GRADED LAGRANGIAN SUBMANIFOLDS

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1. Introduction

Floer theory assigns, in favourable circumstances, an abelian group $HF(L_0, L_1)$ to a pair (L_0, L_1) of Lagrangian submanifolds of a symplectic manifold (M, ω). This group is a qualitative invariant, which remains unchanged under suitable deformations of L_0 or L_1. Following Floer [7] one can equip $HF(L_0, L_1)$ with a canonical relative $\mathbb{Z}/N$-grading, where $1 \leq N \leq \infty$ is a number which depends on (M, ω), L_0 and L_1 (for $N = \infty$ we set $\mathbb{Z}/N = \mathbb{Z}$). *Relative* mostly means that the grading is unique up to an overall shift, although there are also cases with more complicated behaviour. In this paper we take a different approach to the grading: we consider Lagrangian submanifolds equipped with certain extra structure (these are what we call graded Lagrangian submanifolds). This extra structure removes the ambiguity and defines an *absolute* $\mathbb{Z}/N$-grading on Floer cohomology. There is also a parallel notion of graded symplectic automorphism, which bears the same relation to the corresponding version of Floer theory. Both concepts were first discovered by Kontsevich (at least for $N = \infty$). Somewhat later, the present author came upon them independently.

One way to approach the definition of graded Lagrangian submanifold is to start with the case $N = 2$. It is well-known that orientations of L_0 and L_1 determine an absolute $\mathbb{Z}/2$-grading $HF(L_0, L_1) = HF^0(L_0, L_1) \oplus HF^1(L_0, L_1)$. One can reformulate this as follows: consider the natural fibre bundles $\mathcal{L}, \mathcal{L}^{or} \longrightarrow M$ whose fibres are the unoriented resp. oriented Lagrangian Grassmannians of the tangent spaces TM_x. Any Lagrangian submanifold L comes with a canonical section $s_L : L \longrightarrow \mathcal{L}|L$, and an orientation of L is the same as a lift of this section to $\mathcal{L}^{or}$. Hence the right objects for a Floer theory with an absolute $\mathbb{Z}/2$-grading are pairs $(L, \tilde{L})$ consisting of a Lagrangian submanifold and a lift $\tilde{L} : L \longrightarrow \mathcal{L}^{or}$ of s_L. In order to define the absolute $\mathbb{Z}/N$-grading one proceeds in the same way, only that $\mathcal{L}^{or}$ must be replaced by a $\mathbb{Z}/N$-covering of $\mathcal{L}$ of a certain kind. Such coverings, which we call Maslov coverings, need not exist in general, and they are also not unique. In fact, choosing an N-fold Maslov covering is equivalent to lifting the structure group of TM from $\mathrm{Sp}(2n)$ to a certain finite extension $\mathrm{Sp}^N(2n)$; and the particularly simple situation for $N = 2$ is due to the fact that $\mathrm{Sp}^2(2n) \cong \mathrm{Sp}(2n) \times \mathbb{Z}/2$.

In itself this 'graded symplectic geometry' is not particularly deep, but it does make Floer cohomology into a more powerful invariant. To put it bluntly, the advantage of the new framework is this: in passing to graded Lagrangian submanifolds there is a choice of $\mathbb{Z}/N$ for any Lagrangian submanifold L (the choice of the lift of s_L). In comparison, if one uses only the relative grading, there is a $\mathbb{Z}/N$-ambiguity for

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any *pair* of Lagrangian submanifolds, and this greater amount of choice entails a loss of information. We illustrate this through three applications, which form the main part of this paper.

(a) Lagrangian submanifolds of $\mathbb{C}\mathrm{P}^n$. We prove that any Lagrangian submanifold $L \subset \mathbb{C}\mathrm{P}^n$ must satisfy $H^1(L;\mathbb{Z}/(2n+2)) \neq 0$ (the actual result is slightly sharper, see Theorem 3.1).

(b) Symplectically knotted Lagrangian spheres. The paper [28] provides examples of compact symplectic four-manifolds (with boundary) M with the following property: there is a family of embedded Lagrangian two-spheres $L^{(k)} \subset M$, $k \in \mathbb{Z}$, such that any two of them are isotopic as smooth submanifolds, but no two are isotopic as Lagrangian submanifolds. In such a situation we say that M contains infinitely many symplectically knotted Lagrangian two-spheres. The examples in [28] were constructed using a special class of symplectic automorphisms, called generalized Dehn twists, and the main step in the proof was a Floer cohomology computation using Pozniak's [22] Morse-Bott type spectral sequence. Both the construction and the proof can be generalized to produce Lagrangian n-spheres with the same property for all even n.

Here, using the method of graded Lagrangian submanifolds, we will first reprove the result from [28] and its generalization in a considerably simpler way. Then, by a more complicated construction, we produce similar examples of Lagrangian n-spheres for all odd $n \geq 5$. The reason why the remaining case $n = 3$ cannot be settled in the same way is topological, and seems to have nothing to do with Floer theory.

We can also improve on [28] in a different direction, by showing that suitable $K3$ and Enriques surfaces contain infinitely many symplectically knotted Lagrangian two-spheres. These are the first known examples of closed symplectic manifolds with this property. As a by-product one obtains that for these manifolds the map $\pi_0(\mathrm{Aut}(M,\omega)) \longrightarrow \pi_0(\mathrm{Diff}(M))$ has infinite kernel, sharpening a result of [26]. Unfortunately, at the present state of development in Floer theory, it is impossible for technical reasons to carry out a similar argument in dimensions > 4.

(c) Weighted homogeneous singularities. Let $p \in \mathbb{C}[x_0,\dots,x_n]$, $n \geq 1$, be a weighted homogeneous polynomial with an isolated critical point at the origin. One can introduce the Milnor fibre of p, which is a compact symplectic manifold (M^{2n},ω) with boundary, and the symplectic monodromy $f \in \mathrm{Aut}(M,\partial M,\omega)$ of the Milnor fibration. This refines the usual notion of geometric monodromy by taking into account the symplectic geometry of the situation. We will show that $[f] \in \pi_0(\mathrm{Aut}(M,\partial M,\omega))$ has infinite order whenever the sum of the weights is $\neq 1$. It is not known whether the condition on the weights is really necessary.

It should be mentioned (although this will not be used later on) that this application and the previous one are related. In fact, generalized Dehn twists are maps modelled on the monodromy of the quadratic singularity $p(x) = x_0^2 + \dots + x_n^2$, and the construction of odd-dimensional knotted Lagrangian spheres is inspired by the monodromy of the singularity $p(x) = x_0^2 + \dots + x_{n-1}^2 + x_n^3$ of type (A_2).

The importance of 'graded symplectic geometry' for these applications varies. For (a) and (c) its role is that of a convenient language. In fact one could replace it by monodromy considerations in the style of [27] without changing the essence of the argument. For (a) there is also a more algebraic argument, based on the fact that $HF(L, L)$ is a module over the quantum cohomology $QH^*(\mathbb{C}P^n)$. The situation in (b) is different, since the 'graded' framework allows us to state a basic geometric property of generalized Dehn twists (Lemma 5.7) which it seems hard to encode in any other way.

Notation and conventions. All manifolds are usually assumed to be connected. The automorphism group of a symplectic manifold (M, ω) will be denoted by $\mathrm{Aut}(M, \omega)$. If M is compact, we equip this group with the C^∞-topology. If M is a symplectic manifold with nonempty boundary, $\mathrm{Aut}(M, \partial M, \omega) \subset \mathrm{Aut}(M, \omega)$ is the subgroup of automorphisms ϕ which are equal to the identity on some neighbourhood (depending on ϕ) of the boundary. Lagrangian submanifolds are always assumed to be compact; if the symplectic manifold has a boundary, any Lagrangian submanifold is assumed to lie in the interior. $\mathrm{Lag}(M, \omega)$ stands for the space of Lagrangian submanifolds of M, with the C^∞-topology. S^1 will often be identified with $\mathbb{R}/\mathbb{Z}$. (Co)homology groups have $\mathbb{Z}$-coefficients unless otherwise stated.

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2. Basic notions

2a. **Linear algebra.** By a $\mathbb{Z}/N$-covering ($1 \leq N \leq \infty$) of a space X we mean a covering X^N with covering group $\mathbb{Z}/N$. Such coverings are classified up to isomorphism by $H^1(X; \mathbb{Z}/N)$. For connected X, this correspondence associates to a homomorphism $\pi : \pi_1(X) \longrightarrow \mathbb{Z}/N$ the covering $X^N = \tilde{X} \times_\pi \mathbb{Z}/N$, where $\tilde{X}$ is the universal cover. If X is a connected Lie group, all $\mathbb{Z}/N$-coverings of it (even the non-connected ones) have canonical Lie group structures.

Let (V^{2n}, β) be a symplectic vector space, $\mathrm{Sp}(V, \beta)$ the linear symplectic group, and $\mathcal{L}(V, \beta)$ the Lagrangian Grassmannian, which parametrizes linear Lagrangian subspaces of V. Both $\mathrm{Sp}(V, \beta)$ and $\mathcal{L}(V, \beta)$ are connected with infinite cyclic fundamental group. Moreover, there are preferred generators $\delta(V, \beta) \in H^1(\mathrm{Sp}(V, \beta))$ and $C(V, \beta) \in H^1(\mathcal{L}(V, \beta))$ (the second one is called the Maslov class) so that one can canonically identify the fundamental groups with $\mathbb{Z}$. $\mathrm{Sp}(V, \beta)$ acts transitively on $\mathcal{L}(V, \beta)$, and any orbit is a map $\mathrm{Sp}(V, \beta) \longrightarrow \mathcal{L}(V, \beta)$ which takes $C(V, \beta)$ to $2\delta(V, \beta)$. For $1 \leq N \leq \infty$, let $\mathcal{L}^N(V, \beta)$ be the $\mathbb{Z}/N$-covering of $\mathcal{L}(V, \beta)$ which corresponds to the image of $C(V, \beta)$ in $H^1(\mathcal{L}(V, \beta); \mathbb{Z}/N)$. The $\mathbb{Z}/N$-action on $\mathcal{L}^N(V, \beta)$ will be denoted by ρ. Define $\mathrm{Sp}^N(V, \beta)$ to be the group of pairs $(\Phi, \tilde{\Phi})$ consisting of $\Phi \in \mathrm{Sp}(V, \beta)$ and a $\mathbb{Z}/N$-equivariant diffeomorphism $\tilde{\Phi}$ of $\mathcal{L}^N(V, \beta)$, which is a lift of the action of Φ on $\mathcal{L}(V, \beta)$. This is a Lie group and fits into an exact sequence

$$1 \longrightarrow \mathbb{Z}/N \longrightarrow \mathrm{Sp}^N(V, \beta) \longrightarrow \mathrm{Sp}(V, \beta) \longrightarrow 1,$$

where $\mathbb{Z}/N$ is the central subgroup of pairs $(\Phi,\tilde{\Phi}) = (\mathrm{id},\rho(k))$. It follows that $\mathrm{Sp}^N(V,\beta)$ must be isomorphic to some $\mathbb{Z}/N$-covering of $\mathrm{Sp}(V,\beta)$. The next Lemma identifies that covering.

Lemma 2.1. *$\mathrm{Sp}^N(V,\beta)$ is isomorphic (as a Lie group) to the $\mathbb{Z}/N$-covering of $\mathrm{Sp}(V,\beta)$ associated to the image of $2\delta(V,\beta)$ in $H^1(\mathrm{Sp}(V,\beta);\mathbb{Z}/N)$.*

Proof. Let $\widetilde{\mathrm{Sp}}(V,\beta)$ be the universal cover of $\mathrm{Sp}(V,\beta)$. Take a loop $\phi : [0;1] \longrightarrow \mathrm{Sp}(V,\beta)$ with $\phi(0) = \phi(1) = \mathrm{id}$ and $\langle \delta(V,\beta), [\phi]\rangle = 1$, and let $\tilde{\phi}(1) \in \widetilde{\mathrm{Sp}}(V,\beta)$ the endpoint of the lift $\tilde{\phi}$ of ϕ with $\tilde{\phi}(0) = \mathrm{id}$. The action of $\mathrm{Sp}(V,\beta)$ on $\mathcal{L}(V,\beta)$ can be lifted uniquely to an action of $\widetilde{\mathrm{Sp}}(V,\beta)$ on $\mathcal{L}^N(V,\beta)$. This action commutes with the $\mathbb{Z}/N$-action ρ, and $\tilde{\phi}(1)$ acts in the same way as $\rho(2)$. Therefore one obtains a homomorphism

$$\widetilde{\mathrm{Sp}}(V,\beta) \times_\pi \mathbb{Z}/N \longrightarrow \mathrm{Sp}^N(V,\beta),$$

where $\pi : \pi_1(\mathrm{Sp}(V,\beta)) = \mathbb{Z} \longrightarrow \mathbb{Z}/N$ is multiplication by two. It is not difficult to see that this is an isomorphism, which proves the desired result. $\square$

As an example consider the case $N = 2$. One can identify $\mathcal{L}^2(V,\beta)$ with the oriented Lagrangian Grassmannian $\mathcal{L}^{or}(V,\beta)$. Since $\mathrm{Sp}(V,\beta)$ acts naturally on $\mathcal{L}^{or}(V,\beta)$ one has $\mathrm{Sp}^2(V,\beta) \cong \mathbb{Z}/2 \times \mathrm{Sp}(V,\beta)$. More generally, one can try to compare $\mathrm{Sp}^N(V,\beta)$ with the more obvious covering $\mathrm{Sp}^N(V,\beta)'$ of $\mathrm{Sp}(V,\beta)$ obtained from the mod N reduction of $\delta(V,\beta)$. One finds that $\mathrm{Sp}^N(V,\beta)' \cong \mathrm{Sp}^N(V,\beta)$ if N is finite and odd, and that

$$\mathrm{Sp}^{2N}(V,\beta) \cong \mathrm{Sp}^N(V,\beta)' \times_{\mathbb{Z}/N} \mathbb{Z}/2N. \tag{2.1}$$

In particular, $\mathrm{Sp}^N(V,\beta)$ is connected iff N is finite and odd, and has two connected components otherwise.

Let J be a β-compatible complex structure on V, and g the corresponding inner product. Recall that the unitary group $U(V,J,g) \subset \mathrm{Sp}(V,\beta)$ is a deformation retract, and that $\delta(V,\beta)$ is represented by the determinant $U(V,J,g) \longrightarrow S^1$. Let $U^N(V,J,g)$ be the $\mathbb{Z}/N$-covering of $U(V,J,g)$ determined by the mod N reduction of $2\delta(V,\beta)$. These coverings are clearly deformation retracts of $\mathrm{Sp}^N(V,\beta)$, and they are explicitly given by

$$U^N(V,J,g) = \begin{cases} \{(\Phi,q) \in U(V,J,g) \times S^1 \mid \det(\Phi)^2 = q^N\} & \text{for } N < \infty, \\ \{(\Phi,t) \in U(V,J,g) \times \mathbb{R} \mid \det(\Phi)^2 = e^{2\pi i t}\} & \text{for } N = \infty. \end{cases} \tag{2.2}$$

In future we will abbreviate $\mathcal{L}(\mathbb{R}^{2n},\omega_{\mathbb{R}^{2n}})$ by $\mathcal{L}(2n)$. Similarly we will write $C(2n)$, $\mathcal{L}^N(2n)$, $\mathrm{Sp}(2n)$, $\mathrm{Sp}^N(2n)$, and $U^N(n)$.

2b. **Graded symplectic geometry.** Let (M^{2n},ω) be a symplectic manifold, possibly with boundary. Let $P \longrightarrow M$ be the principal $\mathrm{Sp}(2n)$-bundle associated to the symplectic vector bundle (TM,ω), and $\mathcal{L} \longrightarrow M$ the natural fibre bundle whose fibres are the Lagrangian Grassmannians $\mathcal{L}_x = \mathcal{L}(TM_x,\omega_x)$. One can identify $\mathcal{L} = P \times_{\mathrm{Sp}(2n)} \mathcal{L}(2n)$.

(a) An $\mathrm{Sp}^N(2n)$-structure ($1 \leq N \leq \infty$) on M is a principal $\mathrm{Sp}^N(2n)$-bundle $P^N \longrightarrow M$ together with an isomorphism $P^N \times_{\mathrm{Sp}^N(2n)} \mathrm{Sp}(2n) \cong P$.

(b) An *N-fold Maslov covering* is a $\mathbb{Z}/N$-covering $\mathcal{L}^N \longrightarrow \mathcal{L}$ whose restriction to $\mathcal{L}_x = \mathcal{L}(TM_x, \omega_x)$, for any $x \in M$, is isomorphic to $\mathcal{L}^N(TM_x, \omega_x)$. The $\mathbb{Z}/N$-action on $\mathcal{L}^N$ will always be denoted by ρ.
(c) A global Maslov class mod N is a class $C^N \in H^1(\mathcal{L}; \mathbb{Z}/N)$ whose restriction to any fibre is the mod N reduction of the ordinary Maslov class.

There are canonical bijections between (isomorphism classes of) these three kinds of objects. If P^N is an $\mathrm{Sp}^N(2n)$-structure then the associated fibre bundle with fibre $\mathcal{L}^N(2n)$ is an N-fold Maslov covering. Conversely, in the presence of a Maslov covering $\mathcal{L}^N$, the transition maps of any system of local trivializations of (TM, ω) have canonical lifts from $\mathrm{Sp}(2n)$ to $\mathrm{Sp}^N(2n)$ which satisfy the cocycle condition, hence define an $\mathrm{Sp}^N(2n)$-structure. The connection between Maslov coverings and global Maslov classes is obvious. Now assume that we have chosen an ω-compatible almost complex structure J on M, and consider the line bundle $\Delta = \Lambda^n(TM, J)^{\otimes 2}$.

(d) An N-th root ($1 \leq N < \infty$) of Δ is a complex line bundle Z together with an isomorphism $r : Z^{\otimes N} \longrightarrow \Delta$. Two pairs (Z, r), (Z', r') are called equivalent if there is an isomorphism $j : Z \longrightarrow Z'$ such that $r'j = r$. In addition, we define an ∞-th root to be a trivialization of Δ, and two of them are called equivalent iff they are homotopic.

There is a canonical bijection between $\mathrm{Sp}^N(2n)$-structures and equivalence classes of N-th roots of Δ; it is defined as follows. Let P_U be the principal $U(n)$-bundle associated to (TM, ω, J). A $U^N(n)$-structure on M is a principal $U^N(n)$-bundle together with an isomorphism of the associated $U(n)$-bundle with P_U. Because $U^N(n) \subset \mathrm{Sp}^N(2n)$ is a deformation retract, there is a canonical bijection between $U^N(n)$-structures and $\mathrm{Sp}^N(2n)$-structures. On the other hand, by looking at (2.2) one sees that a $U^N(n)$-structure is just a choice of N-th root of Δ. Among the equivalent notions (a)–(d) we will most frequently work with Maslov coverings, since that is convenient for dealing with Lagrangian submanifolds.

Lemma 2.2. *(M^{2n}, ω) admits an N-fold Maslov covering iff $2c_1(M, \omega)$ goes to zero in $H^2(M; \mathbb{Z}/N)$. The isomorphism classes of such coverings (provided that any exist) form an affine space over $H^1(M; \mathbb{Z}/N)$.*

Proof. This is immediate if one uses an almost complex structure and the description (d). Alternatively one can use (a) and an argument based on the exact sequence

$$0 \to H^1(M; \mathbb{Z}/N) \to H^1(M; \mathrm{Sp}^N(2n)) \to H^1(M; \mathrm{Sp}(2n)) \to H^2(M; \mathbb{Z}/N)$$

of non-abelian cohomology groups, just as in the classification of spin structures in [14, Appendix A]. □

Let $\mathcal{L}^N$ be an N-fold Maslov covering on (M, ω). For every Lagrangian submanifold $L \subset M$ there is a natural section $s_L : L \longrightarrow \mathcal{L}|L$, $s_L(x) = TL_x \in \mathcal{L}(TM_x, \omega_x)$. An *$\mathcal{L}^N$-grading* of L is a lift $\tilde{L} : L \longrightarrow \mathcal{L}^N$ of s_L. The pair $(L, \tilde{L})$ is called an *$\mathcal{L}^N$-graded Lagrangian submanifold.* We write $\mathrm{Lag}^{\mathrm{gr}}(M, \omega; \mathcal{L}^N)$ for the set of such pairs, and equip it with the topology which comes from the space of compact submanifolds of $\mathcal{L}^N$ (by considering the image of $\tilde{L}$). This topology defines the notion of an isotopy of graded Lagrangian submanifolds. Clearly, if $\tilde{L}$ is an $\mathcal{L}^N$-grading of L then so is $\rho(k) \circ \tilde{L}$ for any $k \in \mathbb{Z}/N$. This defines a free $\mathbb{Z}/N$-action on $\mathrm{Lag}^{\mathrm{gr}}(M, \omega; \mathcal{L}^N)$.

Lemma 2.3. *The forgetful map* $\mathrm{Lag}^{\mathrm{gr}}(M,\omega;\mathcal{L}^N) \longrightarrow \mathrm{Lag}(M,\omega)$ *is a* $\mathbb{Z}/N$*-covering of its image. The image itself consists of those* L *such that* $s_L^*(C^N) \in H^1(L;\mathbb{Z}/N)$ *is zero, where* C^N *is the global Maslov class corresponding to* $\mathcal{L}^N$*. In particular, a Lagrangian submanifold with* $H^1(L;\mathbb{Z}/N) = 0$ *always admits an* $\mathcal{L}^N$*-grading.*

Proof. By definition, L admits an $\mathcal{L}^N$-grading iff $s_L^*(\mathcal{L}^N) \longrightarrow L$ is a trivial covering, which is equivalent to $s_L^*(C^N) = 0$. The rest is obvious. □

Let $\mathcal{L}^N$ be an N-fold Maslov covering on (M,ω) and ϕ a symplectic automorphism. There is a natural map $\phi^{\mathcal{L}} : \mathcal{L} \longrightarrow \mathcal{L}$, $\phi^{\mathcal{L}}(\Lambda) = D\phi(\Lambda)$, which covers ϕ. An $\mathcal{L}^N$-*grading* of ϕ is a $\mathbb{Z}/N$-equivariant diffeomorphism $\tilde\phi$ of $\mathcal{L}^N$ which is a lift of $\phi^{\mathcal{L}}$. The pair $(\phi,\tilde\phi)$ is called an $\mathcal{L}^N$*-graded symplectic automorphism.* Such pairs form a group which we denote by $\mathrm{Aut}^{\mathrm{gr}}(M,\omega;\mathcal{L}^N)$. If M is compact, we equip this group with the topology induced from embedding it into $\mathrm{Diff}(\mathcal{L}^N)^{\mathbb{Z}/N}$. The pairs $(\phi,\tilde\phi) = (\mathrm{id},\rho(k))$ form a central subgroup $\mathbb{Z}/N \subset \mathrm{Aut}^{\mathrm{gr}}(M,\omega;\mathcal{L}^N)$. $\mathcal{L}^N$-graded symplectic automorphisms act naturally on $\mathcal{L}^N$-graded Lagrangian submanifolds by $(\phi,\tilde\phi)(L,\tilde L) = (\phi(L),\tilde\phi\circ\tilde L\circ\phi^{-1})$.

Lemma 2.4. *Let* $\mathcal{L}^N$ *be an* N*-fold Maslov covering. The forgetful homomorphism* $\mathrm{Aut}^{\mathrm{gr}}(M,\omega;\mathcal{L}^N) \longrightarrow \mathrm{Aut}(M,\omega)$ *fits into an exact sequence*

$$1 \longrightarrow \mathbb{Z}/N \longrightarrow \mathrm{Aut}^{\mathrm{gr}}(M,\omega;\mathcal{L}^N) \longrightarrow \mathrm{Aut}(M,\omega) \xrightarrow{\partial} H^1(M;\mathbb{Z}/N).$$

Here ∂ *is not a group homomorphism, but it satisfies* $\partial(\phi\psi) = \psi^*\partial(\phi) + \partial(\psi)$*, so that its kernel is a subgroup of* $\mathrm{Aut}(M,\omega)$*. If* M *is compact then this is a sequence of topological groups, with* $\mathbb{Z}/N$ *and* $H^1(M;\mathbb{Z}/N)$ *discrete.*

Proof. By definition, a symplectic automorphism ϕ admits an $\mathcal{L}^N$-grading iff the two Maslov coverings $(\phi^{\mathcal{L}})^*(\mathcal{L}^N)$ and $\mathcal{L}^N$ are isomorphic. By Lemma 2.2 the difference between these two coverings can be measured by a class in $H^1(M;\mathbb{Z}/N)$. We define $\partial(\phi)$ to be this class. The rest is easy. □

Remark 2.5. Assume that M has nonempty boundary. Then one can consider the subgroup $\mathrm{Aut}^{\mathrm{gr}}(M,\partial M,\omega;\mathcal{L}^N) \subset \mathrm{Aut}^{\mathrm{gr}}(M,\omega;\mathcal{L}^N)$ consisting of pairs with $\phi \in \mathrm{Aut}(M,\partial M,\omega)$ and where $\tilde\phi|\mathcal{L}_x^N = \mathrm{id}$ for any $x \in \partial M$ (here $\mathcal{L}_x^N$ is the part of $\mathcal{L}^N$ which lies over $\mathcal{L}_x$). The central elements $(\mathrm{id},\rho(k))$, $k \neq 0$, do not lie in $\mathrm{Aut}^{\mathrm{gr}}(M,\partial M,\omega;\mathcal{L}^N)$. In fact one has an exact sequence (with ∂ defined in a similar way as before)

$$1 \longrightarrow \mathrm{Aut}^{\mathrm{gr}}(M,\partial M,\omega;\mathcal{L}^N) \longrightarrow \mathrm{Aut}(M,\partial M,\omega) \xrightarrow{\partial} H^1(M,\partial M;\mathbb{Z}/N).$$

The minimal Chern number N_M of (M,ω) is defined to be the positive generator of the group $\langle c_1(M), H_2(M)\rangle \subset \mathbb{Z}$. Similarly, the relative minimal Chern number N_L of a Lagrangian submanifold $L \subset M$ is the positive generator of the group $\langle 2c_1(M,L), H_2(M,L)\rangle$, where now $2c_1(M,L) \in H^2(M,L)$ is the relative first Chern class. These numbers, or variants of them, are familiar from the definition of the relative grading on Floer cohomology. Their relationship to the concepts introduced here, at least in the case $H_1(M) = 0$, is as follows.

Lemma 2.6. *A symplectic manifold* (M,ω) *with* $H_1(M) = 0$ *admits a Maslov covering* $\mathcal{L}^N$ *of order* N *iff* N *divides* $2N_M$*. Moreover, this covering is unique up to isomorphism. A Lagrangian submanifold* $L \subset M$ *admits an* $\mathcal{L}^N$*-grading iff* N *divides* N_L*.*

Proof. Because $H_1(M) = 0$, it follows from the universal coefficient sequence that $2c_1(M)$ goes to zero in $H^2(M; \mathbb{Z}/N)$ iff $\langle 2c_1(M), x\rangle$ is a multiple of N for any $x \in H_2(M)$. In view of Lemma 2.2, this proves the first part.

Take a compact oriented surface Σ and a map $w : (\Sigma, \partial\Sigma) \longrightarrow (M, L)$. The number $\langle 2c_1(M, L), [w]\rangle \in \mathbb{Z}$ can be computed as follows: choose a trivialization of the pullback $w^*\mathcal{L}$, that is to say, a fibre bundle map

$$\begin{array}{ccc} \mathcal{L}(2n) \times \Sigma & \xrightarrow{\tau} & \mathcal{L} \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{w} & M. \end{array}$$

One has $\tau^{-1} \circ s_L \circ (w|\partial\Sigma)(x) = (\lambda(x), x)$ for some map $\lambda : \partial\Sigma \longrightarrow \mathcal{L}(2n)$, and $\langle 2c_1(M, L), [w]\rangle = \langle C(2n), \lambda_*[\partial\Sigma]\rangle$.

Let $\mathcal{L}^N$ be the unique Maslov covering of some order N on (M, ω) and C^N its global Maslov class. The pullback $\tau^*(C^N) \in H^1(\mathcal{L}(2n) \times \Sigma; \mathbb{Z}/N)$ is of the form $C(2n) + y$ for some $y \in H^1(\Sigma; \mathbb{Z}/N)$. Hence in $\mathbb{Z}/N$ one has

$$\begin{aligned} &\langle s_L^*(C^N), w_*[\partial\Sigma]\rangle = \\ &= \langle C^N, (s_L \circ w|\partial\Sigma)_*[\partial\Sigma]\rangle = \langle \tau^*(C^N), (\tau^{-1} \circ s_L \circ w|\partial\Sigma)_*[\partial\Sigma]\rangle = \\ &= \langle C(2n), \lambda_*[\partial\Sigma]\rangle + \langle y, [\partial\Sigma]\rangle = \langle 2c_1(M, L), [w]\rangle. \end{aligned}$$

If $N|N_L$ then $\langle 2c_1(M, L), [w]\rangle$ is always a multiple of N. Since one can choose w in such a way that $w_*[\partial\Sigma]$ is an arbitrary class of $H_1(L)$, it follows that $s_L^*(C^N) = 0$, which means that L admits an $\mathcal{L}^N$-grading. The converse is equally simple. ☐

In future we will use the following notation. Instead of $(\phi, \tilde{\phi})$ and $(L, \tilde{L})$ we will often write only $\tilde{\phi}$ and $\tilde{L}$. The action of $\mathrm{Aut}^{\mathrm{gr}}(M, \omega; \mathcal{L}^N)$ on $\mathrm{Lag}^{\mathrm{gr}}(M, \omega; \mathcal{L}^N)$ will be written as $\tilde{\phi}(\tilde{L})$. We will denote $(\mathrm{id}, \rho(-k)) \in \mathrm{Aut}^{\mathrm{gr}}(M, \omega; \mathcal{L}^N)$ by $[k]$ and call it the k-fold shift operator. The graded Lagrangian submanifold $\rho(-k) \circ \tilde{L}$, which is obtained from $\tilde{L}$ by the action of $[k]$, will be denoted by $\tilde{L}[k]$. The similarity with homological algebra is intentional, and the sign in the definition of $[k]$ has been introduced with that in mind.

2c. **Examples.** We will now complement the basic definitions by several examples and remarks, some of which will be used later on.

Example 2.7. Since $\mathrm{Sp}^2(2n) \cong \mathrm{Sp}(2n) \times \mathbb{Z}/2$, an $\mathrm{Sp}^2(2n)$-structure is just the choice of a real line bundle ξ on M. The corresponding two-fold Maslov covering, which we denote by $\mathcal{L}^{or,\xi}$, is the space of pairs (Λ, o), where $\Lambda \in \mathcal{L}$ is a Lagrangian subspace of TM_x and o is an orientation of the vector space $\Lambda \otimes_{\mathbb{R}} \xi_x$. An $\mathcal{L}^{or,\xi}$-grading of a Lagrangian submanifold $L \subset M$ is the same as an orientation of $TL \otimes (\xi|L)$. An $\mathcal{L}^{or,\xi}$-grading of a symplectic automorphism ϕ is the same as a bundle isomorphism $\phi^*(\xi) \longrightarrow \xi$. In particular, the trivial line bundle yields a two-fold Maslov covering $\mathcal{L}^{or}$ for which a grading of a Lagrangian submanifold is just an orientation, and such that $\mathrm{Aut}^{\mathrm{gr}}(M, \omega; \mathcal{L}^{or}) \cong \mathrm{Aut}(M, \omega) \times \mathbb{Z}/2$.

Example 2.8. One can associate to any spin structure on M an $\mathrm{Sp}^4(2n)$-structure. The reason, in the notation of (2.1), is that the restriction of the universal cover of $GL^+(2n)$ to the subgroup $\mathrm{Sp}(2n)$ is again a nontrivial double cover, hence isomorphic to $\mathrm{Sp}^2(2n)'$, and that $\mathrm{Sp}^4(2n) \cong \mathrm{Sp}^2(2n)' \times_{\mathbb{Z}/2} \mathbb{Z}/4$. Note that not all $\mathrm{Sp}^4(2n)$-structures arise in this way.

Example 2.9. The following discussion relates our point of view to that of Kontsevich. Let (V,β) be a $2n$-dimensional symplectic vector space, J a compatible complex structure, and g the corresponding inner product. Set $\Delta(V,J) = \Lambda^n(V,J)^{\otimes 2}$, and let $S\Delta(V,J) \subset \Delta(V,J)$ be the unit circle (with respect to the metric induced by g). One can define a fibration with simply-connected fibres

$$\det{}^2 : \mathcal{L}(V,\beta) \longrightarrow S\Delta(V,J)$$

by $\det^2(\Lambda) = (e_1 \wedge \cdots \wedge e_n)^{\otimes 2}$, where (e_j) is any orthonormal basis of $(\Lambda, g|\Lambda)$. After choosing an element $\Theta \in \Delta(V,J)^*$ of unit length one can identify $S\Delta(V,J)$ with S^1. In this way one obtains a map $\det^2_\Theta : \mathcal{L}(V,\beta) \longrightarrow S^1$. The Maslov class $C(V,\beta)$ is equal to the pullback of the standard generator $[S^1]$. Hence $\mathcal{L}^\infty(V,\beta)$ is isomorphic to the pullback of the standard covering $\mathbb{R} \longrightarrow S^1$.

Now let (M,ω) be a symplectic manifold and J a compatible almost complex structure. Assume that $2c_1(M,\omega) = 0$, which means that $\Delta = \Lambda^n(TM,J)^{\otimes 2}$ is trivial. Choose a section Θ of Δ^* (in other words, a quadratic complex n-form) which has length one everywhere. As before this determines a map $\det^2_\Theta : \mathcal{L} \longrightarrow S^1$, and one can define an ∞-fold Maslov covering by

$$\mathcal{L}^\infty = \{(\Lambda,t) \in \mathcal{L} \times \mathbb{R} \mid \det{}^2_\Theta(\Lambda) = e^{2\pi i t}\}. \tag{2.3}$$

An $\mathcal{L}^\infty$-grading of a Lagrangian submanifold L is just a lift of the map $\det^2_\Theta \circ s_L : L \longrightarrow S^1$ to $\mathbb{R}$. This approach is particularly useful in complex geometry: let (M,ω,J) be a Calabi-Yau manifold, take a covariantly constant holomorphic n-form θ of unit length, and set $\Theta = \theta^{\otimes 2}$. A Lagrangian submanifold $L \subset M$ is called special if $(\mathrm{im}\ \theta)|L = 0$. This condition is equivalent to $\det^2_\Theta \circ s_L \equiv 1 \in S^1$. It follows that special Lagrangian submanifolds have a canonical $\mathcal{L}^\infty$-grading.

Example 2.10. Let (M,ω) be a symplectic manifold which admits a Lagrangian distribution $S \subset TM$. Then one can define an ∞-fold Maslov covering $\mathcal{L}^\infty$ simply by putting together the universal covers of $\mathcal{L}(TM_x,\omega_x)$ with base point S_x, for all x. The global Maslov class of this covering is represented by the fibrewise Maslov cycle $I = \bigcup_{x \in M}\{\Lambda \in \mathcal{L}_x \mid \Lambda \cap S_x \neq 0\}$ with its canonical co-orientation. Any Lagrangian submanifold which is either tangent or transverse to S admits an $\mathcal{L}^\infty$-grading (there is even a preferred one). Typical examples are cotangent bundles $M = T^*X$ with either the vertical distribution (tangent spaces along the fibres) or the horizontal one (with respect to some Riemannian metric on X).

Example 2.11. Let M be an oriented closed surface of genus $g \geq 2$, and ω a volume form on it. Using Lemma 2.2 one can see that (M,ω) admits many different Maslov coverings of any order which divides $4g-4$. However, in a sense there is no really good choice of covering:

Proposition 2.12. *For every Maslov covering $\mathcal{L}^N$ of order $N > 2$ on M, there is an automorphism $\phi \in \mathrm{Aut}(M,\omega)$ which does not admit an $\mathcal{L}^N$-grading.*

Proof. Choose a Maslov covering $\mathcal{L}^N$ and let C^N be the corresponding global Maslov class. One can associate to any oriented embedded curve $L \subset M$ a

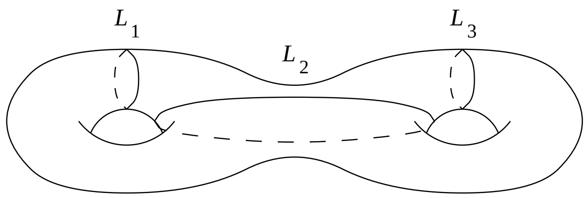

FIGURE 1.

number $R(\mathcal{L}^N, L) = \langle s_L^*(C^N), [L]\rangle \in \mathbb{Z}/N$. L admits an $\mathcal{L}^N$-grading iff it has zero rotation number, and a symplectic automorphism $\phi \in \mathrm{Aut}(M,\omega)$ admits an $\mathcal{L}^N$-grading iff $R(\mathcal{L}^N, L) = R(\mathcal{L}^N, \phi(L))$ for all curves L. We need two more facts, whose proofs we leave to the reader: (a) if $\Sigma \subset M$ is a surface whose boundary is formed by the curves $L_1, \dots, L_k$ then $\sum_j R(\mathcal{L}^N, L_j) \equiv -2\chi(\Sigma)$ mod N. (b) If t_L is the positive Dehn twist along a curve L, then $R(\mathcal{L}^N, t_L(L')) = R(\mathcal{L}^N, L') - ([L']\cdot[L])R(\mathcal{L}^N, L)$ for any L'. Now consider the curves L_k shown in fig. 1. By (a) we have $R(\mathcal{L}^N, L_1) + R(\mathcal{L}^N, L_2) + R(\mathcal{L}^N, L_3) = 2$. Hence there is a $\nu \in \{1,2,3\}$ such that $R(\mathcal{L}^N, L_\nu) \neq 0$. Take a curve L' with $[L']\cdot[L_\nu] = \pm 1$. Property (b) shows that $R(\mathcal{L}^N, t_{L_\nu}(L')) \neq R(\mathcal{L}^N, L')$, which means that t_{L_ν} does not admit an $\mathcal{L}^N$-grading. This is for $g = 2$, but one sees immediately that the argument generalizes to all $g > 2$. $\square$

In contrast, for $M = T^2$ there is exactly one ∞-fold Maslov covering $\mathcal{L}^\infty$ (namely, the one coming from the standard trivialization of TM) with the property that any $\phi \in \mathrm{Aut}(M,\omega)$ admits an $\mathcal{L}^\infty$-grading.

2d. **Two kinds of index.** Let (V,β) be a $2n$-dimensional symplectic vector space. The *Maslov index for paths* [29] [23] assigns a half-integer $\mu(\lambda_0,\lambda_1) \in \frac{1}{2}\mathbb{Z}$ to any pair of paths $\lambda_0, \lambda_1 : [a;b] \longrightarrow \mathcal{L}(V,\beta)$. $\mu(\lambda_0,\lambda_1)$ is an integer iff $\dim(\lambda_0(a)\cap\lambda_1(a)) \equiv \dim(\lambda_0(b)\cap\lambda_1(b))$ mod 2. We will now adapt this invariant to our situation. Fix some $1 \leq N \leq \infty$. Let $\tilde\Lambda_0, \tilde\Lambda_1 \in \mathcal{L}^N(V,\beta)$ be a pair of points whose images in $\mathcal{L}(V,\beta)$ intersect transversely. Choose two paths $\tilde\lambda_0, \tilde\lambda_1 : [0;1] \longrightarrow \mathcal{L}^N(V,\beta)$ with $\tilde\lambda_0(0) = \tilde\lambda_1(0)$ and $\tilde\lambda_0(1) = \tilde\Lambda_0$, $\tilde\lambda_1(1) = \tilde\Lambda_1$. Let λ_0, λ_1 be the projections of these paths to $\mathcal{L}(V,\beta)$. The *absolute Maslov index* of $(\tilde\Lambda_0, \tilde\Lambda_1)$ is defined by

$$\tilde\mu(\tilde\Lambda_0, \tilde\Lambda_1) = n/2 - \mu(\lambda_0,\lambda_1) \in \mathbb{Z}/N.$$

One can easily show that this is independent of all choices. From the standard properties of μ [23] one derives the following facts:

(i) Let $\lambda_0, \lambda_1 : [a;b] \longrightarrow \mathcal{L}(V,\beta)$ be two paths such that $\lambda_0(a)\cap\lambda_1(a) = \lambda_0(b)\cap\lambda_1(b) = 0$. Lift them to paths $\tilde\lambda_0, \tilde\lambda_1$ in $\mathcal{L}^N(V,\beta)$. Then
$$\tilde\mu(\tilde\lambda_0(b), \tilde\lambda_1(b)) - \tilde\mu(\tilde\lambda_0(a), \tilde\lambda_1(a)) = -\mu(\lambda_0,\lambda_1).$$
(ii) $\tilde\mu(\rho(k)\tilde\Lambda_0, \rho(l)\tilde\Lambda_1) = \tilde\mu(\tilde\Lambda_0, \tilde\Lambda_1) - k + l$.
(iii) For any $(\Phi, \tilde\Phi) \in \mathrm{Sp}^N(V,\beta)$ one has $\tilde\mu(\tilde\Phi(\tilde\Lambda_0), \tilde\Phi(\tilde\Lambda_1)) = \tilde\mu(\tilde\Lambda_0, \tilde\Lambda_1)$.
(iv) $\tilde\mu(\tilde\Lambda_1, \tilde\Lambda_0) = n - \tilde\mu(\tilde\Lambda_0, \tilde\Lambda_1)$.
(v) Let $\Lambda, \Lambda^\perp \subset V$ be two complementary Lagrangian subspaces. Let B be a nondegenerate quadratic form on Λ, and $A : \Lambda \longrightarrow \Lambda^\perp$ the unique linear map such that $B(v) = \beta(v, Av)$. Take the path $\lambda : [0;1] \longrightarrow \mathcal{L}(V,\beta)$ given by

$\lambda(t) = \mathrm{graph}(tA)$, and lift it to a path $\tilde{\lambda}$ in $\mathcal{L}^N(V,\beta)$. Then $\tilde{\mu}(\tilde{\lambda}(1), \tilde{\lambda}(0))$ is the Morse index of B (mod N).

(vi) For $N = 2$, if one identifies $\mathcal{L}^2(V,\beta) = \mathcal{L}^{or}(V,\beta)$ then $(-1)^{\tilde{\mu}}$ agrees with the intersection number of oriented Lagrangian subspaces up to a constant $(-1)^{n(n+1)/2}$.

The application to graded symplectic geometry is as follows. Let (M,ω) be a symplectic manifold with an N-fold Maslov covering $\mathcal{L}^N$, and $(L_0, \tilde{L}_0)$, $(L_1, \tilde{L}_1)$ a pair of $\mathcal{L}^N$-graded Lagrangian submanifolds which intersect transversally. Then one can associate to any point $x \in L_0 \cap L_1$ an absolute index mod N,

$$\tilde{I}(\tilde{L}_0, \tilde{L}_1; x) \stackrel{\mathrm{def}}{=} \tilde{\mu}(\tilde{L}_0(x), \tilde{L}_1(x)).$$

The *Conley-Zehnder index* associates to any path $\phi : [a;b] \longrightarrow \mathrm{Sp}(V,\beta)$ a half-integer $\zeta(\phi) \in \frac{1}{2}\mathbb{Z}$. It can be reduced to the Maslov index for paths as follows: take $(V', \beta') = (V, -\beta) \oplus (V, \beta)$ and consider the two paths in $\mathcal{L}(V', \beta')$ given by $\lambda_0(t) = \mathrm{graph}(\phi(t))$ and $\lambda_1(t) = \Delta$ (the diagonal). Then $\zeta(\phi) = \mu(\lambda_0, \lambda_1)$ (here we are following [23, Remark 5.4]; it seems that the definition in [25] has the opposite sign). Now let $(\Phi, \tilde{\Phi}) \in \mathrm{Sp}^N(V,\beta)$ be a point such that $\det(1-\Phi) \neq 0$. Choose a path $\tilde{\phi} : [0;1] \longrightarrow \mathrm{Sp}^N(V,\beta)$ from $(\mathrm{id}, \rho(k)) \in \mathrm{Sp}^N(V,\beta)$, for some $k \in \mathbb{Z}/N$, to $(\Phi, \tilde{\Phi})$, and project it to a path ϕ in $\mathrm{Sp}(V,\beta)$. The *absolute Conley-Zehnder index* is $\tilde{\zeta}(\Phi, \tilde{\Phi}) = n - \zeta(\phi) - k \in \mathbb{Z}/N$. This is independent of all choices and has the following properties:

(i) it is invariant under conjugation.
(ii) $\tilde{\zeta}(\Phi, \tilde{\Phi} \circ \rho(l)) = \tilde{\zeta}(\Phi, \tilde{\Phi}) - l$.
(iii) $\tilde{\zeta}(\Phi^{-1}, \tilde{\Phi}^{-1}) = 2n - \tilde{\zeta}(\Phi, \tilde{\Phi})$.
(iv) Let B be a nondegenerate quadratic form on V, and $A : V \longrightarrow V$ the linear map given by $\omega(x, Ax) = B(x)$. Take the path $\phi(t) = \exp(tA)$ in $\mathrm{Sp}(V,\beta)$ and lift it to a path $\tilde{\phi}$ in $\mathrm{Sp}^N(V,\beta)$ with $\tilde{\phi}(0) = (\mathrm{id}, \mathrm{id})$. Then $\tilde{\zeta}(\phi(t), \tilde{\phi}(t))$ is equal to the Morse index of B (mod N) for sufficiently small $t > 0$.

Given a symplectic manifold (M,ω), an N-fold Maslov covering $\mathcal{L}^N$, and an $\mathcal{L}^N$-graded symplectic automorphism $(\psi, \tilde{\psi})$ which has nondegenerate fixed points, one can associate to any fixed point x an absolute index $\tilde{Z}(\tilde{\psi}; x) = \tilde{\zeta}(D\psi_x, \tilde{\psi}(x)) \in \mathbb{Z}/N$.

2e. **Lagrangian surgery.** We will now discuss an example which shows that the absolute Maslov index appears even in elementary questions about graded Lagrangian submanifolds.

Take an embedding $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ with $\gamma(t) = t$ for $t \leq -1/2$, $\gamma(t) = it$ for $t \geq 1/2$, and $\gamma(\mathbb{R}) \cap -\gamma(\mathbb{R}) = \emptyset$. Then $H = \bigcup_{t \in \mathbb{R}} \gamma(t) S^{n-1} \subset \mathbb{C}^n$ is a Lagrangian submanifold with respect to the standard symplectic form. Outside the unit ball $B^{2n} \subset \mathbb{C}^n$ one has $H \cap (\mathbb{C}^n \setminus B^{2n}) = (\mathbb{R}^n \cup i\mathbb{R}^n) \setminus B^{2n}$. H is called a *Lagrangian handle.* It is used in the following way: let (M^{2n}, ω) be a symplectic manifold and $L_1, L_2 \subset M$ a pair of Lagrangian submanifolds which intersect transversely in a single point $\{x_0\} = L_1 \cap L_2$. There is always an embedding $j : B^{2n} \longrightarrow M$ with $j(0) = x_0$, $j^{-1}(L_1) = \mathbb{R}^n \cap B^{2n}$, $j^{-1}(L_2) = i\mathbb{R}^n \cap B^{2n}$, and $j^*\omega = \epsilon\, \omega_{\mathbb{C}^n}$ for some $\epsilon > 0$. Then one can form the embedded connected sum

$$L_1 \# L_2 = (L_1 \setminus \mathrm{im}(j)) \cup (L_2 \setminus \mathrm{im}(j)) \cup j(H \cap B^{2n})$$

which is again a Lagrangian submanifold. This process, which is independent of all choices up to Lagrangian isotopy, is usually called *Lagrangian surgery.* It has been studied by Polterovich [21] and others. Our conventions are those of [28, Appendix] and differ from Polterovich's. Note that Lagrangian surgery is not symmetric: $L_1 \# L_2$ and $L_2 \# L_1$ are the two possibilities of resolving the self-intersection of $L_1 \cup L_2$ (this can be seen clearly already in the case $n = 1$).

Take the ∞-fold Maslov covering $\mathcal{L}^\infty$ on $\mathbb{C}^n$ induced by the quadratic complex n-form $\Theta = (dz_1 \wedge \cdots \wedge dz_n)^{\otimes 2}$ as in (2.3). An $\mathcal{L}^\infty$-grading of a Lagrangian submanifold $L \subset \mathbb{C}^n$ is the same as a map $\tilde{L} : L \longrightarrow \mathbb{R}$ such that $e^{2\pi i \tilde{L}(x)} = \det^2_\Theta(TL_x)$ for all x. For $L_1 = \mathbb{R}^n$ and $L_2 = i\mathbb{R}^n$ we choose the gradings $\tilde{L}_1 \equiv 0$, $\tilde{L}_2 \equiv 1 - n/2$. Then the absolute index at the origin is $\tilde{I}(\tilde{L}_1, \tilde{L}_2; 0) = 1$.

Lemma 2.13. *There is an $\mathcal{L}^\infty$-grading $\tilde{H}$ of the Lagrangian handle H which agrees with $\tilde{L}_1$ on $\mathbb{R}^n \setminus B^{2n}$ and with $\tilde{L}_2$ on $i\mathbb{R}^n \setminus B^{2n}$.*

Proof. Take $y_0 \in S^{n-1}$, and let $y_0^\perp \subset \mathbb{R}^n$ be its orthogonal complement. The tangent space of H at a point $y = \gamma(t) y_0$ is $TH_y = \mathbb{R}\gamma'(t) y_0 \oplus \gamma(t) y_0^\perp$. Therefore

$$\det\nolimits^2_\Theta(TH_y) = \frac{\gamma'(t)^2 \gamma(t)^{2n-2}}{|\gamma'(t)^2 \gamma(t)^{2n-2}|} \in S^1. \tag{2.4}$$

As t goes from $-\infty$ to ∞, $\gamma(t)^2/|\gamma(t)|^2$ makes half a turn clockwise from 1 to -1, and $\gamma'(t)^2/|\gamma'(t)|^2$ makes half a turn counterclockwise from 1 to -1. It follows that one can find a map $\alpha \in C^\infty(\mathbb{R}, \mathbb{R})$ with $\alpha(t) = 0$ for $t \leq -1/2$ and $\alpha(t) = 1 - n/2$ for $t \geq 1/2$, such that $e^{2\pi i \alpha(t)}$ equals the r.h.s. of (2.4). Then $\tilde{H}(e^{it} y_0) = \alpha(t)$ is a grading of H with the desired property. □

From these local considerations one immediately obtains the following graded version of Lagrangian surgery.

Lemma 2.14. *Let (M, ω) be a symplectic manifold with an ∞-fold Maslov covering $\mathcal{L}^\infty$. Let $(L_1, \tilde{L}_1)$ and $(L_2, \tilde{L}_2)$ be two $\mathcal{L}^\infty$-graded Lagrangian submanifolds which intersect transversely and in a single point $x_0 \in M$. If $\tilde{I}(\tilde{L}_1, \tilde{L}_2; x_0) = 1$ then the surgery $\Sigma = L_1 \# L_2$ has an $\mathcal{L}^\infty$-grading $\tilde{\Sigma}$ which agrees with $\tilde{L}_1$ on $\Sigma \cap L_1$, and with $\tilde{L}_2$ on $\Sigma \cap L_2$.* □

2f. **Floer cohomology.** We can now introduce the absolute grading on Floer cohomology. The exposition in this section is formal, in the sense that the conditions which are necessary to make Floer cohomology well-defined will be suppressed. Our justification is that there is no relation between these conditions and the problem of grading. Concretely, this means that if the ordinary Floer cohomology $HF(L_0, L_1)$ of two Lagrangian submanifolds is defined, and L_0, L_1 admit gradings $\tilde{L}_0, \tilde{L}_1$, then the graded version $HF^*(\tilde{L}_0, \tilde{L}_1)$ is also defined. The discussion of the properties of $HF^*(\tilde{L}_0, \tilde{L}_1)$ should be understood in the same way: they hold in the same generality as their ungraded analogues.

Let L_0, L_1 be a pair of transversely intersecting Lagrangian submanifolds in a symplectic manifold (M^{2n}, ω), and $\mathbf{J} = (J_t)_{0 \leq t \leq 1}$ a smooth family of ω-compatible almost complex structures. For any two points $x_-, x_+ \in L_0 \cap L_1$, let $\mathcal{B}(x_-, x_+)$ be the set of smooth maps $u : \mathbb{R} \times [0; 1] \longrightarrow M$ with $u(\mathbb{R} \times \{j\}) \subset L_j$ for $j = 0, 1$, and $\lim_{s \to \pm\infty} u(s, \cdot) = x_\pm$. Consider the subspace $\mathcal{M}(x_-, x_+; \mathbf{J}) \subset \mathcal{B}(x_-, x_+)$ of maps which satisfy Floer's equation $\partial_s u + J_t(u) \partial_t u = 0$. It has a natural action of $\mathbb{R}$ by translation in the first variable. For generic choice of $\mathbf{J}$, $\mathcal{M}(x_-,x_+;\mathbf{J})$ has a natural structure of a smooth manifold, with connected components of different dimensions. Let $\mathcal{M}_k(x_-,x_+;\mathbf{J})$ be the k-dimension part of $\mathcal{M}(x_-,x_+;\mathbf{J})$. In the simplest situation, such as in the original work of Floer and in that of Oh [19], suitable assumptions on (M,ω) and L_0, L_1 ensure that the quotients $\mathcal{M}_1(x_-,x_+;\mathbf{J})/\mathbb{R}$ are finite sets. Then, writing $n(x_-,x_+;\mathbf{J}) \in \mathbb{Z}/2$ for the number of points mod 2 in $\mathcal{M}_1(x_-,x_+;\mathbf{J})/\mathbb{R}$, one defines a chain group $(CF(L_0,L_1),\partial_{\mathbf{J}})$ as follows: $CF(L_0,L_1)$ is the $\mathbb{Z}/2$-vector space freely generated by the points of $L_0 \cap L_1$, and $\partial_{\mathbf{J}}\langle x_+\rangle = \sum_{x_-} n(x_-,x_+;\mathbf{J})\langle x_-\rangle$. One finds that $\partial_{\mathbf{J}}^2 = 0$, and the Floer cohomology is $HF(L_0,L_1;\mathbf{J}) = \ker \partial_{\mathbf{J}}/\mathrm{im}\ \partial_{\mathbf{J}}$ (this is Floer *co*homology because we have exchanged the usual roles of x_- and x_+ in the definition of $\partial_{\mathbf{J}}$).

Let $\mathcal{L}^N$ be an N-fold Maslov covering on M, and assume that L_0, L_1 admit $\mathcal{L}^N$-gradings $\tilde{L}_0, \tilde{L}_1$. For $i \in \mathbb{Z}/N$, let $CF^i(\tilde{L}_0,\tilde{L}_1) \subset CF(L_0,L_1)$ be the subspace generated by elements $\langle x\rangle$ where $\tilde{I}(\tilde{L}_0,\tilde{L}_1;x) = i$. This defines a $\mathbb{Z}/N$-grading on $CF(L_0,L_1)$. Floer's index theorem [7] together with property (i) of the absolute Maslov index implies that $\partial_{\mathbf{J}}$ has degree one. Hence there is an induced grading on Floer cohomology. We will refer to the $\mathbb{Z}/N$-graded group $HF^*(\tilde{L}_0,\tilde{L}_1;\mathbf{J})$ as *graded Floer cohomology.*

Let $\mathbf{L} = (L_t)_{0\leq t\leq 1}$ be an exact isotopy of Lagrangian submanifolds (exact means that it can be embedded in a Hamiltonian isotopy of M) and L a Lagrangian submanifold which intersects both L_0 and L_1 transversely. For any almost complex structures $\mathbf{J}^-$, $\mathbf{J}^+$ such that $HF(L,L_0;\mathbf{J}^-)$ and $HF(L,L_1;\mathbf{J}^+)$ are defined, there is a canonical isomorphism

$$q(\mathbf{L},\mathbf{J}^-,\mathbf{J}^+) : HF(L,L_0;\mathbf{J}^-) \longrightarrow HF(L,L_1;\mathbf{J}^+). \tag{2.5}$$

Now assume that we have a Maslov covering $\mathcal{L}^N$, and that L_0 and L admit $\mathcal{L}^N$-gradings $\tilde{L}_0, \tilde{L}$. Then the isotopy (L_t) can be lifted to an isotopy $(\tilde{L}_t)$ of $\mathcal{L}^N$-graded Lagrangian submanifolds, and the map (2.5) has degree zero with respect to the gradings of $HF^*(\tilde{L},\tilde{L}_0;\mathbf{J}^-)$ and $HF^*(\tilde{L},\tilde{L}_1;\mathbf{J}^+)$. To complete the construction of graded Floer cohomology one follows the usual strategy: first, using the isomorphisms (2.5) for constant isotopies, one shows that graded Floer cohomology is independent of the choice of almost complex structure (we will therefore omit $\mathbf{J}$ from the notation from now on). Secondly, using again (2.5) but this time for C^1-small isotopies, one extends the definition of graded Floer cohomology to Lagrangian submanifolds which do not intersect transversally. Clearly, this extended definition is still invariant under exact isotopies of graded Lagrangian submanifolds (in both variables). Some other properties of graded Floer cohomology are: the shifting formula

$$HF^j(\tilde{L}_0[k],\tilde{L}_1[l]) \cong HF^{j-k+l}(\tilde{L}_0,\tilde{L}_1),$$

invariance under graded symplectic automorphisms,

$$HF^j(\tilde{\phi}(\tilde{L}_0),\tilde{\phi}(\tilde{L}_1)) \cong HF^j(\tilde{L}_0,\tilde{L}_1),$$

and Poincaré duality:

$$HF^j(\tilde{L}_1,\tilde{L}_0) \cong HF^{n-j}(\tilde{L}_0,\tilde{L}_1)^\vee.$$

These follow immediately from the properties of $\tilde{\mu}$ listed in section 2d. We also need to mention the isomorphism between $HF(L,L)$ and the ordinary cohomology

$H^*(L;\mathbb{Z}/2)$. This isomorphism holds in Floer's original setup [8], but it can fail in more general situations, see [20]. Whenever it holds, so does the graded version

$$HF^j(\tilde{L},\tilde{L}) \cong \bigoplus_{i\in\mathbb{Z}} H^{j+iN}(L;\mathbb{Z}/2).$$

The other version of Floer cohomology, that for symplectic automorphisms, will not be used in this paper. Nevertheless, it seems appropriate to outline briefly the parallel story about the grading. Let ϕ be an automorphism of (M,ω) which has non-degenerate fixed points. One considers a chain group $CF(\phi)$ which has one basis element $\langle x\rangle$ for any fixed point $x \in M$ of ϕ, and a boundary operator $\partial_{\mathbf{J}}$ defined as before, but this time using the space of maps $u : \mathbb{R}^2 \longrightarrow M$ which satisfy

$$\begin{cases} \partial_s u + J_t(u)\partial_t u = 0, \\ u(s,t) = \phi(u(s,t+1)), \end{cases}$$

where $\mathbf{J} = (J_t)_{t\in\mathbb{R}}$ is a family of compatible almost complex structures satisfying a suitable periodicity condition. If (M,ω) has a Maslov covering $\mathcal{L}^N$ and ϕ an $\mathcal{L}^N$-grading $\tilde{\phi}$ then, using the absolute Conley-Zehnder index $\tilde{\zeta}$, one can define a $\mathbb{Z}/N$-grading on $CF(\phi)$. The index theorem of [25, section 4] implies that $\partial_{\mathbf{J}}$ has degree one. Therefore one obtains a $\mathbb{Z}/N$-graded Floer cohomology group $HF^*(\tilde{\phi})$. These groups are invariant under Hamiltonian isotopies and satisfy $HF^*(\tilde{\phi}\circ[k]) = HF^{*-k}(\tilde{\phi})$, $HF^*(\tilde{\phi}^{-1}) = HF^{2n-*}(\phi)^\vee$, $HF^*(\mathrm{id}) = H^*(M)$. For the special case of monotone symplectic manifolds, a more extensive account of this kind of Floer theory (without mention of the absolute grading) can be found in [5].

3. Lagrangian submanifolds of $\mathbb{C}\mathrm{P}^n$

In this section we use graded Floer cohomology to obtain some restrictions on the topology of Lagrangian submanifolds of $\mathbb{C}\mathrm{P}^n$. Note that, by starting with the familiar embeddings of $\mathbb{R}\mathrm{P}^n$ and T^n and applying Lagrangian surgery, one can construct many different Lagrangian submanifolds of $\mathbb{C}\mathrm{P}^n$.

Theorem 3.1. *Any Lagrangian submanifold $L \subset \mathbb{C}\mathrm{P}^n$ satisfies*
(a) *$H^1(L;\mathbb{Z}/(2n+2)) \neq 0$,*
(b) *$H^1(L;\mathbb{Z}/(2n+2)) \not\cong (\mathbb{Z}/2)^g$ for any $g \geq 2$,*
(c) *if $H^1(L;\mathbb{Z}/(2n+2)) \cong \mathbb{Z}/2$ then $H^i(L;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for all $i = 0,\dots,n$.*

By Lemma 2.6, $\mathbb{C}\mathrm{P}^n$ admits a unique Maslov covering $\mathcal{L}^N$ of any order N which divides $2N_M = 2n+2$. Consider the Hamiltonian circle action given by $\sigma(t) = \mathrm{diag}(e^{2\pi it},1,\dots,1) \in U(n+1)$. One can lift σ uniquely to a map $\tilde{\sigma} : [0;1] \longrightarrow \mathrm{Aut}^{\mathrm{gr}}(M,\omega;\mathcal{L}^N)$ with $\tilde{\sigma}(0) = \mathrm{id}$. By looking at any fixed point one sees that $\tilde{\sigma}(1) = [-2]$. It follows that every $\mathcal{L}^N$-graded Lagrangian submanifold $\tilde{L}$ is graded Lagrangian isotopic to $\tilde{L}[-2]$, by an isotopy which is also exact. This implies that $HF^*(\tilde{L},\tilde{L})$, whenever defined, is periodic with period two. From this fact we will derive Theorem 3.1.

Before explaining the proof in detail, we need to recall the Floer cohomology for monotone symplectic manifolds as developed by Oh. The basic references are [19] and [20]. Technical issues are discussed further in [18] and [13]; see also [15]. We will present the theory in a slightly simplified form. Let (M^{2n},ω) be a closed symplectic manifold which is monotone, that is, $[\omega] = \lambda\, c_1(M,\omega)$ for some $\lambda > 0$. For any

Lagrangian submanifold $L \subset M$, let $N_L \geq 1$ be the number defined in section 2b. Oh shows that $HF(L_0, L_1)$ is well-defined, and invariant under Lagrangian isotopy, for all pairs (L_0, L_1) such that $N_{L_0}, N_{L_1} \geq 3$ and $H^1(L_0; \mathbb{R}) = H^1(L_1; \mathbb{R}) = 0$. Moreover one has

Theorem 3.2 (Oh [20, Theorem 5.1]). *Let $L \subset M$ be a Lagrangian submanifold with $H^1(L; \mathbb{R}) = 0$ and $N_L \geq 3$.*
(a) *If $N_L \geq n+2$ then $HF(L, L) \cong \bigoplus_i H^i(L; \mathbb{Z}/2)$.*
(b) *If $N_L = n+1$ then $HF(L, L)$ is either $\bigoplus_i H^i(L; \mathbb{Z}/2)$ or $\bigoplus_{i \neq 0,n} H^i(L; \mathbb{Z}/2)$.*

The proof of this goes as follow [20, p. 332]. Let H be a Morse function on L, and L' a small Lagrangian perturbation of L constructed using H and a Darboux chart. We may assume that H has only one local minimum x_- and local maximum x_+. The intersection points of L and L' are the critical points of H. One can write the boundary operator on $CF(L, L')$ as $\partial_{\mathbf{J}} = \partial_0 + \partial_1 + \dots$, where ∂_k takes critical points of Morse index i to those of Morse index $i + 1 - kN_L$. Floer [8] proved that, for a suitable choice of $\mathbf{J}$, ∂_0 can be identified with the boundary operator in a Morse cohomology complex for H. Therefore the homology of $(CF(L, L'), \partial_0)$, which is sometimes called the local Floer cohomology of L, is always isomorphic to $H^*(L; \mathbb{Z}/2)$. If $N_L \geq n+2$ then for dimension reasons $\partial_k = 0$ for all $k > 0$, which proves (a). If $N_L = n+1$ then $\partial_k = 0$ for $k \geq 2$, $\partial_1 \langle x \rangle = 0$ for all $x \neq x_+$, and $\partial_1 \langle x_+ \rangle$ can be either zero or $\langle x_- \rangle$. This leads to the two possibilities in (b).

Now let $\mathcal{L}^N$ be a Maslov covering of order $N \geq 3$ on M. Lemma 2.6 shows that any $\mathcal{L}^N$-graded Lagrangian submanifold $(L, \tilde{L})$ automatically satisfies $N_L \geq N \geq 3$. Hence the graded Floer cohomology groups $HF^*(\tilde{L}_0, \tilde{L}_1)$ are well-defined for $\mathcal{L}^N$-graded Lagrangian submanifolds with zero first Betti number. Moreover, as a look at the proof shows, the obvious graded analogue of Theorem 3.2 holds.

Proof of Theorem 3.1. (a) Assume that $L \subset \mathbb{C}\mathrm{P}^n$ $(n \geq 2)$ is a Lagrangian submanifold with $H^1(L; \mathbb{Z}/(2n+2)) = 0$. This implies that $H^1(L; \mathbb{R}) = 0$. By Lemma 2.3, L admits a grading $\tilde{L}$ with respect to the unique Maslov covering $\mathcal{L}^{2n+2}$ of order $2n+2$. Hence the graded Floer cohomology $HF^*(\tilde{L}, \tilde{L})$ is well-defined. Lemma 2.6 shows that $N_L \geq 2n+2$, and by applying Theorem 3.2(a) one finds that

$$HF^i(\tilde{L}, \tilde{L}) = \begin{cases} H^i(L; \mathbb{Z}/2) & 0 \leq i \leq n, \\ 0 & n+1 \leq i \leq 2n+1. \end{cases} \tag{3.1}$$

As discussed above, the circle action σ on $\mathbb{C}\mathrm{P}^n$ provides a graded Lagrangian isotopy between $\tilde{L}$ and $\tilde{L}[-2]$, which implies that $HF^*(\tilde{L}, \tilde{L}) \cong HF^*(\tilde{L}, \tilde{L}[-2]) = HF^{*-2}(\tilde{L}, \tilde{L})$. This is a contradiction, since (3.1) is not two-periodic.

(b,c) Let $L \subset \mathbb{C}\mathrm{P}^n$ $(n \geq 2)$ be a Lagrangian submanifold with $H^1(L; \mathbb{Z}/(2n+2)) = (\mathbb{Z}/2)^g$ for some $g \geq 1$. This implies that $H^1(L) = 0$, that $H^1(L; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^g$, and that the homomorphism $\rho : H^1(L; \mathbb{Z}/(2n+2)) \longrightarrow H^1(L; \mathbb{Z}/(n+1))$ induced by the projection $\mathbb{Z}/(2n+2) \longrightarrow \mathbb{Z}/(n+1)$ is zero.

Let $C^{2n+2} \in H^1(\mathcal{L}; \mathbb{Z}/(2n+2))$ and $C^{n+1} \in H^1(\mathcal{L}; \mathbb{Z}/(n+1))$ be the global Maslov classes of the Maslov coverings $\mathcal{L}^{2n+2}, \mathcal{L}^{n+1}$ on $\mathbb{C}\mathrm{P}^n$. Clearly C^{n+1} is obtained from C^{2n+2} by reducing mod $(n+1)$. We conclude that $s_L^*(C^{n+1}) = \rho(s_L^*(C^{2n+2})) = 0$, which means that L admits an $\mathcal{L}^{n+1}$-grading $\tilde{L}$. The same argument as before shows that the $\mathbb{Z}/(n+1)$-graded Floer cohomology $HF^*(\tilde{L}, \tilde{L})$ is two-periodic.

Lemma 2.6 says that $N_L \geq n+1$. By Theorem 3.2(b) there are two possibilities for the Floer cohomology. One is that $HF^*(\tilde{L},\tilde{L}) = H^*(L;\mathbb{Z}/2)$. In this case it follows that $H^*(L;\mathbb{Z}/2)$, with the grading reduced mod $n+1$, is two-periodic. Since $H^0(L;\mathbb{Z}/2) = \mathbb{Z}/2$ and $H^n(L;\mathbb{Z}/2) = \mathbb{Z}/2$, the periodicity leads to $H^i(L;\mathbb{Z}/2) = \mathbb{Z}/2$ for all i, which proves both (b) and (c). It remains to consider the other possibility, which is

$$HF^i(\tilde{L},\tilde{L}) = \begin{cases} H^i(L;\mathbb{Z}/2) & 0 < i < n, \\ 0 & i = 0, n. \end{cases}$$

This contradicts the two-periodicity, because $HF^1(\tilde{L},\tilde{L}) = H^1(L;\mathbb{Z}/2) = (\mathbb{Z}/2)^g$ while $HF^{-1}(\tilde{L},\tilde{L}) = 0$. Hence this possibility cannot occur. $\square$

4. A class of symplectic automorphisms

4a. **The basic result.** Let (M,ω,α) be a compact symplectic manifold with contact type boundary, and (ϕ_t^K) an S^1-action on ∂M which preserves α. This means that the symplectic form $\omega \in \Omega^2(M)$ and the contact one-form $\alpha \in \Omega^1(\partial M)$ are related by $d\alpha = \omega|\partial M$, and that the Reeb vector field R of α satisfies $\omega(N,R) > 0$, where N is any vector field pointing outwards along ∂M; in addition, we are given a vector field K on ∂M with $L_K\alpha = 0$ and whose flow (ϕ_t^K) is one-periodic. One can always find a collar $j : (-\epsilon;0] \times \partial M \hookrightarrow M$, for some $\epsilon > 0$, such that $j^*\omega = d(e^r\alpha)$. Choose a function $H \in C^\infty(M,\mathbb{R})$ with $H(j(r,x)) = e^r(i_K\alpha)(x)$ for all $r \geq -\epsilon/2$. Then the Hamiltonian flow (ϕ_t^H) satisfies $\phi_t^H(j(r,x)) = j(r,\phi_t^K(x))$, and in particular $\phi_{\pm 1}^H(r,x) = (r,x)$, for all $r \geq -\epsilon/2$. Set

$$\chi_K = \phi_{-1}^H \in \mathrm{Aut}(M,\partial M,\omega). \tag{4.1}$$

It is easy to see that the class $[\chi_K] \in \pi_0(\mathrm{Aut}(M,\partial M,\omega))$ is independent of the choice of H and j. Note that by taking a suitable choice, one can achieve that χ_K is the identity outside an arbitrarily small neighbourhood of ∂M. The question we are interested in is: *when is $[\chi_K]$ nontrivial, and more generally, what is its order in $\pi_0(\mathrm{Aut}(M,\partial M,\omega))$?* Answering this can be easy or difficult, depending on the specific situation. We list a few easy cases:

Examples 4.1. (a) $[\chi_K]$ is trivial whenever (ϕ_t^K) can be extended to a Hamiltonian circle action on M. The simplest example is when M is the unit ball in $\mathbb{C}^n$ and (ϕ_t^K) is any $\mathbb{C}$-linear circle action on S^{2n-1}.

(b) $[\chi_K]$ is often nontrivial for topological reasons. The simplest relevant topological invariant is the variation homomorphism $\mathrm{var}(f) : H^*(M) \longrightarrow H_*(M,\partial M)$ associated to any continuous map $f : M \longrightarrow M$ with $f|\partial M = \mathrm{id}$. If $\mathrm{var}(f) \neq 0$ then f cannot be homotoped to the identity rel ∂M. In the special case when the circle action (ϕ_t^K) is free one can compute $\mathrm{var}(\chi_K^m)$ (mod torsion) for any $m \in \mathbb{Z}$ from the formula

$$\int_M \mathrm{var}(\chi_K^m)(a) \cup b = \pm m \int_{\partial M/S^1} p(a) \cup p(b). \tag{4.2}$$

Here $a, b \in H^*(M)$, and p is the map $H^*(M) \longrightarrow H^*(\partial M) \longrightarrow H^{*-1}(\partial M/S^1)$. The sign in (4.2) depends on orientation conventions, and is irrelevant for our purpose. For example, let $H \subset \mathbb{CP}^n$ $(n \geq 2)$ be a smooth complex hypersurface of degree d. If one takes a suitable tubular neighbourhood U of H, the complement $M = \mathbb{CP}^n \setminus U$

has the structure of a symplectic manifold with contact type boundary, and ∂M is a circle bundle over H. Let χ_K be the automorphism of M obtained from the obvious circle action on ∂M. Using (4.2) one can see that $[\chi_K]$ has infinite order whenever n is odd or $d \geq 3$. On the other hand, $[\chi_K]$ is trivial for $d = 1$, see (a). The remaining case ($d = 2$ and n even) is more difficult; we will solve it later.

(c) Take M to be a compact surface with $(\partial M, \alpha) = (S^1, dt)$. Consider the standard circle action on ∂M. Then χ_K is a positive Dehn twist along a curve parallel to ∂M. $[\chi_K]$ is trivial if M is a disc, and has infinite order (for topological reasons) otherwise.

From now on assume that ∂M is connected, $H^1(M) = 0$, and that $2c_1(M, \omega) = 0$. Then there is a unique ∞-fold Maslov covering $\mathcal{L}^\infty$ on M. There are two ways to choose an $\mathcal{L}^\infty$-grading for χ_K. One way is to use Remark 2.5 which says that there is a unique grading $\tilde{\chi}_K \in \mathrm{Aut}^{\mathrm{gr}}(M, \partial M, \omega; \mathcal{L}^\infty)$. Alternatively one can lift $(\phi_t^H)_{-1 \leq t \leq 0}$ to an isotopy of $\mathcal{L}^\infty$-graded symplectic automorphisms $(\tilde{\phi}_t^H)$ with $\tilde{\phi}_0^H = \mathrm{id}$, and then $\tilde{\phi}_{-1}^H$ is again a grading of χ_K. These two approaches do not necessarily lead to the same grading. Let $\sigma_K \in \mathbb{Z}$ be the difference, that is to say

$$\tilde{\chi}_K = \tilde{\phi}_{-1}^H \circ [\sigma_K]. \tag{4.3}$$

Lemma 4.2. *Let L be a Lagrangian submanifold of M which admits an $\mathcal{L}^\infty$-grading $\tilde{L}$. If the class $[\chi_K^m] \in \pi_0(\mathrm{Aut}(M, \partial M, \omega))$ is trivial for some $m \geq 1$, then $\tilde{L}$ is isotopic to $\tilde{L}[m\sigma_K]$ as an $\mathcal{L}^\infty$-graded Lagrangian submanifold.*

Proof. Since the statement is independent of the choices made in the definition of χ_K, we can assume that the embedding j satisfies $L \cap \mathrm{im}(j) = \emptyset$. This means that $\phi_t^H(L) = L$ for all t. Since $\tilde{\phi}_0^H = \mathrm{id}$, it follows that $\tilde{\phi}_t^H(\tilde{L}) = \tilde{L}$ for all t. Because of (4.3) one has $\tilde{\chi}_K^m(\tilde{L}) = \tilde{L}[m\sigma_K]$. By assumption there is an isotopy (ψ_t) in $\mathrm{Aut}(M, \partial M, \omega)$ from $\psi_0 = \mathrm{id}$ to $\psi_1 = \chi_K^m$. One can lift this to an isotopy $(\tilde{\psi}_t)$ of $\mathcal{L}^\infty$-graded symplectic automorphisms with $\tilde{\psi}_0 = \mathrm{id}$. This isotopy will remain inside $\mathrm{Aut}^{\mathrm{gr}}(M, \partial M, \omega; \mathcal{L}^\infty)$, which implies that $\tilde{\psi}_1 = \tilde{\chi}_K^m$. Hence $\tilde{\psi}_t(\tilde{L})$ is an isotopy of $\mathcal{L}^\infty$-graded Lagrangian submanifolds from $\tilde{L}$ to $\tilde{\chi}_K^m(\tilde{L}) = \tilde{L}[m\sigma_K]$. □

Theorem 4.3. *Let (M, ω, α) be a compact symplectic manifold with contact type boundary, and (ϕ_t^K) a circle action on ∂M which preserves α. Assume that ∂M is connected, that $H^1(M) = 0$, that $2c_1(M, \omega) = 0$, and that $[\omega] \in H^2(M; \mathbb{R})$ is zero. Furthermore, assume that M contains a Lagrangian submanifold L with $H^1(L) = 0$. Let χ_K be the automorphism of M defined in (4.1), and σ_K the number from (4.3). If $\sigma_K \neq 0$ then $[\chi_K] \in \pi_0(\mathrm{Aut}(M, \partial M, \omega))$ is an element of infinite order.*

Proof. The assumption $[\omega] = 0$ implies that the Floer cohomology $HF(L_0, L_1)$ of any pair of Lagrangian submanifolds L_0, L_1 with $H^1(L_0) = H^1(L_1)$ is well-defined and invariant under Lagrangian isotopy. For $L_0 = L_1$ one has $HF(L_0, L_1) \cong H^*(L_0; \mathbb{Z}/2)$. Of course, the same properties are true for graded Floer cohomology. Choose an $\mathcal{L}^\infty$-grading $\tilde{L}$ of L. If $[\chi_K]$ was trivial then by Lemma 4.2 $\tilde{L}$ would be graded Lagrangian isotopic to $\tilde{L}[\sigma_K]$, and hence

$$HF^*(\tilde{L}, \tilde{L}) \cong HF^*(\tilde{L}, \tilde{L}[\sigma_K]) = HF^{*+\sigma_K}(\tilde{L}, \tilde{L}).$$

Because $HF^*(\tilde{L}, \tilde{L})$ is nonzero and concentrated in finitely many degrees, this implies that σ_K must be zero. Conversely, if $\sigma_K \neq 0$ then $[\chi_K]$ must be nontrivial.

Since the same argument works for the iterates χ_K^m, it also follows that $[\chi_K]$ has infinite order. $\square$

Let $C^\infty \in H^1(\mathcal{L})$ be the global Maslov class of $\mathcal{L}^\infty$. Take a point $x \in \partial M$ and a Lagrangian subspace $\Lambda \in \mathcal{L}_x$, and define $\lambda : S^1 \longrightarrow \mathcal{L}$ by $\lambda(t) = D\phi_t^H(\Lambda)$. One can easily show that

$$\sigma_K = -\langle C^\infty, [\lambda]\rangle. \tag{4.4}$$

This formula is useful for determining σ_K, which is important in applications of Theorem 4.3.

Remarks 4.4. (a) The conditions in Theorem 4.3 were chosen for their simplicity, and are not the most general ones. For instance, an inspection of the proof shows that the assumption $H^1(M) = 0$ can be omitted (however, for $H^1(M) \neq 0$ the shift σ_K may depend on the choice of Maslov covering). The condition $[\omega] = 0$ is there to ensure that there is a well-behaved Floer theory, and can also be weakened considerably. In contrast, the existence of the Lagrangian submanifold L, and the assumption that $2c_1(M,\omega) = 0$, are both essential parts of our argument.

(b) There is an alternative approach which dispenses with Lagrangian submanifolds altogether, and uses instead the Floer cohomology of the automorphism χ_K. This approach is more difficult to carry out than the one used here, but it is possibly more general.

4b. **Manifolds with periodic geodesic flow.** Let (N^n, g) be a Riemannian manifold such that any geodesic of length one is closed. Take $M = T_1^*N = \{\xi \in T^*N \mid |\xi|_g \leq 1\}$ with the standard symplectic form $\omega \in \Omega^2(M)$ and contact form $\alpha \in \Omega^1(\partial M)$. The flow of the Reeb vector field R of α is just the geodesic flow on N, and our assumption is that $\phi_1^R = \mathrm{id}$. Hence one can define an automorphism $\chi_R \in \mathrm{Aut}(M, \partial M, \omega)$. Extend (ϕ_t^R) radially to a Hamiltonian circle action σ on $M \setminus N$ (the moment map of σ is $\mu(\xi) = |\xi|$). Choose a function $r \in C^\infty([0;1], \mathbb{R})$ with $r(t) = 0$ for $t \leq 1/3$, $r(t) = -1$ for $t \geq 2/3$. Then a possible choice for χ_R is

$$\chi_R(\xi) = \begin{cases} \sigma_{r(|\xi|)}(\xi) & \xi \notin N, \\ \xi & \xi \in N. \end{cases} \tag{4.5}$$

M admits a Lagrangian distribution $S \subset TM$, formed by the tangent spaces along the fibres of the projection $M \longrightarrow N$. As indicated in Example 2.10, one can use S to define an ∞-fold Maslov covering $\mathcal{L}^\infty$. In order to satisfy the conditions of Theorem 4.3, we will assume that $H^1(N)$ is zero, so that there is only one ∞-fold Maslov covering. The global Maslov class C^∞ is represented by the fibrewise Maslov cycle $I = \bigcup_\xi \{\Lambda \in \mathcal{L}_\xi \mid \Lambda \cap S_\xi \neq 0\}$, and one can write (4.4) as

$$\sigma_R = -[I] \cdot [\lambda].$$

By definition $\lambda(t) = D\sigma_t(\Lambda)$ for some $\Lambda \in \mathcal{L}_\xi$, $\xi \in \partial M$. It is convenient to choose Λ in the following way: split TM_ξ into its horizontal and vertical parts, both of which are naturally isomorphic to TN_x, where $x \in N$ is the base point of ξ. Then take $\Lambda = \{(\eta_1, \eta_2) \in TN_x \times TN_x \mid \eta_1 = g(\xi, \eta_2)\xi\}$. This has the consequence that $\lambda(t) \in I$ iff $c(t)$ and $c(0)$ are conjugate points. It is a familiar result, see [6, section 4] or [24, section 6], that the local intersection number at a point $\lambda(t) \in I$ is equal to the multiplicity $m_c(t) > 0$ of $c(t)$ as a conjugate point of $c(0)$. Therefore

$$\sigma_R = -\sum_t m_c(t), \tag{4.6}$$

where the sum is over all conjugate points $t \in S^1$. Since $m_c(0) = n-1$, σ_R is always negative ($N = S^1$ is impossible because we have assumed that $H^1(N) = 0$). M always contains a Lagrangian submanifold (the zero-section) with zero first Betti number. Moreover, $[\omega] = 0$. By applying Theorem 4.3 one obtains

Corollary 4.5. *Let (N, g) be a closed Riemannian manifold with $H^1(N) = 0$ such that any geodesic of length one is closed. Then, for $M = T_1^*N$ and the Reeb vector field R, $[\chi_R] \in \pi_0(\mathrm{Aut}(M, \partial M, \omega))$ has infinite order.* □

The main examples of manifolds with periodic geodesic flow are compact globally symmetric spaces of rank one [2]. These spaces are two-point homogeneous, which means that the isometry group $\mathrm{Iso}(N, g)$ acts transitively on ∂M. Hence all geodesics have the same minimal period. Symmetric spaces also have the property that any geodesic path $c : [0; T] \longrightarrow N$ with $c(T) = c(0)$ is a closed geodesic [12, p. 144]. It follows that the energy functional on the *based* loop space ΩN is a Morse-Bott function. Its critical point set consists of one point (the constant path) and infinitely many copies of S^{n-1}. The Morse index of the point is zero. The Morse indices of the $(n-1)$-spheres can be computed by comparing (4.6) with Morse's index theorem: they are $-\sigma_R - n + 1$, $-2\sigma_R - n + 1$, etc. This means that for $n \geq 3$, ΩN has a CW-decomposition with one cell of dimension $-\sigma_R - n + 1$ and other cells of dimension $\geq -\sigma_R - n + 3$. It follows that ΩN is precisely $(-\sigma_R - n)$-connected, and that N is precisely $(-\sigma_R - n + 1)$-connected. This approach, complemented by explicit computations for $\mathbb{RP}^2$ and S^2, yields the following values for σ_R:

N	S^m	$\mathbb{RP}^m$	$\mathbb{CP}^m$	$\mathbb{HP}^m$	F_4/Spin_9
σ_R	$2-2m$	$1-m$	$-2m$	$-4m+2$	-22

Here F_4/Spin_9 is the exceptional symmetric space diffeomorphic to the Cayley plane (which is 16-dimensional and 7-connected). For $N = S^{2m+1}$ or $\mathbb{RP}^{2m+1}$ the result of Corollary 4.5 can be obtained more easily, without using any symplectic geometry, by computing the variation $\mathrm{var}(\chi_R)$. For $N = \mathbb{RP}^{2m}$ the space M can be identified with the complement of a neighbourhood of a quadric $H \subset \mathbb{CP}^m$, thus settling the remaining case in Example 4.1(b). The most interesting example is that of $\mathbb{CP}^m$, since there we can show that the non-vanishing of $[\chi_R]$ is a genuinely symplectic phenomenon:

Proposition 4.6. *Let $M = T_1^*\mathbb{CP}^m$, $m \geq 1$. Then χ_R can be deformed to the identity in the group $\mathrm{Diff}(M, \partial M)$ of diffeomorphisms which act trivially on ∂M.*

Lemma 4.7. *Let $M = T_1^*\mathbb{CP}^m$. Then there is a smooth family σ^s, $0 \leq s \leq 1$, of circle actions on ∂M with $\sigma_t^0 = \phi_t^R$, and such that σ^1 extends to a circle action on the whole of M.*

Proof. Since $\mathbb{CP}^m = S^{2m+1}/S^1$, $T^*\mathbb{CP}^m$ is a symplectic quotient $T^*S^{2m+1}//S^1$. Explicitly $M = \{(u, v) \in \mathbb{C}^{m+1} \times \mathbb{C}^{m+1} \mid |u| = 1,\ |v| \leq 1,\ \langle u, v\rangle_{\mathbb{C}} = 0\}/S^1$, where S^1 acts by $t \cdot (u, v) = (e^{2\pi i t}u, e^{2\pi i t}v)$. The diagonal $SU(2)$-action on $\mathbb{C}^{m+1} \times \mathbb{C}^{m+1}$ descends to an action of $PU(2) = SU(2)/\pm 1$ on ∂M, which we will denote by ρ. Assume that we have rescaled the standard symplectic form to make the Reeb flow one-periodic, that is, $\omega = \pi/2 \sum_j (dv_j \wedge d\bar{u}_j + d\bar{v}_j \wedge du_j)$. Then the Reeb flow is $\phi^R_t = \rho(\exp(tA))$ with $A = \left(\begin{smallmatrix} 0 & -\pi \\ \pi & 0 \end{smallmatrix}\right) \in su_2$. The family σ^s of circle actions is defined by $\sigma^s_t = \rho(\exp(tA^s))$, where (A^s) is any path in $su(2)$ from $A^0 = A$ to $A^1 = \mathrm{diag}(\pi i, -\pi i)$ such that $\exp(A^s) = -1$ for all s. Because ∂M is a quotient one can write $\sigma^1_t(u,v) = (u, e^{-2\pi i t}v)$, and this shows that σ^1 extends to a circle action on all of M. □

Proof of Proposition 4.6. Clearly, using the family σ^s of circle actions one can deform χ_R inside $\mathrm{Diff}(M,\partial M)$ to $\chi'_R(\xi) = \sigma^1_{r(|\xi|)}(\xi)$. To deform this to the identity one uses the isotopy $\psi_s(\xi) = \sigma^1_{(1-s)r(|\xi|)-s}(\xi)$. □

Remark 4.8. It seems likely that the maps χ_R on $T_1^*\mathbb{C}\mathrm{P}^m$ are not only differentiably isotopic to the identity but also 'fragile' in the sense of [26]. We have not checked the details.

4c. **Weighted homogeneous polynomials.** A polynomial $p \in \mathbb{C}[x_0,\dots,x_n]$, $n \geq 1$, is called weighted homogeneous if there are integers $\beta_0,\dots,\beta_n,\beta > 0$ such that $p(z^{\beta_0}x_0,\dots,z^{\beta_n}x_n) = z^\beta p(x_0,\dots,x_n)$. The numbers $w_i = \beta_i/\beta$ are called the weights of p. Throughout this section $p(x)$ will be a weighted homogeneous polynomial with an isolated critical point at $x = 0$; because of the homogeneity, this implies that p has no other critical points. By definition, the link of the singular point is $L = p^{-1}(0) \cap S^{2n+1}$. This is in fact a contact submanifold of S^{2n+1} with respect to the standard contact form $\alpha_{S^{2n+1}} = \frac{i}{4}\sum_j z_j d\bar{z}_j - \bar{z}_j dz_j$. Let $B^{2n+2} \subset \mathbb{C}^{n+1}$ be the closed unit ball. Fix a cutoff function ψ with $\psi(t^2) = 1$ for $t \leq 1/3$ and $\psi(t^2) = 0$ for $t \geq 2/3$. For $z \in \mathbb{C} \setminus \{0\}$ set $F_z = \{x \in B^{2n+2} \mid p(x) = \psi(|x|^2)z\}$.

Lemma 4.9. *There is an $\epsilon > 0$ such that for all $0 < |z| < \epsilon$, F_z is a symplectic submanifold of B^{2n+2} with boundary $\partial F_z = L$.*

Proof. The Zariski tangent space of F_z is $(TF_z)_x = \ker L(x,z)$ with $L(x,z) : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$, $L(x,z)\xi = dp(x)\xi - 2z\,\psi'(|x|^2)\langle x,\xi\rangle$. Clearly $L(x,z)$ is onto in the following three cases: (1) $z = 0$ and $x \neq 0$, (2) $0 < |x| \leq 1/3$, (3) $|x| \geq 2/3$. On the other hand, the set of those (x,z) for which $L(x,z)$ is onto must be open. This implies that $L(x,z)$ is onto for all $(x,z) \in F_z$, provided that $z \neq 0$ is sufficiently small. An argument of the same kind shows that $(TF_z)_x$ is a symplectic subspace of $\mathbb{C}^{n+1}$ for all small $z \neq 0$. □

We fix a z_0 with $0 < |z_0| < \epsilon$ and call $M = F_{z_0}$ the Milnor fibre of the singular point $0 \in p^{-1}(0)$. Clearly $(M, \omega = \omega_{\mathbb{C}^{2n+2}}|M, \alpha = \alpha_{S^{2n+1}}|L)$ is a compact symplectic manifold with contact type boundary. The choice of z_0 is not really important since one can prove (by a standard argument using Moser's technique) that any two choices give symplectically isomorphic Milnor fibres. During the following discussion, we will repeatedly make use of our right to pass to a smaller z_0. The next Lemma shows that M is diffeomorphic to what is traditionally called the Milnor fibre.

Lemma 4.10. *There is an $\epsilon > 0$ such that for all $0 < |z| < \epsilon$, F_z is diffeomorphic to $p^{-1}(z) \cap B^{2n+2}$.*

Proof. For $(z,t) \in (\mathbb{C} \setminus \{0\}) \times [0;1]$ consider $G_{(z,t)} = \{x \in B^{2n+2} \mid p(x) = t\psi(|x|^2)z + (1-t)z\}$. Using the same argument as before, one can prove that these are smooth manifolds for all sufficiently small z. If we fix such a z, the $G_{(z,t)}$ form a differentiable fibre bundle over $[0;1]$. Hence $G_{(z,1)} = F_z$ and $G_{(z,0)} = p^{-1}(z) \cap B^{2n+2}$ are diffeomorphic. □

Corollary 4.11. *M is $(n-1)$-connected. In fact, it is homotopy equivalent to a nontrivial wedge of n-spheres.*

This follows from Lemma 4.10 and a classical result of Milnor [16, Theorem 6.5]. Denote by $A = \{x \in M \mid 2/3 \leq |x| \leq 1\}$ the part of M on which it agrees with $p^{-1}(0)$. Let $\theta_{\mathbb{C}^{n+1}}$ be the complex n-form on $\mathbb{C}^{n+1}$ given by $(\theta_{\mathbb{C}^{n+1}})_x(v_1, \dots, v_n) = \det_{\mathbb{C}}(\overline{dp_x} \wedge v_1 \wedge \dots \wedge v_n)$.

Lemma 4.12. *There is an ω-compatible almost complex structure J on M, and a nowhere vanishing J-complex n-form θ, which have the following properties: the restriction of J to A agrees with the standard complex structure on $A \subset p^{-1}(0)$, and $\theta|A = \theta_{\mathbb{C}^{n+1}}|A$.*

Proof. Consider the $2n$-dimensional complex vector bundle $K \longrightarrow M$ with fibres $K_x = \ker(dp_x)$. $\theta_{\mathbb{C}^{n+1}}$ defines a complex n-form on every fibre of K, and these n-forms are easily seen to be nonzero. Both K and TM are subbundles of the trivial bundle $\mathbb{C}^{n+1} \times M$. The proof of Lemma 4.9 shows that, by choosing z_0 sufficiently small, one can make these two subbundles arbitrarily close. This means that the orthogonal projection $P_K : TM \longrightarrow K$ is a bundle isomorphism, and that the pullback of the complex structure on K is an almost complex structure J' on TM which is ω-tame. The pullback $\theta' = P_K^*(\theta_{\mathbb{C}^{n+1}}|K)$ is a J'-complex n-form on M which is nowhere zero. Since P_K is the identity over A, J' and θ' have all the properties required in the Lemma, except that J' may not be ω-compatible. However, one can easily find an ω-compatible almost complex structure J which agrees with J' on A. Moreover, J and J' can be deformed into each other through almost complex structures, and the deformation can be chosen constant on A. This implies that there is a bundle automorphism $Q : TM \longrightarrow TM$ such that $Q^*(J') = J$, and which is the identity on A. Now set $\theta = Q^*(\theta')$. □

Corollary 4.13. $c_1(M, \omega) = 0$.

This is clear, since there is a nowhere vanishing complex n-form.

Lemma 4.14. *(M, ω) always contains an embedded Lagrangian n-sphere.*

Sketch of proof. The method which produces such spheres is to deform p by adding a linear term λ, such that $p + \lambda$ has only nondegenerate critical points. The Lagrangian spheres appear as vanishing cycles associated to these critical points. We will now explain one version of this argument; for variations on this theme see [26, section 1.4]. Choose some small $\delta > 0$, and let $D = \{(z, \lambda) \in \mathbb{C} \times (\mathbb{C}^{n+1})^* \mid |z| < \delta, |\lambda| < \delta\}$. For $(z, \lambda) \in D$ define

$$\tilde{F}_{(z,\lambda)} = \{x \in B^{2n+2} \mid p(x) + \lambda(x) = \psi(|x|^2)z\}.$$

If δ is sufficiently small, there are two possibilities for each (z, λ): either (1) $\tilde{F}_{(z,\lambda)}$ is a smooth symplectic submanifold of B^{2n+2}, or (2) the complex hypersurface $p(x) + \lambda(x) = z$ has a singular point x with $|x| \leq 1/3$. The subset $\Delta \subset D$ where (2) occurs is a complex hypersurface. Hence $D \setminus \Delta$ is connected. By an application of Moser's technique, it follows that all the symplectic manifolds $\tilde{F}_{(z,\lambda)}$ occurring in case (1) are isomorphic to the Milnor fibre $M = \tilde{F}_{(z_0,0)}$.

Take a generic small λ. Then $p + \lambda$ has only nondegenerate (Morse-type) critical points. These critical points lie close to the origin, and there is always at least one of them (this is a well-known fact, which follows from considering the Milnor number μ of the singularity). Choose a critical point x of $p+\lambda$, such that $|x| < 1/4$, and set $z = p(x) + \lambda(x)$. Since x is non-degenerate, one can write

$$(p + \lambda)(x + y) = z + Q(y) + (\text{higher order terms in } y)$$

where Q is a nondegenerate complex quadratic form. A careful application of Moser's technique shows that for all $0 < \epsilon_2 \ll \epsilon_1 \ll |\lambda|$, one can embed $U = \{y \in \mathbb{C}^{n+1} \mid |y| \leq \epsilon_1,\ Q(y) = \epsilon_2\}$ symplectically into $\tilde{F}_{(z+\epsilon_2,\lambda)}$. Now U is symplectically isomorphic to a neighbourhood of the zero-section in T^*S^n, hence contains a Lagrangian n-sphere. It follows that $\tilde{F}_{(z+\epsilon_2,\lambda)}$, and hence the Milnor fibre, contain a Lagrangian n-sphere. □

The Milnor fibration associated to the singular point $0 \in p^{-1}(0)$ is obtained by putting together the manifolds F_z for all $|z| = |z_0|$:

$$p : F = \bigcup_{z=|z_0|} F_z \times \{z\} \longrightarrow |z_0|S^1.$$

The proof of Lemma 4.9 shows that this is a smooth and proper fibration. Moreover, if we pull back $\omega_{\mathbb{C}^{n+1}}$ to F via the obvious projection, we obtain a closed two-form Ω whose restriction to any fibre is a symplectic form. Such a two-form defines a connection $TF^h \subset TF$, given by the Ω-orthogonal complements of the tangent spaces along the fibres:

$$TF^h_{(x,z)} = \{X \in TF_{(x,z)} \mid \Omega(X, Y) = 0 \text{ for all } Y \text{ such that } Dp(Y) = 0\}.$$

The fact that Ω is closed implies that the parallel transport maps of this connection are symplectic isomorphisms between the fibres. In our case, since the fibres are manifolds with boundary, one has to check that the parallel transport maps exist. But this it clear because near the boundary $\partial F = \bigcup_z \partial F_z \times \{z\}$ one has a natural trivialization $A \times |z_0|\, S^1 \subset F$, and the connection is compatible with this trivialization. This also implies that if we go once around the base $|z_0|S^1$, the parallel transport yields a map $f \in \mathrm{Aut}(M, \partial M, \omega)$. We call f the *symplectic monodromy* of our singularity. One can show that the class $[f] \in \pi_0(\mathrm{Aut}(M, \partial M, \omega))$ is, in a suitable sense, independent of the choices of ψ and z_0.

Up to now, the fact that p is weighted homogeneous has not been of any importance. We will now begin to exploit this particular feature of our situation. Let σ be the complex circle action on $\mathbb{C}^{n+1}$ with multiplicities $\beta_0, \dots, \beta_n$, and K the vector field generating it. $\sigma|L$ is a circle action preserving α. Hence one can construct the associated automorphism $\chi_K \in \mathrm{Aut}(M, \partial M, \omega)$ of the Milnor fibre. We will use a particular choice which is $\chi_K = \phi^H_{-1}$, where $H \in C^\infty(M, \mathbb{R})$ is given by $H(x) = \pi \sum_i \beta_i |x_i|^2$. From now on assume that $n \geq 2$. Since $H^1(M)$ is zero (Corollary 4.11), ∂M is connected (this follows from Corollary 4.11 by a Poincaré duality consideration), and $2c_1(M, \omega) = 0$ (Corollary 4.13), one can define the shift σ_K of χ_K.

Lemma 4.15. $\sigma_K = 2(\beta - \sum_j \beta_j)$.

Proof. Let J, θ be as in Lemma 4.12. As explained in Example 2.9, $\Theta = \theta^{\otimes 2}/|\theta|^2$ defines an ∞-fold Maslov covering on M, whose global Maslov class is represented by the map $\det^2_\Theta : \mathcal{L} \longrightarrow S^1$. Together with (4.4) this means that $-\sigma_K$ is the degree of the map $c : S^1 \longrightarrow S^1$, $c(t) = \det^2_\Theta(D\phi^H_t(\Lambda))$ for some $\Lambda \in \mathcal{L}_x$, $x \in \partial M \subset A$. The restriction of ϕ^H_t to A is simply the circle action σ, and $\theta|A$ agrees with $\theta_{\mathbb{C}^{n+1}}$. An explicit computation shows that $\sigma_t^*(\theta_{\mathbb{C}^{n+1}}) = e^{2\pi i t(\beta_0 + \cdots + \beta_n - \beta)}\theta_{\mathbb{C}^{n+1}}$. Therefore c has the form $c(t) = e^{4\pi i t(\beta_0+\cdots+\beta_n-\beta)}c(0)$. □

Lemma 4.16. *χ_K is equal to the β-th iterate f^β of the symplectic monodromy.*

Proof. Let $\tilde{X} \in C^\infty(TE^h)$ be the unique horizontal lift of the vector field $X(z) = 2\pi i z$ on $|z_0|S^1$. Its flow (μ_t) maps F_z to $F_{e^{2\pi i t}z}$ symplectically. By definition, the symplectic monodromy is $f = \mu_1|F_{z_0}$. Now let ρ be the circle action on F given by $\rho_t(x,z) = (\sigma_t(x), e^{2\pi i\beta t}z)$. We denote the Killing vector field of ρ by Y. Since ρ preserves Ω, the connection TE^h is ρ-invariant. It follows that μ_t commutes with ρ for any t. Therefore $\eta_t = \rho_{-t} \circ \mu_{\beta t}$ is the flow on F generated by $\beta\tilde{X} - Y$. Note that η_t maps any fibre F_z to itself symplectically. Let H be the function which we have used to define χ_K. Clearly $d(H|F_{z_0}) = -i_Y\Omega|F_{z_0}$. Since $i_{\tilde{X}}\Omega$ vanishes on each fibre F_z one also has

$$d(-H|F_{z_0}) = (-i_{\beta\tilde{X}-Y}\Omega)|F_{z_0}.$$

This means that $(\eta_t|F_{z_0})$ is the Hamiltonian flow of $-H|F_{z_0}$. Hence by definition $\chi_K = \eta_1|F_{z_0}$. On the other hand, by the definition of η_t, $f^\beta = \eta_1|F_{z_0}$. □

We can now prove our main result about symplectic monodromy.

Theorem 4.17. *Let $p \in \mathbb{C}[x_0, \dots, x_n]$ be a weighted homogeneous polynomial with an isolated critical point at 0. Assume that $n \geq 2$ and that the sum of the weights w_i is not one. Then the symplectic monodromy f defines a class of infinite order in $\pi_0(\mathrm{Aut}(M, \partial M, \omega))$.*

Proof. Since $f^\beta = \chi_K$, it is sufficient to prove that $[\chi_K]$ has infinite order. Lemma 4.15 shows that $\sigma_K = 2\beta(1 - \sum_i w_i) \neq 0$. Therefore one only needs to apply Theorem 4.3. The necessary assumptions about M have all been proved above except for $[\omega] = 0$, which is obvious from the definition. □

Examples and comments 4.18. (a) Let $p \in \mathbb{C}[x_0, x_1, x_2]$ be one of the standard models for the du Val (or simple) singularities [1]. These models are weighted homogeneous, and the sum of the weights is > 1. For example, if $p(x) = x_0^2 + x_1^3 + x_1x_2^3$ is the singularity of type (E_7) then $w_0 + w_1 + w_2 = 1/2 + 1/3 + 2/9 = 19/18$. Hence Theorem 4.17 applies, showing that $[f] \in \pi_0(\mathrm{Aut}(M, \partial M, \omega))$ has infinite order. In contrast, it follows from Brieskorn's simultaneous resolution [3] that the class of f in $\pi_0(\mathrm{Diff}(M, \partial M))$ has *finite order*. Hence, at least in these cases, Theorem 4.17 expresses a genuinely symplectic phenomenon.

(b) It is possible that the assumption $\sum_i w_i \neq 1$ might be removed. In fact, there are many cases when $\sum_i w_i = 1$ and in which the monodromy has infinite order for topological reasons, for example $p(x_0, x_1, x_2, x_3) = x_0^4 + x_1^4 + x_2^4 + x_3^4$.

(c) Let $p \in \mathbb{C}[x_0, x_1]$ be a weighted homogeneous polynomial with an isolated critical point. Then M is a connected surface and not a disc. The iterate f^β of the monodromy can be written as a composition of positive Dehn twists along the connected components of ∂M. Using this one can show easily that $[f] \in \pi_0(\mathrm{Diff}(M, \partial M))$ is always of infinite order. This means that Theorem 4.17 holds also for $n = 1$.

5. Knotted Lagrangian spheres

5a. **Generalized Dehn twists.** This section contains the basic definitions and some facts, both topological and symplectic, which are used later on. Throughout (M, ω) will be a compact symplectic manifold of dimension $2n$. By a Lagrangian sphere in M we will mean a Lagrangian embedding $S^n \hookrightarrow M$. Such embeddings will be denoted by the letters $l, l_1, l_2, \dots$ and their images by $L, L_1, L_2, \dots$. An (A_k)-*configuration*, $k \geq 2$, is a collection of Lagrangian spheres $(l_1, \dots, l_k)$ which are pairwise transverse, such that $L_i \cap L_j = \emptyset$ for $|i - j| \geq 2$ and $|L_i \cap L_{i\pm1}| = 1$. The name comes from the relationship with singularity theory. In fact the Milnor fibre of the (A_k)-singularity, which is the hypersurface

$$M = \{x \in \mathbb{C}^{n+1} \mid x_0^{k+1} + x_1^2 + \dots + x_n^2 = \epsilon, |x| \leq 1\} \tag{5.1}$$

for sufficiently small $\epsilon \neq 0$, contains such a configuration. This was proved in [28, Proposition 8.1] for $n = 2$, and the general case can be treated in the same way (here we have used the classical form of the Milnor fibre, rather than the definition adopted in section 4c; this does not really matter, since the Milnor fibre as defined there also contains an (A_k)-configuration).

Consider $U = T_1^*S^n = \{\xi \in T^*S^n \mid |\xi| \leq 1\}$ with its standard symplectic structure ω_U. The complement of the zero-section $S^n \subset U$ carries a Hamiltonian circle action σ whose moment map is the length $|\xi|$. In the coordinates $U = \{(u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |v| = 1,\ |u| \leq 1,\ \langle u, v\rangle = 0\}$ with $\omega_U = \sum_j du_j \wedge dv_j$ one has $\sigma_t(u, v) = (\cos(2\pi t)u - \sin(2\pi t)v|u|, \cos(2\pi t)v + \sin(2\pi t)u/|u|)$. Note that $\sigma_{1/2}(u, v) = (-u, -v)$ extends smoothly over the zero-section. Therefore one can define a diffeomorphism τ of U by setting

$$\tau(\xi) = \sigma_{\psi(|\xi|)}(\xi),$$

where ψ is a function with $\psi(t) = 1/2$ for $t \leq 1/3$ and $\psi(t) = 0$ for $t \geq 2/3$. τ is equal to the identity near ∂U, and it acts on the zero-section as the antipodal map. An explicit computation shows that τ is symplectic. We call it a *model generalized Dehn twist.* Now let l be any Lagrangian sphere in (M, ω). One can always find an embedding $j : U \longrightarrow M$ such that $j|S^n = l$ and $j^*\omega = \delta\, \omega_U$ for some $\delta > 0$. By extending $j\tau j^{-1}$ trivially over $M \setminus \mathrm{im}(j)$ one defines a symplectic automorphism τ_l of M, which we call a *generalized Dehn twist along* l. It is not difficult to see that the class $[\tau_l]$ in $\pi_0(\mathrm{Aut}(M, \omega))$ (or in $\pi_0(\mathrm{Aut}(M, \partial M, \omega))$, if M has a boundary) is independent of the choice of j and ψ. For this reason, we will often speak of τ_l as *the* generalized Dehn twist along l. For $n = 1$ these maps are just the ordinary (positive) Dehn twists along curves on a surface.

Remark 5.1. It is an open question whether $[\tau_l]$ depends only on the image L. If l and l' are two embeddings with the same image, and such that $l^{-1} \circ l'$ is isotopic to the identity in $\mathrm{Diff}(S^n)$, one can easily prove that $[\tau_l] = [\tau_{l'}]$. Moreover, the same holds if $l^{-1} \circ l'$ is an element of $O(n + 1)$ (this shows that, just as in the case $n = 1$, the choice of orientation of L is not important). For $n \leq 3$ it is known that $\pi_0(\mathrm{Diff}^+(S^n)) = 1$ [17] [4] which implies that $[\tau_l]$ does indeed depend only on L, but in higher dimensions $\pi_0(\mathrm{Diff}^+(S^n))$ is often nonzero.

We will first look at generalized Dehn twists from a topological point of view. Since these maps are a symplectic form of the classical Picard-Lefschetz transformations, their action on homology is given by the familiar formula

$$(\tau_l)_*(c) = c - (-1)^{n(n-1)/2}(c \cdot [L])[L]. \tag{5.2}$$

Using the fact that $[L] \cdot [L] = (-1)^{n(n-1)/2}\chi(L)$ for any n-dimensional Lagrangian submanifold, one obtains the following

Lemma 5.2. (a) *If n is even then $(\tau_l)_*$ has order two. If n is odd then $(\tau_l)_*$ has infinite order iff $[L]$ is not a torsion class (otherwise $(\tau_l)_* = \mathrm{id}$).*
(b) *Assume that n is odd, and that l_1, l_2 are two Lagrangian spheres with $[L_1]\cdot[L_2] = \pm 1$. Set $g = (\tau_{l_2}\tau_{l_1})^3$. Then g^2 induces the identity on homology.* □

For $n = 2$ it is known [28, Lemma 6.3] that τ_l^2 is actually isotopic to the identity in $\mathrm{Diff}(M)$. It seems natural to ask whether this holds for other even n; there is also the analogous problem for the map g^2 defined in (b). In both cases the answer is unknown to the author. However, there are some weaker topological results which are easier to obtain, and which are sufficient for our purpose.

Lemma 5.3. (a) *Assume that n is even, and that there is an (A_2)-configuration of Lagrangian spheres (l_1, l_2) in M. Then $l_1^{(k)} = \tau_{l_2}^{2k} \circ l_1$ is isotopic to l_1 through smooth embeddings $S^n \hookrightarrow M$ for any $k \in \mathbb{Z}$.*
(b) *Assume that n is odd and ≥ 5, and that there is an (A_3)-configuration of Lagrangian spheres (l_1, l_2, l_3) in M. Then $l_1^{(k)} = g^{2k} \circ l_1$, where $g = (\tau_{l_2}\tau_{l_3})^3$, is isotopic to l_1 through smooth embeddings $S^n \hookrightarrow M$ for any $k \in \mathbb{Z}$.*

Proof. (a) If $n = 2$ then $\tau_{l_2}^2$ is isotopic to the identity in $\mathrm{Diff}(M)$, which implies our result. Hence we can assume that $n \geq 4$. let $W \subset M$ be a regular neighbourhood of $L_1 \cup L_2$. Since W retracts onto $L_1 \cup L_2$, it is $(n-1)$-connected with $H_n(W) = \mathbb{Z}\langle L_1\rangle \oplus \mathbb{Z}\langle L_2\rangle$. We can assume that τ_{l_2} has been chosen in such a way that it preserves W. Lemma 5.2(a) shows that all the embeddings $l_1^{(k)}$ represent the same homology class in W. Hence, by Hurewicz's theorem, they are homotopic as continuous maps $S^n \longrightarrow W$. The proof is completed by applying a result of Haefliger [9] which shows that any two homotopic embeddings $S^n \longrightarrow W$ are differentiably isotopic.

(b) is proved in the same way. The condition $n \geq 5$ is necessary in order to use Haefliger's result. The result is easily seen to be false for $n = 1$, but the author was unable to decide the remaining case $n = 3$. In that dimension, there are obstructions for two homotopic embeddings $S^3 \longrightarrow W$ to be differentiably isotopic. These obstructions are completely understood in principle, see [10, Corollary B], but not so easy to compute in practice. □

Now consider an (A_2)-configuration (l_1, l_2) of Lagrangian spheres in M. From (5.2) it follows that, in any dimension, $[\tau_{l_2}(L_1)] = \pm[\tau_{l_1}^{-1}(L_2)] \in H_n(M)$. In fact the following stronger result is true:

Lemma 5.4. *$\tau_{l_1}(L_2)$ and $\tau_{l_2}^{-1}(L_1)$ are isotopic as (unoriented) Lagrangian submanifolds of (M, ω). In fact, both of them are Lagrangian isotopic to the surgery $L_1 \# L_2$.*

This was proved in [28, Appendix] for $n = 2$, and the argument given there carries over to arbitrary n. For future use we need to recall one aspect of the proof: both

$\tau_{l_1}(L_2)$ and $L_1 \# L_2$ agree with L_2 away from a neighbourhood of L_1 which, by an appropriate choice of τ_{l_1} and of the surgery, can be made arbitrarily small. The Lagrangian isotopy constructed in [28] between them remains constant outside this neighbourhood. Similarly, the isotopy between $L_1 \# L_2$ and $\tau_{l_2}^{-1}(L_1)$ is constant outside a neighbourhood of L_2.

Lemma 5.5. *Assume that (l_1, l_2) form an (A_2)-configuration of Lagrangian spheres in M, and set $g = (\tau_{l_1}\tau_{l_2})^3$. Then $g(L_1)$ is Lagrangian isotopic to L_1, and $g(L_2)$ is Lagrangian isotopic to L_2.*

Proof. Using Lemma 5.4 and the obvious fact that $\tau_l(L) = L$ for any l, one sees that $(\tau_{l_1}\tau_{l_2})^3(L_2) \simeq (\tau_{l_1}\tau_{l_2})^2\tau_{l_2}^{-1}(L_1) \simeq \tau_{l_1}\tau_{l_2}\tau_{l_1}(L_1) \simeq \tau_{l_1}\tau_{l_1}^{-1}(L_2) = L_2$, where $\simeq$ stands for Lagrangian isotopy. The proof for L_1 is similar. $\square$

5b. **The graded point of view.** From now on we assume that (M, ω) satisfies $2c_1(M, \omega) = 0$ and that its dimension $2n$ is at least four (the case of classical Dehn twists, $n = 1$, is more complicated because T^*S^1 admits infinitely many different Maslov coverings). Choose an ∞-fold Maslov covering $\mathcal{L}^\infty$ on M. Lemma 2.3 implies that any Lagrangian submanifold $L \subset M$ with $H^1(L) = 0$ admits an $\mathcal{L}^\infty$-grading. Let l be a Lagrangian sphere in M and τ_l the generalized Dehn twist along it defined using some embedding $j : U \hookrightarrow M$. By definition τ_l is the identity outside $\mathrm{im}(j)$.

Lemma 5.6. *There is a unique $\mathcal{L}^\infty$-grading $\tilde{\tau}_l$ of τ_l which acts trivially on the part of $\mathcal{L}^\infty$ which lies over $M \setminus \mathrm{im}(j)$.*

Proof. The uniqueness is obvious. To prove the existence, consider the local model $U = T_1^*S^n$. Since $c_1(U, \omega_U) = 0$ and $H^1(U) = 0$, U admits a unique ∞-fold Maslov covering $\mathcal{L}_U^\infty$. Remark 2.5 says that the model generalized Dehn twist τ has a unique $\mathcal{L}_U^\infty$-grading $\tilde{\tau}$ which acts trivially on the part of $\mathcal{L}_U^\infty$ which lies over ∂U. Now, for an arbitrary Lagrangian sphere l in M and Maslov covering $\mathcal{L}^\infty$, one can identify $\mathcal{L}^\infty|\mathrm{im}(j)$ with $\mathcal{L}_U^\infty$ and then extend $\tilde{\tau}$ by the identity to an $\mathcal{L}^\infty$-grading $\tilde{\tau}_l$ of τ_l. $\square$

We emphasize that this preferred grading $\tilde{\tau}_l$ does not depend on the choice of a grading for L.

Lemma 5.7. *The preferred $\mathcal{L}^\infty$-grading $\tilde{\tau}_l$ satisfies $\tilde{\tau}_l(\tilde{L}) = \tilde{L}[1-n]$ for any $\mathcal{L}^\infty$-grading $\tilde{L}$ of L.*

Proof. Clearly it is enough to prove the corresponding fact for the local model $\tilde{\tau}$. We find it convenient to use complex coordinates $U = \{\xi \in \mathbb{C}^{n+1} \mid |\mathrm{re}\ \xi| \leq 1, |\mathrm{im}\ \xi| = 1, \langle \mathrm{re}\ \xi, \mathrm{im}\ \xi \rangle_{\mathbb{R}} = 0\}$. The tangent spaces are $TU_\xi = \{\eta \in \mathbb{C}^{n+1} \mid \langle \mathrm{im}\ \xi, \mathrm{im}\ \eta \rangle_{\mathbb{R}} = 0,\ \omega_{\mathbb{C}^{n+1}}(\bar{\xi}, \eta) = 0\}$. Hence if Λ is a Lagrangian subspace of TU_ξ, $\Lambda \oplus \mathbb{R}\bar{\xi}$ is a Lagrangian subspace of $\mathbb{C}^{n+1}$. This stabilization defines a map $r : \mathcal{L} \longrightarrow \mathcal{L}(2n+2)$ whose restriction to any fibre $\mathcal{L}_\xi$ induces an isomorphism of the fundamental groups. It follows that as our Maslov covering $\mathcal{L}_U^\infty$, we can take the pullback $r^*(\mathcal{L}^\infty(2n+2))$ of the universal cover of $\mathcal{L}(2n+2)$.

τ^2 is the time-one map of the Hamiltonian flow $\phi_t(\xi) = \sigma_{t(2\psi(|\xi|)-1)}(\xi)$. Lift (ϕ_t) to an isotopy $(\tilde{\phi}_t)$ of $\mathcal{L}_U^\infty$-graded symplectic automorphisms, starting with $\tilde{\phi}_0 = \mathrm{id}$. By definition, ϕ_t is equal to the identity in a neighbourhood of the zero-section

$S^n \subset U$ for any t. This implies that $\tilde{\phi}_t(\tilde{S}^n) = \tilde{S}^n$ for any t and any $\mathcal{L}^\infty_U$-grading $\tilde{S}^n$ of S^n. On the other hand, $\tilde{\phi}_1$ acts as some shift $[k]$ on the part of $\mathcal{L}^\infty$ which lies over ∂U. Using the fact that $\mathcal{L}^\infty_U$ is defined as a pullback, it is not difficult to see that k can be computed as follows: choose a point $\xi \in \partial U$ and a Lagrangian subspace $\Lambda \subset TU_\xi$. Then $\lambda(t) = r(D\phi_t(\Lambda))$ is a loop in $\mathcal{L}(2n+2)$, and one has

$$k = -\langle C(2n+2), [\lambda]\rangle.$$

To determine k it is convenient to take Λ tangent to ∂U, because $\phi_t|\partial U$ agrees with the inverse of the standard diagonal circle action on $\mathbb{C}^{n+1}$. Then $\lambda(t) = e^{-2\pi i t}\Lambda \oplus e^{2\pi i t}(\mathbb{R}\bar{\xi})$, which means that $k = 2n-2$. We now know that $\tilde{\phi}_1 \circ [2-2n]$ is an $\mathcal{L}^\infty_U$-grading of τ^2 which acts trivially on the boundary of U. Therefore it must be the square $\tilde{\tau}^2$ of the preferred grading $\tilde{\tau}$. Hence $\tilde{\tau}^2(\tilde{S}^n) = \tilde{\phi}_1(\tilde{S}^n[2-2n]) = \tilde{S}^n[2-2n]$ which proves that $\tilde{\tau}(\tilde{S}^n) = \tilde{S}^n[1-n]$. □

The next two results are graded analogues of Lemma 5.4 and Lemma 5.5.

Lemma 5.8. *Assume that (l_1, l_2) form an (A_2)-configuration in M. Choose $\mathcal{L}^\infty$-gradings $\tilde{L}_1$, $\tilde{L}_2$ such that the index at the only intersection point $\{x_0\} = L_1 \cap L_2$ is $\tilde{I}(\tilde{L}_1, \tilde{L}_2; x_0) = 1$. Then $\tilde{\tau}_{l_1}(\tilde{L}_2)$ and $\tilde{\tau}_{l_2}^{-1}(\tilde{L}_1)$ are isotopic as $\mathcal{L}^\infty$-graded Lagrangian submanifolds.*

Proof. Let $\Sigma = L_1 \# L_2$ be the Lagrangian surgery, and $\tilde{\Sigma}$ the $\mathcal{L}^\infty$-grading from Lemma 2.14. Recall that $\tau_{l_1}(L_2)$ and Σ agree outside a neighbourhood of L_1. The gradings $\tilde{\tau}_{l_1}(\tilde{L}_2)$ and $\tilde{\Sigma}$ also agree there: this follows from the fact that $\tilde{\tau}_{l_1}$ is trivial away from L_1, and that the grading $\tilde{\Sigma}$ agrees with $\tilde{L}_2$ on $\Sigma \cap L_2$. In the proof of Lemma 5.4 we have used an isotopy from $\tau_{l_1}(L_2)$ to Σ which is concentrated near L_1. By considering a point which remains fixed, one sees that when one lifts this to an isotopy of graded Lagrangian submanifolds, it connects $\tilde{\tau}_{l_1}(\tilde{L}_2)$ with $\tilde{\Sigma}$. The same kind of argument shows that $\tilde{\Sigma}$ is isotopic to $\tilde{\tau}_{l_2}^{-1}(\tilde{L}_1)$. □

Lemma 5.9. *Assume that (l_1, l_2) is an (A_2)-configuration in M, and define $\tilde{g} = (\tilde{\tau}_{l_1}\tilde{\tau}_{l_2})^3$. Then for any $\mathcal{L}^\infty$-gradings $\tilde{L}_1$, $\tilde{L}_2$ one has*

$$\tilde{g}(\tilde{L}_1) \simeq_{\mathrm{gr}} \tilde{L}_1[4-3n], \quad \tilde{g}(\tilde{L}_2) \simeq_{\mathrm{gr}} \tilde{L}_2[4-3n],$$

where $\simeq_{\mathrm{gr}}$ stands for isotopy of $\mathcal{L}^\infty$-graded Lagrangian submanifolds.

Proof. Clearly, if the result holds for some grading of L_1, L_2 then it holds for all gradings. Hence we can assume that the absolute index $\tilde{I}(\tilde{L}_1, \tilde{L}_2; x_0)$ at $\{x_0\} = L_1 \cap L_2$ is 1. Lemma 5.8 says that $\tilde{\tau}_{l_1}(\tilde{L}_2) \simeq_{\mathrm{gr}} \tilde{\tau}_{l_2}^{-1}(\tilde{L}_1)$. Applying the same result to $\tilde{L}_2$ and $\tilde{L}_1[n-2]$, which satisfy $\tilde{I}(\tilde{L}_2, \tilde{L}_1[n-2]; x_0) = n - \tilde{I}(\tilde{L}_1[n-2], \tilde{L}_2; x_0) = n - \tilde{I}(\tilde{L}_1, \tilde{L}_2; x_0) - n + 2 = 1$, yields $\tilde{\tau}_{l_2}(\tilde{L}_1) \simeq_{\mathrm{gr}} \tilde{\tau}_{l_1}^{-1}(\tilde{L}_2)[2-n]$. Using these two equations and Lemma 5.6 one computes

$$\begin{aligned}(\tilde{\tau}_{l_1}\tilde{\tau}_{l_2})^3(\tilde{L}_2) &= (\tilde{\tau}_{l_1}\tilde{\tau}_{l_2})^2\tilde{\tau}_{l_1}(\tilde{L}_2)[1-n] \simeq_{\mathrm{gr}} (\tilde{\tau}_{l_1}\tilde{\tau}_{l_2})^2\tilde{\tau}_{l_2}^{-1}(\tilde{L}_1)[1-n] = \\ &= \tilde{\tau}_{l_1}\tilde{\tau}_{l_2}(\tilde{L}_1)[2-2n] \simeq_{\mathrm{gr}} \tilde{\tau}_{l_1}\tilde{\tau}_{l_1}^{-1}(\tilde{L}_2)[4-3n] = \tilde{L}_2[4-3n].\end{aligned}$$

The result for $\tilde{L}_1$ is proved in the same way. □

5c. **Symplectically knotted Lagrangian spheres.** After these preparations, we can now apply the 'graded' techniques to the construction of symplectically knotted Lagrangian spheres. The two parallel results obtained in this way (for even-dimensional and odd-dimensional spheres, respectively) are

Theorem 5.10. *Let (M^{2n},ω) be a compact symplectic manifold with contact type boundary, with n even, which satisfies $[\omega]=0$ and $2c_1(M,\omega)=0$. Assume that M contains an (A_3)-configuration (l_0,l_1,l_2) of Lagrangian spheres. Then M contains infinitely many symplectically knotted Lagrangian spheres. More precisely, if one defines $L_1^{(k)} = \tau_{l_2}^{2k}(L_1)$ for $k\in\mathbb{Z}$, then all the $L_1^{(k)}$ are isotopic as smooth submanifolds of M, but no two of them are isotopic as Lagrangian submanifolds.*

Theorem 5.11. *Let (M^{2n},ω) be a compact symplectic manifold with contact type boundary, with $n\geq 5$ odd, which satisfies $[\omega]=0$ and $2c_1(M,\omega)=0$. Assume that M contains an (A_4)-configuration (l_0,l_1,l_2,l_3) of Lagrangian spheres. Then M contains infinitely many symplectically knotted Lagrangian spheres. More precisely, if one defines $L_1^{(k)} = g^{2k}(L_1)$ for $k\in\mathbb{Z}$, where $g=(\tau_{l_2}\tau_{l_3})^3$, then all the $L_1^{(k)}$ are isotopic as smooth submanifolds of M, but no two of them are isotopic as Lagrangian submanifolds.*

Examples of manifolds satisfying these conditions are the Milnor fibres (5.1). Together these two Theorems prove the existence of infinitely many knotted Lagrangian n-spheres in any dimension $n\neq 1,3$. As mentioned in the Introduction, the case $n=2$ has been proved before in [28], and the proof given there would work in all even dimensions. Nevertheless, our new proof is substantially simpler.

Proof of Theorem 5.10. Let $\{x_0\} = L_0\cap L_1$ and $\{x_1\} = L_1\cap L_2$. Choose an ∞-fold Maslov covering $\mathcal{L}^\infty$ on M, and $\mathcal{L}^\infty$-gradings $\tilde{L}_0,\tilde{L}_1,\tilde{L}_2$ in such a way that $\tilde{I}(\tilde{L}_0,\tilde{L}_1;x_0) = \tilde{I}(\tilde{L}_1,\tilde{L}_2;x_1) = 0$. Set $\tilde{L}_1^{(k)} = \tilde{\tau}_{l_2}^{2k}(\tilde{L}_1)$. The assumptions on M imply that Floer cohomology (and its graded version) are well-defined for all (graded) Lagrangian submanifolds of M with vanishing first Betti number. Because $L_0\cap L_1$ and $L_1\cap L_2$ intersect in a single point, which has index zero, it follows from the definition of graded Floer cohomology that

$$HF^*(\tilde{L}_0,\tilde{L}_1) = \mathbb{Z}/2^{[0]},\quad HF^*(\tilde{L}_1,\tilde{L}_2) = \mathbb{Z}/2^{[0]}.$$

Here $\mathbb{Z}/2^{[k]}$ stands for the $\mathbb{Z}$-graded group which is $\mathbb{Z}/2$ in degree k and zero in other degrees. Since $L_0\cap L_2=\emptyset$, one can choose τ_{l_2} in such a way that it acts trivially in a neighbourhood of L_0. This implies that $L_0\cap L_1^{(k)} = L_0\cap L_1 = \{x_0\}$. Because the preferred grading $\tilde{\tau}_{l_2}$ also acts trivially near L_0, one has $\tilde{I}(\tilde{L}_0,\tilde{L}_1^{(k)};x_0) = \tilde{I}(\tilde{L}_0,\tilde{L}_1;x_0)=0$ and hence (for any k)

$$HF^*(\tilde{L}_0,\tilde{L}_1^{(k)}) = \mathbb{Z}/2^{[0]}. \tag{5.3}$$

On the other hand, using the invariance of graded Floer cohomology under graded symplectic automorphisms together with Lemma 5.6, one finds that

$$\begin{aligned} HF^*(\tilde{L}_1^{(k)},\tilde{L}_2) &\cong HF^*(\tilde{L}_1,\tilde{\tau}_{l_2}^{-2k}(\tilde{L}_2)) = \\ &= HF^*(\tilde{L}_1,\tilde{L}_2[2k(n-1)]) = \mathbb{Z}/2^{[2k(1-n)]}. \end{aligned} \tag{5.4}$$

Now assume that for some $L_1^{(k)}$, for some k, is Lagrangian isotopic to L_1. This implies that $\tilde{L}_1^{(k)}$ is graded Lagrangian isotopic to $\tilde{L}_1[r]$ for some $r\in\mathbb{Z}$. Using the isotopy invariance property of Floer cohomology one obtains

$$HF^*(\tilde{L}_0, \tilde{L}_1^{(k)}) \cong HF^*(\tilde{L}_0, \tilde{L}_1[r]) = \mathbb{Z}/2^{[-r]} \text{ and } HF^*(\tilde{L}_1^{(k)}, \tilde{L}_2) \cong \mathbb{Z}/2^{[r]}.$$

Comparing the first part of this equation with (5.3) yields $r = 0$. Comparing the second part with (5.4) yields $r = 2k(1-n) \neq 0$ and hence that $k = 0$. This means that $L_1^{(k)}$ is not Lagrangian isotopic to L_1 unless $k = 0$. Because of the way in which the $L_1^{(k)}$ are defined, it follows that no two of them are Lagrangian isotopic. The topological part of the theorem follows from Lemma 5.3(a). □

Proof of Theorem 5.11. Since this is very similar to the proof of Theorem 5.10, we will be more brief. Take an ∞-fold Maslov covering $\mathcal{L}^\infty$ and gradings $\tilde{L}_0, \tilde{L}_1, \tilde{L}_2$ such that $HF^*(\tilde{L}_0, \tilde{L}_1) = HF^*(\tilde{L}_1, \tilde{L}_2) = \mathbb{Z}/2^{[0]}$. Define $\tilde{g} = (\tilde{\tau}_{l_2}\tilde{\tau}_{l_3})^3$ and $\tilde{L}_1^{(k)} = \tilde{g}^{2k}(\tilde{L}_1)$. One can choose τ_{l_2}, τ_{l_3} in such a way that g and $\tilde{g}$ act trivially near L_0. This implies that $HF^*(\tilde{L}_0, \tilde{L}_1^{(k)}) = \mathbb{Z}/2^{[0]}$ for any k. Using Lemma 5.9 one computes $HF^*(\tilde{L}_1^{(k)}, \tilde{L}_2) \cong HF^*(\tilde{L}_1, \tilde{g}^{-2k}(\tilde{L}_2)) \cong HF^*(\tilde{L}_1, \tilde{L}_2[2k(3n-4)]) = \mathbb{Z}/2^{[2k(4-3n)]}$. The assumption that $\tilde{L}_1^{(k)}$, for some $k \neq 0$, is graded Lagrangian isotopic to $\tilde{L}_1[r]$, for some $r \in \mathbb{Z}$, leads to the contradiction $0 = r = 2k(4-3n)$. The topological part of the theorem is Lemma 5.3(b). □

5d. $K3$ **and Enriques surfaces.** Let (M, ω) be a closed symplectic four-manifold such that $c_1(M)$ is a torsion class. We want to consider Lagrangian spheres in M. This is a borderline case for Floer cohomology, where the conventional methods do not yield a completely satisfactory theory. The problems appear when one tries to prove that the Floer group is independent of the choice of almost complex structure. We will now explain what parts of Floer's construction can be salvaged from this breakdown. Throughout the whole of this section, *all Lagrangian submanifolds are assumed to be two-spheres.* For such an $L \subset M$, let $\mathcal{J}^{reg}(L)$ be the set of ω-compatible almost complex structures J such that there are no non-constant J-holomorphic maps $\mathbb{CP}^1 \to M$ or $(D^2, \partial D^2) \to (M, L)$.

Lemma 5.12. *$\mathcal{J}^{reg}(L)$ is a dense subset of the space of all ω-compatible almost complex structures.*

Proof. The virtual dimension of the space of J-holomorphic spheres in a homology class A is $4 + 2\langle c_1(M), A\rangle - 6 = -2$. Similarly, the virtual dimension of the space of J-holomorphic discs representing $B \in H_2(M, L)$ is $2 + \langle 2c_1(M, L), B\rangle - 3 = -1$. Here $2c_1(M, L) \in H^2(M, L)$ is the relative first Chern class, and we have used the assumption that $H^1(L) = 0$ to conclude that $c_1(M, L)$ is a torsion class. Standard transversality results say that for generic J there are no *simple* J-holomorphic spheres and no *somewhere injective* J-holomorphic discs. The first result implies that there are no non-constant J-spheres at all, because any such sphere covers a simple one. Similarly, the second result implies that there are no J-holomorphic discs. However, this time the argument is more complicated: it relies on the structure theorem of Kwon-Oh [13] or on the simpler form given by Lazzarini [15]. □

Let $L_0, L_1 \subset M$ be two Lagrangian submanifolds which intersect transversally. Fix $J_0 \in \mathcal{J}^{reg}(L_0)$, $J_1 \in \mathcal{J}^{reg}(L_1)$. We define $\mathcal{J}^{reg}(L_0, J_0, L_1, J_1)$ to be the space of smooth families $\mathbf{J} = (J_t)_{0 \leq t \leq 1}$ of compatible almost complex structures, connecting the given J_0, J_1, which satisfy the following conditions: (a) there are no non-constant J_t-holomorphic maps $\mathbb{CP}^1 \to M$ for any t; (b) any solution of Floer's equation, $u \in \mathcal{M}(x_-, x_+; \mathbf{J})$ for $x_-, x_+ \in L_0 \cap L_1$, is regular.

Lemma 5.13. *$\mathcal{J}^{reg}(L_0, J_0, L_1, J_1)$ is a dense subset of of the space of all families $\mathbf{J}$ which connect J_0 with J_1.*

The proof is again a combination of standard transversality arguments. For $\mathbf{J} \in \mathcal{J}^{reg}(L_0, J_0, L_1, J_1)$ one can define the Floer cohomology $HF(L_0, J_0, L_1, J_1; \mathbf{J})$ in the familiar way, using a suitable Novikov ring Λ as coefficient ring. The next step in the construction is

Lemma 5.14. *$HF(L_0, J_0, L_1, J_1; \mathbf{J})$ is independent of $\mathbf{J} \in \mathcal{J}^{reg}(L_0, J_0, L_1, J_1)$ up to canonical isomorphism.*

The proof uses the continuation equation $\partial_s u + \hat{J}_{s,t}(u)\partial_t u = 0$ for a two-parameter family $\hat{\mathbf{J}} = (\hat{J}_{s,t})$ of almost complex structures such that $\hat{J}_{s,0} = J_0$ and $\hat{J}_{s,1} = J_1$ for all $s \in \mathbb{R}$. There will be $\hat{J}_{s,t}$-holomorphic spheres (for isolated values of s, t) even if $\hat{\mathbf{J}}$ is generic, but these can be dealt with as in [11]. Following the usual custom, we will from now on omit $\mathbf{J}$ from the notation of Floer homology.

The problematic issue mentioned above is whether $HF(J_0, L_0, J_1, L_1)$ is independent of J_0, J_1. We will use only a special case, in which the result is obvious:

Lemma 5.15. *Assume that $L_0, L_1 \subset M$ intersect transversally in a single point. Then $HF(L_0, J_0, L_1, J_1) \cong \Lambda$ for all $J_0 \in \mathcal{J}^{reg}(L_0)$, $J_1 \in \mathcal{J}^{reg}(L_1)$.* □

By definition, Floer cohomology is invariant under symplectic automorphisms, in the sense that $HF(\phi(L_0), \phi_*(J_0), \phi(L_1), \phi_*(J_1)) \cong HF(L_0, J_0, L_1, J_1)$. The next result is a weak form of isotopy invariance.

Lemma 5.16. *Assume that L_0, L_1 are transverse and choose $J_0 \in \mathbf{J}^{reg}(L_0)$, $J_1 \in \mathbf{J}^{reg}(L_1)$. Let $\phi \in \mathrm{Aut}(M, \omega)$ be a map which is Hamiltonian isotopic to the identity, and such that $\phi(L_1)$ intersects L_0 transversally. Then $\phi_*(J_1) \in \mathcal{J}^{reg}(\phi(L_1))$ and*

$$HF(L_0, J_0, \phi(L_1), \phi_*(J_1)) \cong HF(L_0, J_0, L_1, J_1).$$

Outline of the proof. take a function $H \in C^\infty([0;1] \times M, \mathbb{R})$ with $H_t = 0$ for $t \leq 1/3$ or $t \geq 2/3$, such that the Hamiltonian isotopy (ϕ_t^H) generated by it satisfies $\phi_1^H = \phi^{-1}$. Let X_t be the Hamiltonian vector field of H_t, and $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ a cutoff function with $\psi(s) = 0$ for $s \leq 0$ and $\psi(s) = 1$ for $s \geq 1$. One considers finite energy solutions $u : \mathbb{R} \times [0;1] \longrightarrow M$ of the equation

$$\left\{ \begin{array}{l} \partial_s u + \hat{J}_{s,t}(u)(\partial_t u - \psi(s)X_t(u)) = 0, \\ u(s,0) \in L_0, \quad u(s,1) \in L_1. \end{array} \right. \tag{5.5}$$

Here $\hat{\mathbf{J}} = (\hat{J}_{s,t})$ is a two-parameter family of ω-compatible almost complex structures such that $\hat{J}_{s,0} = J_0$, $\hat{J}_{s,1} = J_1$ for all s. If one writes $u(s,t) = \phi_t^H(w(s,t))$ then (5.5) in the region $s \geq 1$ reads

$$\left\{ \begin{array}{l} \partial_s w + \hat{J}'_{s,t}(w)\partial_t w = 0, \\ w(s,0) \in L_0, \quad w(s,1) \in \phi(L_1). \end{array} \right.$$

where $\hat{J}'_{s,t} = (\phi_t^H)_*^{-1}\hat{J}_{s,t}$, in particular $\hat{J}'_{s,1} = \phi_* J_1$. Following the usual strategy, one can use solutions of (5.5) to define a map

$$HF(L_0, J_0, L_1, J_1) \longrightarrow HF(L_0, J_0, \phi(L_1), \phi_*(J_1)). \tag{5.6}$$

The main technical point is that there can be no bubbling off of holomorphic discs, since the almost complex structures $\hat{J}_{s,t}$ for $t = 0, 1$ do not admit such discs. Standard arguments of a similar kind show that (5.6) is an isomorphism. □

If $2c_1(M) = 0$, one can consider graded Lagrangian submanifolds, and obtains graded Floer groups $HF^*(\tilde{L}_0, J_0, \tilde{L}_1, J_1)$ with properties analogous to those above.

Theorem 5.17. *Let (M, ω) be a symplectic four-manifold with $2c_1(M, \omega) = 0$ and which contains an (A_3)-configuration (l_0, l_1, l_2) of Lagrangian two-spheres. Define $L_1^{(k)} = \tau_{l_2}^{2k}(L_1)$. Then all the $L_1^{(k)}$ are isotopic as smooth submanifolds, but no two of them are isotopic as Lagrangian submanifolds.*

Proof. This is the same argument as in Theorem 5.10, except that one has to be more careful about the properties of Floer cohomology. Choose an ∞-fold Maslov covering $\mathcal{L}^\infty$ and gradings $\tilde{L}_k$ such that $\tilde{I}(\tilde{L}_0, \tilde{L}_1; x_0) = \tilde{I}(\tilde{L}_1, \tilde{L}_2; x_1) = 0$, where x_0, x_1 are the unique intersection points. Fix some $k \neq 0$ and write $L_1' = \tau_{l_2}^{2k}(L_1)$, $\tilde{L}_1' = \tilde{\tau}_{l_2}^{2k}(\tilde{L}_1)$. Assume that $L_1 \simeq L_1'$, so that $\tilde{L}_1[r] \simeq_{\mathrm{gr}} \tilde{L}_1'$ for some $r \in \mathbb{Z}$. One can embed the Lagrangian isotopy into a Hamiltonian isotopy (ϕ_t) of M. The graded analogue of Lemma 5.16 shows that

$$\begin{aligned} HF^*(\tilde{L}_0, J_0, \tilde{L}_1, J_1) &\cong HF^{*-r}(\tilde{L}_0, J_0, \tilde{L}_1', (\phi_1)_*(J_1)), \\ HF^*(\tilde{L}_1, J_1, \tilde{L}_2, J_2) &\cong HF^{*+r}(\tilde{L}_1', (\phi_1)_*(J_1), \tilde{L}_2, J_2) \end{aligned}$$

for all $J_m \in \mathcal{J}^{reg}(L_m)$, $m = 0, 1, 2$. On the other hand, by using the graded version of Lemma 5.15 and arguing as in the proof of Theorem 5.10 one finds that

$$\begin{aligned} &HF^*(\tilde{L}_0, J_0, \tilde{L}_1, J_1) = HF^*(\tilde{L}_1, J_1, \tilde{L}_2, J_2) = \Lambda^{[0]}, \\ &HF^*(\tilde{L}_0, J_0, \tilde{L}_1', (\phi_1)_*(J_1)) = \Lambda^{[0]}, \quad HF^*(\tilde{L}_1', (\phi_1)_*(J_1), \tilde{L}_2, J_2) = \Lambda^{[-2k]}, \end{aligned}$$

which leads to a contradiction. □

Corollary 5.18. *For (M, ω) as in Theorem 5.17, the map $\pi_0(\mathrm{Aut}(M, \omega)) \longrightarrow \pi_0(\mathrm{Diff}(M))$ has infinite kernel.* □

Proof. We have already quoted the fact that $\tau_{l_2}^2$ is trivial in $\pi_0(\mathrm{Diff}(M))$ [28, Lemma 6.3]. Theorem 5.17 obviously implies that $[\tau_{l_2}^2] \in \pi_0(\mathrm{Aut}(M, \omega))$ has infinite order. □

Example 5.19. Recall that an Enriques surface is an algebraic surface with fundamental group $\mathbb{Z}/2$, whose universal cover is a $K3$ surface. Enriques surfaces satisfy $2c_1 = 0$. We will now construct an Enriques surface which contains an (A_3)-configuration of Lagrangian spheres. Consider the quartic surface $X \subset \mathbb{CP}^3$ defined by

$$(x_0^2 + x_1^2)^2 + x_0^2 x_2^2 + x_2^4 + x_3^4 = 0. \tag{5.7}$$

X has two singular points $[1 : \pm i : 0 : 0]$ of type (A_3). Let Y be the minimal resolution of singularities of X. It is a $K3$ surface and hence has a holomorphic symplectic form Ω. Complex conjugation defines an anti-holomorphic involution on

X, which is free because (5.7) has no nonzero real solutions. Because of the uniqueness of minimal resolutions, this involution lifts to an anti-holomorphic involution on Y, which we denote by ι. Obviously ι is again free. Since the holomorphic symplectic form is unique up to a constant, it satisfies $\iota^*\Omega = z\bar{\Omega}$ for some $z \in S^1$. By rescaling Ω we can assume that $z = 1$. Then $\omega = \mathrm{re}\,\Omega$ descends to a real symplectic form on the quotient $M = Y/\iota$. The singular points of X give rise to two disjoint (A_3)-configuration of rational curves in Y, which are exchanged by ι. These curves are Lagrangian with respect to ω, hence descend to a single (A_3)-configuration of Lagrangian spheres on M. To see that M is Kähler one argues as follows: let $\beta \in \Omega^{1,1}$ be a positive form on Y such that $\iota^*\beta = -\beta$. Such a form can be constructed e.g. by averaging. Let g be the unique Ricci-flat Kähler metric which has the same Kähler class as β. The uniqueness theorem for such metrics implies that g is ι-invariant. The metric g is hyperkähler, which means that there are complex structures J and K such that $\Omega(v,w) = g(v,Jw) + ig(v,Kw)$. It follows that J is ι-invariant and hence descends to a complex structure on M which is compatible with ω. This means that M is Kähler and in fact an Enriques surface.

Example 5.20. $K3$ surfaces have $c_1 = 0$. As an example of a $K3$ surface containing an (A_3)-configuration of Lagrangian spheres one can take the manifold $(Y, \mathrm{re}\,\Omega)$ constructed in the previous example.

References

[1] V. I. Arnol'd, *Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities*, Funct. Anal. Appl. **6** (1972), 254–272.
[2] A. Besse, *Manifolds all of whose geodesics are closed*, Erg. der Mathematik und ihrer Grenzgebiete, vol. 93, Springer, 1978.
[3] E. Brieskorn, *Die Auflösung rationaler Singularitäten holomorpher Abbildungen*, Math. Ann. **178** (1968), 255–270.
[4] J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), Lecture Notes in Math., vol. 53, Springer, 1968.
[5] S. Dostoglou and D. Salamon, *Self dual instantons and holomorphic curves*, Annals of Math. **139** (1994), 581–640.
[6] J. Duistermaat, *On the Morse index in variational calculus*, Advances in Math. **21** (1976), 173–195.
[7] A. Floer, *A relative Morse index for the symplectic action*, Comm. Pure Appl. Math. **41** (1988), 393–407.
[8] ———, *Witten's complex and infinite dimensional Morse theory*, J. Differential Geom. **30** (1989), 207–221.
[9] A. Haefliger, *Plongements différentiables des variétés dans variétés*, Comm. Math. Helv. **36** (1962), 47–82.
[10] ———, *Knotted spheres and related geometric problems*, Proceedings of the ICM, Moscow, 1966, pp. 437–445.
[11] H. Hofer and D. Salamon, *Floer homology and Novikov rings*, The Floer memorial volume (H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, eds.), Progress in Mathematics, vol. 133, Birkhäuser, 1995, pp. 483–524.
[12] W. Klingenberg, *Riemannian geometry*, De Gruyter, 1982.
[13] D. Kwon and Y.-G. Oh, *Structure of the image of (pseudo)-holomorphic discs with totally real boundary conditions*, Preprint (final version).
[14] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton Univ. Press, 1989.
[15] L. Lazzarini, *Existence of a somewhere injective pseudo-holomorphic disc*, Preprint, December 1998.
[16] J. Milnor, *Singular points of complex hypersurfaces*, Princeton Univ. Press, 1968.
[17] J. Munkres, *Differentiable isotopies of the 2-sphere*, Notices of the Amer. Math. Soc. **5** (1958), 582.

[18] Y.-G. Oh, *On the structure of pseudo-holomorphic discs with totally real boundary conditions*, Preprint (revised version, May 1996).

[19] ———, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs I*, Comm. Pure Appl. Math. **46** (1993), 949–994.

[20] ———, *Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings*, Int. Math. Res. Notices (1996), 305–346.

[21] L. Polterovich, *Surgery of Lagrange submanifolds*, Geom. Funct. Anal. **1** (1991), 198–210.

[22] M. Poźniak, *Floer homology, Novikov rings and clean intersections*, Ph.D. thesis, University of Warwick, 1994.

[23] J. Robbin and D. Salamon, *The Maslov index for paths*, Topology **32** (1993), 827–844.

[24] ———, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27** (1995), 1–33.

[25] D. Salamon and E. Zehnder, *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*, Comm. Pure Appl. Math. **45** (1992), 1303–1360.

[26] P. Seidel, *Floer homology and the symplectic isotopy problem*, Ph.D. thesis, Oxford University, 1997.

[27] ———, π_1 *of symplectic automorphism groups and invertibles in quantum homology rings*, Geom. Funct. Anal. **7** (1997), 1046–1095.

[28] ———, *Lagrangian two-spheres can be symplectically knotted*, Preprint, 1998.

[29] C. Viterbo, *Intersection des sous-variétés Lagrangiennes, fonctionelles d'action et indice des systèmes Hamiltoniens*, Bull. Soc. Math. France **115** (1987), 361–390.