INTERSECTION NUMBERS AND RANK ONE COHOMOLOGICAL FIELD THEORIES IN GENUS ONE

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Abstract. We obtain a simple, recursive presentation of the tautological (κ, ψ, and λ) classes on the moduli space of curves in genus 0 and 1 in terms of boundary strata (graphs). We derive differential equations for the generating functions for their intersection numbers which allow us to prove a simple relationship between the genus zero and genus one potentials. As an application, we describe the moduli space of normalized, restricted, rank one cohomological field theories in genus one in coordinates which are additive under taking tensor products. Our results simplify and generalize those of Kaufmann, Manin, and Zagier.

Recently, there has been a great deal of interest in the topology of the moduli space of curves. Much of this interest has been due to the important role that these spaces (and their cousins, the moduli space of stable maps) play in the theory of Gromov-Witten invariants and quantum cohomology [22, 32, 30] whose origins in the physical literature are called a topological gravity [32]. They furnish nontrivial examples of cohomological field theories (CohFTs) [22, 25], in genus zero (and conjecturally for higher genera). Often, this structure is enough to completely determine the Gromov-Witten invariants themselves. The moduli spaces of curves are endowed with tautological classes whose generating functions for their associated intersection numbers obey a system of differential equations which often possess remarkable properties [32, 21]. In this paper, we apply a mixture of algebraic geometry and combinatorics to find a simple presentation of these classes in genus 0 and 1 to obtain a generalization of some equations due to Witten and Dijkgraaf [32, 5]. These generating functions parameterize the potentials associated to the space of all normalized, restricted, rank one cohomological field theories in genus one and endow this space with coordinates which are additive with respect to tensor product in the category of CohFTs. This paper is motivated by the work of Kaufmann, Manin, and Zagier [19].

The moduli space of genus g curves with n marked points, $\mathcal{M}_{g,n} := \{\,[C; x_1, x_2, \ldots, x_n]\,\}$, is the moduli space of configurations of n marked

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points, on a smooth, complex curve (Riemann surface) C of genus g. We assume throughout that the stability condition $2-2g-n<0$ is satisfied. This moduli space has a compactification $\overline{\mathcal{M}}_{g,n}$ (due to Deligne-Knudsen-Mumford) which is the moduli space of stable curves of genus g with n marked points where the boundary divisor $\overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ is the locus of degenerate curves. The space $\overline{\mathcal{M}}_{g,n}$ is a stratified, complex orbifold (stack) of complex dimension $3g-3+n$ where each stratum is indexed by a decorated graph (stable graph) which denotes the type of degeneration that curves in that stratum have. A stable graph represents a cohomology class on $\overline{\mathcal{M}}_{g,n}$ by taking the closure of the corresponding stratum and applying Poincaré duality to the associated (rational) homology class.

The space $\overline{\mathcal{M}}_{g,n}$ is endowed with tautological cohomology classes whose study was initiated by Mumford [28]. Let $\mathcal{L}_i$ be the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over a point $[C;x_1,x_2,\ldots,x_n]$ is $T^*_{x_i}C$. Then $\psi_{(g,n),i} = c_1(\mathcal{L}_i)$, the first Chern class of $\mathcal{L}_i$. The classes $\kappa_{(g,n),i}$ in $H^{2i}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})^{S_n}$ are defined by $\kappa_{(g,n),i} := \pi_*(c_1(\omega_{g,n}(D))^{i+1})$ where $\omega_{g,n}$ is the cotangent bundle to the fibers of the universal curve $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, D is the sum of the images of the canonical sections, and π_* is fiber integration [1]. Integrals of products of these classes (intersection numbers) are of great geometric interest and are the main object of study in this paper. In particular, $\frac{1}{2\pi^2}\,\kappa_{(g,n),1}$ is the class of the Weil-Petersson symplectic form [1]. Zograf in [33] obtained a recursion formula for the classical Weil-Petersson volumes in genus zero. The intersection numbers of the κ classes studied in [19] are called higher Weil-Petersson volumes.

In the first part of this paper we study a generating function $H(\mathbf{t};\mathbf{s}) \in \mathbb{C}[[\mathbf{t},\mathbf{s}]]$ which incorporates all intersection numbers of the ψ and κ classes. Here $\mathbf{t} = (t_0,t_1,\ldots)$ and $\mathbf{s} = (s_1,s_2,\ldots)$ are formal variables. The function $H_g(\mathbf{t};\mathbf{s})$ denotes the summand of $H(\mathbf{t};\mathbf{s})$ corresponding to genus g. This function has the property that $H(\mathbf{t},\mathbf{0}) = F(\mathbf{t})$, the generating function for the ψ intersection numbers defined by Witten in [32, 31]. On the other hand, setting $t_i = 0$ for $i \geq 1$ gives a generating function for the intersection numbers of the κ classes. This function is closely related to that considered in [19].

We give a simple, recursive presentation of powers of the κ and ψ classes in terms of boundary strata in genus zero and genus one and derive a simple system of differential equations for H in genus zero and one which completely determine those intersection numbers. Taking appropriate limits in genus zero, we obtain equations due to Witten [32] for the ψ classes and equations for the κ classes which are equivalent of those in [19] and much simpler. (A differential equation for the classical Weil-Petersson volumes in genus zero was first obtained in [27].) This simplification arises because our presentation of the κ classes in terms of boundary strata is simpler. Furthermore, the genus one equation can be solved to obtain the relation

$$H_1 = \frac{1}{24} \log H_0''' , \tag{1}$$

where $'$ denotes the partial derivative with respect to t_0, for all values of $\mathbf{s}$ and $\mathbf{t}$ generalizing the result of Dijkgraaf and Witten saying that $F_1 = \frac{1}{24} \log F_0'''$ [5]. We were informed that (1) was known to Zograf [26] in the special case where $t_j = 0$ for all $j \geq 1$ and $s_i = 0$ for all $i \geq 2$.

Witten [32] conjectured (and Kontsevich [21] proved) that $F(\mathbf{s})$ was the logarithm of a tau function of the KdV hierarchy after rescaling the variables. This tau function was completely characterized [5, 17, 24] by being annihilated by a sequence of differential operators L_n for $n \geq -1$ satisfying $[L_m, L_n] = (m-n)L_{m+n}$, the relations of the Virasoro algebra.[1] The equations corresponding to L_{-1} and L_0 were proven by Witten in [32, 31] and are essentially the so-called puncture and dilaton equations. We derive the analog of the puncture and dilaton equations for intersection numbers of κ and ψ classes. These equations are valid for all genera and do not use the presentation of the tautological classes in terms of boundary strata. Solving these equations provides another proof of (1). It is not clear which one of these approaches will prove to be most useful in higher genera.

In the second part of our paper, we apply the previous to describe the moduli space of normalized, restricted, rank one cohomological field theories (CohFTs) in genus one generalizing the results of Kaufmann, Manin, and Zagier in genus zero [19].

A (complete) CohFT of rank r [22, 25] is an r dimensional vector space with metric (V, h) together with a collection of linear maps $H_\bullet(\overline{\mathcal{M}}_{g,n}) \to T^n V$ which are equivariant under the action of the permutation groups and which satisfy some compatibility conditions arising from inclusion of strata on $\overline{\mathcal{M}}_{g,n}$ In the language of Getzler and Kapranov [13], the maps form a morphism of modular operads. Restricting to $g = 0$, a CohFT (V, h) is equivalent to endowing (V, h) with the structure of a (formal) Frobenius manifold [6, 16, 25]. The most

[1] This result is all the more interesting because of the recent conjecture in [7] which predicts a Virasoro algebra playing a similar role in the the case where the target manifold is nontrivial.

spectacular examples of such theories arise when (V, h) is the cohomology ring of certain smooth, projective varieties with its intersection pairing and the morphisms come from the Gromov-Witten invariants associated to the manifold thereby endowing the cohomology ring with a deformed cup product giving it the structure of quantum cohomology. In many cases, *e.g.* $\mathbb{CP}^n$ or Grassmannians, the structure of a CohFT is strong enough to completely determine, recursively, the number of rational curves in the manifold counted with multiplicity (see [22, 10]).

A CohFT in genus zero can be described in terms of a certain generating function (potential) associated to the structure morphisms which must satisfy the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations (see [10, 22]). These equations encode the relations between the boundary strata in $\overline{\mathcal{M}}_{0,n}$ due to Keel in genus zero. Recently, Getzler [11] derived equations which are the analogs of WDVV in genus one by proving new relations which plays an analogous role in genus one to those of Keel in genus zero. His equation allowed him to predict the elliptic Gromov-Witten invariants of $\mathbb{CP}^2$ and $\mathbb{CP}^3$ (see also [2]).

Kaufmann, Manin, and Zagier proved [19] that the moduli space of normalized, rank one, CohFTs in genus zero has coordinates $\mathbf{s}$ such that the tensor product in the category of CohFTs is additive in these coordinates. We prove that the moduli space of normalized, rank one, restricted, CohFTs in genus one has similar coordinates $(\mathbf{s}, u)$. Here the variable u arises because in genus one we need to introduce another tautological cohomology class (called λ) due to Mumford. Our proof involves a mixture of techniques from Kaufmann, Manin, and Zagier [19], Getzler [11], and results from the first part of this paper.

In section 1, we review the geometry of the moduli space of stable curves, its stratification in terms of stable graphs, tautological cohomology classes, their intersection numbers, and associated generating functions. In section 2, we obtain a simple presentation for these classes in terms of stable graphs in genus 0 and 1 and derive differential equations satisfied by the generating functions associated to their intersection numbers. In section 3, we derive the analogues of the puncture and dilaton equations. In section 4, we write closed form expressions for these intersection numbers. In section 5, we use analytic properties of the generating function to prove an asymptotic formula for the Weil-Petersson volumes of the moduli space of genus 1 curves as the number of punctures becomes very large. Finally in section 6, we describe the moduli space of rank 1, restricted, CohFTs in genus one.

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1. Moduli Space of Curves

Notation . In this paper we always consider cohomology with the rational coefficients: $H^\bullet(X)$ stands for $H^\bullet(X;\mathbb{Q})$. We denote the set $\{1,\dots,n\}$ by $[n]$. If I is a finite set we denote its cardinality by $|I|$.

1.1. **Basic Definitions.** Let $\mathcal{M}_{g,n}$ be the moduli space of smooth curves of genus g with n marked points, where $2g-2+n>0$, i.e. $\mathcal{M}_{g,n}=\{\,[\Sigma;x_1,x_2,\dots,x_n]\,\}$ where Σ is a genus g Riemann surface and $x_1,x_2,\dots,x_n$ are distinct marked points on Σ. Two such configurations are equivalent if they are related by a biholomorphic map. The moduli space $\mathcal{M}_{g,n}$ has a natural compactification due to Deligne, Knudsen, and Mumford denoted by $\overline{\mathcal{M}}_{g,n}=\{\,[C;x_1,x_2,\dots,x_n]\,\}$ which is the moduli space of stable curves of genus g with n punctures in which $\mathcal{M}_{g,n}$ sits as a dense open subset. The spaces $\overline{\mathcal{M}}_{g,n}$ are connected, compact, complex orbifolds (in fact, stacks) with complex dimensions $3g-3+n$. The complement of $\mathcal{M}_{g,n}$ in $\overline{\mathcal{M}}_{g,n}$ is a divisor with normal crossings and consists of those stable curves which have double points.

The moduli space $\overline{\mathcal{M}}_{g,n}$ forms the base of a universal family. Let $\pi:\mathcal{C}_{g,n}\to\overline{\mathcal{M}}_{g,n}$ be the *universal curve* which can be identified with $\mathcal{C}_{g,n}=\overline{\mathcal{M}}_{g,n+1}$ where π is the projection obtained by forgetting the $(n+1)^{\text{st}}$ puncture and followed by collapsing any resulting unstable irreducible components of the curve, if any, to a point. The universal curve $\mathcal{C}_{g,n}\to\overline{\mathcal{M}}_{g,n}$ is furthermore endowed with canonical sections $\sigma_1,\sigma_2,\dots,\sigma_n$ such that σ_i maps $[C;x_1,x_2,\dots,x_n]\mapsto[C';x'_1,x'_2,\dots,x'_{n+1}]$ where C' is obtained from C by attaching a three punctured sphere to x_i at one of its punctures to create a double point, then labeling the remaining two punctures on the sphere x'_i and x'_{n+1}, and finally setting all other $x'_j=x_j$. The sections σ_i are well-defined since $\overline{\mathcal{M}}_{0,3}$ is a point. The image of $\sigma_i:\overline{\mathcal{M}}_{g,n}\to\mathcal{C}_{g,n}$ gives rise to a divisor D_i in $\mathcal{C}_{g,n}$ for all $i=1,2,\dots,n$.

1.2. **Natural Stratification.** In the sequel it will be convenient to consider markings by arbitrary finite sets rather than by just $[n]$. If I

is a finite set we denote by $\overline{\mathcal{M}}_{g,I} \cong \overline{\mathcal{M}}_{g,|I|}$ the corresponding moduli space.

The natural stratification of $\overline{\mathcal{M}}_{g,n}$ is best described in terms of graphs, and therefore we start with fixing the notation concerning graphs. We will consider only connected graphs. Each graph Γ can be described in terms of its set of vertices $V(\Gamma)$, set of edges $E(\Gamma)$, and set of tails $S(\Gamma)$. Each edge has two endpoints belonging to $V(\Gamma)$ which are allowed to be the same. Each tail has only one endpoint. If $v \in V(\Gamma)$,we denote by $n(v)$ the number of half-edges emanating from v, where each edge gives rise to two half-edges, and each tail to one half-edge.

The natural stratification of $\overline{\mathcal{M}}_{g,n}$ is determined by the type of the degeneration of the curve representing a point in the moduli space, and its strata can be labeled by stable graphs. A *stable graph* consists of a triple (Γ, g, μ), where Γ is a connected graph as above, $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$, and μ is a bijection between $S(\Gamma)$ and a given set I. Moreover, one requires that for each vertex v, the stability condition $2g(v)-2+n(v) > 0$ is satisfied. If $[C; x_1, x_2, \ldots, x_n]$ is a stable, n-pointed curve one obtains the corresponding stable graph, called its *dual graph*, by collapsing each irreducible component to a point (vertex), connecting any two vertices if their corresponding components share a double point and attaching a tail to a vertex for each marked point on that component.

We define the *genus* $g(\Gamma)$ *of* Γ to be $b_1(\Gamma) + \sum_{v\in V(\Gamma)} g(v)$, where $b_1(\Gamma)$ is the first Betti number of Γ. We denote by $\mathcal{G}_{g,n}$ the set of the equivalence classes of stable graphs of genus g with n tails labeled by $[n]$. There is a natural action of the symmetric group S_n on $\mathcal{G}_{g,n}$. Associating a stable curve to its dual graph provides an S_n-equivariant bijection between the strata of the natural stratification of $\overline{\mathcal{M}}_{g,n}$ and the elements of $\mathcal{G}_{g,n}$.

Let $\overline{\mathcal{M}}_\Gamma$ be (the closure of) the moduli space of stable curves whose dual graph is Γ. It is a closed irreducible subvariety of codimension $|E(\Gamma)|$ of $\overline{\mathcal{M}}_{g(\Gamma),S(\Gamma)}$. Moreover, it is isomorphic to a quotient of the cartesian product

$$\prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}$$

by $\mathrm{Aut}(\Gamma)$, where the automorphisms of a stable graph (Γ, g, μ) are required to preserve g and μ. This quotient morphism can be made canonical if one creates a pair of labels for each edge of Γ and labels the $n(v)$ half-edges emanating from v by the corresponding elements of $S(\Gamma)$ with the labels corresponding to the edges.

Each $\overline{\mathcal{M}}_\Gamma$ determines the fundamental class, in the sense of orbifolds, lying in $H^\bullet(\overline{\mathcal{M}}_{g(\Gamma),n(\Gamma)})$. The pull back of this class under the morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is represented by a subvariety of $\overline{\mathcal{M}}_{g,n+1}$ corresponding to the $|V(\Gamma)|$ graphs each of which is obtained by attaching a tail numbered $n+1$ to a vertex of Γ. It is also easy to push down these fundamental classes. Let $\overline{\mathcal{M}}_{\Gamma'}$ represents an element of $H^\bullet(\overline{\mathcal{M}}_{g,n+1})$. The image of this element under $\pi_* : H^{\bullet+2}(\mathcal{C}_{g,n}) \to H^\bullet(\overline{\mathcal{M}}_{g,n})$, induced by the fiber integration, is zero if after removing the $(n+1)^{\rm st}$ tail from Γ' the graph remains stable. In the other case, when the removal of the $(n+1)^{\rm st}$ tail destabilizes Γ', the image is obtained by stabilization, i.e., contracting the edge connecting the unstable vertex with the rest of the graph.

1.3. **Tautological Classes.** We will now describe three types of tautological cohomology classes (ψ, κ, and λ) associated to the universal curve. Consider the universal curve $\mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. The cotangent bundle to its fibers (in the orbifold sense) forms the holomorphic line bundle $\omega_{g,n}$. Let $\mathcal{L}_{(g,n),i} \to \overline{\mathcal{M}}_{g,n}$ be given by the pullback $\mathcal{L}_{(g,n),i} = \sigma_i^*\omega_{(g,n)}$. The tautological classes $\psi_{(g,n),i}$ in $H^2(\overline{\mathcal{M}}_{g,n})$ are defined by

$$\psi_{(g,n),i} := c_1(\mathcal{L}_{(g,n),i})$$

where c_1 denotes the first Chern class.[2]

The tautological classes $\kappa_{(g,n),i}$ in $H^{2i}(\overline{\mathcal{M}}_{g,n})$ for $i = 0, 1, \dots , (3g - 3 + n)$ are defined as follows. Consider the bundle $\omega_{g,n}(D) \to \mathcal{C}_{g,n}$ consisting of $\omega_{g,n}$ twisted by the divisor $D = \sum_{i=1}^n D_i$, then

$$\kappa_{(g,n),i} := \pi_*(c_1(\omega_{g,n}(D))^{i+1}).$$

In particular, $\kappa_{(g,n),0} = 2g - 2 + n \in H^0(\overline{\mathcal{M}}_{g,n})$ is the negative of the Euler characteristic of a smooth curve of genus g with n points removed. We also have the equality $\omega_{g,n}(D) = \mathcal{L}_{(g,n+1),n+1}$ [1, 15]. Therefore

$$\kappa_{(g,n),i} = \pi_*(\psi^{i+1}_{(g,n+1),n+1}). \tag{2}$$

The tautological λ classes are defined to be

$$\lambda_{(g,n),i} := c_l(\pi_*\,\omega_{g,n}) \in H^{2i}(\overline{\mathcal{M}}_{g,n}),$$

where $l = 1, \dots , g$ because $\pi_*\,\omega_{g,n}$ is an orbifold bundle of rank g. (There are no λ classes in genus 0 and we define $\lambda_{(0,n),i} := 0$.) One can easily see that $\lambda_{(g,n+1),i} = \pi^*\lambda_{(g,n),i}$. Therefore, all of the λ classes are pull backs of the λ classes on $\overline{\mathcal{M}}_{1,1}$ and $\overline{\mathcal{M}}_{g,0}$, $g \geq 2$. They can be

[2]The Chern classes are in the sense of orbifolds and are therefore rational.

expressed in terms of the κ classes, the ψ classes, and the cohomology classes lying at the boundary [28]. In particular,

$$\kappa_{(g,n),1} = 12\lambda_{(g,n),1} - \delta_{(g,n)} + \sum_{i=1}^{n} \psi_{(g,n),i}, \tag{3}$$

where $\delta_{(g,n)}$ is the fundamental class of $\overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ [3, 9, 28]. (This formula was brought to our attention by E. Getzler.)

We will drop subscripts associated to the genus and the number of punctures if there is no ambiguity.

Notation . Let $\mathcal{S}_k$ be the set of infinite sequences of non-negative integers $\mathbf{m} = (m_k, m_{k+1}, m_{k+2}, \dots)$ such that $m_i = 0$ for all i sufficiently large. We denote by $\boldsymbol{\delta}_a$ the infinite sequence which has only one non-zero entry 1 at the a^{th} place. For $\mathbf{m} = (m_0, m_1, m_2, \dots) \in \mathcal{S}_0$ and $\mathbf{t} = (t_0, t_1, t_2, \dots)$, a family of independent formal variables, we will use notation of the type

$$|\mathbf{m}| := \sum_{i\geq 0} i\, m_i, \quad \|\mathbf{m}\| := \sum_{i\geq 0} m_i, \quad \mathbf{m}! := \prod_{i\geq 0} m_i!, \quad \mathbf{t}^{\mathbf{m}} := \prod_{i\geq 0} t_i^{m_i}.$$

We say that $\mathbf{l} \leq \mathbf{m}$ if $l_i \leq m_i$ for all i. If $\mathbf{l} \leq \mathbf{m}$ we let

$$\binom{\mathbf{m}}{\mathbf{l}} := \prod_{i\geq 0} \binom{m_i}{l_i}$$

We will use the same notation when $\mathbf{m} \in \mathcal{S}_1$.

1.4. **Generating Functions.** Witten [31] defined a generating function which incorporates all of the information about the integrals of products of the ψ classes. In order to describe this function we need to introduce the following notation. Let

$$\langle \tau_{d_1}\tau_{d_2}\dots\tau_{d_n}\rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\psi_2^{d_2}\dots\psi_n^{d_n},$$

where g is determined by the equation $3g - 3 + n = d_1 + d_2 + \cdots + d_n$. If there exists no such g, then the left hand side is by definition zero. In case we want to mention the genus explicitly we will write $\langle \tau_{d_1}\tau_{d_2}\dots\tau_{d_n}\rangle_g$. Note that this expression is symmetric with respect to $d_1, d_2, \dots, d_n$ since ψ_i's are interchanged under the action of the symmetric group S_n. Therefore one can write it as $\langle \tau_0^{m_0}\tau_1^{m_1}\tau_2^{m_2}\dots\rangle$, where the set $\{d_1, \dots, d_n\}$ contains m_0 zeros, m_1 ones, etc. The generating function is defined by

$$F(t_0, t_1, t_2, \dots) := \langle \exp \sum_{j=0}^{\infty} t_j\tau_j\rangle = \sum_{\{m_i\}\in\mathcal{S}_0} \prod_{i=0}^{\infty} \langle \tau_0^{m_0}\tau_1^{m_1}\tau_2^{m_2}\dots\rangle \frac{t_i^{m_i}}{m_i!}.$$

We will also use the notation

$$\sum_{\mathbf{m}\in\mathcal{S}_0} \langle \boldsymbol{\tau}^{\mathbf{m}} \rangle \frac{\mathbf{t}^{\mathbf{m}}}{\mathbf{m}!}$$

for the last expression. Note that one can also write $F(\mathbf{t}) = \sum_{g=0}^{\infty} F_g(\mathbf{t})$, where $F_g(\mathbf{t}) := \sum_{\mathbf{m}\in\mathcal{S}_0} \langle \boldsymbol{\tau}^{\mathbf{m}} \rangle_g \frac{\mathbf{t}^{\mathbf{m}}}{\mathbf{m}!}$.

Witten conjectured in [32] and Kontsevich proved in [21] that F is the logarithm of a τ-function in the KdV-hierarchy.

In [19] Kaufmann, Manin, and Zagier considered a similar generating function for κ classes. If one defines

$$\langle \boldsymbol{\kappa}^{\mathbf{p}} \rangle_g = \langle \kappa_1^{p_1} \kappa_2^{p_2} \dots \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^{p_1} \kappa_2^{p_2} \dots ,$$

then their generating function is

$$K_g(x;\mathbf{s}) = K_g(x; s_1, s_2, \dots) := \sum_{\mathbf{p}\in\mathcal{S}_1} \langle \boldsymbol{\kappa}^{\mathbf{p}} \rangle_g \frac{x^{|\mathbf{p}|}}{|\mathbf{p}|!} \frac{\mathbf{s}^{\mathbf{p}}}{\mathbf{p}!}.$$

Note that here it is important to indicate the genus. The number of punctures n is then determined from $3g - 3 + n = |\mathbf{p}|$.

We introduce the generating function H which incorporates both of the ψ and κ classes. We shall see that F and K enjoy similar properties which arise because H obeys those properties.

First we introduce the following notation. Let $\mathbf{m} \in \mathcal{S}_0$ and $\mathbf{p} \in \mathcal{S}_1$. Define

$$\langle \boldsymbol{\tau}^{\mathbf{m}} \boldsymbol{\kappa}^{\mathbf{p}} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \psi_2^{d_2} \dots \psi_n^{d_n} \kappa_1^{p_1} \kappa_2^{p_2} \dots ,$$

where the set $\{d_1, \dots , d_n\}$ contains m_0 zeros, m_1 ones, etc., and (g, n) is determined by the equations $n = \|m\|$, $3g - 3 + n = |\mathbf{m}| + |\mathbf{p}|$. If no such g exists we set the expression above to zero. As before, we write $\langle \boldsymbol{\tau}^{\mathbf{m}} \boldsymbol{\kappa}^{\mathbf{p}} \rangle_g$ when we want to fix g.

Definition 1.1. Let $\mathbf{t} = (t_0, t_1, \dots)$ and $\mathbf{s} = (s_1, s_2, \dots)$ be independent families of independent formal variables. We define

$$H(\mathbf{t};\mathbf{s}) := \sum_{\mathbf{m}\in\mathcal{S}_0, \mathbf{p}\in\mathcal{S}_1} \langle \boldsymbol{\tau}^{\mathbf{m}} \boldsymbol{\kappa}^{\mathbf{p}} \rangle \frac{\mathbf{t}^{\mathbf{m}}}{\mathbf{m}!} \frac{\mathbf{s}^{\mathbf{p}}}{\mathbf{p}!}.$$

One can split H into the sum of H_g, $g = 0, 1, \dots$. Each H_g lies in a kernel of a certain scaling differential operator, i.e., it satisfies the *charge conservation equation*. The multiplication of H by $\|\mathbf{m}\|$ is equivalent to applying the operator $\mathcal{E} := \sum t_i \partial_i$, and by $|\mathbf{m}|$ is equivalent to applying $\sum i\, t_i \partial_i$. Therefore one has

$$[3(1-g) + \sum_{i=0}^{\infty}(i-1)\, t_i \partial_i + \sum_{i=1}^{\infty} i\, s_i d_i]\, H_g = 0, \tag{4}$$

where $\partial_i := \partial/\partial t_i$ and $d_i := \partial/\partial s_i$.

Clearly $H(\mathbf{t};\mathbf{0}) = F(\mathbf{t})$. In order to relate H to K one fixes a genus g, sets $t_1 = t_2 = \cdots = 0$, and $t_0 = x$. The infinite sequence $\mathbf{m}$ reduces to $m_0 = n = |\mathbf{p}| + 3 - 3g$. It follows that

$$H_g(x, \mathbf{0}; \mathbf{s}) = \sum_{\mathbf{p}\in\mathcal{S}_1} \langle \boldsymbol{\kappa}^{\mathbf{p}} \rangle_g \, \frac{x^{|\mathbf{p}|+3-3g}}{(|\mathbf{p}|+3-3g)!}\, \frac{\mathbf{s}^{\mathbf{p}}}{\mathbf{p}!}.$$

In this paper we are primarily interested in genus 0 and 1. Then $K_0(x;\mathbf{s}) = H_0'''(x,\mathbf{0};\mathbf{s})$ and $K_1(x;\mathbf{s}) = H_1(x,\mathbf{0};\mathbf{s})$, where the prime denotes the partial derivative with respect to x.

2. Presentation of the Tautological Classes via Graphs

In this section we will mainly focus on H_0 and H_1, the generating functions for the intersection numbers of the κ and ψ classes in genus 0 and genus 1. First we show that H_0 satisfies a system of nonlinear differential equations. These equations when, restricted to the ψ classes, were first obtained by Witten [31, 32]. When restricted to the κ classes in genus zero, equivalent but much more complicated equations were obtained by Kaufmann, Manin, and Zagier [19], the difference being accounted for by our simple, recursive presentation of powers of κ and ψ classes in genus zero and one in terms of boundary divisors. We prove that H_0 satisfies differential equations of the same form as those obtained by Witten for just the ψ classes. We then obtain a system of differential equations relating H_0 and H_1, and the explicit formula (1).

Here we present a geometric approach using the explicit presentation of the ψ and κ classes in terms of the boundary strata.

2.1. Basic Relation.

In the beginning we want to state some general facts which hold for all genera. Let $\pi : \mathcal{C}_{g,n} = \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve. To simplify the notation in this subsection, we denote $\psi_{(g,n),i}$ by ψ_i, $\psi_{(g,n+1),i}$ by $\widehat{\psi}_i$, and we use the same convention for the κ classes. Recall from Sec. 1 that D_i denotes the image of the i^{th} canonical section $\sigma_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$. We will denote by the same letter its dual cohomology class in $H^2(\overline{\mathcal{M}}_{g,n+1})$.

Lemma 2.1. *For each $k \geq 1$*

$$\widehat{\psi}_i^a = \pi^*\psi_i^a + \sigma_{i*}\psi_i^{a-1}, \quad k \geq 1. \tag{5}$$

Proof. It is easy to show that the equation $\widehat{\psi}_i^a = \pi^*\psi_i^a + \pi^*\psi_i^{a-1}D_i$ from [31] can be rewritten as

$$\widehat{\psi}_i^a = \pi^*\psi_i^a + (-1)^{a-1}D_i^a, \quad k \geq 1.$$

Applying the functor $\sigma_{i*}\sigma_i^*$, which is the multiplication by D_i, to the above equation, and using that $\sigma_i^*\widehat{\psi}_i = 0$ one gets

$$(-1)^a D_i^{a+1} = \sigma_{i*}\psi_i^a, \quad k \geq 0.$$

These two equalities imply the lemma. □

2.2. **Explicit Presentation in Genus 0, 1.** In genus 0 and genus 1 equation (5) allows us to express inductively all powers of ψ classes, and therefore κ classes, in terms of the boundary strata. Because the ψ classes are interchanged under the action of the symmetric group this is enough to compute $\psi_{(g,n),1}$, $g = 0, 1$, and all its powers. In the calculations below we will use the properties stated in Sec. 1 regarding the pull backs and push forwards of the cohomology classes represented by graphs.

Let us introduce the following notation for the rest of this section. On graphs we denote by $\bullet$ vertices of genus 0, and by $\circ$ vertices of genus 1. We always assume that the ψ classes are associated to the marked point labeled 1, and subsequently omit it from the notation. We denote $\psi_{(1,n),1}$ by $\psi_{(n)}$, and we denote $\psi_{(0,J),1}$ by $\phi_{(J)}$, where J is a finite set, $1 \in J$. Similarly, we denote $\kappa_{(1,n),a}$ by $\kappa_{(n),a}$, and $\kappa_{(0,J),a}$ by $\omega_{(J),a}$.

We also adopt the following convention. Let Γ be a stable graph. According to Sec. 1, Γ determines a canonical finite quotient map from a product of moduli spaces to $\overline{\mathcal{M}}_\Gamma$ provided certain choices have been made. We denote by ρ_Γ the composition

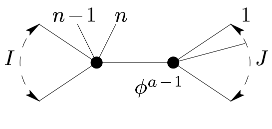
$$\rho_\Gamma : \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \longrightarrow \overline{\mathcal{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}_{g(\Gamma),S(\Gamma)}, \tag{6}$$

where the first arrow is the quotient morphism, and the second arrow is the inclusion.

Let $\gamma_v \in H^\bullet(\overline{\mathcal{M}}_{g(v),n(v)})$. We denote $\frac{1}{|\mathrm{Aut}(\Gamma)|}\rho_{\Gamma*}(\otimes_v\gamma_v)$ by the picture of Γ where each vertex v is in addition labeled by the cohomology class γ_v. We may omit the label of v if $\gamma(v)$ is the fundamental class of $\overline{\mathcal{M}}_{g(v),n(v)}$. In particular, the fundamental class of $\overline{\mathcal{M}}_\Gamma$ (in the orbifold sense) is represented by Γ with all additional labels omitted. The only possible ambiguity would arise if $\otimes_v\gamma_v$ were not invariant under $\mathrm{Aut}(\Gamma)$, but this situation will not arise in this paper.

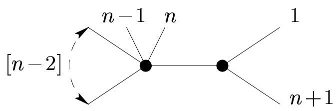
In the pictures the dashed line with two arrows indicates the (sub)set of the tails emanating from a particular vertex. If ϕ or ω labels a vertex of a graph we will omit the subscript from the notation because it is determined by the graph. (We also assume that ϕ is associated to the marked point labeled by 1.) The power ϕ^0 represents the fundamental class.

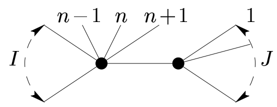
Proposition 2.2. *If $n \geq 4$, $a \geq 1$, then the following holds in $H^\bullet(\overline{\mathcal{M}}_{0,n})$*

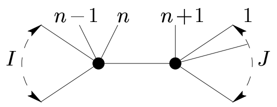
(7) $$\phi^a_{(n)} = \sum_{\substack{I \sqcup J = [n-2] \\ 1 \in J}}$$ 

Remark . The class $\phi^a_{(n)}$ is invariant under the subgroup $S_{n-1} \subset S_n$ whose elements fix 1. Therefore instead of $n-1, n$ we can choose any two labels a, b, $2 \leq a < b \leq n$ to be distinguished.

Proof. The statement is true when $n = 4$ and $a = 1$ because $\int_{\overline{\mathcal{M}}_{0,4}} \psi_1 = 1$. We shall first prove by induction that the statement is true for all $n \geq 4$ and $a = 1$. Let us consider the projection $\pi : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ which "forgets" the $n+1^{\text{st}}$ marked point. By the induction hypothesis we assume that the statement is true for some n and $a = 1$. Applying π^* one gets:

$\phi_{(n+1)}$ − 

$= \sum_{\substack{I \sqcup J = [n-2] \\ 1 \in J}} \Bigg($ 

$+$ 

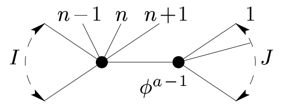
$\Bigg)$

The left hand side is equal to $\phi_{(n+1)} - D_1 = \pi^*\phi_{(n)}$. Moving D_1 to the right hand side and relabeling the tails marked $n-1, n$ by $n, n+1$ respectively one gets the statement in case of $n+1$, $a = 1$.

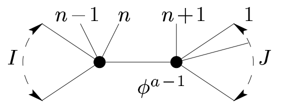
Now assume that $a \geq 2$. If $n = 4$, then the statement of the proposition is trivially satisfied because all terms vanish by dimensional considerations. We assume that the statement is true for a pair (n, a) and all pairs (n', a'), where $a' < a$, and prove it for $(n+1, a)$. Applying π^*

one gets

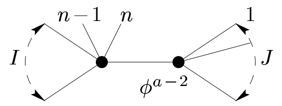
$$\phi^a_{(n+1)} - \sigma_{1*}\phi^{a-1}_{(n)} = \sum_{\substack{I\sqcup J=[n-2]\\ 1\in J}} \Bigg($$



$$+$$



$$- \sigma_{1*}\Big($$



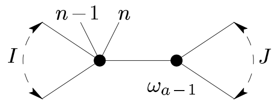
$$\Big)\Bigg)$$

According to the induction hypothesis the terms with σ_{1*} on the left hand side and the right hand side cancel each other. Thus, we get the statement of the proposition for the pair $(n+1, a)$. $\square$

Let π_1 be the morphism $\overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ forgetting the first marked point. Applying it to (7) with $a+1$, using (2), and renumbering the labels $\{2, \dots, n+1\}$ by the elements of $[n]$ we get the following

Corollary 2.3. *If $n \geq 4$, $a \geq 1$, then the following holds in $H^\bullet(\overline{\mathcal{M}}_{0,n})$*

(8)
$$\omega_{(n),a} = \sum_{I\sqcup J=[n-2]}$$



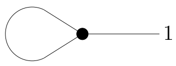
$\square$

In (8) if $a = 1$ one should use that $\omega_{(J\sqcup *),0}$ associated to the right vertex is equal to $|J|-1$ times the fundamental class (cf. Sec. 1). This agrees with π_{1*} of (7) when $a = 1$.

Now we establish a relation between the ψ and κ classes in genus 0 and 1. The proof is virtually identical to that of Prop. 2.2, and we will not reproduce it.

It is shown in [4, VI.4] that

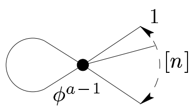
$$\psi_{(1)} = \frac{1}{12}$$



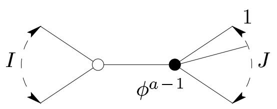
Note that we take the coefficient is $\frac{1}{12}$ rather than $\frac{1}{24}$ due to the non-trivial automorphism of the graph. Using this and (5) one obtains the following

Proposition 2.4. *If $n \geq 1$, $a \geq 1$, then the following holds in $H^\bullet(\overline{\mathcal{M}}_{1,n})$*

(9)
$$\psi^a_{(n)} = \frac{1}{12}$$



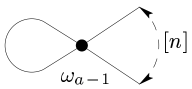
$$+ \sum_{\substack{I\sqcup J=[n]\\ 1\in J}}$$

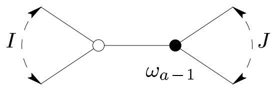


$\square$

Pushing down the above along $\pi_1 : \overline{\mathcal{M}}_{1,n+1} \to \overline{\mathcal{M}}_{1,n}$, and renumbering the labels one gets

Corollary 2.5. *If $n \geq 1$, $a \geq 1$, then the following holds in $H^\bullet(\overline{\mathcal{M}}_{1,n})$*

(10) $\kappa_{(n),a} = \dfrac{1}{12}$ 

$+ \sum_{I \sqcup J = [n]}$ 

□

2.3. **Recursion Relations and Differential Equations.** Now we derive the corresponding recursion relations and differential equations for the intersection numbers of the products of the ψ and κ classes using the explicit graph presentations above. In order to obtain the recursion relations we use a method from [19] to integrate the product of the ψ and κ classes over the Poincaré dual of a chosen ψ or κ class. In order to do this we need to know how the tautological classes restrict to the strata of the natural stratification. The restriction of a ψ class to a boundary stratum is obvious.

In order to restrict products of the κ classes we use Lemma 1.3 from [19] where the authors show the following restriction property for the κ classes. (They show it in the case of genus 0, but their proof is in fact valid for all genera.) Let (Γ, g, μ) be a stable graph, ρ_Γ is the corresponding morphism defined by (6), and $\boldsymbol{\kappa}^{\mathbf{p}}$ is a product of the κ classes on $\overline{\mathcal{M}}_{g(\Gamma),S(\Gamma)}$. Then

$$\int_{\prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}} \frac{\rho_\Gamma^*(\boldsymbol{\kappa}^{\mathbf{p}})}{\mathbf{p}!} = \sum_{\substack{\mathbf{p}^v : v \in V(\Gamma) \\ \sum \mathbf{p}^v = \mathbf{p}}} \prod_{v \in V(\Gamma)} \frac{\langle \boldsymbol{\kappa}^{\mathbf{p}^v} \rangle_{g(v)}}{\mathbf{p}^v!}$$

The argument uses a fact proved in [1] that the collection $\kappa_{(g,n),a}$ for each fixed a forms a *logarithmic cohomological field theory* (cf. Sec. 6), i.e., the κ classes satisfy the relation

$$\rho_\Gamma^*(\kappa_a) = \sum_{v \in V(\Gamma)} \kappa_{g(v),n(v)}. \tag{11}$$

We start with genus 0. Recall that $H(\mathbf{t}; \mathbf{s})$ is the generating function incorporating the intersection numbers for the products of the ψ and κ classes defined in Sec. 1, ∂_a, d_a are partial derivative with respect to t_a, $a \geq 0$, s_a, $a \geq 1$, and $\mathcal{E} = \sum_{i=0}^{\infty} t_i \partial_i$.

Theorem 2.6. *For each* $\mathbf{m} \in \mathcal{S}_1$, $\mathbf{p} \in \mathcal{S}$, $k, l \geq 0$, *and* $a \geq 1$ *one has*

$$\begin{aligned}
&\langle \boldsymbol{\tau}^{\mathbf{m}+\boldsymbol{\delta}_k+\boldsymbol{\delta}_l+\boldsymbol{\delta}_a} \boldsymbol{\kappa}^{\mathbf{p}} \rangle_0 \\
&\quad = \sum_{\substack{\mathbf{m}'+\mathbf{m}''=\mathbf{m} \\ \mathbf{p}'+\mathbf{p}''=\mathbf{p}}} \binom{\mathbf{m}}{\mathbf{m}'} \binom{\mathbf{p}}{\mathbf{p}'} \langle \boldsymbol{\tau}^{\mathbf{m}'+\boldsymbol{\delta}_k+\boldsymbol{\delta}_l+\boldsymbol{\delta}_0} \boldsymbol{\kappa}^{\mathbf{p}'} \rangle_0 \langle \boldsymbol{\tau}^{\mathbf{m}''+\boldsymbol{\delta}_{a-1}+\boldsymbol{\delta}_0} \boldsymbol{\kappa}^{\mathbf{p}''} \rangle_0, \\
&\langle \boldsymbol{\tau}^{\mathbf{m}+\boldsymbol{\delta}_k+\boldsymbol{\delta}_l} \boldsymbol{\kappa}^{\mathbf{p}+\boldsymbol{\delta}_a} \rangle_0 \\
&\quad = \sum_{\substack{\mathbf{m}'+\mathbf{m}''=\mathbf{m} \\ \mathbf{p}'+\mathbf{p}''=\mathbf{p}}} \binom{\mathbf{m}}{\mathbf{m}'} \binom{\mathbf{p}}{\mathbf{p}'} \langle \boldsymbol{\tau}^{\mathbf{m}'+\boldsymbol{\delta}_k+\boldsymbol{\delta}_l+\boldsymbol{\delta}_0} \boldsymbol{\kappa}^{\mathbf{p}'} \rangle_0 \langle \boldsymbol{\tau}^{\mathbf{m}''+\boldsymbol{\delta}_0} \boldsymbol{\kappa}^{\mathbf{p}''+\boldsymbol{\delta}_{a-1}} \rangle_0.
\end{aligned}$$

Equivalently, for each $k, l \geq 0$ *the function* $H_0(\mathbf{t}; \mathbf{s})$ *satisfies*

$$\begin{aligned}
\partial_a \partial_k \partial_l H_0 &= (\partial_k \partial_l \partial_0 H_0)(\partial_{a-1} \partial_0 H_0) \quad \textit{when } a \geq 1, \\
d_1 \partial_k \partial_l H_0 &= (\partial_k \partial_l \partial_0 H_0)((\mathcal{E} - 1)\partial_0 H_0), \\
d_a \partial_k \partial_l H_0 &= (\partial_k \partial_l \partial_0 H_0)(d_{a-1} \partial_0 H_0) \quad \textit{when } a \geq 2.
\end{aligned}$$

This system together with $H_0(t_0, \mathbf{0}; \mathbf{0}) = \frac{t_0^3}{6}$ *uniquely determines* H_0. □

Proof. The recursion relations are a direct consequence of (7), (8), and the restriction properties of the ψ and κ classes described above.

In order to derive the differential equations from the recursion relations one notices that the increment of m_a or p_a by one in a recursion relation corresponds to taking the partial derivative with respect to t_a or s_a.

The operator $\mathcal{E}$ appears because the second recursion relation when $a = 1$ produces ω_0. The corresponding moduli space is $\overline{\mathcal{M}}_{0,J\sqcup *}$. As $|J \sqcup *| = \|\mathbf{m}''\| + 1$, it follows that $\omega_0 = |J| - 1 = \|\mathbf{m}''\| - 1$, and we use that the multiplication by m_i can be expressed by $t_i \partial_i$. □

Remark . Setting $\mathbf{s} = \mathbf{0}$ in the first equation one gets differential equations satisfied by F_0 (cf. [32]).

Remark . Setting $k = l = 0$, $t_0 = x$, and $t_1 = t_2 = \ldots = 0$ in the second equation one gets differential equations satisfied by $H(x, \mathbf{0}; \mathbf{s})$ whose third derivative with respect to x is $K_0(x; \mathbf{s})$. These equations are a simple consequence of the results in [19, Sec. 1].

Now we turn to genus 1. We use the explicit presentations (9) and (10) and take into account the automorphism groups of the graphs to obtain the following

Theorem 2.7. *For each* $\mathbf{m} \in \mathcal{S}_0$, $\mathbf{p} \in \mathcal{S}_1$, *and* $a \geq 1$ *one has*

$$\begin{aligned}
\langle \boldsymbol{\tau}^{\mathbf{m}+\boldsymbol{\delta}_a} \boldsymbol{\kappa}^{\mathbf{p}} \rangle_1 &= \frac{1}{24} \langle \boldsymbol{\tau}^{\mathbf{m}+2\boldsymbol{\delta}_0+\boldsymbol{\delta}_{a-1}} \boldsymbol{\kappa}^{\mathbf{p}} \rangle_0 \\
&\quad + \sum_{\substack{\mathbf{m}'+\mathbf{m}''=\mathbf{m} \\ \mathbf{p}'+\mathbf{p}''=\mathbf{p}}} \binom{\mathbf{m}}{\mathbf{m}'} \binom{\mathbf{p}}{\mathbf{p}'} \langle \boldsymbol{\tau}^{\mathbf{m}'+\boldsymbol{\delta}_0} \boldsymbol{\kappa}^{\mathbf{p}'} \rangle_1 \langle \boldsymbol{\tau}^{\mathbf{m}''+\boldsymbol{\delta}_0+\boldsymbol{\delta}_{a-1}} \boldsymbol{\kappa}^{\mathbf{p}''} \rangle_0, \\
\langle \boldsymbol{\tau}^{\mathbf{m}} \boldsymbol{\kappa}^{\mathbf{p}+\boldsymbol{\delta}_a} \rangle_1 &= \frac{1}{24} \langle \boldsymbol{\tau}^{\mathbf{m}+2\boldsymbol{\delta}_0} \boldsymbol{\kappa}^{\mathbf{p}+\boldsymbol{\delta}_{a-1}} \rangle_0 \\
&\quad + \sum_{\substack{\mathbf{m}'+\mathbf{m}''=\mathbf{m} \\ \mathbf{p}'+\mathbf{p}''=\mathbf{p}}} \binom{\mathbf{m}}{\mathbf{m}'} \binom{\mathbf{p}}{\mathbf{p}'} \langle \boldsymbol{\tau}^{\mathbf{m}'+\boldsymbol{\delta}_0} \boldsymbol{\kappa}^{\mathbf{p}'} \rangle_1 \langle \boldsymbol{\tau}^{\mathbf{m}''+\boldsymbol{\delta}_0} \boldsymbol{\kappa}^{\mathbf{p}''+\boldsymbol{\delta}_{a-1}} \rangle_0.
\end{aligned}$$

Equivalently, the functions $H_1(\mathbf{t};\mathbf{s})$ *and* $H_0(\mathbf{t};\mathbf{s})$ *satisfy*

$$\begin{aligned}
\partial_a H_1 &= \frac{1}{24} \partial_{a-1}\partial_0\partial_0 H_0 + (\partial_0 H_1)(\partial_{a-1}\partial_0 H_0) \quad \textit{when } a \geq 1, \\
d_1 H_1 &= \frac{1}{24} \mathcal{E}\partial_0\partial_0 H_0 + (\partial_0 H_1)((\mathcal{E}-1)\partial_0 H_0), \\
d_a H_1 &= \frac{1}{24} d_{a-1}\partial_0\partial_0 H_0 + (\partial_0 H_1)(d_{a-1}\partial_0 H_0) \quad \textit{when } a \geq 2.
\end{aligned}$$

This system together with the system and the initial conditions from Thm. 2.6 uniquely determines the pair H_0, H_1. □

Remark . The first recursion relation when $\mathbf{p} = \mathbf{0}$ was obtained by Witten in [32].

The system of differential equations above can be solved explicitly for H_1 in terms of H_0 to derive (1).

Corollary 2.8. *The functions* H_1 *and* H_0 *are related by*

$$H_1 = \frac{1}{24} \log \partial_0^3 H_0.$$

Proof. Because of uniqueness it is enough to check that $\frac{1}{24}\log \partial_0^3 H_0$ satisfies the differential equation in Thm. 2.7. This is a straight forward calculation which makes use of the differential equations from Thm. 2.6. □

Remark . Setting $\mathbf{s} = \mathbf{0}$ we recover a result from [5, Sec. 2.2]. Setting $t_0 = x, t_1 = t_2 = \ldots = 0$ we get $K_1 = \frac{1}{24}\log K_0$.

3. Puncture and Dilaton Equations

In this section we introduce an approach which does not use explicit presentations of ψ and κ classes in terms of graphs. Instead we introduce the analogues of the puncture and dilaton equations. These equations generalize the classical puncture and dilaton equations obtained by Witten [32, 31]. (We shall explain these equations below.) This will allow us to write differential equations for H, and then, using these differential equations, prove that H_0 and H_1 satisfy (1).

3.1. **Recursion Relations.** In Sec. 1 we introduce the notation incorporating the intersection numbers of both of the ψ and κ classes. Now we shall to prove certain recursion relations for these numbers.

Lemma 3.1. *The following recursion relations are satisfied:*

$$\langle \boldsymbol{\tau}^{\mathbf{m}+\boldsymbol{\delta}_0}\boldsymbol{\kappa}^{\mathbf{p}}\rangle = \sum_{i=1}^{\infty} m_i \langle \boldsymbol{\tau}^{\mathbf{m}+\boldsymbol{\delta}_{i-1}-\boldsymbol{\delta}_i}\boldsymbol{\kappa}^{\mathbf{p}}\rangle + \sum_{\substack{\mathbf{j}=\mathbf{0}\\ |\mathbf{j}|>0}}^{\mathbf{p}} \binom{\mathbf{p}}{\mathbf{j}} \langle \boldsymbol{\tau}^{\mathbf{m}}\boldsymbol{\kappa}^{\mathbf{p}-\mathbf{j}+\boldsymbol{\delta}_{|\mathbf{j}|-1}}\rangle, \tag{12}$$

and for each $a \geq 1$

$$\langle \boldsymbol{\tau}^{\mathbf{m}+\boldsymbol{\delta}_a}\boldsymbol{\kappa}^{\mathbf{p}}\rangle = \sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{p}} \binom{\mathbf{p}}{\mathbf{j}} \langle \boldsymbol{\tau}^{\mathbf{m}}\boldsymbol{\kappa}^{\mathbf{p}-\mathbf{j}+\boldsymbol{\delta}_{|\mathbf{j}|+a-1}}\rangle. \tag{13}$$

Proof. We continue to use the notation from 2.1, i.e., π is the universal curve over $\overline{\mathcal{M}}_{g,n}$, $(\widehat{\psi}_i,\ \widehat{\kappa}_i)$ and $(\psi_i,\ \kappa_i)$ are classes upstairs and downstairs respectively, σ_i is the i^{th} canonical section of π, and D_i is its image.

It was shown in [31] that $\widehat{\psi}_i^a = \pi^*\psi_i^a + \pi^*\psi_i^{a-1}D_i$ and in [1] that $\widehat{\kappa}_i = \pi^*\kappa_i + \widehat{\psi}_{n+1}^i$. Note also that $\widehat{\psi}_i D_i = 0$, $\widehat{\psi}_{n+1}D_i = 0$ for $i = 1, \dots, n$, and $D_iD_j = 0$ when $i \neq j$. Using this one derives that

$$\begin{aligned}
\pi_*(\widehat{\psi}_1^{d_1}\dots\widehat{\psi}_n^{d_n}\widehat{\kappa}_1^{p_1}\widehat{\kappa}_2^{p_2}\dots)&\\
&= \pi_*\Big((\pi^*\psi_1^{d_1} + \pi^*\psi_1^{d_1-1}D_1)\dots(\pi^*\psi_n^{d_n} + \pi^*\psi_n^{d_n-1}D_n)\\
&\qquad\quad \times (\pi^*\kappa_1 + \widehat{\psi}_{n+1})^{p_1}(\pi^*\kappa_2 + \widehat{\psi}_{n+1}^2)^{p_2}\dots\Big)\\
&= \sum_{i:d_i\neq 0} \psi_1^{d_1}\dots\psi_i^{d_i-1}\dots\psi_n^{d_n}\kappa_1^{p_1}\kappa_2^{p_2}\dots\\
&\qquad\quad + \sum_{\substack{\mathbf{j}=\mathbf{0}\\ |\mathbf{j}|>0}}^{\mathbf{p}} \binom{\mathbf{p}}{\mathbf{j}} \psi_1^{d_1}\dots\psi_n^{d_n}\kappa_{|\mathbf{j}|-1}\kappa_1^{p_1-j_1}\kappa_2^{p_2-j_2}\dots
\end{aligned}$$

Similarly one can show that

$$\begin{aligned}
\pi_*(\widehat{\psi}_1^{d_1}\dots\widehat{\psi}_n^{d_n}\widehat{\psi}_{n+1}^a\widehat{\kappa}_1^{p_1}\widehat{\kappa}_2^{p_2}\dots)&\\
&= \sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{p}} \binom{\mathbf{p}}{\mathbf{j}} \psi_1^{d_1}\dots\psi_n^{d_n}\kappa_{|\mathbf{j}|+a-1}\kappa_1^{p_1-j_1}\kappa_2^{p_2-j_2}\dots
\end{aligned}$$

Recall that $\kappa_0 = 2g - 2 + n$.

One can further integrate the push forward formulas above to obtain the statement of the lemma. $\square$

Remark . Recursion relations (12) and (13) do not mix intersection numbers in different genera.

Remark . If $\mathbf{p} = \mathbf{0}$, then the second sum in the first relation vanishes, and we obtain the classical puncture equation. If $\mathbf{p} = \mathbf{0}$ and $a = 1$ in the second relation, then we get the classical dilaton equation. Note that both classical equations involve only ψ classes.

Remark . This is clear recursion relations (12) and (13) allow to eliminate $\boldsymbol{\tau}$ from the intersection number, i.e., to express all mixed intersection numbers through the intersection numbers on $\overline{\mathcal{M}}_{0,3}$, $\overline{\mathcal{M}}_{1,1}$, and the intersection numbers of the κ classes on $\overline{\mathcal{M}}_{g,0}$, $g \geq 2$.

In [19, Cor. 2.3] the authors obtained an explicit expression for the intersection numbers of the κ classes through the intersection numbers of the ψ classes. This should be related to (13), but we do not know how to derive their formula from it.

3.2. **Differential Operators.** Now we derive differential equations for H using recursions (12) and (13). Recall that ∂_i, d_i denote the partial derivatives with respect to t_i, s_i.

Theorem 3.2. *The function* $\exp(H(\mathbf{t};\mathbf{s}))$ *is annihilated by the following differential operators:*

$$\begin{aligned}
-\partial_0 &+ \sum_{\mathbf{j}:\,|\mathbf{j}|\geq 2} \frac{\mathbf{s}^{\mathbf{j}}}{\mathbf{j}!} d_{|\mathbf{j}|-1} + \sum_{i=0}^{\infty} t_i \partial_{i-1} \\
&+ s_1 \Big(\sum_{i=0}^{\infty} \frac{2i+1}{3} t_i \partial_i + \sum_{i=1}^{\infty} \frac{2}{3} i\, s_i d_i \Big) + \frac{1}{2} t_0^2 \delta_{g,0} + \frac{1}{24} s_1 \delta_{g,1}, \\
-\partial_1 &+ \sum_{\mathbf{j}:\,|\mathbf{j}|\geq 1} \frac{\mathbf{s}^{\mathbf{j}}}{\mathbf{j}!} d_{|\mathbf{j}|} + \Big(\sum_{i=0}^{\infty} \frac{2i+1}{3} t_i \partial_i + \sum_{i=1}^{\infty} \frac{2}{3} i\, s_i d_i \Big) + \frac{1}{24} \delta_{g,1}, \\
-\partial_a &+ \sum_{\mathbf{j}} \frac{\mathbf{s}^{\mathbf{j}}}{\mathbf{j}!} d_{|\mathbf{j}|+a-1} \quad \textit{when } a \geq 2.
\end{aligned}$$

Remark . The differential operators above do not mix genus, and therefore they annihilate each $\exp(H_g(\mathbf{t};\mathbf{s}))$ separately.

Remark . The function $\exp(F(\mathbf{t}))$ is annihilated by differential operators L_i, $i \geq -1$, which, after a rescaling of variables, satisfy the Virasoro relations [5]. The first two differential operators in the statement of the theorem are analogues of L_{-1} and L_0 respectively, which encode the puncture and dilaton equations, respectively.

Proof. The differential operators above are the direct translation of the recursion relations (12) and (13). The addition of $\boldsymbol{\delta}_i$ to $\mathbf{m}$ or $\mathbf{p}$ translates into taking the corresponding partial derivative. The subtraction of $\boldsymbol{\delta}_i$ from $\mathbf{m}$, and multiplying the term by m_i translates into the multiplication by t_i. One should also change the summation index to obtain the second summand of each differential operator.

The terms in parentheses in the first two equations come from the value of $\kappa_0 = 2g-2+n$. We use (4) in order to express this number in terms of differential operators. Finally, the last terms in the first two equations correspond to the initial conditions $\langle\tau_0^3\rangle_0 = 1$, $\langle\kappa_1\rangle_1 = \frac{1}{24}$, and $\langle\tau_1\rangle_1 = \frac{1}{24}$. □

The theorem above leads to another proof of Cor. 1. First one notes that H_0 and H_1 are uniquely determined by the differential operators above. Therefore it suffices to check that $\frac{1}{24}\partial_0^3 H_0$ satisfies the genus 1 equations. This is a direct calculation.

4. Explicit Expressions

In this section we write the closed form expressions for the intersection numbers in genus 1 as sums of the multinomial coefficients.

Notation . If $\mathbf{b} = (b_1, \dots, b_k)$ is a vector with integer entries we denote by $[\mathbf{b}]$ the multinomial coefficient $\frac{(b_1+\cdots+b_k)!}{b_1!\dots b_k!}$, and we set it to zero if at least one entry is negative. Recall also that $\|\mathbf{b}\|$ denotes the sum $b_1 + \dots + b_k$.

In genus 0 the intersection numbers of the ψ classes are very simple: $\langle\tau_{b_1}\dots\tau_{b_k}\rangle_0 = [\mathbf{b}]$. In order to state our result in genus 1 we define for each $k \geq 1$ a function $f_k : \mathbb{Z}^k_{\geq 1} \to \mathbb{Z}_{\geq 1}$ by

$$f_k(\mathbf{b}) = f_k(b_1, \dots, b_k) := \langle\tau_0^{\|\mathbf{b}\|-k}\tau_{b_1}\dots\tau_{b_k}\rangle_1.$$

Clearly each f_k is invariant under the permutations of its arguments.

Proposition 4.1. *For each $k \geq 1$*

$$f_k(\mathbf{b}) = \frac{1}{24}[\mathbf{b}] - \frac{1}{24}\sum_{\substack{\boldsymbol{\varepsilon}\in\{0,1\}^k\\ \|\boldsymbol{\varepsilon}\|\geq 2}} (\|\boldsymbol{\varepsilon}\|-2)!\,[\mathbf{b}-\boldsymbol{\varepsilon}]. \tag{14}$$

Proof. The intersection numbers of the ψ classes in genus 1 are determined by the classical puncture equation and the classical dilaton equation. (See the second remark after Lemma 3.1.) Reformulated in terms of the collection $\{f_k\}$ these equations say that this collection is uniquely determined by the following properties:

- $f_1(b_1) \equiv \frac{1}{24}$,
- f_k is invariant under the permutation of the arguments,
- $f_k(\mathbf{b}) = \sum_{i=1}^k f_k(\mathbf{b} - \boldsymbol{\delta}_i)$ when $b_i \geq 2$ for all i,
- $f_k(b_1, \dots, b_{k-1}, 1) = (b_1 + \cdots + b_{k-1}) f_{k-1}(b_1, \dots, b_{k-1})$.

The first three properties are obviously satisfied by the expression given in the statement of the proposition. A direct computation verifies the last property. □

An explicit expression for the intersection numbers of the κ classes in genus 1 can be obtained by substitution of (14) into Cor. 2.3 from [19]. This expression is quite complicated. We do not know how to simplify this expression, and therefore we do not present the resulting formula for the κ classes here.

5. Asymptotic Formulas for Volumes of $\overline{\mathcal{M}}_{1,n}$

In this section, we derive an asymptotic formula for the Weil-Petersson volumes of $\overline{\mathcal{M}}_{1,n}$ in the limit that n becomes very large extending the proof of a similar result for genus zero in [19]. We do so by using analytic properties of the generating function $K(x; \mathbf{s})$ in the case where all $s_i = 0$ for all $i \geq 2$. In some sense, these results are complementary to those of Penner [29] who obtains similar formulas for the case where the genus becomes very large.

The class of the Weil-Petersson symplectic form on $\overline{\mathcal{M}}_{g,n}$ is precisely $\frac{1}{2\pi^2}\kappa_{(g,n),1}$. The symplectic volume of $\overline{\mathcal{M}}_{g,n}$ is called the *Weil-Petersson volume of* $\overline{\mathcal{M}}_{g,n}$. For this reason, the intersection numbers associated to the κ classes are sometimes called *higher Weil-Petersson volumes.* To avoid unnecessary factors, we shall work instead with the quantity

Definition 5.1. $w_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^{3g-3+n}$.

Theorem 5.2 ([19]). *The genus zero Weil-Petersson volumes satisfy the asymptotic relation as $n \to \infty$*

$$w_{0,n+3} \sim \frac{\gamma_0\, 2^{\frac{3}{2}}}{C\sqrt{\pi}} \frac{2^{2n} n^{2n+\frac{1}{2}}}{C^n e^{2n}},$$

where $\gamma_0 \approx 2.40482555777\dots$ is the smallest zero of the Bessel function J_0 and $C = -2\gamma_0 J_0'(\gamma_0) \approx 2.496918339\dots$.

They proved this by noticing that the function $H_0''(x, \mathbf{0}; \mathbf{s})$ is invertible, and when all $s_i = 0$ except for s_1 the inverse function satisfies Bessel's equation, after a change of variables. Combining their results with ours for genus one, we obtain the following.

Theorem 5.3. *The genus one Weil-Petersson volumes satisfy the asymptotic relation as $n \to \infty$*

$$w_{1,n} \sim \frac{\pi}{24} \frac{(2n)^{2n}}{C^n e^{2n}},$$

where C is same constant as in the above.

Proof. One uses the asymptotic formulas for the genus zero case and our result that the generating functions are related by (1) . ☐

Remark . The theorem above supports a conjecture of Itzykson regarding the existence of such an asymptotic formula for all genera with a constant C independent of the genus (cf. [19, p. 765]).

6. The Moduli Space of Cohomological Field Theories

The moduli space of normalized, rank one cohomological field theories of genus zero was described in Kontsevich, Manin, and Zagier [19]. The generating function associated to the κ classes endows this moduli space with coordinates which behave nicely with respect to taking tensor products of cohomological field theories, a notion introduced in [23] (see also [18]).

In this section, we introduce the notion of a restricted, normalized, cohomological field theory in genus one and describe the moduli space of rank one theories of this kind. Such CohFTs turn out to be almost completely determined by their genus zero part using the relations between the boundary strata of $\overline{\mathcal{M}}_{1,n}$ recently obtained by Getzler [11]. The analogous set of coordinates are constructed for this moduli space but to do so, we must introduce the λ classes, as well.

6.1. Cohomological Field Theories.

Consider $\mathcal{G}_{g,n}$, the set of stable graphs of genus g and n tails labeled with the set $[n]$. Each $\mathcal{G}_{g,n}$ is acted upon by the permutation group S_n which permutes the labels on the tails. There are composition maps

$$\mathcal{G}_{g_1,n_1} \times \mathcal{G}_{g_2,n_2} \to \mathcal{G}_{g_1+g_2,n_1+n_2-2}$$

taking $(\Gamma, \Gamma') \mapsto \Gamma \circ_{(i_1,i_2)} \Gamma'$ for all i_1 in $[n_1]$ and i_2 in $[n_2]$ given by grafting the tail i_1 of Γ with tail i_2 of Γ' and then relabeling the remaining tails with elements of the set $[n_1 + n_2 - 2]$ by inserting orders. There are another set of composition maps $\mathcal{G}_{g,n} \to \mathcal{G}_{g+1,n-2}$ taking $\Gamma \mapsto \mathrm{tr}_{(i_1,i_2)}\Gamma$ for all distinct pairs i_1 and i_2 in $[n]$ in which the tails i_1 and i_2 of Γ are grafted together. These composition maps are equivariant with respect to the action of the permutation groups. Let $\mathbb{C}[\mathcal{G}_{g,n}]$ be the vector space over $\mathbb{C}$ with a basis $\mathcal{G}_{g,n}$ then the compositions and permutations group actions can be extended $\mathbb{C}$-linearly. Let $\mathbb{C}[\mathcal{G}]$ denote the direct sum of $\mathbb{C}[\mathcal{G}_{g,n}]$ for all stable pairs g, n.

The collection $\{ \mathbb{C}[\mathcal{G}_{g,n}] \}$ (or, for that matter, $\{ \mathcal{G}_{g,n} \}$) together with the composition maps and actions of the permutation groups described above forms an example of a *modular operad*, a notion due to by Getzler and Kapranov [13]. By restricting to just the genus zero subcollection $\{ \mathbb{C}[\mathcal{G}_{0,n}] \}$ and forgetting about the composition maps tr, we obtain an example of a *cyclic operad* [14].

Similarly, the homology groups $H_\bullet(\overline{\mathcal{M}}_{g,n})$ are endowed with an action of S_n which relabels the punctures on the stable curve and there are composition maps

$$\circ_{i_1,i_2} : H_{p_1}(\overline{\mathcal{M}}_{g_1,n_1}) \otimes H_{p_2}(\overline{\mathcal{M}}_{g_2,n_2}) \to H_{p_1+p_2}(\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2-2})$$

for all i_1 in $[n_1]$, i_2 in $[n_2]$ and $\mathrm{tr}_{(i_1,i_2)} : H_p(\overline{\mathcal{M}}_{g,n}) \to H_p(\overline{\mathcal{M}}_{g+1,n-2})$ for all distinct i_1 and i_2 in $[n]$, both of which are induced from the inclusion of strata. These composition maps are equivariant under the action of the permutation groups.

The natural maps $\alpha_{g,n} : \mathbb{C}[\mathcal{G}_{g,n}] \to H_\bullet(\overline{\mathcal{M}}_{g,n})$ mapping $\Gamma \mapsto [\overline{\mathcal{M}}(\Gamma)]$ where $\overline{\mathcal{M}}(\Gamma) := \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}$ preserves the above structures and gives rise to the sequence of morphisms

$$0 \longrightarrow \langle \mathcal{R}_{g,n} \rangle \longrightarrow \mathbb{C}[\mathcal{G}_{g,n}] \xrightarrow{\alpha_{g,n}} H_\bullet(\overline{\mathcal{M}}_{g,n}) \tag{15}$$

where the kernel of $\alpha_{g,n}$ is denoted by $\langle \mathcal{R}_{g,n} \rangle$, the ideal in $\mathbb{C}[\mathcal{G}]$ generated by some space of relations $\mathcal{R}_{g,n}$.

Definition 6.1. The modular operad $\mathcal{H} := \{ \mathcal{H}_{g,n} \}$ is the collection of

$$\mathcal{H}_{g,n} := \frac{\mathbb{C}[\mathcal{G}_{g,n}]}{\langle \mathcal{R}_{g,n} \rangle}$$

The canonical diagonal maps $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g,n}$ induce maps $H_\bullet(\overline{\mathcal{M}}_{g,n}) \to H_\bullet(\overline{\mathcal{M}}_{g,n}) \otimes H_\bullet(\overline{\mathcal{M}}_{g,n})$ making $H_\bullet(\overline{\mathcal{M}}_{g,n})$ into a Hopf modular operad in the natural way [13]. This endows $\mathcal{H}_{g,n}$ with the structure of a Hopf modular operad, as well.

In the case of $g = 0$, the results of [22] and [20] implies that $\alpha_{0,n}$ is surjective and $\mathcal{H}_{0,n}$ is isomorphic to $H_\bullet(\overline{\mathcal{M}}_{0,n})$. Furthermore, the relations $\mathcal{R}_{0,n}$ are those due to Keel [20] which come from a lift of the basic codimension one relations $\mathcal{R}_{0,4}$ on $\overline{\mathcal{M}}_{0,4}$ via the canonical forgetful map $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,4}$.

In the case of $g = 1$, $\alpha_{1,n}$ is known not to be surjective since $\overline{\mathcal{M}}_{1,n}$ has odd dimensional homology classes. However, Getzler [11] has shown that the space of relations $\mathcal{R}_{1,n}$, in addition to those coming from Keel's relations, contains the lifts of two other relations. The first is the lift of the basic codimension one relation on $\overline{\mathcal{M}}_{1,2}$ which contains no genus one vertices – this may be regarded as the image of Keel's relations under the self-sewing morphism $\mathrm{tr}_{(3,4)} : \mathbb{C}[\mathcal{G}_{0,4}] \to \mathbb{C}[\mathcal{G}_{1,2}]$. The second relation, which contains genus one vertices, is between codimension two strata and is of the form

$$12\delta_{2,2} - 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} + \delta_{0,3} + \delta_{0,4} - 2\delta_{\beta} = 0 \tag{16}$$

where each term is an S_4-invariant combination of graphs of a given topological type and each graph Γ represents the homology class $[\overline{\mathcal{M}}_\Gamma]$. (See [11] for details.) Getzler also states [11] that he has shown [12] that $\alpha_{1,n}$ maps surjectively onto the even dimensional homology of $\overline{\mathcal{M}}_{1,n}$ and that the relations mentioned above do in fact generate all of $\mathcal{R}_{1,n}$.

A cohomological field theory is essentially a representation, in the sense of operads, of $H_\bullet(\overline{\mathcal{M}}_{g,n})$. In order to define such an object, we need to define the appropriate notion of the endomorphisms of a vector space is in this context. Let V be a vector space over $\mathbb{C}$ with a symmetric, nondegenerate bilinear form h of degree zero. Let $\mathcal{E}nd(V)_{g,n} := T^n V$ be the n^{th} tensor power of V for all nonnegative integers g, n such that $2g-2+n>0$ where T^0V is understood to be $\mathbb{C}$. S_n acts upon $\mathcal{E}nd(V)_{g,n}$ by permuting the tensor factors and the composition maps $\mathcal{E}nd(V)_{g_1,n_1} \otimes \mathcal{E}nd(V)_{g_2,n_2} \to \mathcal{E}nd(V)_{g_1+g_2,n_1+n_2-2}$ taking $(\mu, \mu') \mapsto \mu \circ_{(i_1,i_2)} \mu'$ for all i_1 in $[n_1]$ and i_2 in $[n_2]$ given by applying the inverse of h to the the corresponding tensor factors of μ and μ', and inserting the remaining factors in the usual way. Similarly, the composition $\mathcal{E}nd(V)_{g,n} \to \mathcal{E}nd(V)_{g+1,n-2}$ taking $\mu \mapsto \mathrm{tr}_{(i_1,i_2)}\mu$ for all distinct pairs i_1 and i_2 in $[n]$ corresponds to applying the inverse of h to the appropriate pair of tensor factors of μ.

Definition 6.2 (Cohomological Field Theory). A *(complete) cohomological field theory (CohFT) of rank* r, (V, h), is a morphism of modular operads $\mu_{g,n} : H_\bullet(\overline{\mathcal{M}}_{g,n}) \to \mathcal{E}nd(V)_{g,n}$ where (V, h) is an r-dimensional vector space with an invariant, symmetric bilinear form h. A *CohFT of genus* g are maps $\mu_{g',n} : H_\bullet(\overline{\mathcal{M}}_{g',n}) \to \mathcal{E}nd(V)_{g',n}$ which are defined only for $g' \leq g$ which satisfy all the axioms of a CohFT in which no higher genus maps appear. A *restricted CohFT* is a morphism $\mu_{g,n} : \mathcal{H}_{g,n} \to \mathcal{E}nd(V)_{g,n}$.

A CohFT can also be described described dually in terms of maps $\mathcal{E}nd(V)_{g,n} \to H^\bullet(\overline{\mathcal{M}}_{g,n})$.

Notice that a restricted CohFT of genus zero is the same as a CohFT of genus zero since $H_\bullet(\overline{\mathcal{M}}_{0,n}) = \mathcal{H}_{0,n}$.

Remark . In the language of [31, 32], a *topological gravity (coupled to topological matter)* is a CohFT and the morphisms $\mu_{g,n}$ are the *correlation functions* of the theory. The genus zero CohFT is said to be *tree level* while a genus one CohFT is said to be *one loop*.

Remark . The natural Hopf structure on $H_\bullet(\overline{\mathcal{M}}_{g,n})$ endows the category of CohFTs with a tensor product as is usual in representation theory.

An restricted CohFT is completely determined by a generating function called its potential. If the CohFT is not restricted then one can still define the notion of a potential (essentially since the modular operad $H_\bullet(\overline{\mathcal{M}}_{g,n})$ is the quotient of some free modular operad) but we will not need to work in such generality.

Definition 6.3. The *potential* $\Phi = \sum_{g=0}^{\infty} \Phi_g$ of a restricted CohFT $\mu : \mathcal{H} \to \mathcal{E}nd(V)$ of rank r is defined by choosing a basis $\{ e_1, \ldots , e_r \}$ for V where $I_{g,n}(e_{a_1}, e_{a_2}, \ldots , e_{a_n})$ is the number obtained by using h to pair $\mu_{g,n}([\overline{\mathcal{M}}_{g,n}])$ with $e_{a_1} \otimes e_{a_2} \otimes \ldots \otimes e_{a_n}$ and

$$\Phi_g(\mathbf{x}) := \sum_{n=0}^{\infty} I_{g,n}(e_{a_1}, e_{a_2}, \ldots , e_{a_n}) \frac{x^{a_1} x^{a_2} \ldots x^{a_n}}{n!}.$$

(where the summation convention has been used) which is regarded as an element in $\mathbb{C}[[x^1, \ldots , x^r]]$.

Theorem 6.4. *A element Φ_0 in $\mathbb{C}[[x^1, \ldots , x^r]]$ is the potential of a rank r, genus zero CohFT (V, h) if and only if* [22, 25] *it satisfies the WDVV equation*

$$(\partial_a\partial_b\partial_e\Phi_0)\, h^{ef}\, (\partial_f\partial_c\partial_d\Phi_0) = (-1)^{|x_a|(|x_b|+|x_c|)}\, (\partial_b\partial_c\partial_e\Phi_0)\, h^{ef}\, (\partial_f\partial_a\partial_d\Phi_0),$$

where $h_{a,b} := h(e_a, e_b)$, h^{ab} is in inverse matrix to h_{ab}, ∂_a is derivative with respect to x^a, and the summation convention has been used.

If (Φ_0, Φ_1) is the potential associated to a restricted, rank r CohFT of genus one then Φ_0 must satisfy the WDVV equation and (Φ_0, Φ_1) must satisfy Getzler's equation from proposition (3.14) *in* [11]. □

The WDVV equation can be read off from the basic codimension one relation on $\overline{\mathcal{M}}_{0,4}$. Similarly, Getzler's equation can be seen from his relation (equation 16). The second statement will become an if and only if after the proof in [12] appears.

6.2. **Rank One Cohomological Field Theories.** Let (V, h) be a rank one CohFT with a fixed unit vector e. The morphisms $\mathcal{H}_{g,n} \to \mathcal{E}nd(V)_{g,n}$ are completely determined by the collection of numbers

$\{ I_{g,n} \}$ where $I_{g,n} := \mu_{g,n}([\overline{\mathcal{M}}_{g,n}])(\underbrace{e, e, \dots, e}_{n})$ which must satisfy relations between themselves reflecting the way that the boundary strata in $\overline{\mathcal{M}}_{g,n}$ fit together. The potential in this case is

$$\Phi_g = \sum_{n=0}^{\infty} I_{g,n} \frac{x^n}{n!},$$

where $I_{g,n}$ is defined to vanish for pairs (g, n) which are not stable.

We will see that tautological classes on the moduli space of curves give rise to complete rank one CohFTs. In order to describe the moduli space of restricted, rank one CohFTs of genus one, we need to introduce a combination of the λ classes which behave nicely with respect to restriction.

Definition 6.5. For all stable pairs, (g, n), let $\Lambda_{g,n}$ be an element in $H^\bullet(\overline{\mathcal{M}}_{g,n})[\mathbf{s}, \mathbf{u}]$ (where $\mathbf{s} = (s_1, s_2, \dots)$ and $\mathbf{u} = (u_1, u_2, u_3, \dots)$) then let

$$\Lambda_{g,n} := \exp\Big(\sum_{i=1}^{\infty} (s_i \, \kappa_{(g,n),i} \, + \, u_i \, \gamma_{(g,n),i} \,) \Big)$$

where $\gamma_{(g,n),i} := \mathrm{ch}_{2i-1}(\pi_* \, \omega_{g,n})$. Here ch_i is the i^{th} Chern character, and $\pi_* \, \omega_{g,n}$ is the pushforward of the relative dualizing sheaf. (Notice that $\mathrm{ch}_{2i}(\pi_* \, \omega_{g,n})$ vanishes for all i [25, 8].)

The classes $\gamma_{(g,n),i}$ are polynomials in the λ classes. In particular, $\gamma_1 = \lambda_1$.

Theorem 6.6. *The collection $\Lambda := \{ \Lambda_{g,n} \}$ gives rise to a complete, rank one CohFT for all values of $\boldsymbol{u}$ and $\boldsymbol{s}$ by integrating the cohomology classes $\Lambda_{g,n}$ over the homology classes on $\overline{\mathcal{M}}_{g,n}$. Furthermore, the tensor product of the CohFT associated to parameter values $(\mathbf{s_1}, \mathbf{u_1})$ and $(\mathbf{s_2}, \mathbf{u_2})$ is the CohFT associated to $(\mathbf{s_1} + \mathbf{s_2}, \mathbf{u_1} + \mathbf{u_2})$. Similarly,*

Proof. In the case where u_i vanishes for all i, this was proven in [23] where it was realized that the κ classes form a logarithmic CohFT following the work of Arbarello and Cornalba [1] (see equation 11).

The proof for the case where all the s_i vanish is as follows. Consider the bundles $E_{g,n} := \pi_* \, \omega_{g,n}$ on $\overline{\mathcal{M}}_{g,n}$ from section 1.3. If Γ is a graph of genus g with n tails, then it determines the morphism $\rho_\Gamma : \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g(\Gamma),S(\Gamma)}$. The pull back of $E_{g,n}$ under ρ_Γ differs from $\oplus_{v \in V(\Gamma)} E_{g(v),n(v)}$ by a trivial bundle. It follows that for each $k \geq 1$ the collection of the Chern characters $\gamma_k = ch_{2k-1} E_{g,n}$ forms a logarithmic CohFT (see [8, 9]).

The first part of the theorem follows by combining these two results. The proof that the coordinates $(\mathbf{s}, \mathbf{u})$ are additive with respect to taking tensor products follows from the definition of coproduct which is induced from the diagonal map. □

Corollary 6.7. *The potential of the rank one CohFT associated to* Λ *(for given values of* $\mathbf{s}$ *and* $\mathbf{u}$*) is precisely the generating function* χ_g *for the intersection numbers of* κ_i *and* γ_i *classes*

$$\chi_g(x; \mathbf{s}, u) := \left\langle \exp(x\,\tau_0 \;+\; \sum_{i=1}^{\infty} (\,s_i\,\kappa_i + u_i\,\gamma_i\,)) \right\rangle_g = \sum_{n=0}^{\infty} I_{g,n}\,\frac{x^n}{n!}$$

where

$$I_{g,n} = \sum_{\mathbf{r},\mathbf{m}} \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!}\,\frac{\mathbf{u}^{\mathbf{r}}}{\mathbf{r}!}\,\langle\,\boldsymbol{\kappa}^{\mathbf{m}}\,\tau_0^n\,\boldsymbol{\gamma}^{\mathbf{r}}\,\rangle_g\,.$$

It is understood that $I_{g,n} := 0$ *for unstable pairs* (g, n)*,*

Notice that the CohFT arising Λ have the property that $I_{0,3} = 1$ for all values of $\mathbf{u}$ and $\mathbf{s}$. This motivates the following definition which will play an important role in what follows.

Definition 6.8. A rank one, CohFT of genus g is said to be *invertible* if $I_{0,3}$ is nonzero and *normalized* if $I_{0,3} = 1$.

6.3. **Cohomological Field Theories in Genus Zero and One.** Let us recall the results of Kaufmann, Manin, and Zagier for rank one CohFTs of genus zero [19]. A rank one CohFT in genus zero is uniquely determined by its potential $\Phi_0(x) = \sum_{n=3}^{\infty} I_{0,n}\frac{x^n}{n!}$. Furthermore, any function $\Phi_0(x)$ in $x^3\,\mathbb{C}[[x]]$ arises from some rank one, CohFT of genus zero since the WDVV equation is trivially satisfied for rank one theories. Therefore, the moduli space of CohFTs of genus zero are parameterized by the independent variables $I_{0,n}$ for $n \geq 3$. What is nontrivial, however, is the behavior of these potentials under tensor product. In particular, the coordinates $I_{0,n}$ do not behave nicely under tensor product. However, the generating function associated to the genus zero κ classes $H_0(x, \mathbf{0}; \mathbf{s})$ (which is equal to χ_0 with $u = 0$) allows them to introduce coordinates on the space of normalized, rank one CohFTs which behave nicely under tensor products.

Theorem 6.9 ([19])**.** *The moduli space of normalized, rank one CohFTs in genus zero are parameterized by* $\mathbf{s}$ *with potential* $\Phi_0(x; \mathbf{s}) = H_0(x, \mathbf{0}; \mathbf{s})$ *in* $\mathbb{C}[\mathbf{s}][[x]]$*, our generating function for the intersection numbers of* κ *classes in genus zero. Furthermore, taking tensor products is additive with respect to the coordinates* $\mathbf{s}$*.*

We now treat the case of genus one and discover that the κ classes are not sufficient to describe the entire moduli space of normalized, rank one CohFTs. We will see that one needs to introduce the first λ classes.

Theorem 6.10. *If the pair (Φ_0, Φ_1) is a potential associated to a restricted, rank one CohFT of genus one then the following equation holds in $\mathbb{C}[[x]]$*

$$-(\Phi_0^{(3)})^2\Phi_1^{(2)} + \Phi_0^{(3)}\Phi_0^{(4)}\Phi_1^{(1)} - \frac{1}{12}(\Phi_0^{(4)})^2 + \frac{1}{24}\Phi_0^{(3)}\Phi_0^{(5)} = 0$$

where $\Phi_g^{(l)}$ is the l^{th} derivative of Φ_g.

Proof. Equation 6.10 above is nothing more than the equation due to Getzler in theorem 6.4 for the case of rank one theories. Our equation can be seen from equation 16 directly by associating to each graph

$$\Gamma \mapsto \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} \frac{\partial^{n(v)}\Phi_{g(v)}}{\partial x^{n(v)}}$$

and then extending linearly to linear combinations of graphs. One will obtain -36 times the equation above. $\square$

Unlike the case of genus zero where the WDVV equation is trivially satisfied, solutions to this equation fall into two classes depending upon whether $\Phi_0^{(3)}$ is invertible in the ring of formal power series $\mathbb{C}[[x]]$.

Theorem 6.11. *The pair (Φ_0, Φ_1) is a potential associated to an invertible, restricted, rank one CohFT of genus one if and only if $\Phi_0(x)$ is of the form $I_{0,3}\,\frac{x^3}{6} + x^4\,\mathbb{C}[[x]]$ for $I_{0,3}$ nonzero and*

$$\Phi_1 = \frac{1}{24}\log \Phi_0''' + B\Phi_0''$$

where B is an arbitrary constant. Therefore, an invertible, restricted CohFT of genus one is uniquely determined by arbitrary values of $I_{0,n}$ for all $n \geq 4$, $I_{0,3} \neq 0$, and $I_{1,1}$.

If the restricted, rank one, CohFT is not invertible then $\Phi_0 = 0$ and Φ_1 obeys no constraints. Therefore, the space of such theories is parameterized by all values of $I_{1,n}$ for all $n \geq 1$.

Proof. If the pair (Φ_0, Φ_1) is a potential associated to an invertible restricted, rank one CohFT then since Φ_0''' has an inverse in $\mathbb{C}[[x]]$, one can solve equation 6.10 explicitly.

The converse is more difficult in the absence of the proof that the lifts of the relations described above genus 1 span the entire space of relations $\mathcal{R}_{1,n}$. However, we will not need this statement but will explicitly construct restricted, normalized, rank one CohFTs in genus one which realize all solutions to equation 6.11 above. This we do in the next subsection.

Since $I_{1,1} = I_{0,4} + BI_{0,3}$, when $I_{0,3}$ nonzero varying B is the same as varying $I_{1,1}$ and leaving all of the $I_{0,n}$ unchanged.

In the case that the restricted, rank one CohFT is not invertible then our result follows from the equation. ☐

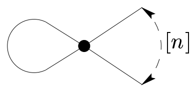
We shall not discuss noninvertible CohFTs any further in this paper. From now on, we shall restrict ourselves to normalized CohFTs.

It is worth observing that by incorporating the ψ classes, one can use the previous result to obtain yet another proof of the formula $H_1 = \frac{1}{24}\log H_0'''$. It is not clear which of these approaches will prove most useful in higher genera.

6.4. **Potentials in Genus Zero and One.** In this subsection, we construct potentials for a class of normalized, restricted, rank one CohFTs in genus one explicitly and show that they span the entire space of solutions to equation 6.11 completing the proof of that theorem. These potentials are generating functions associated to the κ classes and λ_1. This will give rise to coordinates which are additive under tensor product in analogy with the case of genus zero in [19].

We begin with a useful lemma.

Lemma 6.12. *The tautological class λ_1 on $\overline{\mathcal{M}}_{1,n}$ can be written in terms of boundary classes as follows:*

$\lambda_1 = \frac{1}{12}$ 

Proof. The proof follows from the fact that $\lambda_{(1,n),1} = \pi^*\lambda_{(1,1),1}$ via the forgetful map $\pi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,1}$. One uses (3) to express $\lambda_{(1,1),1}$ in terms of boundary classes. ☐

In the sequel, let $\widetilde{\chi}_g(\mathbf{s}, u)$ be equal to the generating function $\chi_g(\mathbf{s}, \mathbf{u})$ where all values of u_i are set to zero except for $u := u_1$.

Theorem 6.13. *The intersection numbers above satisfy the following:*

$$\widetilde{\chi}_0(x; \mathbf{s}, u) = H_0(x, \mathbf{0}; \mathbf{s})$$

and

$$\widetilde{\chi}_1(x; \mathbf{s}, u) = \frac{u}{24} H_0''(x, \mathbf{0}; \mathbf{s}) + \frac{1}{24}\log H_0'''(x, \mathbf{0}; \mathbf{s})$$

where $'$ denotes differentiation with respect to x.

Proof. Using that λ_1 vanishes on $\overline{\mathcal{M}}_{0,n}$, the presentation of λ_1 on $\overline{\mathcal{M}}_{1,n}$ in terms of boundary strata above, and the fact that the κ classes and λ_1 forms a logarithmic CohFT, we obtain the equations

$$\langle \boldsymbol{\kappa}^{\mathbf{m}} \lambda_1^r \tau_0^n \rangle_0 = 0.$$

and

$$\langle \boldsymbol{\kappa}^{\mathbf{m}} \lambda_1^r \tau_0^n \rangle_1 = \begin{cases} \frac{1}{24} \left\langle \boldsymbol{\kappa}^{\mathbf{m}} \tau_0^{n+2} \right\rangle_0 & \text{if } r = 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

Rewriting these identities in terms of $\widetilde{\chi}_g$, using the fact that $\widetilde{\chi}_g(\mathbf{s}, x, 0) = H_g(\mathbf{s}, x)$ and theorem 1, we obtain the desired result. □

Proof. (completion of theorem 6.11) By setting $B := \frac{u}{24}$ in the previous theorem and $\Phi_g = \widetilde{\chi}_g$ for $g = 0, 1$, we conclude the proof of theorem 6.11 since theorem 6.9 implies that by forgetting Φ_1, we obtain all possible CohFTs in genus zero. Furthermore, by varying u, one obtains all possible values of $I_{1,1}$ without changing the values of $I_{0,n}$. □

Remark . The relations between the intersection numbers obtained in the previous proof can be encoded in the differential equations

$$\frac{\partial}{\partial u}\widetilde{\chi}_0 = 0 \qquad \text{and} \qquad \frac{\partial}{\partial u}\widetilde{\chi}_1 = \frac{1}{24}\frac{\partial^2}{\partial x^2}\widetilde{\chi}_0 \tag{17}$$

Putting everything together, we arrive at the following theorem.

Theorem 6.14. *The moduli space of normalized, restricted, rank one CohFTs of genus one is parameterized by coordinates* $(\mathbf{s}, u)$ *via potentials* $(\widetilde{\chi}_0, \widetilde{\chi}_1)$ *where* $\widetilde{\chi}_0(x)$ *belongs to* $\frac{x^3}{6} + x^4\,\mathbb{C}[\mathbf{s}, u][[x]]$ *and* $\widetilde{\chi}_1(x)$ *belongs to* $x\,\mathbb{C}[\mathbf{s}, u][[x]]$ *satisfying theorem 6.13. The tensor product is additive in the coordinates* $(\mathbf{s}, u)$.

Given two rank one, normalized CohFTs in genus zero, it is not obvious how to write down the potential of the tensor product CohFT explicitly in terms of the potentials of the tensor factors. In [19], the authors show that the operation of tensor product corresponds to multiplication of the formal Laplace transforms of the two potentials associated to the tensor factors. Because a rank one, normalized, restricted CohFTs in genus one is determined by its genus zero potential and the value of u, an explicit expression for the potential associated to the tensor product of two such theories follows from the genus zero result of [19] and theorem 6.13.

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