(i) $\mathcal{CP}^2$ at most 9 times; or
(ii) $\mathcal{CP}^1 \times \mathcal{CP}^1$ or Σ_2 at most 8 times.

Although these blow-ups may be performed at infinitely-near points, they cannot be too arbitrary because $C^2 \geq -2$ for all smooth irreducible curves C on S. We need to distinguish those blow-ups which ensure that $-K_S$ is nef from those which do not.

We first look at blow-ups of $\mathcal{CP}^2$. We need the notion of almost general position according to Demazure.

Let $S_r \xrightarrow{b_r} S_{r-1} \xrightarrow{b_{r-1}} \cdots \longrightarrow S_1 \xrightarrow{b_1} S_0 = \mathcal{CP}^2$ be a succession of blow-ups of $\mathcal{CP}^2$, may be at infinitely-near points, such that b_i is a blow-up of S_{i-1} at a single point x_i and $0 \leq r \leq 8$. Let $\Sigma = \{x_1, \cdots, x_{r-1}\}$ and write $\varphi_i = b_1 \circ \cdots \circ b_i$.

For each fixed i, define $E_j(\varphi_{i-1})$ to be the set-theoretic inverse image of x_j under the map φ_{i-1} for $1 \leq j \leq i-1$. Notice that $E_j(\varphi_{i-1})$ is a divisor on S_{i-1} whose support may contain more that 1 irreducible component.

Let C be an effective divisor on $S_0 = \mathcal{CP}^2$. We define $\text{mult}_{x_i}(C)$ to be the multiplicity at x_i of the strict transform of C under the map φ_{i-1}. We say that x_i lies on C if $\text{mult}_{x_i}(C) > 0$.

We note the following condition

$(*)$: For each $x_i \in \Sigma$, $1 \leq i \leq r-1$, x_i does not lie on any irreducible component of $E_j(\varphi_{i-1})(1 \leq j \leq i-1)$ not of the form $(\varphi_{i-1})^{-1}(x_j)$ for some j.

Definition 2.2.4 (Demazure [4], p.39)

With the above definitions and notations, we say that Σ is in almost general position if

(i) Σ satisfies condition $(*)$,
(ii) no 4 points of Σ lie on a line of $\mathcal{CP}^2$,
(iii) no 7 points of Σ lie on an irreducible conic of $\mathcal{CP}^2$.

If $\Sigma = \{x_1, \cdots, x_r\}$, $r \leq 8$, is a set of distinct points on $\mathcal{CP}^2$ and if Σ is in general position, then it is also in almost general position. We need the following theorem of Demazure.

Theorem 2.2.5 (Demazure [4], p.39)

Let $S_r \xrightarrow{b_r} S_{r-1} \xrightarrow{b_{r-1}} \cdots \longrightarrow S_1 \xrightarrow{b_1} S_0 = \mathcal{CP}^2$ be a succession of blow-ups of $\mathcal{CP}^2$ with $\Sigma = \{x_1, \cdots, x_r\}$, where $x_i \in S_{i-1}$ is the center of the blow-up b_i, and $r \leq 8$. Then the followings are equivalent:
(i) Σ is in almost general position;
(ii) the anticanonical system of S_r has no fixed components;
(iii) the anticanonical system of S_r contains a smooth irreducible curve;
(iv) for each effective divisor D on S_r, $(-K_{S_r}) \cdot D \geq 0$.

By virtue of this theorem, we conclude that if S is a blow-up of $\mathcal{CP}^2$ at r points in almost general position, $0 \leq r \leq 8$, then $-K_S$ is nef.

Now let S_9 be a rational surface obtained by blowing up $\mathcal{CP}^2$ nine times, may be at infinitely-near points, such that $-K_{S_9}$ is nef. Let $\sigma : S_9 \to S_8$ be a blow-down of any (-1) curve on S_9, resulting in a smooth rational surface S_8. Since $-K_{S_9}$ is nef, so is $-K_{S_8}$ by Proposition 2.2.2. Therefore S_8 is a blow-up of $\mathcal{CP}^2$ at 8 points in almost general position and S_9 is obtained by blowing up some point $s \in S_8$. To determine which point of S_8 is allowed to be blown up, we need some more information about the linear system $|-K_{S_8}|$.

Recall that the linear system $|-K_{S_8}|$ has no fixed components but has a unique base point s_0, and that for any point s on S_8 distinct from s_0, there exists a unique member C of $|-K_{S_8}|$ passing through s (cf. Demazure [4], p.40, Proposition 2 and p.55). These notations will be fixed throughout the following discussions. We want to investigate members of $|-K_{S_8}|$.

Proposition 2.2.6

Let S_8 and $s_0 \in S_8$ be as above. Then

(i) any member of $|-K_{S_8}|$ is non-singular at s_0;

(ii) any two distinct members of $|-K_{S_8}|$ intersect transversely at s_0;

(iii) all members of $|-K_{S_8}|$ are connected;

(iv) general members of $|-K_{S_8}|$ are smooth irreducible elliptic curves.

Proof

(i) Since for any point s on S_8 distinct from s_0, there exists a unique member of $|-K_{S_8}|$ passing through s, we deduce that any 2 distinct members of $|-K_{S_8}|$ do not have common components and must intersect at s_0 only. Let C be an arbitrary member of $|-K_{S_8}|$ and D a smooth irreducible member of $|-K_{S_8}|$ guranteed by Theorem 2.2.5 (iii). We have $1 = (-K_{S_8})(-K_{S_8}) = C \cdot D = (C \cdot D)_{s_0}$. We also have $\text{mult}_{s_0}(C) \geq 1$ and $\text{mult}_{s_0}(D) = 1$. Therefore $1 = (C \cdot D)_{s_0} \geq \text{mult}_{s_0}(C) \cdot \text{mult}_{s_0}(D) = \text{mult}_{s_0}(C)$. Thus $\text{mult}_{s_0}(C) = 1$ which implies that C is non-singular at s_0.

(ii) Follows directly from the equality $1 = C \cdot C' = (C \cdot C')_{s_0} = \text{mult}_{s_0}(C) \cdot \text{mult}_{s_0}(C')$ using (i), where C and C' are any two distinct members of $|-K_{S_8}|$.

(iii) Let C be an arbitrary member of $|-K_{S_8}|$. If C is irreducible, C is already connected. If C is reducible, then C can be written as $C = \xi + \Gamma$, where ξ is a special exceptional divisor and Γ is a fundamental cycle (Demazure [4], p.55). ξ is irreducible and Γ is connected (ibid, p.53, Corollaire 2 and p.54, Proposition 3). Also, we have $\xi \cdot \Gamma = \xi(C - \xi) = \xi(-K_{S_8} - \xi) = (-K_{S_8}) \cdot \xi - \xi^2 = 1 - (-1) = 2 > 0$, by definition of special exceptional divisor. Since both ξ and Γ are effective divisors having no common components, we must have $\xi \cap \Gamma \neq \emptyset$. Thus $C = \xi + \Gamma$ is connected.

(iv) Follows directly from Bertini theorem, (i) and the genus formula.$\diamondsuit$

Remark 2.2.7

In particular, if C is a reducible member of $|-K_{S_8}|$, we can write $C = C_0 + \sum_i n_i C_i$ where C_0 is irreducible and is distinct from each $C_i (i \geq 1)$. Moreover, C_0 is non-singular at s_0 and no C_i passes through s_0 for $i \geq 1$.

Proposition 2.2.8

Let $\sigma : S_9 \to S_8$ be the blow-up of S_8 at the unique base-point s_0 of $|-K_{S_8}|$. Then S_9 is a relatively minimal elliptic surface fibered over $\mathcal{CP}^1$ without multiple fibers. Moreover, $|-K_{S_9}|$ is base-point free.

Proof

Since s_0 is the unique base-point of $|-K_{S_8}|$, by blowing up S_8 at s_0, we obtain a holomorphic map $p : S_9 \to \mathcal{CP}^1$. Fibers of p are just strict transforms under σ of members of $|-K_{S_8}|$. Therefore general fibers of p are smooth elliptic curves. Also, all fibers of p are connected by virtue of Proposition 2.2.6 (iii) and Remark 2.2.7. Hence S_9 is an elliptic surface. The exceptional $\mathcal{CP}^1$ of the blow-up σ is a section of p. Therefore p has no multiple fibers.

Let F be an arbitrary fiber of p. Then $F = \widehat{C}$ for some $C \in |-K_{S_8}|$. We have $F = \widehat{C} = \pi^*(C) - E \sim \pi^*(-K_{S_8}) - E = -K_{S_9}$, where E is the exceptional curve of the blow-up σ. Let $F = \sum_i n_i C_i$ be the irreducible decomposition of F. Let F' be another fiber of p disjoint from F. Then $F' \cdot C_i = 0$, so that $K_{S_9} \cdot C_i = 0$ as well. Therefore none of the C_i is an exceptional curve of the first kind and thus $p : S_9 \to \mathcal{CP}^1$ is relatively minimal.

Since the base curve of p is $\mathcal{CP}^1$ and p does not have multiple fibers, any 2 fibers of p are linearly equivalent. But we have proved that $-K_{S_9} \sim$ any arbitrary fiber F. Hence $|-K_{S_9}|$ is base-point free.$\diamondsuit$

Observe that fibers of $p : S_9 \to \mathcal{CP}^1$ are just strict transforms of members of $|-K_{S_8}|$ under σ. Therefore we immediately arrive at the following corollary.

Corollary 2.2.9

Let C be a member of $|-K_{S_8}|$. Then C is of one of the following types:

(i) a non-singular irreducible elliptic curve;

(ii) a rational curve with a node not at s_0;

(iii) a rational curve with a cusp not at s_0;

(iv) $C_0 + \sum_i n_i C_i$ where C_0 is a (-1) curve and passes through s_0, C_i's $(i \geq 1)$ are mutually distinct smooth rational curves with $C_i^2 = -2$ and no C_i for $i \geq 1$ passes through s_0. Moreover, g.c.d.$(n_i) = 1$ and C_0 is distinct from all C_i for $i \geq 1$.

Proof

The strict transform of an arbitrary member C of $|-K_{S_8}|$ becomes a fiber of the elliptic surface $p : S_9 \to \mathcal{CP}^1$, whose fibers are already classified by Kodaira ([10]). If C is irreducible, so is $\widehat{C}$ which is a fiber of p. Therefore C must be either (i), (ii) or (iii). If C is reducible, we can write $C = C_0 + \sum_i n_i C_i$ by Remark 2.2.7. The blow-up σ does not change C_i for $i \geq 1$ because none of them passes through s_0. Therefore each C_i is a (-2) curve with g.c.d.$(n_i) = 1$, as p has no multiple fibers. Also, C_0 passes through s_0 and $\widehat{C}_0$ is a (-2) curve. Therefore C_0 itself must be a (-1) curve.◊

Now we look at the blow-up $\sigma : S_9 \to S_8$ of S_8 at a point s on S_8 distinct from s_0. Recall that s lies on a unique member of $|-K_{S_8}|$.

If s lies on an irreducible member C of $|-K_{S_8}|$ and if C is singular at s, then $\text{mult}_s(C) \geq 2$, so that

$$\begin{aligned}(-K_{S_9}) \cdot \widehat{C} &= (\sigma^*(-K_{S_8}) - E)(\sigma^*(C) - \text{mult}_s(C) \cdot E) \\ &= -K_{S_8} \cdot C - \text{mult}_s(C) \\ &= c_1^2(S_8) - \text{mult}_s(C) \\ &= 1 - \text{mult}_s(C) < 0,\end{aligned}$$

where E is the exceptional curve of the blow-up σ. Thus $-K_{S_9}$ is not nef.

On the other hand, if s lies on a (-2) curve C_i which is an irreducible component of a reducible member C of $|-K_{S_8}|$, then the strict transform of C_i will be a (-3) curve on S_9. Thus again $-K_{S_9}$ is not nef.

Before we go on, we digress to recall some notions which will be useful later.

Definition 2.2.10 (Sakai [15], p.106, Mumford [13], p.330)

Let $C = \sum_i n_i C_i$ be the irreducible decomposition of a curve C on a smooth projective surface S. C is called a curve of fiber type if $C \cdot C_i = 0$ for all i. C is called a curve of canonical type if $C \cdot C_i = K_S \cdot C_i = 0$ for all i. If moreover C is connected and g.c.d.$(n_i) = 1$, then C is called an indecomposable curve of canonical type.

We record the following easy consequence.

Propsition 2.2.11

A curve C of fiber type on a smooth projective surface S is nef.

Proof

Take an arbitrary irreducible curve D on S. If $D = C_i$ for some i, then $C \cdot D = C \cdot C_i = 0$. If D is distinct from all C_i, then $D \cdot C_i \geq 0$ for all i. Therefore, $C \cdot D = \sum_i n_i C_i \cdot D \geq 0.\Diamond$

On S_8, we define
$\Lambda_1 = \{s \in S_8 | s \text{ is a singular point of some irreducible member of } |-K_{S_8}|\}$,
$\Lambda_2 = \{F | F \text{ is a } (-2) \text{ curve contained in some reducible member of } |-K_{S_8}|\}$.
Denote $\Lambda = \Lambda_1 \cup \Lambda_2$. Notice that $s_0 \notin \Lambda$.

Proposition 2.2.12

Let $\sigma : S_9 \rightarrow S_8$ be the blow-up of S_8 at a point $s \in S_8 \setminus \Lambda$. Then $-K_{S_9}$ is nef.

Proof

If $s = s_0$, $|-K_{S_9}|$ is base-point free by Proposition 2.2.8 and therefore is nef.

If $s \neq s_0$, $s \in C$ for a unique $C \in |-K_{S_8}|$. We separate into 2 cases:

(i) C is irreducible : then C is non-singular at s, $\widehat{C}$ is irreducible on S_9 and $\widehat{C} \cdot \widehat{C} = C \cdot C - 1 = 0$. Therefore $\widehat{C}$ is a curve of fiber type and hence is nef. But $\widehat{C} = \sigma^*(C) - E \sim -K_{S_9}$, where E is the exceptional curve of the blow-up. Thus $-K_{S_9}$ is nef as well.

(ii) C is reducible : then $C = C_0 + \sum_i n_i C_i$, $s \in C_0$ which is a (-1) curve. We have

$$\begin{aligned} \sigma^*(C) &= \sigma^*(C_0) + \sum_i n_i \sigma^*(C_i) \\ &= \widehat{C}_0 + E + \sum_i n_i \sigma^*(C_i) \\ &= \widehat{C} + E, \end{aligned}$$

where E is the exceptional curve of the blow-up and

$$\begin{aligned} \widehat{C} &= \widehat{C}_0 + \sum_i n_i \sigma^*(C_i) \\ &= \sigma^*(C) - E \sim -K_{S_9}. \end{aligned}$$

We only need to prove that $\widehat{C}$ is a curve of fiber type. We have

$$\begin{aligned}\widehat{C}\cdot\widehat{C}_0 &= (\widehat{C}_0+\sum_i n_i\sigma^*(C_i))\cdot\widehat{C}_0\\ &= (\widehat{C}_0)^2+\sum_i n_i\sigma^*(C_i)(\sigma^*(C_0)-E)\\ &= -2+\sum_i n_iC_i\cdot C_0\\ &= -2+(C-C_0)\cdot C_0\\ &= -2+(-K_{S_8})\cdot C_0+1=0.\end{aligned}$$

Also, for any $i\geq 1$,

$$\begin{aligned}\widehat{C}\cdot\sigma^*(C_i) &= (\sigma^*(C_0)-E)\cdot\sigma^*(C_i)+\sum_j n_j\sigma^*(C_j)\cdot\sigma^*(C_i)\\ &= C_0\cdot C_i+\sum_j n_jC_j\cdot C_i\\ &= C\cdot C_i\\ &= (-K_{S_8})\cdot C_i\\ &= 0,\end{aligned}$$

because each C_i is a (-2) curve.$\diamondsuit$

Remark 2.2.13

In the above proof, we observe that if we blow-up S_8 at $s\neq s_0$ with $s\in C$ for some $C\in|-K_{S_8}|$, then $\widehat{C}$ is always a curve of fiber type on S_9. Moreover, since $\widehat{C}\sim -K_{S_9}$, we have $-K_{S_9}\cdot C_i=\widehat{C}\cdot C_i=0$ for any irreducible component C_i of C. Thus $\widehat{C}$ is in fact a curve of canonical type. In addition, $\widehat{C}$ is indecomposable since C itself is indecomposable by Corollary 2.2.9.

To sum up, we have proved the following

Proposition 2.2.14

Let S be a rational surface obtained by a succession of blow-ups of $\mathcal{CP}^2$, may be at infinitely-near points. If $-K_S$ is nef, then S is one of the followings:
(i) a blow-up of $\mathcal{CP}^2$ at r points in almost general position, $0\leq r\leq 8$;

(ii) a blow-up of S_8 at a point $s \in S_8 \setminus \Lambda$.

Next we turn to blow-ups of $\mathcal{CP}^1 \times \mathcal{CP}^1$. It will be shown that these are exactly those blow-ups of $\mathcal{CP}^2$ we have just considered.

Proposition 2.2.15

Let S be a smooth projective surface obtained by a succession of blow-ups of $\mathcal{CP}^1 \times \mathcal{CP}^1$, may be at infinitely-near points, such that $-K_S$ is nef. Then S is isomorphic to some surface on the list of Proposition 2.2.14.

Proof

Write $S \cong \Sigma_0^m \xrightarrow{b_m} \Sigma_0^{m-1} \xrightarrow{b_{m-1}} \cdots \longrightarrow \Sigma_0^1 \xrightarrow{b_1} \Sigma_0 \cong \mathcal{CP}^1 \times \mathcal{CP}^1$, where b_i is a blow-up of Σ_0^{i-1} at a single point. It is well-known that Σ_0^1 is isomorphic to $\mathcal{CP}^2$ blown-up at 2 distinct points, so that S itself may be regarded as a blow-up of $\mathcal{CP}^2$, may be at infinitely-near points. As $-K_S$ is nef, the assertion follows from Proposition 2.2.14.$\Diamond$

For blow-ups of Σ_2, the situation is quite similar. As before, we denote by E_∞ the ∞-section of $p : \Sigma_2 \to \mathcal{CP}^1$ with $(E_\infty)^2 = -2$.

If $\sigma : S \to \Sigma_2$ is the blow-up of Σ_2 at a point $x \in E_\infty$, the strict transform $\widehat{E_\infty}$ of E_∞ will be a smooth irreducible curve with self-intersectionn -3. Thus $-K_S$ is not nef.

On the other hand, if $\sigma : S \to \Sigma_2$ is the blow-up of Σ_2 at a point $x \notin E_\infty$, then $-K_S$ is nef. Indeed, suppose $x \in F_\lambda$ for some fiber F_λ of the projection $p : \Sigma_2 \to \mathcal{CP}^1$. The strict transform $\widehat{F_\lambda}$ of F_λ is a (-1) curve, intersecting both $\widehat{F_\lambda}$ and E transversely, where E is the exceptional curve of the blow-up. We can blow down $\widehat{F_\lambda}$, obtaining the first Hirzebruch surface Σ_1 which can further be blown down to $\mathcal{CP}^2$. In other words, S can be obtained by blowing up $\mathcal{CP}^2$ at p and q, where $p \in \mathcal{CP}^2$ and q is infinitely-near to p. Thus $-K_S$ is nef.

Now we can state the following proposition.

Proposition 2.2.16

Let S be a projective surface obtained by a succession of blow-ups of Σ_2, may be at infinitely-near points, such that $-K_S$ is nef. Then S is isomorphic to some surface on the list of Proposition 2.2.14.

Proof

Write $S \cong \Sigma_2^m \xrightarrow{b_m} \Sigma_2^{m-1} \xrightarrow{b_{m-1}} \cdots \longrightarrow \Sigma_2^1 \xrightarrow{b_1} \Sigma_2 \cong \mathcal{CP}^1 \times \mathcal{CP}^1$, where b_i is a blow-up of Σ_2^{i-1} at a single point. Since S has nef anticanonical bundle, so does Σ_2^i for all i. In particular, b_1 is a blow-up of Σ_2 at some point $x \notin E_\infty$. By the preceeding discussion, Σ_2^1 is obtained by blowing up $\mathcal{CP}^2$ at 2 points p and q, where $p \in \mathcal{CP}^2$ and q is infinitely-near to p. Now proceed as in the proof of Proposition 2.2.15.$\diamondsuit$

Theorem 2.2.17

Let $\pi : X \to S$ be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$ and $q(X) = 0$. Then S is among one of the followings:

(i) $\mathcal{CP}^1 \times \mathcal{CP}^1$;

(ii) Σ_2;

(iii) blow-ups of $\mathcal{CP}^2$ at r points in almost general position, $0 \le r \le 8$;

(iv) blow-ups of S_8 at points on $S_8 \setminus \Lambda$.

Proof

Follows from Propositions 1.2.1, 2.1.3, 2.2.14, 2.2.15 and 2.2.16. $\diamondsuit$

§3 The case $q(X) \geq 1$

We shall now treat elliptic 3-folds $\pi : X \to S$ with $K_X \cong \mathcal{O}_X$ and $q(X) \geq 1$. We first recall a theorem of Kawamata.

Theorem (Kawamata [8], Theorem 15)

Let M be a smooth projective manifold with $\kappa(M) = 0$ and $q(M) = \dim_{\mathcal{C}}(M) - 1$. Then the Albanese mapping $\alpha : M \to Alb(M)$ is surjective and has connected fibers. Moreover, $h^0(M, K_M) = 0$.

It follows from this that if M is a smooth projective manifold with $K_M \cong \mathcal{O}_M$, then $q(M) \neq \dim_{\mathcal{C}}(M) - 1$. Therefore, in considering elliptic 3-folds $\pi : X \to S$ with $K_X \cong \mathcal{O}_X$, the case $q(X) = 2$ does not occur.

In the following subsections we shall consider the cases $q(X) = 1$ and $q(X) = 3$.

§3.1 $q(X) = 1$

Given an elliptic 3-fold $\pi : X \to S$ with $K_X \cong \mathcal{O}_X$ and $q(X) = 1$, the inequality proved in Proposition 1.1.2 gives $q(S) \leq 1 \leq q(S) + p_g(S)$. Let S_{min} be a minimal model of S. We still have $q(S_{min}) \leq 1 \leq q(S_{min}) + p_g(S_{min})$ because these are birational invariants. Also, $\kappa(S_{min}) \leq 0$ by $C_{3,1}$ ([17]). By Enriques-Kodaira classification, we have the following possibilities:

(i) S_{min} is a projective K3 surface;
(ii) S_{min} is a ruled surface of genus 1;
(iii) S_{min} is a hyperelliptic surface.

Observe that $c_1^2(S_{min}) = 0$. On the other hand, Proposition 1.2.1 implies that $-K_S$ is nef, so that $c_1^2(S) \geq 0$. Thus we must have $S \cong S_{min}$. Therefore S is either (i), (ii) or (iii) listed as above.

We want to show that S cannot be a hyperelliptic surface. We start with an elementary result.

Proposition 3.1.1

Let X be a smooth projective 3-fold with $K_X \cong \mathcal{O}_X$ and $q(X) = 1$. Then the universal covering space $\widetilde{X}$ of X is biholomorphic to $\mathcal{C}\times$ a projective K3 surface. Moreover, if $\alpha : X \to Alb(X)$ is the Albanese mapping of X, then α is a holomorphic fiber bundle with constant fiber a projective K3 surface.

Proof

By a result of Matsushima ([12],Theorem 3), there exist an abelian variety A and a connected projective manifold V such that

(i) $c_1(V) = 0$ and $q(V) = 0$;

(ii) $A \times V$ is a regular covering space of X and the group of covering transformations is solvable.

Since $\dim X = 3$, we must have $A \cong$ an elliptic curve and $V \cong$ a projective K3 surface. Hence the universal covering $\widetilde{X}$ of X is biholomorphic to $\mathcal{C}\times$ a projective K3 surface.

Let $\alpha : X \longrightarrow Alb(X)$ be the Albanese mapping of X. By combining a result of Kawamata ([8], Theorem 1) and a result of Bogomolov([2], Theorem 2), α is a holomorphic fiber bundle with constant fiber S and $K_S \cong \mathcal{O}_S$. Thus S is either a projective K3 surface or an abelian surface. Let G be the identity component of the group of all holomorphic transformations of X. By an argument of Matsushima ([12], p.479), G is an elliptic curve and $G \times S$ is a finite covering space of X. If $S \cong$ abelian surface, the universal covering space of X would be biholomorphic to $\mathcal{C}^3$, which is not possible. Therefore S must be a projective K3 surface.$\diamondsuit$

From this, we have the following

Proposition 3.1.2

Let $\pi : X \longrightarrow S$ be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$ and $q(X) = 1$. Then S cannot be a hyperelliptic surface.

Proof

Suppose on the contrary that S were a hyperelliptic surface. Consider the composite $\varphi = p \circ \pi : X \longrightarrow S \longrightarrow E$, where $p : S \longrightarrow E$ is the canonical projection of S onto an elliptic curve E. It is easy to see that φ is still a fibration. We want to show that φ is just the Albanese mapping $\alpha : X \longrightarrow Alb(X)$ of X.

By the universal property of Albanese mapping, there exists a morphism $h : Alb(X) \longrightarrow E$ such that for all $x \in X$, we have $h(\alpha(x)) + a = \varphi(x)$ for some fixed $a \in E$. Notice that $Alb(X)$ is an elliptic curve, from which we conclude that h is an n-sheeted unramified covering by Hurwitz theorem, $n \geq 1$. Since both φ and α have connected fibers, we must have $n = 1$. Hence h is an isomorphism and thus $\alpha = \varphi$. It follows that φ is a holomorphic fiber bundle with constant fiber a projective K3 surface by Proposition 3.1.1. Now for any $e \in E$, $\varphi^{-1}(e) = \pi^{-1}(p^{-1}(e))$ is a K3 surface fibered over $p^{-1}(e) \cong$ elliptic curve via π, which is absurd. Therefore S cannot be a hyperelliptic surface.$\diamondsuit$

Thus we are left with possibilities (i) and (ii). Now we can prove the main theorem of this subsection.

Theorem 3.1.3

Let $\pi : X \to S$ be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$ and $q(X) = 1$. Then S is either a projective K3 surface or a ruled surface of genus 1 of the following types (in Atiyah's notations):

(i) a $\mathcal{C}^*$-bundle which comes from a decomposable rank 2 holomorphic vector bundle $V \cong \mathcal{O}_E \oplus \mathcal{L}$ over an elliptic curve E, where $\mathcal{L}$ is a line bundle on E with deg$\mathcal{L}$ $= 0$;

(ii) the A_0-bundle;

(iii) the A_{-1}-bundle.

Proof

We have seen that with the given hypothesis, S is either a projective K3 surface or a ruled surface of genus 1. In case S is a ruled surface of genus 1, we can write $p : S \cong \mathcal{P}(V) \to E$ where E is an elliptic curve and $\mathcal{P}(V)$ is the associated projective bundle of a normalized rank 2 holomorphic vector bundle V on E. Let F be a fiber of p and let C_0 be the canonical section of p with $C_0^2 = -e = \deg V$. We know that K_S is numerically equivalent to $-2C_0 - eF$. By hypothesis and Proposition 1.2.1, $-K_S$ is nef. Thus we have

$$0 \leq (-K_S) \cdot F = 2, \text{ and}$$
$$0 \leq (-K_S) \cdot C_0 = -e.$$

Also, a result of Nagata ([14]) implies that $e \geq -\text{genus}(E) = -1$. Hence $e = -1$ or 0.

If $e = -1$, then V is indecomposable and S corresponds to the A_{-1} -bundle ([6], p.377).

If $e = 0$, V may be indecomposable or decomposable. If V is indecomposable, S corresponds to the A_0-bundle. If V is decomposable, then $V \cong \mathcal{O}_E \oplus \mathcal{L}$, where $\mathcal{L}$ is a holomorphic line bundle on E and $0 = e = -\deg(\mathcal{O}_E \oplus \mathcal{L}) = -\deg\mathcal{L}$ (ibid, p.376).$\diamondsuit$

We can say something about the singular fibers of π in these cases.

Proposition 3.1.4

Let $\pi : X \to S$ be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$ and $q(X) = 1$. If S is a projective K3 surface, then π is a holomorphic fiber bundle with constant fiber an elliptic curve. If S is a ruled surface of genus 1, then the composite map $\varphi = p \circ \pi : X \to S \to E$ is a holomorphic fiber bundle with constant fiber a projective elliptic K3 surface without multiple fibers.

Proof

In case S is a projective K3 surface, the assertion follows from Bogomolov ([2], Theorem 2). In case S is a ruled surface of genus 1, by arguing exactly as in Proposition 3.1.2, we see that φ is just the Albanese mapping of X and is therefore a holomorphic fiber bundle over E with constant fiber a projective K3 surface S fibered over $\mathcal{CP}^1$. Because $K_S \cong \mathcal{O}_S$, S is an elliptic surface without multiple fibers.$\diamondsuit$

In particular, we conclude that for elliptic 3-folds $\pi : X \to S$ with $K_X \cong \mathcal{O}_X$ and $q(X) = 1$, the singular fibers of π are just those which were already classified by Kodaira([10]).

§3.2 $q(X) = 3$

In this case, we have the following result.

Theorem 3.2.1

Let $\pi : X \to S$ be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$ and $q(X) = 3$. Then S is an abelian surface and π is a holomorphic fiber bundle with constant fiber an elliptic curve.

Proof

By the inequality of Proposition 1.1.2, we have $q(S_{min}) \leq 3 \leq q(S_{min}) + p_g(S_{min})$. Also, $\kappa(S_{min}) \leq 0$ ([17]) and $c_1^2(S_{min}) \geq 0$(Proposition 1.2.1). Therefore the only possibility is $S \cong S_{min} \cong$ abelian surface. The last assertion follows from Bogomolov ([2], Theorem 2).$\diamondsuit$

§4 Construction of Examples

As we have explained in the Introduction, we shall focus on constructing examples of elliptic 3-folds $\pi : X \to S$ with $K_X \cong \mathcal{O}_X$ and $q(X) = 0$. We shall present an approach which works for almost all surfaces on the list of Theorem 2.2.17.

We begin with some preliminaries.

Proposition 4.1

Let $f : M \to N$ be a holomorphic map between complex manifolds M and N and let L be a holomorphic line bundle on N. If the linear system $|L|$ is base-point free, then so is the induced linear system $|f^*L|$.

Proof

Suppose on the contrary that $|f^*L|$ had a base-point $x \in M$. Write $y = f(x)$. For any section $s \in \Gamma(N, L)$, we would have $s(y) = s(f(x)) = (f^*s)(x) = 0$, where f^*s is the induced section of s. Thus y would be a base-point of $|L|$, a contradiction.$\diamondsuit$

Proposition 4.2

Let L_1 and L_2 be two holomorphic line bundles on a complex manifold M. If the linear systems $|L_1|$ and $|L_2|$ are base-point free, then so is $|L_1 \otimes L_2|$.

Proof

Given any point x on M, there exist a section s of L_1 and a section t of L_2 such that $s(x) \neq 0$ and $t(x) \neq 0$. Then $s \otimes t$ is a section of $L_1 \otimes L_2$ and $(s \otimes t)(x) = s(x) \cdot t(x) \neq 0$. Thus $|L_1 \otimes L_2|$ is base-point free.$\diamondsuit$

Proposition 4.3

Let $L_i \to S_i$ be holomorphic line bundles over complex manifolds S_i, $i = 1, 2$. If the linear systems $|L_i|$, $i = 1, 2$, are base-point free, then so is the linear system $|p^*L_1 \otimes q^*L_2|$ on $S_1 \times S_2$, where p and q are the projections onto S_1 and S_2 respectively.

Proof

Combine Propositions 4.1 and 4.2.$\diamondsuit$

Now let L be a holomorphic line bundle on a smoooth projective surface S. If the linear system $|L|$ is base-point free, we denote by $\varphi_L : S \to \mathcal{CP}^N$ the holomorphic map defined by choosing a basis of $\Gamma(S, L)$. We need the following proposition.

Proposition 4.4

Let L_1 and L_2 be holomorphic line bundles on smooth projective surfaces S_1 and S_2 respectively, such that the linear systems $|L_1|$ and $|L_2|$ are base-point free. Denote by $L = p^*L_1 \otimes q^*L_2$ the corresponding line bundle on $S_1 \times S_2$. If the holomorphic map $\varphi_{L_1} : S_1 \to \mathcal{CP}^N$ is one to one (e.g. if $|L_1|$ separates points on S_1), then the holomorphic map given by $f = \varphi_L : S_1 \times S_2 \to \mathcal{CP}^N$ satisfies $\dim f(S_1 \times S_2) \geq 2$.

Proof

We have $\Gamma(S_1 \times S_2, L) \cong \Gamma(S_1, L_1) \otimes \Gamma(S_2, L_2)$. Let $\{s_i | i = 1, \cdots, m\}$ be a basis of $\Gamma(S_1, L_1)$ and let $\{t_j | j = 1, \cdots, n\}$ be a basis of $\Gamma(S_2, L_2)$. Fix a point $y \in S_2$. For each t_j, either $t_j(y) = 0$ or $t_j(y) = a_j \in \mathcal{C} \setminus \{0\}$. Consider the sections $s_i \otimes t_j|_{S_1 \times \{y\}} = s_i(x)t_j(y)$, $x \in S_1$. We may re-arrange indices such that $t_1(y) = 0, \cdots, t_p(y) = 0, t_{p+1}(y) = a_{p+1} \neq 0, \cdots, t_n(y) = a_n \neq 0$. Then on $S_1 \times \{y\}$, the sections $\{s_i \otimes t_j\}_{i,j}$ becomes $[0 : \cdots : 0; a_{p+1}s_1 : \cdots : a_{p+1}s_m; \cdots ; a_n s_1 : \cdots : a_n s_m]$. Hence the map $f|_{S_1 \times \{y\}} : S_1 \times \{y\} \to \mathcal{CP}^N$ takes values in $\mathcal{CP}^{(n-p)m-1}$ by forgetting about the zeros. If we can show that $f|_{S_1 \times \{y\}}$ is one-to-one, then we will have $\dim f(S_1 \times S_2) \geq \dim f(S_1 \times \{y\}) \geq 2$.

Suppose on the contrary that $f|_{S_1 \times \{y\}}$ were not one-to-one. Then there would exist distinct points $x, \widetilde{x} \in S_1$ such that (x, y) and $(\widetilde{x}, y)$ had the same image in $\mathcal{CP}^{(n-p)m-1}$ under $f|_{S_1 \times \{y\}}$. Hence there would exist $\eta \neq 0$ such that $s_i(\widetilde{x}) = \eta s_i(x)$ for all $i = 1, \cdots, m$, which would imply that φ_{L_1} is not one-to-one, a contradiction.$\diamondsuit$

Using this, we immediately have the following result.

Theorem 4.5

Let S_1 be a rational surface with $-K_{S_1}$ very ample and let S_2 be a rational surface with $|-K_{S_2}|$ base-point free. Then a general divisor X in the linear system $|p^*(-K_{S_1}) \otimes q^*(-K_{S_2})|$ is a Calabi-Yau 3-fold. Denote by $i : X \to S_1 \times S_2$ the inclusion map. Then the composite map $\pi_1 = p \circ i$ (resp. $\pi_2 = q \circ i$) is an elliptic 3-fold X fibered over S_1 (resp. S_2) with $K_X \cong \mathcal{O}_X$ and $q(X) = 0$.

Proof

Given the hypothesis of the theorem, we conclude from Proposition 4.4 and Bertini theorem that a general divisor X in the linear system $|p^*(-K_{S_1})\otimes q^*(-K_{S_2})|$ is a connected smooth projective manifold. As $K_{S_1\times S_2}\cong p^*(K_{S_1})\otimes q^*(K_{S_2})$, $K_X\cong \mathcal{O}_X$ follows from the adjunction formula. We have an exact sequence

$0\to \mathcal{O}_{S_1\times S_2}(-X)\to \mathcal{O}_{S_1\times S_2}\to \mathcal{O}_X\to 0$ on $S_1\times S_2$.

Check that $\mathcal{O}_{S_1\times S_2}(-X)\cong K_{S_1\times S_2}$. The corresponding long exact sequence of cohomology groups is

$\cdots\to H^1(S_1\times S_2,\mathcal{O}_{S_1\times S_2})\to H^1(X,\mathcal{O}_X)\to H^2(S_1\times S_2,K_{S_1\times S_2})\to\cdots$.

Since both S_1 and S_2 are rational, we conclude from Künneth formula that both $H^1(S_1\times S_2,\mathcal{O}_{S_1\times S_2})$ and $H^2(S_1\times S_2,K_{S_1\times S_2})$ vanish. Hence $H^1(X,\mathcal{O}_X)=0$ and therefore X is a Calabi-Yau 3-fold.

We now prove that $\pi_1: X\to S_1$ is a fibration. The proof for π_2 is similar. We will use the notations established in the proof of Proposition 4.4. Holomorphicity and properness of π_1 are obvious. For any point $p\in S_1$, $\pi_1^{-1}(p)=(\{p\}\times S_2)\cap X$ is connected since X is connected. Hence π_1 has connected fibers. To show that π_1 is surjective, we suppose that the contrary were true. Then there would exist some point $p\in S_1$ such that $\pi_1^{-1}(p)=(\{p\}\times S_2)\cap X$ is empty. Since X is the zero set of a section $s\in\Gamma(S_1\times S_2,p^*(-K_{S_1})\otimes q^*(-K_{S_2}))$, this would mean that $s(p,y)\neq 0$ for all $y\in S_2$. Write $s=\sum_{i,j}a_{ij}s_i\otimes t_j$. Then, on $\{p\}\times S_2$,

$$\begin{aligned}0\neq s(p,y)&=\sum_{i,j}a_{ij}s_i(p)t_j(y)\\&=\sum_j b_jt_j(y),\end{aligned}$$

where $b_j=\sum_i a_{ij}s_i(p)$. Notice that not all b_j are zero because the left-hand side is not zero. Thus $\sum_j b_jt_j$ would be a non-trivial section of $-K_{S_2}$, which does not vanish at any point y on S_2. Thus $-K_{S_2}$ would be a trivial line bundle. This is not possible because S_2 is rational.◇

In order that this theorem may be useful, we need to make sure that there exist rational surfaces whose anticanonical system is base-point free. This is the content of the following proposition.

Proposition 4.6 (Demazure [4], p.55)

Let S be a projective surface obtained by blowing up r points in almost general position on $\mathcal{CP}^2$, $0 \le r \le 7$. Then $|-K_S|$ is base-point free.

Proof

By Theorem 2.2.5, $|-K_S|$ contains a smooth irreducible curve C. By adjunction fromula, genus$(C) = g(C) = 1$. Consider the linear system $|-K_S|C|$ on C. We have $\deg(-K_S|C) = (-K_S) \cdot C = 9 - r \ge 2 = 2g(C)$, using $0 \le r \le 7$. Therefore $|-K_S|C|$ has no base- points ([6], p.308, Corollary 3.2(a)).

From the exact sequence

$0 \to \mathcal{O}_S(-C - K_S) \to \mathcal{O}_S(-K_S) \to \mathcal{O}_C(-K_S) \to 0$, we have the long exact sequence

$\cdots \to H^0(S, \mathcal{O}_S(-K_S)) \to H^0(C, -K_S|C) \to H^1(S, \mathcal{O}_S(-C - K_S))$. As $C \sim -K_S$ and S is rational, $H^1(S, \mathcal{O}_S(-C - K_S))$ vanishes. Therefore the restriction map $H^0(S, \mathcal{O}_S(-K_S)) \to H^0(C, -K_S|C)$ is surjective.

Now if $p \in S$ were a base-point of $|-K_S|$, p would be contained in C by definition. But every section of $-K_S|C$ on C extends to a section of $-K_S$ on S, so that $p \in C$ would be a base-point of $-K_S|C$, a contradiction.$\diamondsuit$

It is well-known that if S is a projective surface obtained by blowing up r points in general position on $\mathcal{CP}^2$, $0 \le r \le 6$, then $-K_S$ is very ample. The surface $\mathcal{CP}^1 \times \mathcal{CP}^1$ also has very ample anticanonical bundle. In addition, the anticanonical system of Σ_2 is base-point free. Therefore, Theorem 4.5 and Proposition 4.6 enable us to construct numerous examples of elliptic 3-folds $\pi : X \to S$ with $K_X \cong \mathcal{O}_X$ and $q(X) = 0$, where S is $\mathcal{CP}^1 \times \mathcal{CP}^1$, Σ_2 or blow-ups of $\mathcal{CP}^2$ at r points in almost general position, $0 \le r \le 7$. We remark that elliptic 3-folds constructed in this way have topological Euler numbers $e(X) = -2(12 - e(S_1))(12 - e(S_2))$, as a simple computation with Chern classes shows.

For projective surfaces S_8 obtained by blowing up $\mathcal{CP}^2$ at 8 points in almost general position, we have seen that $|-K_{S_8}|$ has a unique base-point s_0. Thus the above construction cannot be applied directly. We get around this difficulty by blowing up S_8 at s_0, obtaining a rational surface S_9. We have proved that $|-K_{S_9}|$ is base-point free (Propostion 2.2.8). Therefore the above construction applies to give examples of elliptic 3-folds $\pi : X \to S_9$ with $K_X \cong \mathcal{O}_X$ and $q(X) = 0$. Let $\sigma : S_9 \to S_8$ be the blow-up map. Then the composite $\sigma \circ \pi : X \to S_8$ will be an elliptic 3-fold with $K_X \cong \mathcal{O}_X$ and $q(X) = 0$ fibered over S_8.

It remains to treat those surfaces obtained by blowing up S_8 at a point s of S_8 distinct from s_0. Let $\sigma : S_9 \to S_8$ be such a blow-up. Denote by $\widehat{C}$ the strict transform of the unique curve $C \in |-K_{S_8}|$ containing s. With these notations, we have the following observation.

Proposition 4.7

$|-K_{S_9}|$ is base-point free iff $N_{\widehat{C}}$ is trivial, where $N_{\widehat{C}}$ is the normal bundle of $\widehat{C}$ in S_9.

Proof

Write $\widehat{C} = \sum_i n_i C_i$. By Remark 2.2.13, $\widehat{C}$ is an indecomposable curve of canonical type. Consider the restriction of $N_{\widehat{C}}$ to each irreducible component C_i of $\widehat{C}$. We have

$$\begin{aligned}\deg(N_{\widehat{C}} \otimes \mathcal{O}_{C_i}) &= \deg(\mathcal{O}_{\widehat{C}}(\widehat{C}) \otimes \mathcal{O}_{C_i}) \\ &= \deg(\mathcal{O}_{S_9}(\widehat{C}) \otimes \mathcal{O}_{C_i}) \\ &= \widehat{C} \cdot C_i = 0.\end{aligned}$$

Therefore, by a result of Mumford ([13], p.332), $N_{\widehat{C}}$ is trivial if and only if $h^0(\widehat{C}, N_{\widehat{C}})$ is non-zero.

Now suppose that $|-K_{S_9}|$ is base-point free. If $h^0(S_9, -K_{S_9}) = 1$, $-K_{S_9}$ would have a nowhere vanishing section which would imply that $-K_{S_9}$ is trivial, a contradiction. Therefore $h^0(S_9, -K_{S_9}) \geq 2$ in view of Proposition 2.2.1. From the short exact sequence $0 \to \mathcal{O}_{S_9} \to \mathcal{O}_{S_9}(\widehat{C}) \to N_{\widehat{C}} \to 0$, we have

$0 \to H^0(S_9, \mathcal{O}_{S_9}) \to H^0(S_9, \mathcal{O}_{S_9}(\widehat{C})) \to H^0(\widehat{C}, N_{\widehat{C}}) \to 0$ because S_9 is rational. Therefore

$$\begin{aligned}h^0(\widehat{C}, N_{\widehat{C}}) &= h^0(S_9, \mathcal{O}_{S_9}(\widehat{C})) - 1 \\ &= h^0(S_9, -K_{S_9}) - 1 \geq 1,\end{aligned}$$

as $\widehat{C} \sim -K_{S_9}$. Hence $N_{\widehat{C}}$ is trivial.

Conversely, suppose that $N_{\widehat{C}}$ is trivial, then $h^0(\widehat{C}, N_{\widehat{C}}) = 1$ because $\widehat{C}$ is connected. Notice that $N_{\widehat{C}} \sim -K_{S_9|\widehat{C}}$ as $\widehat{C} \sim -K_{S_9}$. Therefore the restriction map $H^0(S_9, -K_{S_9}) \to H^0(\widehat{C}, -K_{S_9}|\widehat{C})$ is surjective by the exact sequence above. If $|-K_{S_9}|$ had a base-point $b \in S_9$, b would be contained in $\widehat{C}$ by definition. For any non-trivial section $\hat{w}$ of $-K_{S_9}|\widehat{C}$, there exists a non-trivial section w of $-K_{S_9}$ such that w restricts to $\hat{w}$ on $\widehat{C}$. Therefore $\hat{w}(b) = w(b) = 0$. But this is not possible since $-K_{S_9}|\widehat{C} \sim N_{\widehat{C}}$ and $N_{\widehat{C}}$ is trivial by hypothesis. Thus $|-K_{S_9}|$ is base-point free.$\diamondsuit$

For such S_9, $\kappa^{-1}(S_9) \geq 0$ because we always have $h^0(S_9, -K_{S_9}) \geq 1$ (Proposition 2.2.1). On the other hand, since $-K_{S_9}$ is nef and $(-K_{S_9})^2 = c_1^2(S_9) = 0$, $\kappa^{-1}(S_9) < 2$ ([15], p.105). Hence $\kappa^{-1}(S_9) = 0$ or 1. If fact, we have ([16], p.407)

$$\kappa^{-1}(S_9) = \begin{cases} 0, & \text{if } N_{\widehat{C}} \text{ is not a torsion element in } Pic(\widehat{C}) \\ 1, & \text{if } N_{\widehat{C}} \text{ is a torsion element in } Pic(\widehat{C}). \end{cases}$$

Unfortunately our construction does not apply to these S_9. It is not known whether there exist elliptic 3-folds X fibered over them with $K_X \cong \mathcal{O}_X$ and $q(X) = 0$.

To conclude,we have shown that elliptic 3-folds $\pi : X \longrightarrow S$ with $K_X \cong \mathcal{O}_X$ and $q(X) = 0$ exist for all surfaces S listed in our classification theorem 2.2.17 except for those S_9's obtained by blowing up S_8 at points $s \in S_8 \setminus \Lambda$ distinct from s_0.

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