Elliptic Polylogarithms: An Analytic Theory

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April 25, 1994

Abstract

In this article we introduce a natural "elliptic" generalization of the classical polylogarithms, study the properties of these functions and their relations with Eisenstein series.

Introduction

The notion of the elliptic polylogarithm functions as a natural generalization of the usual polylogarithms was introduced in [BL, 4.8]. In this article we study the properties of these functions. Part of these properties are equivalent to some theorems of [BL] but I will prove them purely analytically.

The paper goes as follows. In the first section we introduce some version of usual polylogarithms which are more convenvient for generalization and describe their properties. In the second section we define the elliptic polylogarithms and prove the simplest facts about them. The modular properties of elliptic polylogarithms and their relations with classical Eisenstein series are discussed in the third section.

I wish to thank A.Beilinson, A.Goncharov, Yu.I.Manin, M.Rovinskii and D.Zagier for stimulating discussions. I thank the Massachusetts Institute for Technology and the Max-Planck-Institut für Mathematik Bonn for their hospitality during my stay there at visiting positions. This work was partially supported by AMS grants for the former Soviet Union.

I'll use some standard notations : $H = (\tau \in \mathbb{C}, \Im\tau > 0)$,
$\mathbf{e}\ (t) := \exp(2\pi i t), z = \mathbf{e}\ (\xi), q = \mathbf{e}\ (\tau), w = \mathbf{e}\ (\eta)$;

$$\begin{aligned}\theta(\xi,\tau) &= \sum_{j=-\infty}^{\infty} \exp(2\pi i(\frac{1}{2}(j+\frac{1}{2})^2\tau + (j+\frac{1}{2})\xi)) \\ &= q^{1/8}(z^{\frac{1}{2}} - z^{-\frac{1}{2}})\prod_{j=1}^{\infty}(1-q^j)(1-q^jz)(1-q^jz^{-1}); \\ \eta(\tau) &= q^{\frac{1}{24}}\prod_{j=1}^{\infty}(1-q^j).\end{aligned}$$

1 Debye polylogarithms

Definition 1.1 *The n-th Debye polylogarithm $\Lambda_n(\xi)$ is the multivalued analytic function on $\mathbb{C}\setminus\mathbb{Z}$ given by integral*

$$\Lambda_n(\xi) = -\int_{i\infty}^{\xi} \frac{t^{n-1}}{(n-1)!}\frac{dt}{\exp(-2\pi it)-1}.$$

The convergence of the integral on the upper halfplane is clear from the bound

$$\left|\frac{t^{n-1}}{\exp(-2i\pi t)-1}\right| < C_n \exp(-2\pi\Im t), \Im t > 1.$$

I will use two single-valued branches $\Lambda_n^+(\xi)$ and $\Lambda_n^-(\xi)$ which are defined on the plane $\mathbb{C}$ without $(-\infty,0)\cup(1,\infty)$ and $(-\infty,-1)\cup(0,\infty)$ respectively and are the analytic continuation to the lower halfplane across $(0,1)$ or $(-1,0)$.

Recall the definition of the classical (Euler) polylogarithms as the analytic continuations of the series

$$Li_n(z) = \sum_{i=1}^{\infty}\frac{z^i}{i^n}.$$

It is clear that $Li_1(z) = -\log(1-z)$ and $Li_n(z) = \int_0^z Li_{n-1}(t)d\log(t)$.

The relation between $\Lambda_.(*)$ and $Li_.(*)$ is the following:

Proposition 1.1

$$a) \qquad \Lambda_n(\xi) = \sum_{k=1}^{n}\frac{\xi^{n-k}}{(n-k)!}(-2\pi i)^{-k}Li_k(z);$$

$$b) \qquad Li_n(z) = (-2\pi i)^n\sum_{k=1}^{n}\frac{(-\xi^{n-k})}{(n-k)!}\Lambda_k(\xi).$$

Recall that $z = \mathbf{e}\ (\xi) = \exp(2\pi i\xi)$.

I give the basic properties of $\Lambda_.(*)$ without the proofs which are very simple.

Proposition 1.2

$$
\begin{array}{llll}
a) & \Lambda_n^+(\xi) = \Lambda_n^-(\xi) & \textit{if}\ \Im(\xi) > 0; & \\
 & \Lambda_n^+(\xi) = \Lambda_n^-(\xi) + \delta_{1,n} & \textit{if}\ \Im(\xi) < 0; & \\
b) & \Lambda_n^{\pm}(\xi + k) = \sum_{i=0}^{n-1} \frac{k^i}{i!}\Lambda_{n-i}^{\pm}(\xi) & \textit{if}\ \Im(\xi) > 0, k \in \mathbf{Z}; & (1) \\
c) & \Lambda_n^+(\xi) + (-1)^n\Lambda_n^-(\xi) = \frac{1}{n!}(\xi^n - (-1)^n B_n); & & (2) \\
d) & (n-1)d\,\Lambda_n(\xi) = \xi d\,\Lambda_{n-1}(\xi); & & (3) \\
e) & |\Lambda_n(\xi)| < C_n \exp(-2\pi\Im\xi)(1 + (\Re\xi)^n). & & (4)
\end{array}
$$

Here the B_n are Bernoulli numbers.

Remark. It is useful to introduce the generating function $\Lambda(\xi; K) = \sum_{n=1}^{\infty} \Lambda_n(\xi)X^{n-1}$. Then

$$\Lambda(\xi; K) = -\int_{i\infty}^{\xi} \frac{\exp(Kt)\,dt}{\exp(-2\pi it) - 1}.$$

Proposition 1.2 can be reformulated in terms of this generating function in a very simple manner.

Proposition 1.3

$$
\begin{array}{llll}
a) & \Lambda^+(\xi; K) = \Lambda^-(\xi; K) & \textit{if}\ \Im(\xi) > 0; & \\
 & \Lambda^+(\xi; K) = \Lambda^-(\xi; K) + 1 & \textit{if}\ \Im(\xi) < 0; & \\
b) & \Lambda(\xi + j; K) = \exp(jK)\Lambda(\xi; K), j \in \mathbf{Z} & \textit{if}\ \Im(\xi) > 0; & (5) \\
c) & \Lambda^+(\xi; K) = \Lambda^-(-\xi; -K) + \frac{\exp(\xi K)}{K} - \frac{\exp(K)}{\exp(K)-1}; & & (6) \\
d) & (\frac{\partial}{\partial K} - \xi)d_\xi\Lambda(\xi; K) = 0. & & (7)
\end{array}
$$

2 Elliptic polylogarithms

The elliptic polylogarithms are single-valued analytic functions on the universal covering of a punctured universal elliptic curve. I will define them as multivalued analytic functions on the partial covering $(\mathbf{C} \times H) \setminus L$, where L is the relative lattice $L := (\xi = m + n\tau | m, n \in \mathbf{Z})$. It is well known that the universal covering of universal elliptic curve is the product $\mathbb{C} \times H$ and its fundamental group is the semidirect product $SL_2(\mathbb{Z}) \ltimes \mathbf{Z}^2$, acting on $\mathbb{C} \times H$ in the usual way:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (0,0)\right)(\xi, \tau) = \left(\frac{\xi}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right);$$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (m,n)\right)(\xi, \tau) = (\xi + m + n\tau, \tau).$$

So the preimage of the zero section is L, and $(\mathbb{C} \times H) \setminus L$ cover the punctured universal elliptic curve.

Definition 2.1 *The (Debye) elliptic polylogarithm $\Lambda_{m,n}(\xi, \tau)$ with index (m,n) is the multivalued analytic function on $(\mathbb{C} \times H) \setminus L$ which is given in the strip $(0 < \Im\xi < \Im\tau)$ by the series*

$$\begin{aligned}\Lambda_{m,n}(\xi, \tau) &= \frac{1}{m!}\Big(\sum_{j=0}^{\infty} j^m \Lambda_n^+(\xi + j\tau) + (-1)^{m+n+1} \sum_{j=1}^{\infty} j^m \Lambda_n^-(-\xi + j\tau) \\ &+ \sum_{k=0}^{n} \frac{\xi^{n-k}\tau^k}{(n-k)!k!} \frac{B_{m+k+1}}{m+k+1} + (-1)^{n+1} \frac{B_n B_{m+1}}{n!(m+1)}\Big)\end{aligned}$$

The convergence of the infinite series is an evident consequence of (4).

This formula defines a single-valued branch of $\Lambda_{m,n}(\xi, \tau)$ on $\mathbb{C} \times H$ without the set

$$\{\xi = j\tau + s | j \in \mathbf{Z}, s \in (-\infty, 0] \cup [0, \infty)\}$$

Remark 1. The origin of this definition is the following. The series $\frac{1}{m!}\sum_{j=-\infty}^{\infty} j^m \Lambda_n^+(\xi + j\tau)$ diverges and one can regularise it by this trick:

$$\begin{aligned}&\frac{1}{m!} \sum_{j=-\infty}^{\infty} j^m \Lambda_n^+(\xi + j\tau) \\ &= \frac{1}{m!}\{\sum_{j=0}^{\infty} j^m \Lambda_n^+(\xi + j\tau) + \sum_{j=-\infty}^{-1} j^m \Lambda_n^+(\xi + j\tau)\} \\ &\overset{(2)}{=} \frac{1}{m!}\{\sum_{j=0}^{\infty} j^m \Lambda_n^+(\xi + j\tau) + \sum_{j=-\infty}^{-1} (-1)^{(m+n+1)} j^m \Lambda_n^-(-\xi - j\tau) \\ &+ \sum_{j=-\infty}^{-1} \frac{1}{n!} j^m ((\xi + j\tau)^n - (-1)^n B_n)\}.\end{aligned}$$

The first and second series converge according to the bound (4) and the third which is a series of polynomials in the variable j, is defined using the formal equality

$$\sum_{j=1}^{\infty} j^m = \zeta(-m) = (-1)^m B_{m+1}/(m+1).$$

Remark 2. A one-valued version of the elliptical polylogarithms was introdused by Bloch [B] in the case $m+n=3$ and by Zagier [Z2] for arbitrary m, n.

The generating function

$$\Lambda(\xi,\tau;X,Y) = \sum_{m\geq 0, n\geq 1} \Lambda_{m,n}(\xi,\tau)(-Y)^{n-1}X^m$$

is equal to

$$\sum_{j=0}^{\infty} \exp(jX)\Lambda^{+}(\xi+j\tau;-Y) + \sum_{j=1}^{\infty} \exp(-jX)\Lambda^{-}(-\xi+j\tau;Y) +$$
$$\frac{\exp(-Y\xi)}{-Y}\left(\frac{1}{\exp(-Y\tau+X)-1} - \frac{1}{-Y\tau+X}\right) + \frac{1}{e^Y-1}\left(\frac{1}{e^X-1} - \frac{1}{X}\right).$$

Evidently $\Lambda_{0,1}(\xi,\tau) = \frac{1}{2\pi i}\log(\theta(\xi,\tau)/\eta(\tau))$.

Many of the transformation properties of the $\Lambda_{m,n}(\xi,\tau)$ become simpler if we introduce the modified generating function

$$\begin{aligned}\underline{\Lambda}(\xi,\tau;X,Y) &= \Lambda(\xi,\tau;X,Y) + \frac{\exp(-Y\xi)}{(-Y)(-Y\tau+X)} + \frac{1}{(\exp Y-1)X} \\ &= \sum_{j=0}^{\infty} \exp(jX)\Lambda^{+}(\xi+j\tau;-Y) + \sum_{j=1}^{\infty} \exp(-jX)\Lambda^{-}(-\xi+j\tau;Y) \\ &\quad + \frac{\exp(-Y\xi)}{-Y}\frac{1}{\exp(-Y\tau+X)-1} + \frac{1}{\exp Y-1}\frac{1}{\exp X-1}.\end{aligned}$$

The reason is that in the domain $\{0 < \Re X, \quad 0 < \Re(-Y\tau+X) < 2\pi\Im\tau\}$ $\sum_{j\in\mathbf{Z}} e^{jX}\Lambda^{+}(\xi+j\tau;-Y)$ converges and equals $\underline{\Lambda}(\xi,\tau;X,Y)$.

Proposition 2.1 *a) Let* $0 < \Re(\xi)\Im(\tau) - \Im(\xi)\Re(\tau) < \Im(\tau)$. *Then*

$$\underline{\Lambda}(\xi+1,\tau;X,Y) = e^{-Y}\left(\underline{\Lambda}(\xi,\tau;X,Y) + \frac{1}{\exp(X)-1}\right). \tag{8}$$

b) Let $0 < \Im(\xi) < \Im(\tau)$. *Then*

$$\underline{\Lambda}(\xi+\tau,\tau;X,Y) = e^X\underline{\Lambda}(\xi,\tau;X,Y) \tag{9}$$

c)

$$(\frac{\partial}{\partial Y} + \tau\frac{\partial}{\partial X} + \xi)d_{\xi,\tau}\underline{\Lambda}(\xi,\tau;X,Y) = 0. \tag{10}$$

Sketch of the proof. Statement a) is obtained after a simple calculation by summing (5) over the arguments $\xi + j\tau$ or $-\xi + j\tau$. Statement b) is the result of substituting $\xi + \tau$ for ξ in the definition and changing the limits of summation in the first and second sums, using (6). Statement c) is obtained after a simple calculation by summing (7) over the arguments $\xi + j\tau$ or $-\xi + j\tau$.

3 Eisenstein - Kronecker series and modular properties of elliptic polylogarithms

We recall a classical result of Kronecker [W] : Denote by L the lattice generated by 1 and τ. Any $\eta \in \mathbb{C}$ determines a character χ_η on L

$$\chi_\eta(\xi) = \exp(2\pi i\frac{\xi\overline{\eta} - \overline{\xi}\eta}{\tau - \overline{\tau}}).$$

Then [W, Z1] the Eisenstein-Kronecker series of weight 1

$$K_1(\xi,\eta,1) = \sum_{w\in L}{}_e\frac{\chi_\eta(w)}{w+\xi}$$

(where $\sum_e$ denotes Eisenstein summation; see [W]) is given by the formula

$$K_1(\xi,\eta,1) \quad = \quad 2\pi i\exp(2\pi i\frac{\xi-\overline{\xi}}{\tau-\overline{\tau}})F(\xi,\eta,\tau), \tag{11}$$

where

$$\begin{aligned} F(\xi,\eta,\tau) &= (1 - \frac{1}{1-z} - \frac{1}{1-w} - \sum_{m,n=1}^{\infty}(z^m w^n - z^{-m}w^{-n})q^{mn}) && (12)\\ &= \frac{\theta'(0,\tau)\theta(\xi+\eta,\tau)}{\theta(\xi,\tau)\theta(\eta,\tau)}. && (13)\end{aligned}$$

The symmetry of $F(\xi,\eta,\tau)$ in ξ and η is a special case of the functional equation for Eisenstein-Kronecker series. The transformation properties of $F(\xi,\eta,\tau)$ are very simple:

$$\begin{aligned} F(\xi+1,\eta,\tau) &= F(\xi,\eta,\tau); && (14)\\ F(\xi+\tau,\eta,\tau) &= \exp(-2\pi i\eta)F(\xi,\eta,\tau); && (15)\\ F(\frac{\xi}{c\tau+d},\frac{\eta}{c\tau+d},\frac{a\tau+b}{c\tau+d}) &= (c\tau+d)\exp(2\pi i\frac{c\xi\eta}{c\tau+d})F(\xi,\eta,\tau). && (16)\end{aligned}$$

One can represent this function as the generating function with respect to variable ξ of the usual Eisenstein series $e_k(\eta,\tau)=\sum_{w\in L}{}'_e\frac{\chi_\eta(w)}{w^k}$:

$$\begin{aligned}\sum_{w\in L}{}_e\frac{\chi_\eta(w)}{w+\xi} &= \frac{1}{\xi}+\sum_{w\in L}{}'_e\sum_{j=0}^{\infty}(-\xi)^j\frac{\chi_\eta(w)}{w^{j+1}}\\ &= \frac{1}{\xi}+\sum_{j=0}^{\infty}(-\xi)^j\sum_{w\in L}{}'_e\frac{\chi_\eta(w)}{w^{j+1}}. \qquad (17)\end{aligned}$$

On the other hand, $F(\xi,\eta,\tau)$ can be expressed as the exponential of the generating function of Eisenstein functions $E_n(\xi,\tau)=\sum_{w\in L}{}_e\frac{1}{(w+\xi)^n}$:

$$F(\xi,\eta,\tau)=\frac{1}{\eta}\exp(-\sum_{i=1}^{\infty}\frac{(-\eta)^i}{i}(E_i(\xi,\tau)-e_i(0,\tau))). \qquad (18)$$

This statment is a simple corollary of Zagier's "Logarithmic Formula" for $F(\xi,\eta,\tau)$ [Z1, Section 3,Theorem(viii)] and the power-series for E_n [W, Chapter III,formula(10)].

Proposition 3.1

$$a)\qquad \frac{\partial}{\partial\xi}\Lambda(\xi,\tau;X,Y)=e^{-Y\xi}F(\xi,\frac{-Y\tau+X}{2\pi i},\tau); \qquad (19)$$

$$b)\quad \frac{\partial}{\partial\tau}\Lambda(\xi,\tau;X,Y)=e^{-Y\xi}\frac{\partial}{\partial X}F(\xi,\frac{-Y\tau+X}{2\pi i},\tau). \qquad (20)$$

The proof is a direct simple calculation.

This proposition together with (18) gives an expression for the derivatives of elliptic polylogarithms as polynomials in Eisenstein functions.

One can define an action of $SL_2(\mathbf{Z})$ on the two-dimensional space generated by X and Y by the standard formulas

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}:\begin{pmatrix}X\\Y\end{pmatrix}\to\begin{pmatrix}aX+bY\\cX+dY\end{pmatrix}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}.$$

Proposition 3.2 *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *belong to* $SL_2(\mathbf{Z})$. *Then*

$$\underline{\Lambda}(\tfrac{\xi}{c\tau+d}, \tfrac{a\tau+b}{c\tau+b}; aX+bY, cX+dY) = \underline{\Lambda}(\xi,\tau;X,Y) + c_M(X,Y).$$

where $c_M(X,Y)$ *is a rational function in* X *and* Y *with rational coefficients.*

Sketch of the proof. We first prove that $c_M(X,Y)$, which can be defined as the difference

$$\underline{\Lambda}(\tfrac{\xi}{c\tau+d}, \tfrac{a\tau+b}{c\tau+b}; aX+bY, cX+dY) - \Lambda(\xi,\tau;X,Y),$$

doesn't depend on ξ and τ. This means that the differential $d_{\xi,\tau}\underline{\Lambda}(\xi,\tau;X,Y)$ satisfy to the following property:

$$d_{\xi,\tau}\underline{\Lambda}(\tfrac{\xi}{c\tau+d}, \tfrac{a\tau+b}{c\tau+b}; aX+bY, cX+dY) - d_{\xi,\tau}\underline{\Lambda}(\xi,\tau;X,Y) = 0$$

One can deduce this equality from (16) using the expresions for derivatives (19) and (20). So we have proved that $c_M(X,Y)$ is a formal function in X and Y with complex coefficients.

To prove the rationality of these coefficients we observe that $M \to c_M(X,Y)$ is a cocycle of $SL_2(\mathbf{Z})$ with coefficients at $\mathbb{C}[[X,Y]]$. So it is enough to check the rationality for generators of $SL_2(\mathbf{Z})$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The calculation of $c_T(X,Y)$ is a simple exercise like the proof of (8)

$$c_T(X,Y) = \frac{1}{\exp(Y)-1}\left(\frac{1}{\exp(X)-1} - \frac{1}{\exp(X+Y)-1}\right).$$

To calculate $c_S(X,Y)$ we use the following trick (in which we use that $SL_2(\mathbf{Z})$ interwine the action of $\mathbf{Z}^2$):

$$\begin{aligned}
c_S(X,Y) &= \underline{\Lambda}(\tfrac{\xi-1}{\tau}, -\tfrac{1}{\tau}; -Y, X) - \underline{\Lambda}(\xi-1,\tau;X,Y) \\
(\text{use}(9)) &= e^Y(\underline{\Lambda}(\tfrac{\xi}{\tau}, -\tfrac{1}{\tau}; -Y, X)) \\
(\text{use}(8)) &\quad -(e^Y\underline{\Lambda}(\xi,\tau;X,Y) - \frac{1}{\exp(X)-1}) \\
&= e^Y(\underline{\Lambda}(\xi,\tau;X,Y) + c_S(X,Y)) \\
&\quad -(e^Y\underline{\Lambda}(\xi,\tau;X,Y) - \frac{1}{\exp(X)-1}) \\
&= e^Y c_S(X,Y) + \frac{1}{\exp(X)-1}
\end{aligned}$$

We have deduced an equation for $c_S(X,Y)$ with an evident solution which is a rational function series with rational coefficients:

$$c_S(X,Y) = -(\frac{1}{\exp(Y)-1})(\frac{1}{\exp(X)-1}) \tag{21}$$

Now we discribe the relation between elliptic polylogarithms and indefinite Eichler-Shimura integrals of Eisenstein series.

Let Γ be a congruence subgroup and $G(\tau)$ a modular form of weight k with respect to Γ. Then the vector-valued differential form $G(\tau)(-Y\tau + X)^{k-2}d\tau$ is Γ-invariant. The indefinite Eichler-Shimura integral $\mathcal{G}(\tau, X, Y)$ of $G(\tau)$ is the indefinite integral of this form :

$$d_\tau \mathcal{G}(\tau, X, Y) = G(\tau)(-Y\tau + X)^{k-2}d\tau.$$

Let r and s be rational . Then the Eisenstein series $e_k^{r,s}(\tau) = e_k(r+s\tau, \tau)$ is a modular form of weight k for some congruence subgroup.

Proposition 3.3 *Define for rational r and s the function of three variables :*

$$\Xi^{r,s}(\tau, X, Y) \quad = \quad e^{sX+rY}\underline{\Lambda}(r+s\tau, \tau; X, Y).$$

Then $\Xi^{r,s}(\tau, X, Y)$ is the modified generating function for the indefinite Eichler-Shimura integrals of Eisenstein series $e_k^{r,s}(\tau)$:

$$\Xi^{r,s}(\tau, X, Y) = \frac{1}{XY} + \sum_{j=1}^{\infty}(-1)^j j(2\pi i)^{-(j-1)}\mathcal{E}_{j+1}^{r,s}; \tag{22}$$

$$d_\tau \mathcal{E}_k^{r,s} = e_k^{r,s}(\tau)(-Y\tau + X)^{k-2}d\tau.$$

The proof is a direct calculation of $d_\tau \Xi^{r,s}(\tau, X, Y)$ using (19), (20) and (17).

4 A Hodge sheaf

I explain in this section the Hodge-theoretical interpretation of previous results very briefly cf [BL, 4].

The Hodge sheaf of relative homologies $\mathcal{H} = R^1 p_*(\mathbf{Q}(1))$ of the universal elliptic curve

$$p: \mathbf{C} \times H/(SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2) \to H/SL_2(\mathbf{Z})$$

on $H/SL_2(\mathbf{Z})$ can be realised in the following manner. The upper-plane H is the modular space of elliptic curves with a symplectic frame (A, B) in the first homology group (A is the class of the loop $\xi+t, 0\leq t\leq 1$ and B is the class of the loop $\xi+t\tau, 0\leq t\leq 1$). The local system $\mathcal{H}_{\mathbf{Q}}$ is the tautological repesentation of $SL_2(\mathbf{Z})=\pi_1(H/SL_2(\mathbf{Z}))$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} B \\ A \end{pmatrix} \to \begin{pmatrix} aB+bA \\ cB+dA \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix}.$$

This means that the $\mathbf{Q}$-sheaf $\mathcal{H}_{\mathbf{Q}}$ on $H/SL_2(\mathbf{Z})$ can be constructed by identifying the vectors B and A in the fiber over $\frac{a\tau+b}{c\tau+d}$ with the vectors $aB+bA$ and $cB+dA$ in the fiber over τ respectively:

$$B|_{\frac{a\tau+b}{c\tau+d}}=(aB+bA)|_\tau, A|_{\frac{a\tau+b}{c\tau+d}}=(cB+dA)|_\tau.$$

The Hodge decomposition $\mathcal{H}_{\mathbf{C}}=F^{-1,0}+F^{0,-1}$ of the fiber over the point τ is

$$F^{-1,0}=\langle -A\tau+B\rangle_{\mathbf{C}}, F^{0,-1}=\langle -A\bar{\tau}+B\rangle_{\mathbf{C}}.$$

and the Hodge filtration is

$$F^0=F^{-1,0}\subset F^{-1}=\mathcal{H}_{\mathbf{C}}.$$

Denote by $f^0_{-1}=-A\tau+B$ the generator of F^0 (superscripts 0 means the Hodge type and subscript -1 means the weight) and by $f^{-1}_{-1}=A$ the additional vector at F^{-1} which formed with the previous one the frame of F^{-1}. These vectors depend holomorphically on τ and

$$f^0_{-1}|_{\frac{a\tau+b}{c\tau+d}}=(c\tau+d)^{-1}f^0_{-1}|_\tau, f^{-1}_{-1}|_{\frac{a\tau+b}{c\tau+d}}=cf^0_{-1}|_\tau+(c\tau+d)f^0_{-1}|_\tau.$$

The symmetric powers $S^n\mathcal{H}$ of $\mathcal{H}$ are the local systems of homogeneous polynomials of degrees n in the variables A and B with the evident action of $SL_2(\mathbf{Z})$. The Hodge filtration

$$F^0\subset F^{-1}\subset\ldots\subset F^{-n}$$

is generated by vectors f^{-i}_{-n} :

$$F^{-i}(S^n\mathcal{H})=\langle f^0_{-n},\ldots,f^{-i}_{-n}\rangle_{\mathbf{C}}; f^{-i}_{-n}=(f^{-1}_{-1})^i(f^0_{-1})^{n-1}.$$

Now we describe the "logarithmic sheaf" G on X. The local system $G_{\mathbf{Q}}$ is the space of formal power series in A and B with the evident action of $SL_2(\mathbf{Z})$ and with the action of $\mathbf{Z}^2=\mathbf{Z}A+\mathbf{Z}B$ by multiplication by exponentials :

$$f(A,B)|_{(\xi+m+n\tau,\tau)}=\exp(mA+nB)f(A,B)|_{(\xi,\tau)}.$$

The weight filtration coincides with the degree filtration on the space of formal power series :

$$W_{-n} = \langle \{A^i B^j | i + j \geq n\} \rangle_{\mathbf{Q}}.$$

The evident action of $\sum_{i=0}^{\infty} S^i(\mathcal{H})$ on G must be compatible with the Hodge structure; so the Hodge filtration can be defined by giving a single vector 1_H in F^0 with unitary constant term, and setting

$$F^{-p} = \langle \{f_{-n}^{-i} 1_H | i \leq p\} \rangle_{\mathbf{C}} .$$

Choose $1_H = \exp(-A\xi)$. It transforms in the following way:

$$A : 1_H \to \exp A \exp(-A(\xi+1)) = \exp(-A\xi) = 1_H;$$

$$B : 1_H \to \exp B \exp(-A(\xi+\tau))$$

$$= \exp(-A\tau + B)\exp(-A\xi) = \exp(f_{-1}^0) 1_H \in F^0;$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : 1_H \to \exp(-(cB + dA)\frac{\xi}{c\tau + d})$$

$$= \exp(-\frac{c(-A\tau + B)}{c\tau + d})\exp(-A\xi) = \exp(-c\frac{f_{-1}^0}{c\tau + d}) 1_H \in F^0.$$

The elliptic polylogarithm $\mathcal{P}$ is the extension of $G(1)$ by $\mathcal{H}$. The pullback of $\mathcal{P}_{\mathbf{Q}}$ to any simply connected variety is the (noncanonical) sum of $\mathcal{H}_{\mathbf{Q}}$ and $G(1) = G \otimes \mathbf{Q}(1)$. So $\mathcal{P}_{\mathbf{Q}}$ is locally the $\mathbf{Q}$-space generated by A, B and $f(A,B)K$, where $f(A,B)$ is a formal power series, the weights of A and B are -1 and the weight of K is -2 (K denote the generator of the Tate module $\mathbf{Q}(1)$). The Hodge filtration on $\mathcal{P}_{\mathbf{C}}|_{\xi,\tau}$ at the point (ξ, τ) is generated by $f_{-n}^{-i} 1_H K \in F^{-i-1}$ (the shift by -1 is the result of Tate's shift on $\mathbf{Q}(1)$, in other words $K \in F^{-1}(\mathbf{Q}(1))$) and two vectors

$$f_{-1}^0 = -A\tau + B + \frac{\exp(-A\xi) - 1}{A} K - \Lambda(\xi, \tau; B, A)(-A\tau + B)K \in F^0;$$

$$f_{-1}^{-1} = A - \Lambda(\xi, \tau; B, A)(-A)K \in F^{-1}.$$

Then the properties(8), (9) and (21) of the elliptic polylogarithms mean that this Hodge filtration is correctly defined on the complexification of some local $\mathbf{Q}$-system on X.Equation (10) is equivalent to the Griffiths transversality condition $\nabla F^j \subset F^{j-1} \otimes \Omega^1$. The formula (22) implies the coincidence of the restriction of $\mathcal{P}$ on torsion points and a sum of Eisenstein extentions.

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