Rigidity of generalized Verma modules

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Abstract

We prove that generalized Verma modules, induced from generic Gelfand-Zetlin modules, and generalized Verma modules, associated with Enright-complete modules, are rigid. Their Loewy length and quotients of the unique Loewy filtrations are calculated for the regular block of the corresponding category $\mathcal{O}(\mathfrak{p}, \Lambda)$.

1 Introduction

Let $\mathfrak{g}$ be a semi-simple complex finite-dimensional Lie algebra with a fixed triangular decomposition, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$ be its parabolic subalgebra. In what follows there will be a distinguished special case, namely when the semi-simple part $\mathfrak{a}$ of the Levi factor $\mathfrak{a}'$ is isomorphic to a direct product of some $\mathfrak{sl}(n_i, \mathbb{C})$. This will be assume all the time when we will discuss generic Gelfand-Zetlin modules. Let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{p}$.

This paper continues the study of certain parabolic generalizations, $\mathcal{O}(\mathfrak{p}, \Lambda)$, of the celebrated BGG category $\mathcal{O}$ of $\mathfrak{g}$-modules ([BGG2]). These categories, associated with $\mathfrak{p}$ and certain admissible category Λ of $\mathfrak{a}$-modules, were introduced in [FKM1], where it was shown that their blocks correspond to the so-called standardly stratified algebras in the sense of [CPS] (and even to the smaller class of properly stratified algebras, which was recently introduced in [Dl]). In particular, there is an analogue of the BGG-reciprocity formula, which involves besides the indecomposable projective modules and simple modules a class of intermediate modules, which are called generalized Verma modules. Later in [FKM2, FKM3] the algebra of the principal block of $\mathcal{O}(\mathfrak{p}, \Lambda)$ was given a combinatorial description, analogous to Soergel's description of $\mathcal{O}$, [S].

The basic example of $\mathcal{O}(\mathfrak{p}, \Lambda)$ was constructed in terms of the so-called generic Gelfand-Zetlin modules (see [DOF1]). In [KM1, KM2] this example was related to a certain subcategory of $\mathcal{O}$, which can be described in terms of Enright's completion functors ([E]). The last categories carry a "strange" abelian structure, which is not inherited from that on $\mathcal{O}$. Where this structure appears from was explained in [KM2] after realizing (Mathieu's version, [M], of) Enright's completion functor via the approximation functor with respect to a suitably chosen injective module in $\mathcal{O}$.

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From [I2, I3] (combined with the reduction to integral case in [S]) it is known that all Verma modules and big projective modules in the principal blocks of $\mathcal{O}$ (this can be viewed as the first extremal case of $\mathcal{O}(\mathfrak{p},\Lambda)$, which corresponds to $\mathfrak{a}=0$) are rigid, i.e. their socle and radical filtrations coincide and form the unique Loewy filtration (the shortest filtration with semi-simple quotients). In [BGS] this rigidity result was reobtained and extended to all Verma modules using the machinery of Koszul rings and some deep geometrical results. Let $\mathcal{O}(\mathfrak{p},\Lambda)$ be the basic example mentioned above. The blocks of $\mathcal{O}(\mathfrak{p},\Lambda)$ contain generalized Verma modules induced from generic Gelfand-Zetlin modules and big projective modules. So, there arise a natural question: are these modules also rigid? By [KM1] there is a full functor, say $\mathcal{F}$, from $\mathcal{O}(\mathfrak{p},\Lambda)$ to $\mathcal{O}$, which sends each generalized Verma module to a Verma module and each big projective (in $\mathcal{O}(\mathfrak{p},\Lambda)$) to a big projective in $\mathcal{O}$. The main problem is that the abelian structure on the image of $\mathcal{F}$ does not coincide with the abelian structure on $\mathcal{O}$. Hence the results of [I2, I3, BGS] can not be applied directly.

The aim of the paper is to give a positive answer to the first part of the question above. Our main result is the following.

Theorem 1. *Let Λ be the admissible category, generated by a simple generic Gelfand-Zetlin module, or the admissible category of Enright-complete modules, and $\mathcal{O}(\mathfrak{p},\Lambda)$ be the corresponding parabolic analogue of $\mathcal{O}$, [FKM1]. Then generalized Verma modules in $\mathcal{O}(\mathfrak{p},\Lambda)$ are rigid.*

In contrast with the classical case, the big projective modules in $\mathcal{O}(\mathfrak{p},\Lambda)$ fail to be rigid in the general case. We present an $sl(3)$-counterexample in Section 4. We also remark that, in the second extremal case, namely $\mathfrak{g}=\mathfrak{a}$, all GVMs are simple, hence rigid of Loewy length 1 and the rigidity of the big projective module (=the unique indecomposable projective in the principal block, =the standard module) was proved already in [KM1].

The paper is organized as follows. The next section contains all necessary preliminary information, in particular, we define $\mathcal{O}(\mathfrak{p},\Lambda)$ and recall how the situation is related to the category $\mathcal{O}$. Section 3 is devoted to the study of Loewy series on generalized Verma modules. Here we prove rigidity, calculate Loewy length and quotients of the unique Loewy filtration. In Section 4 we compute an $\mathfrak{sl}(3,\mathbb{C})$-example, and, in particular, give an example of a non-rigid big projective module in $\mathcal{O}(\mathfrak{p},\Lambda)$. We finish the paper with a short discussion of properties of standard modules in Section 5.

2 Notation and preliminary results

2.1 Generalized Verma modules and generic Gelfand-Zetlin modules

By [KM1], the category constructed here is a special case of the categories considered in the next subsection (we recall that for Gelfand-Zetlin modules we assume that $\mathfrak{a}$ is a product of $\mathfrak{sl}(n_i,\mathbb{C})$). The advantage of this construction is that for these categories the notions of simple objects and simple $\mathfrak{g}$-modules coincide.

Let V be an $\mathfrak{a}'$-module. Extending it trivially to a $\mathfrak{p}$-module we can form the module $M_{\mathfrak{p}}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$, which is called a *generalized Verma module* (GVM in the sequel), provided V is simple.

Let $[l] = (l_{i,j})_{i=1,\dots,n}^{j=1,\dots,i}$ be a *tableau* with complex entries such that $l_{i,j} - l_{i,k} \notin \mathbb{Z}$ for all $i < n$ and all possible j, k, and $V([l])$ be the corresponding *generic Gelfand-Zetlin* $\mathfrak{gl}(n,\mathbb{C})$-*module* (generic GZ-module in the sequel), as defined in [DOF1]. $V([l])$ can be restricted in a natural way to the canonical copy of $\mathfrak{sl}(n,\mathbb{C}) \subset \mathfrak{gl}(n,\mathbb{C})$ and thus we can talk about generic GZ-modules over $\mathfrak{sl}(n,\mathbb{C})$. Now over a direct product of some $\mathfrak{sl}(n_i,\mathbb{C})$ (e.g. over $\mathfrak{a}$) we define generic GZ-modules as tensor product of the above generic GZ-modules over the components.

Fix a simple generic GZ-module, V, over $\mathfrak{a}$ and consider the category Λ of all subquotients in $V \otimes F$, where F runs through all finite-dimensional $\mathfrak{gl}(n,\mathbb{C})$-module. This category extends in a natural diagonalizable way to $\mathfrak{a}'$. The blocks of this category will be module categories over local finite-dimensional associative algebras, [FKM1]. For a simple object, $V \in \Lambda$, we denote by $\tilde{V}$ the projective cover of V.

Following [FKM1] we define $\mathcal{O}(\mathfrak{p},\Lambda)$ as the full subcategory of the category of all $\mathfrak{g}$-modules, which consists of finitely-generated $\mathfrak{n}$-locally finite $\mathfrak{g}$-modules, which decompose into a direct sum of modules from Λ, when viewed as $\mathfrak{a}'$-modules. The blocks of $\mathcal{O}(\mathfrak{p},\Lambda)$ are module categories over properly stratified finite-dimensional associative algebras, [FKM1]. GVMs $M_{\mathfrak{p}}(V)$, V is a simple generic GZ-module over $\mathfrak{a}'$, are objects of $\mathcal{O}(\mathfrak{p},\Lambda)$ and they are also *proper standard modules* for the corresponding properly stratified algebras. The modules $\Delta(V) = M_{\mathfrak{p}}(\tilde{V})$, V is simple in Λ, are *standard modules* for $\mathcal{O}(\mathfrak{p},\Lambda)$.

2.2 Enright completions and $\mathcal{S}$-subcategories in $\mathcal{O}$

Let $\mathfrak{R}$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and π be the basis of $\mathfrak{R}$, which corresponds to our fixed triangular decomposition. Fix some Weyl-Chevalley basis, $\{X_\alpha | \alpha \in \mathfrak{R}\}$, $\{H_\alpha | \alpha \in \pi\}$, in $\mathfrak{g}$. Denote by W the Weyl group of $\mathfrak{R}$ and by s_α the reflection, corresponding to $\alpha \in \mathfrak{R}$. We denote by $l : W \to \mathbb{Z}_+$ the length function and by $\hat{w}$ the unique longest element of W.

For a simple root, α, denote by U_α the localization of $U(\mathfrak{g})$ with respect to $\{X_{-\alpha}^k | k \in \mathbb{N}\}$ (see [M]) and define the completion functor $E_\alpha : \mathcal{O} \to \mathcal{O}$ as the composition of $U_\alpha \otimes_{U(\mathfrak{g})} {}_-$, ${}_-|_{U(\mathfrak{g})}$ and the functor of taking the locally X_α-finite part of a module (see [FKM2, FKM3, KM1, KM2]). Clearly $E_\alpha \circ E_\alpha = E_\alpha$.

For $S \subset \pi$ let $\mathcal{O}(S)$ denote the full subcategory of $\mathcal{O}$, which consists of all modules on which all $X_{-\alpha}$, $\alpha \in S$, act injectively. Then for $\alpha, \beta \in S$ the restrictions of E_α and E_β on $\mathcal{O}(S)$ satisfy braid relations (see [De] or [KM2] for a short proof). Hence, if $s_{\alpha_1} \dots s_{\alpha_k} = w \in W_S = \langle s_\alpha | \alpha \in S \rangle$ the functor $E_w = E_{\alpha_1} \circ \dots \circ E_{\alpha_k} : \mathcal{O}(S) \to \mathcal{O}(S)$ is well-defined. Let $w_S \in W_S$ be the longest element. The functor E_{w_S} is called the S-*completion* functor and a module, $M \in \mathcal{O}$, is called S-*complete* if $M \in E_{w_S}(\mathcal{O}(S))$. The category $\mathcal{O}(S)_{st}$ of all S-complete modules is an abelian category with usual kernels and cokernels defined as follows: if $f : M \to N$, $M, N \in \mathcal{O}(S)_{st}$, then the cokernel of f is $E_{w_S}(N/E_{w_S}(f(M)))$ ([FKM3], see [KM2] for another description in terms of approximation with respect to an injective module).

Now let S be the set of simple roots of $\mathfrak{a}$. Then, by [KM1], for appropriate $\mathfrak{a}$, there exists a blockwise equivalence of categories, $\mathcal{F}:\mathcal{O}(\mathfrak{p},\Lambda)\to\mathcal{O}(S)_{st}$, which sends GVMs to Verma modules. Again by [KM1], the functor $\mathcal{F}^{-1}:\mathcal{O}(S)_{st}\to\mathcal{O}(\mathfrak{p},\Lambda)$ is also well-defined. In particular, we can set $\Lambda=\mathcal{O}(S)_{st}$ and consider $\mathcal{O}(\mathfrak{p},\Lambda)$ in the case of arbitrary $\mathfrak{a}$. We emphasize once more that the advantage of Gelfand-Zetlin modules is that in this case simple objects of $\mathcal{O}(\mathfrak{p},\Lambda)$ are simple $\mathfrak{g}$-modules. Finally, we recall the following properties of E_{w_S}, [FKM3]:

1. for a simple $L\in\mathcal{O}$ the module E_{w_S} is either 0 or simple in $\mathcal{O}(S)_{st}$;
2. $E_{w_S}(M)\subset E_{w_S}(N)$ for $M\subset N$;
3. $E_{w_S}(N/M)\supset E_{w_S}(N)/E_{w_S}(M)$ for $M\subset N$.

2.3 Loewy length and Loewy filtrations

Let M be a module of finite length. A filtration, $0\subset M_1\subset\cdots\subset M_k=M$ is called a *Loewy filtration* if all M_i/M_{i-1} are semi-simple and k is the minimal possible. This k is called the *Loewy length* of M and is denoted by $ll(M)$ (if M is an object of two different categories and we want to underline that it's Loewy length is considered inside, say the category A, we will write $ll_A(M)$). Associated with M there are two Loewy filtrations: the socle filtration $0\subset\operatorname{soc}^1 M\subset\operatorname{soc}^2 M\subset\cdots\subset\operatorname{soc}^{ll(M)}=M$ (here $\operatorname{soc}^{i+1}M/soc^i M=\operatorname{soc}(M/\operatorname{soc}^i M)$) and the radical filtration $0=\operatorname{rad}^{ll(M)}M\subset\cdots\subset\operatorname{rad}^1 M\subset\operatorname{rad}^0 M=M$ (here $\operatorname{rad}^i(M)=\operatorname{rad}(\operatorname{rad}^{i-1}M)$). If $0\subset M_1\subset\cdots\subset M_k=M$ is a Loewy filtration of M, there holds $\operatorname{rad}^{ll(M)-i}M\subset M_i\subset\operatorname{soc}^i M$. The module M is called *rigid* if the socle filtration and the radical filtration coincide and thus there is only one Loewy filtration. We will denote the layers of the socle and the radical filtrations by $\operatorname{soc}_i M=\operatorname{soc}^i M/\operatorname{soc}^{i-1}M$ and $\operatorname{rad}_i M=\operatorname{rad}^i M/\operatorname{rad}^{i+1}M$ correspondingly.

2.4 Soergel's combinatorics of $\mathcal{O}$ and its generalization to $\mathcal{O}(\mathfrak{p},\Lambda)$

For $\lambda\in\mathfrak{h}^*$ we will denote by $M(\lambda)$ the Verma module with highest weight λ, by $L(\lambda)$ the unique simple quotient of $M(\lambda)$, by $P(\lambda)$ the indecomposable projective cover of $L(\lambda)$ and by $I(\lambda)$ the indecomposable injective envelope of $L(\lambda)$. If ρ is the half-sum of all positive root, we recall the standard $\cdot$-action of W on $\mathfrak{h}^*$, defined as follows: $w\cdot(\lambda)=w(\lambda+\rho)-\rho$.

Let χ be an integral central character of $\mathfrak{g}$ and $\mathcal{O}_\chi$ be the block of $\mathcal{O}$, corresponding to χ. By the main result of [S], any indecomposable block of $\mathcal{O}$ is equivalent to $\mathcal{O}_\chi$ for some χ, however, this equivalence may force to change $\mathfrak{g}$. The character χ thus corresponds to an integral W-orbit, $W\cdot\lambda$, where λ is antidominant. Let W_λ denote the stabilizer of λ. Then the simples in $\mathcal{O}_\chi$ are naturally parameterized by cosets $\xi\in W/W_\lambda$. If $w\in\xi$ is the longest element, then we set $L(\xi)=L(w\cdot\lambda)$. Analogously we define $P(\xi)=P(w\cdot\lambda)$, $\Delta(\xi)=\Delta(w\cdot\lambda)$ and $M(\xi)=M(w\cdot\lambda)$. Set $P_\lambda=\oplus_{\xi\in W/W_\lambda}P(\xi)$ and $A_\lambda=\operatorname{End}_{\mathcal{O}}(P_\lambda)$. A_λ is a basic, finite-dimensional, associative and quasi-hereditary algebra. The Verma modules $M(\xi)$ are standard modules with respect to the quasi-hereditary structure. Set $C_\lambda = \mathrm{End}_{\mathcal{O}}(P(\lambda))$. By [S, Endomorphismensatz 7], the algebra C_λ is the subalgebra of W_λ-invariants in the coinvariant algebra C, which is the quotient of $\mathbb{C}[\mathfrak{h}^*]$ modulo the ideal generated by W-invariant, with respect to the usual W-action, polynomials without constant term. In particular, $C_\lambda = C$ if λ is regular. Let e be the primitive idempotent of A_λ, corresponding to $\xi = W_\lambda$ (which means $C_\lambda = \mathrm{End}_{A_\lambda Ae}$). Then, by [S, Struktursatz 9], $A_\lambda \simeq \mathrm{End}(A_\lambda e_{C_\lambda})$. This is usually called the *double centralizer property*. For a primitive idempotent, f, of A_λ we will denote the C_λ-module fAe by D_f.

Now we recall the analogous description of $\mathcal{O}(\mathfrak{p}, \Lambda)$, the latter as defined in Subsection 2.2 ([FKM2, FKM3]). For this we consider the equivalent category $\mathcal{O}(S)_{st}$ and set $\mathcal{O}_\lambda^S = \mathcal{O}(S)_{st} \cap \mathcal{O}_\lambda$. We emphasize that the abelian structure on $\mathcal{O}_\lambda^S$ is not inherited from that on $\mathcal{O}_\lambda$. The simples of $\mathcal{O}_\lambda^S$ are indexed by those $\xi \in W/W_\lambda$ such that the longest representative w of ξ is at the same time the shortest element in a coset from W/W_S. Let $T = \{\xi_1, \dots, \xi_k\}$ be the complete set of parameters of simples in $\mathcal{O}_\lambda^S$. Set $P_\lambda^S = \oplus_{\xi \in T} P(\xi)$ and $A_\lambda^S = \mathrm{End}_{\mathfrak{g}}(P_\lambda^S)$. Denote by e^S the primitive idempotent of A_λ^S corresponding to $P(\lambda)$. Then, by [FKM3, Section 5], we have $A_\lambda^S = \mathrm{End}((A_\lambda^S e^S)_{C_\lambda})$.

3 Rigidity of generalized Verma modules

In this section we prove the main result of this paper, the rigidity of GVMs. Our proof follows the ideas of [BGS]. In fact, we are going to realize GVMs as graded modules over graded algebras and then apply the following lemma ([BGS, Proposition 2.4.1]).

Lemma 1. *Let R be a $\mathbb{Z}$-graded ring, generated by R_0 and R_1, such that R_0 is semi simple. Let M be a graded R-module of finite length. If $\mathrm{soc}(M)$ (resp. $\mathrm{rad}(M)$) is simple, then the socle (resp. radical) filtration on M coincides with the grading filtration, up to shift.*

The coinvariant algebra C is graded in a natural way, and we fix a grading, in which the generators $(\mathfrak{h}^*)$ have degree 2. Hence the algebra C_λ is graded as well. Both modules $A_\lambda e$ and $A_\lambda^S e^S$ are graded C_λ-modules (see Remark after [S, Lemma 7]). Moreover, as $A_\lambda^S = tA_\lambda t$ for some, in general case non-primitive, idempotent t of A_λ, we can naturally consider $A_\lambda^S e^S$ as a graded submodule of $A_\lambda e$. The decomposition $A_\lambda e = \oplus_f D_f$, where f runs through the set of all primitive idempotents of A_λ, is a decomposition of $A_\lambda e$ into a direct sum of indecomposable modules and all summands are graded. If f corresponds to $L(\xi)$, $\xi \in W/W_\lambda$, then the minimal non-zero degree of D_f is exactly $ll(M(\xi))$. Via the double centralizer property we get a positive grading on A_λ, which corresponds to the "mixed" structure on $\mathcal{O}$. By [BGS, Section 4], this grading is Koszul, in particular, $(A_\lambda)_0$ is semi simple and A_λ is generated by $(A_\lambda)_1$ over $(A_\lambda)_0$.

Now we want to move this picture to A_λ^S, which is a positively graded subalgebra of A_λ via the double centralizer property. This trivially implies that $(A_\lambda^S)_0$ is semi simple as well. Unfortunately, in the general case A_λ^S **is not** generated by $(A_\lambda^S)_1$ over $(A_\lambda^S)_0$. We refer the reader to Section 4 for the corresponding $\mathfrak{sl}(3, \mathbb{C})$-example. However, we can find some convenient set of generators for A_λ^S.

Since the algebra C_λ is commutative the double centralizer property implies that it is a subalgebra of A_λ^S and, moreover, is a subalgebra of the center of A_λ^S. We also recall that C_λ appears in [S] as the image of the action of $Z(\mathfrak{g})$ on $P(\lambda)$.

Proposition 1. *A_λ^S is generated by $(A_\lambda^S)_0$, $(A_\lambda^S)_1$ and C_λ.*

Proof. Let f_1 and f_2 be two primitive idempotents of A_λ^S. Our aim is to decompose any graded element from $f_1 A_\lambda^S f_2$ into a product of elements from $(A_\lambda^S)_0$, $(A_\lambda^S)_1$ and C_λ. But any element from $f_1 A_\lambda^S f_2$ corresponds to a map from an indecomposable projective $P(\xi_1)$ to indecomposable projective $P(\xi_2)$ in $\mathcal{O}(\mathfrak{p},\Lambda)$. from the definition of projective modules it follows that the map $\varphi : P(\xi_1) \to P(\xi_2)$ can not be decomposed non-trivially into a product of other maps between projectives only in the case, when $\varphi(P(\xi_1)) \not\in \mathrm{rad}^2(P(\xi_2))$. Hence, it is enough to prove that, under this assumption, we have $\varphi \in (A_\lambda^S)_0$ or $\varphi \in (A_\lambda^S)_1$ or $\varphi \in C_\lambda$. We have to consider several cases.

First, let $\xi_1 = \xi_2$ and φ be an isomorphism. Then the assumption that φ is a graded element implies $\varphi \in (A_\lambda^S)_0$.

Now, let $\xi_1 \neq \xi_2$. Because of the duality on $\mathcal{O}(\mathfrak{p},\Lambda)$ we may assume $\xi_1 \not\leq \xi_2$ with respect to the order of properly stratified structure. Then, necessarily $L(\xi_1)$ is a top of $\mathrm{rad}(P(\xi_2))$. Now we recall that, according to [FKM1], all projectives in $\mathcal{O}(\mathfrak{p},\Lambda)$ are filtered by standard modules. Because of $\xi_1 \not\leq \xi_2$ and the BGG-reciprocity, [FKM1], the standard module $\Delta(\xi_1)$ occurs in this filtration. Hence there should exist a quotient of $P(\xi_2)$, which is an extension of $\Delta(\xi_1)$ by $\Delta(\xi_2)$, such that the unique copy of $L(\xi_1)$ in the top of $\Delta(\xi_1)$ is exactly the one, covered by φ. But this means that φ covers the whole $\Delta(\xi_1)$ as it has a simple top. So, the image of φ does not belong to $\mathrm{rad}(P(\xi_2))$ in the category $\mathcal{O}$ as well. This implies that φ is a degree 1 map in A_λ, as it is graded. Hence $\varphi \in (A_\lambda^S)_1$.

Finally, we consider the case $\xi_1 = \xi_2 = \xi$ and $\varphi(P(\xi)) \subset \mathrm{rad}(P(\xi))$. Again we recall that $P(\xi)$ is filtered by standard modules and there is only one of them (the last one), isomorphic to $\Delta(\xi)$. As all other standard modules have different tops, φ maps the top of $P(\xi)$, which is also the top of $\Delta(\xi)$, into some other copy of $L(\xi)$ in $\Delta(\xi)$. Our claim will easily follow if we prove that the natural action of C_λ on $\Delta(\xi)$ surjects onto $\mathrm{End}_{A_\lambda}(\Delta(\xi))$. Realize $\Delta(\xi)$ as $M_{\mathfrak{p}}(\tilde{V})$, where $\tilde{V}$ is projective in Λ. Then the exactness of induction guarantees $\mathrm{End}_{A_\lambda}(\Delta(\xi)) = \mathrm{End}_\Lambda(\tilde{V})$. The S-Harish-Chandra homomorphism, [DOF2], restricts the action of $Z(\mathfrak{g})$ on $\Delta(\xi)$ to the action of $Z(\mathfrak{a})$ on $\tilde{V}$, and the statement follows from [S, Endomorphismensatz]. This completes the proof. $\square$

Lemma 2. *$(C_\lambda)_{>0}$ annihilates $M(\xi)$.*

Proof. The action of C_λ comes from the action of $Z(\mathfrak{g})$, which acts by scalars on $M(\xi)$, [FC, DOF2]. $\square$

Lemma 3. *$M(\xi)$ is a graded A_λ^S-module.*

Proof. It follows directly from [BGS], as $M(\xi)$ is a graded A_λ-module and A_λ^S is a graded subalgebra of A_λ. But one can also get this from our graded picture and the double-centralizer. Obviously, $A_\lambda^S e^S$ is a positively graded finite-dimensional A_λ^S-module. Moreover, it is filtered by GVMs and, by BGG-reciprocity, [FKM1], each GVM from $\mathcal{O}_\lambda^S$ occurs as a subquotient of $A^S_\lambda e^S$. The biggest GVM is naturally identified with $\mathrm{soc}(A^S_\lambda e^S)$, the latter viewed as C_λ-module, hence graded. The statement follows by induction. $\square$

We are now ready to state our main result.

Theorem 2. *Any GVM $M(\xi)$ in $\mathcal{O}^S_\lambda$ is rigid.*

Proof. By Lemma 3, $M(\xi)$ is a graded module over graded algebra A^S_λ. By Lemma 2, it is even a graded module over a graded algebra $A^S_\lambda/((C_\lambda)_{>0})$. By Proposition 1, the last is generated in degrees 0 and 1. It is well-known that GVMs have simple socles and tops, [FC, MO]. Hence the statement follows from Lemma 1. $\square$

Now we can calculate the Loewy length of GVMs for regular λ. We recall that in this case they are parameterized by $\xi \in W/W_S$. We start with the following easy observation.

Lemma 4. *The inequality $ll_{\mathcal{O}(\mathfrak{p},\Lambda)}(\mathcal{F}^{-1} \circ E_{w_S}(M)) \leq ll_{\mathcal{O}}(M)$ is true for any $M \in \mathcal{O}$.*

Proof. Follows from properties of E_{w_S}, described in Subsection 2.2. $\square$

Now we can calculate the Loewy length of $M(\xi)$.

Lemma 5. *Let λ be regular and $\xi \in W/W_S$. Then $ll(M(\xi)) = l(w^\xi) + 1$.*

Proof. The inequality $ll(M(\xi)) \geq l(w^\xi) + 1$ follows from the analogue of BGG-criterion for inclusion of GVMs, [MO]. By this criterion, each $M(\xi)$ has a filtration, $0 \subset M(\xi_1) \subset \cdots \subset M(\xi_r) = M(\xi)$, where $r = l(w^\xi) + 1$ and all quotients of the filtration are non-zero. As each GVM has a simple top, $M(\xi_i)$ lies in the radical of $M(\xi_{i+1})$ and hence $ll(M(\xi)) \geq r$.

Let us now prove that $ll(M(\xi)) \leq l(w^\xi) + 1$. Consider two Verma modules $M(w_\xi \lambda)$ and $M(w^\xi \lambda)$. By [De] $E_{w_S}(M(w^\xi \lambda)) = M(w_\xi \lambda)$ and, as we have already mentioned, $\mathcal{F}^{-1}(M(w_\xi \lambda)) = M(\xi)$. Hence, by Lemma 4, we have $ll_{\mathcal{O}(\mathfrak{p},\Lambda)}(M(\xi)) \leq ll_{\mathcal{O}}(M(w_\xi \lambda)) = l(w^\xi) + 1$. This completes the proof for GVMs. $\square$

We remark that in the case of singular λ one can also calculate the Loewy length in terms of height of an ideal in the poset W/W_λ. However, this does not give any closed formula as in Lemma 5.

Combining the above results we can describe the layers of the unique Loewy filtration of $M(\xi)$.

Corollary 1. *Let λ be regular, $\xi \in W/W_S$, w the shortest representative of ξ, and $i \in \mathbb{Z}_+$. Then* $\mathrm{soc}_i(M(\xi)) = \mathrm{rad}_{ll(M(\xi))-i+1}(M(\xi)) = \mathcal{F}^{-1} \circ E_{w_S}(\mathrm{soc}_i(M(w \cdot \lambda)))$.

Proof. As it was already mentioned, $\mathcal{F}^{-1} \circ E_{w_S}$, sends any Loewy filtration of $M(w \cdot \lambda)$ to a filtration of $M(\xi)$ with semi-simple subquotients. But by Lemma 5, we have $ll_{\mathcal{O}}(M(w \cdot \lambda)) = ll_{\mathcal{O}(\mathfrak{p},\Lambda)}(M(\xi))$ and both modules are rigid. The statement follows. $\square$

4 $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{C})$ – example

In this section we assume $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ with the standard Cartan subalgebra and $\{\alpha, \beta\}$ is a basis of Δ. We assume that $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{C})$ is the subalgebra corresponding to the root α. We fix λ regular antidominant. Then $\mathcal{O}_\lambda$ contains 6 simple modules indexed by $1 = \lambda$, $2 = s_\alpha \cdot \lambda$, $3 = s_\beta \cdot \lambda$, $4 = s_\alpha s_\beta \cdot \lambda$, $5 = s_\beta s_\alpha \cdot \lambda$ and $6 = s_\alpha s_\beta s_\alpha \cdot \lambda$. The Verma modules have the following radical filtrations (the number on the left is the index i of rad_i):

	$M(1)$	$M(2)$	$M(3)$	$M(4)$	$M(5)$	$M(6)$
0	1	2	3	4	5	6
1		1	1	2 3	2 3	4 5
2				1	1	2 3
3						1

The big projective module $A_\lambda e$ is rigid and filtered by all Vermas with all multiplicities equal to 1. The graded picture of $A_\lambda e$ as C_λ module is the following one, where by i we denote the simple subquotients, which come from $\mathfrak{g}$-simple subquotients of A_λ, isomorphic to $L(i)$. This is what we called D_f before. Here we will call these C_λ-direct summands of $A_\lambda e$ by D_i respectively. The number of the left is the degree of the corresponding graded component.

0	1					
1		2	3			
2	1 1			4	5	
3		2 2	3 3			6
4	1 1			4	5	
5		2	3			
6	1					

In particular, the algebra C_λ, as a graded left module over itself, is isomorphic to D_1. Going to $\mathcal{O}(\mathfrak{p}, \Lambda)$ we take (see [FKM3, KM2]) the category of injectively copresented modules with respect to the direct sum of indecomposable injectives indexed by the shortest representatives in W/W_S. This means that the only indices, corresponding to these shortest representatives, namely 1, 3 and 5, will survive. Hence, the graded structure of $A_\lambda^S e^S$ will be:

0	1		
1		3	
2	1 1		5
3		3 3	
4	1 1		5
5		3	
6	1		

Now let us look at GVMs and standard modules, which we will naturally index by 1, 3 and 5. As all multiplicities are one, all GVMs are uniserial and have the following radical filtrations:

	$M_{\mathfrak{p}}(1)$	$M_{\mathfrak{p}}(3)$	$M_{\mathfrak{p}}(5)$
0	1	3	5
1		1	3
2			1

To obtain $\Delta(5)$ one has to find the trace of D_5 in $A_\lambda^S e^S$. It is easy to see that this will be the following part of $A_\lambda^S e^S$:

2			5
3		3	
4	1		5
5		3	
6	1		

Analogously, $\Delta(3)$ is presented by the following part of $A_\lambda^S e^S$:

1		3
2	1	
3		3
4	1	

By direct calculation one easily gets that these graded filtrations are in fact Loewy filtrations of the corresponding standard modules. The module $\Delta(1)$ is, obviously, a self-extension of $M_{\mathfrak{p}}(1)$. It is presented by the following part of $A_\lambda^S e^S$:

0	1
1	
2	1

And here we see the difference: to obtain the radical filtration of $\Delta(1)$ one has to re-scale the grading. The last causes the fact that the big projective module $A_\lambda^S e^S$ in $\mathcal{O}(\mathfrak{p}, \Lambda)$ is not rigid. Indeed, let us look at the first three graded components of it:

0	1	
1		3
2	1 1	

Clearly, the dimension of C_λ-homomorphisms of degree 1 from D_1 to D_3 is 1. But the dimension of C_λ-homomorphisms of degree 1 from D_3 to D_1 is also 1 (e.g since D_1 is injective). Hence their composition has dimension at most 1, which means that one composition factor of $\mathrm{rad}_1(D_1)$ belongs to $\mathrm{rad}_1(A_\lambda^S e^S)$. The above description of Loewy filtrations of standard modules (it is enough to have it for $\Delta(5)$) implies that the Loewy length of $A_\lambda^S e^S$ is 7. hence the graded filtration is a Loewy one and does not coincide with the radical filtration. Therefore $A_\lambda^S e^S$ is not rigid.

5 What can be said about standard modules?

Based on the example above we formulate the following conjecture:
Conjecture. Standard modules in $\mathcal{O}(\mathfrak{p}, \Lambda)$ are rigid.

We also have to remark that even the calculation of Loewy length of standard modules seems to be a very non-trivial problem. As the above example shows, the answer will be much more difficult than that for GVMs. However, it is not difficult to get the following inequalities:

Lemma 6. *Let $M = M_{\mathfrak{p}}(V)$ be a generalized Verma module and $\Delta = \Delta(\tilde{V})$ be the corresponding standard module. Then*

$$ll(M) + ll(\tilde{V}) - 1 \leq ll(\Delta) \leq ll(M) + 2ll(\tilde{V}) - 1.$$

Proof. To prove the left inequality we recall that $\tilde{V}$ has simple socle, V, and M is a submodule of Δ with simple top $L_{\mathfrak{p}}(V)$ and all other composition factors different from $L_{\mathfrak{p}}(V)$. Hence $M \subset rad_{ll(\tilde{V})-1}(\Delta)$, which implies necessary inequality.

The right inequality follows from the double centralizer property. Indeed, it implies that the big projective is graded as A_λ-module. Moreover, by BGG reciprocity, it is filtered by standard modules, and by induction one derives that all standard modules are in fact graded modules over A_λ. It is straightforward to see that the length of the grading filtration inherited from the big projective module is precisely $ll(\Delta) \leq ll(M) + 2ll(\tilde{V}) - 1$. Indeed, $ll(\Delta) \leq ll(M)$ corresponds to the Verma submodule M of Δ and its grading filtration and $2ll(\tilde{V})$ corresponds to the grading filtration of $\tilde{V}$ as C_λ-module (we recall that C_λ is even-graded). This implies the necessary inequality. □

We note that both extremal cases of equalities are possible. This can be read off from the example in Section 4. Indeed, the left equality holds for $\Delta(1)$ and the right equality holds for both $\Delta(3)$ and $\Delta(5)$. It is easy to construct other examples where $ll(\Delta)$ satisfies two strict inequalities from the above formula. However, for this one has to take $\mathfrak{a}$ of rank greater than 1.

Acknowledgments

The research was done during the visit of the second author to Max-Planck-Institut für Mathematik in Bonn, Germany. The financial support, hospitality, accommodation and perfect working conditions provided by the Institut are gratefully acknowledged. We also thank Prof. Steffen König for helpful discussions.

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