AN ANALYTICAL INDEX FORMULA FOR PSEUDO-DIFFERENTIAL OPERATORS ON WEDGES

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Abstract: We show an analytical index formula of Fedosov type for certain operators on an infinite wedge $W = \mathbb{R}^q \times C$, where C is an (infinite) cone with smooth compact basis. We employ a version of SCHULZE's edge calculus with weighted symbols. The operators under consideration are of the form $I + M + G$; here I is the identity while M and G are zero order pseudo-differential operators taking values in the smoothing Mellin and Green operators, respectively, on the cone.

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Introduction

The classical index formula of ATIYAH and SINGER [1] showed how to express the index of an elliptic operator on a closed compact manifold in terms of characteristic classes induced by its principal symbol and the manifold. FEDOSOV in 1974 proved an analytical index formula, a simple way of expressing the index of a zero order pseudo-differential operator $\mathrm{op}(a)$ on $\mathbb{R}^n$ via its complete (matrix-valued) symbol:

$$\operatorname{ind}\mathrm{op}(a) = c_n \int_{\partial B} \mathrm{tr}\ (a^{-1}da)^{2n-1};$$

here B is a large ball in $T^*\mathbb{R}^n$, and c_n is a constant depending only on n, cf. [5] or [9].

We establish a formula in the same spirit for operators on a wedge. The wedge $W = \mathbb{R}^q \times C$ is the cartesian product of Euclidean space and a cone C, where the base is a smooth compact manifold.

Following a central idea in the approach of SCHULZE for an edge calculus [15], we consider the operators on W as pseudo-differential operators on $\mathbb{R}^q$ taking values in operators on the cone C. Due to the non-compactness however, we work with a special class of 'weighted' pseudo-differential symbols with a precisely controlled behaviour near infinity. For the scalar case, these classes have been introduced by SHUBIN [18], PARENTI [10], and CORDES [2], see also [12].

The operators we are considering are of the form $I + M + G$. Here I is the identity operator, M and G are operator-valued pseudo-differential operators of order zero. They take values in the ideal of smoothing Mellin and Green operators, respectively, on the cone. The operators of the form $M + G$ as above form an ideal in the full edge calculus, consisting of smoothing, but nevertheless (generally) noncompact operators. In a certain sense they may be compared to the singular Green operators in BOUTET DE MONVEL's calculus, which are pseudo-differential along the boundary and smoothing in the normal direction. These operators carry important index information. In fact, many index problems in the full calculus on the wedge can be reduced to one like this, cf. [13]; therefore this paper constitutes an essential step towards an index theory on manifolds with edges.

The paper is based on an approach developed by FEDOSOV [6] for expressing the index of pseudo-differential operators with values in algebras with traces. The natural spaces the operators in this edge calculus act on are weighted variants of the edge Sobolev spaces introduced by SCHULZE. We therefore start with an introduction of the weighted operator-valued symbols and the weighted edge Sobolev spaces and discuss the question of Hilbert-Schmidt and trace class embeddings. We then define the classes of smoothing Mellin and Green operators we are interested in. Next we review the relevant techniques developed by FEDOSOV and employ them to finally derive the analytic index formula in Theorem 3.19.

Acknowledgement: We thank B.-W. Schulze for many helpful discussions. Recently, FEDOSOV, SCHULZE, and TARKHANOV [7] obtained a similar index formula for compactly supported perturbations of the identity by zero order operators in the edge calculus. Their formula, however, still involves the symbol of the parametrix.

1 Operator-valued pseudo-differential operators and Sobolev spaces

1.1 Definition. Let E be a Hilbert space. A set $\boldsymbol{\kappa} = \{\kappa_\lambda;\ \lambda > 0\} \subset \mathcal{L}(E)$ of isomorphisms is called a (strongly continuous) group action on E if

i) $\kappa_\lambda \kappa_\varrho = \kappa_{\lambda\varrho}$ for all $\lambda, \varrho > 0$ (in particular, $\kappa_1 = 1$).

ii) For each $e \in E$ the function $\lambda \mapsto \kappa_\lambda e : \mathbb{R}_+ \to E$ is continuous.

For a group action $\boldsymbol{\kappa}$ on E one can find non-negative constants c and M such that

$$\|\kappa_\lambda\|_{E,E} \leq c \max\{\lambda, \lambda^{-1}\}^M \quad \text{for all } \lambda > 0. \tag{1.1}$$

This can be derived from Banach-Steinhaus' theorem (cf. [8]).
For the following considerations we fix pairs $(E_j, \boldsymbol{\kappa}_j)$, $j = 0, 1, 2$, of Hilbert spaces with corresponding group actions. Furthermore, we choose a smooth and strictly positive function

$$\eta \mapsto [\eta] : \mathbb{R}^q \to \mathbb{R}_+ \quad \text{with} \quad [\eta] = |\eta| \text{ for } |\eta| \geq c$$

for a fixed constant $c > 0$. For abbreviation we set $\kappa(\eta) = \kappa_{[\eta]}$.

1.2 Definition. For $\mu, m \in \mathbb{R}$ let $S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ denote the space of all functions $a \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E_0, E_1))$ satisfying

$$\sup_{y,\eta \in \mathbb{R}^q} \left\{ \|\kappa_1^{-1}(\eta) \partial_\eta^\alpha \partial_y^\beta a(y,\eta) \kappa_0(\eta)\|_{E_0,E_1} [\eta]^{|\alpha|-\mu} [y]^{|\beta|-m} \right\} < \infty \quad \forall \alpha, \beta \in \mathbb{N}_0^q.$$

These semi-norms induce a Fréchet topology on $S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$.

Clearly,

$$\partial_\eta^\alpha \partial_y^\beta S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \subset S^{\mu-|\alpha|, m-|\beta|}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1),$$

$$S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_2) \cdot S^{\mu',m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \subset S^{\mu+\mu', m+m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2).$$

In case $E_1 \hookrightarrow E_2$ and $\boldsymbol{\kappa}_2 = \boldsymbol{\kappa}_1$ on E_1, i.e., $\kappa_{2,\lambda} = \kappa_{1,\lambda}$ on E_1 for all $\lambda > 0$,

$$S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \hookrightarrow S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2).$$

If M_0, M_1 are the constants corresponding to $\boldsymbol{\kappa}_0$, $\boldsymbol{\kappa}_1$ via (1.1), then

$$S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) \hookrightarrow S^{\mu+M_0+M_1, m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)_{(1)},$$

where the subscript (1) indicates that both E_0 and E_1 are equipped with the trivial action $\boldsymbol{\kappa} \equiv 1$.
Let $\mathcal{S}(\mathbb{R}^q, E)$ be the Schwartz space of rapidly decreasing functions taking values in a Hilbert space E. To a given symbol $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ associate a continuous operator

$$\operatorname{op}(a) : \mathcal{S}(\mathbb{R}^q, E_0) \to \mathcal{S}(\mathbb{R}^q, E_1) : u \mapsto [\operatorname{op}(a)u](y) = \int e^{iy\eta} a(y,\eta)(\mathcal{F}u)(\eta)\, đ\eta.$$

Here, $\mathcal{F}$ is the Fourier transform, and $đ\eta = (2\pi)^{-q} d\eta$.

1.3 Theorem. *If $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_2)$ and $b \in S^{\mu',m'}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ then $\mathrm{op}(a)\mathrm{op}(b) = \mathrm{op}(a\#b)$, where*

$$(a\#b)(y,\eta) = \iint e^{-ix\xi} a(y, \eta+\xi) b(y+x, \eta)\, dx đ\xi.$$

For each $N \in \mathbb{N}$,

$$(a\#b)(y,\eta) = \sum_{\alpha<N} \frac{1}{\alpha!} (\partial_\eta^\alpha a)(D_y^\alpha b) \; + \; r_N(y,\eta)$$

with a remainder $r_N \in S^{\mu+\mu'-N, m+m'-N}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2)$ given by

$$r_N(y,\eta) = N \sum_{|\sigma|=N} \int_0^1 \frac{(1-\theta)^{N-1}}{\sigma!} \iint e^{-ix\xi} \partial_\eta^\sigma a(y, \eta+\theta\xi) D_y^\sigma b(y+x, \eta)\, dx đ\xi d\theta.$$

The double-integrals have to be understood as oscillatory integrals.

The adequate Sobolev spaces, the so called abstract edge Sobolev spaces, are defined as follows.

1.4 Definition. Let $\mathcal{W}^s(\mathbb{R}^q, E_0)$, $s \in \mathbb{R}$, denote the space of all distributions $u \in \mathcal{S}'(\mathbb{R}^q, E_0)$ such that $\mathcal{F}u$ is a measurable function and

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q, E_0)} = \left(\int [\eta]^{2s} \|\kappa_0^{-1}(\eta)(\mathcal{F}u)(\eta)\|_{E_0}^2 \, d\eta \right)^{1/2} < \infty.$$

For $\delta \in \mathbb{R}$ we have weighted variants of those spaces, namely

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0) = \{u \in \mathcal{S}'(\mathbb{R}^q, E_0);\, [\cdot]^\delta u \in \mathcal{W}^s(\mathbb{R}^q, E_0)\}.$$

Equipped with the obvious norm they are Hilbert spaces, having $\mathcal{S}(\mathbb{R}^q, E_0)$ as a dense subset. In case of a trivial group action, i.e., $\kappa_0 \equiv 1$, we use write $H^s(\mathbb{R}^q, E_0)$ and $H^{s,\delta}(\mathbb{R}^q, E_0)$. Then,

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0) \hookrightarrow H^{s-M,\delta}(\mathbb{R}^q, E_0), \tag{1.2}$$

where M is the constant in (1.1). Further, if $E_0 \hookrightarrow E_1$ and $\kappa_1 = \kappa_0$ on E_0 we immediately obtain that

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0) \hookrightarrow \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_1). \tag{1.3}$$

1.5 Theorem. *Let E_0, E_1 be Hilbert spaces equipped with arbitrary group actions, and $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$. Then a induces for all $s, \delta \in \mathbb{R}$ continuous operators*

$$\mathrm{op}(a) : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_0) \to \mathcal{W}^{s-\mu, \delta-m}(\mathbb{R}^q, E_1).$$

A proof is given in [17]. Next we extend the above material to the case where E_1 is a Fréchet space, which can be written as a projective limit

$$E_1 = \operatorname*{proj-lim}_{k \in \mathbb{N}} E_1^k$$

with Hilbert spaces $E_1^1 \hookleftarrow E_1^2 \hookleftarrow \ldots$, such that the group action on E_1^1 induces (by restriction) the group actions on each E_1^k. Then we set

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E_1) = \operatorname*{proj-lim}_{k \in \mathbb{N}} \mathcal{W}^{s,\delta}(\mathbb{R}^q, E_1^k)$$

equipped with the topology of the projective limit, and

$$S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1) = \cap_{k \in \mathbb{N}} S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1^k).$$

Theorem 1.4 extends to this situation.

2 Hilbert-Schmidt and trace class operators

2.1 Mappings between L^2-spaces

For Hilbert spaces E_0, E_1 let $\mathcal{L}^1(E_0, E_1)$ denote the Banach space of all trace class operators $E_0 \to E_1$, and, in case of E_0, E_1 being Hilbert spaces, $\mathcal{L}^2(E_0, E_1)$ the Hilbert space of all Hilbert-Schmidt operators $E_0 \to E_1$. For $A \in \mathcal{L}^1(E_0)$ let $\mathrm{Tr}\,(A)$ denote its trace. The following results are well-known:

2.1 Theorem. *An operator $A : L^2(\mathbb{R}^m, E_0) \to L^2(\mathbb{R}^l, E_1)$, with Hilbert spaces E_0 and E_1, is a Hilbert-Schmidt operator iff it has a representation as an integral operator*

$$(Af)(y) = \int k_A(x,y) f(x)\, dx, \qquad f \in L^2(\mathbb{R}^m, E_0),$$

with a kernel $k_A \in L^2(\mathbb{R}^m \times \mathbb{R}^l, \mathcal{L}^2(E_0, E_1))$. In this case

$$\|A\|^2_{\mathcal{L}^2} = \iint \|k_A(x,y)\|^2_{\mathcal{L}^2(E_0,E_1)}\, dxdy.$$

2.2 Theorem. *If E is a Hilbert space and $A \in \mathcal{L}^1(L^2(\mathbb{R}^m, E))$ is an integral operator with a continuous kernel $k_A \in C(\mathbb{R}^m \times \mathbb{R}^m, \mathcal{L}^1(E))$ then*

$$\mathrm{Tr}\,(A) = \int tr\; k_A(x,x)\, dx.$$

Here 'tr' is the trace on $\mathcal{L}^1(E)$.

2.2 The case of abstract edge Sobolev spaces

Let $E_0 \hookrightarrow E_1 \hookrightarrow E_2$ be Hilbert spaces, where each of the embeddings is Hilbert-Schmidt. Then the embedding $H^{s,\delta}(\mathbb{R}^q, E_0) \hookrightarrow H^{s',\delta'}(\mathbb{R}^q, E_2)$ is of trace class whenever $s - s' > q$ and $\delta - \delta' > q$. We need analogous statements for abstract edge Sobolev spaces. As it will be satisfied in later applications, we consider a scale of Hilbert spaces E^r, $r \in \mathbb{R}^l$, which fulfills

(1) $E^r \hookrightarrow E^{r'}$ if $r \geq r'$ (here '$\geq$' holds in each component);

(2) there are mappings $\kappa_\lambda : \cup_r E^r \to \cup_r E^r$, $\lambda > 0$, such that $\boldsymbol{\kappa}_r = \{\kappa_\lambda|_{E^r};\, \lambda > 0\}$ is a group action on E_r, and $\boldsymbol{\kappa}_0$ is a group of unitary operators on E^0;

(3) there is an $r_0 \geq 0$, such that the embedding $E^r \hookrightarrow E^{r'}$ is Hilbert-Schmidt if $r - r' > r_0$;

(4) the mapping $r \mapsto M(r) : \mathbb{R}^l \to \mathbb{R}$, where $M(r)$ is the constant associated with $\boldsymbol{\kappa}_r$ via (1.1), is locally bounded.

Assumption (4) allows us to define

$$N(\varepsilon) := \sup\{M(r) + q;\, r > 2r_0 \text{ and } |r - 2r_0| < \varepsilon\}, \qquad N := \inf\{N(\varepsilon);\, \varepsilon > 0\}. \tag{2.4}$$

2.3 Lemma. *For $s > N$, N as in (2.4), $\delta > q$, and $r > 2r_0$, r_0 as in (3), we have a trace class embedding*

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E^r) \hookrightarrow \mathcal{W}^{0,0}(\mathbb{R}^q, E^0).$$

PROOF: By construction of N we find an $\tilde{\varepsilon} > 0$ such that $s > N(\tilde{\varepsilon})$. Further we can choose an $\tilde{r} \in \mathbb{R}^n$ with $r > \tilde{r} > 2r_0$ and $|\tilde{r} - 2r_0| < \tilde{\varepsilon}$. This implies, in particular, that

$$s > \sup\{M(r) + q;\ r > 2r_0 \text{ and } |r - 2r_0| < \tilde{\varepsilon}\} \geq M(\tilde{r}) + q.$$

By 1.2 and 1.3 we obtain

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E^r) \hookrightarrow \mathcal{W}^{s,\delta}(\mathbb{R}^q, E^{\tilde{r}}) \hookrightarrow H^{s-M(\tilde{r}),\delta}(\mathbb{R}^q, E^{\tilde{r}}) \overset{(*)}{\hookrightarrow} H^{0,0}(\mathbb{R}^q, E^0) = \mathcal{W}^{0,0}(\mathbb{R}^q, E^0),$$

where the embedding $(*)$ is of trace class. Note that the last identity holds since the group action is unitary on E^0. ∎

2.4 Corollary. *Let $a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E^0, E^r)$ with $\mu < -N$, $m < -q$, and $r > 2r_0$; here N is as in (2.4) and r_0 as in (3). Then a induces a trace class operator*

$$\mathrm{op}(a) : \mathcal{W}^{0,0}(\mathbb{R}^q, E^0) \to \mathcal{W}^{0,0}(\mathbb{R}^q, E^0),$$

where the trace is given by

$$\mathrm{Tr}\, \mathrm{op}(a) = (2\pi)^{-q} \iint tr\ a(y, \eta)\, dy d\eta.$$

Here 'tr' is the trace in $\mathcal{L}^1(E^0)$.

PROOF: First, from Lemma 2.3 it is clear that $\mathrm{op}(a)$ is of trace class. Further, there exists an $\tilde{\varepsilon} > 0$ such that $-\mu < -N(\tilde{\varepsilon})$. Hence we can choose an $\tilde{r}$ with $r > \tilde{r} > 2r_0$ and $|\tilde{r} - 2r_0| < \tilde{\varepsilon}$ such that $-\mu < N(\tilde{\varepsilon}) \leq -(M(\tilde{r}) + q)$. In particular, we obtain

$$a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; E^0, E^r) \subset S^{\mu+M(\tilde{r}),m}(\mathbb{R}^q \times \mathbb{R}^q; E^0, E^{\tilde{r}})_{(1)},$$

where the subscript (1) means that both E^0 and $E^{\tilde{r}}$ are equipped with the trivial group action $\boldsymbol{\kappa} \equiv 1$. Since the group action on E^0 is unitary we have $\mathcal{W}^{0,0}(\mathbb{R}^q, E^0) = L^2(\mathbb{R}^q, E^0)$. Furthermore, in view of $\mu + M(\tilde{r}) < -q$ and $\mathcal{L}(E^0, E^{\tilde{r}}) \hookrightarrow \mathcal{L}^1(E^0)$, $\mathrm{op}(a)$ has a kernel

$$k(y, y') = \int e^{i(y-y')\eta} a(y, \eta)\, d\!\!\bar{}\,\eta \in C(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{L}^1(E^0)).$$

Thus Theorem 2.2 yields $\mathrm{Tr}\,(A) = \int \mathrm{tr}\ k(y, y)\, dy = (2\pi)^{-q} \iint \mathrm{tr}\ a(y, \eta)\, dy d\eta.$ ∎

2.3 Application

Here we show that the results of Section 2.2 are applicable to wedge Sobolev spaces.

In this section let X be a smooth compact manifold of dimension n. We fix a covering $\mathcal{U} = \{U_1, \ldots, U_N\}$ of X with coordinate neighbourhoods U_j and charts $\chi_j : U_j \to V_j \subset \mathbb{R}^n$, and $\tilde{\theta}_j : U_j \to \tilde{V}_j \subset S^n$, the unit sphere in $\mathbb{R}^{n+1}$. To the latter diffeomorphisms associate

$$\theta_j : \mathbb{R}_+ \times U_j \to \mathbb{R}^{1+n} : (t, x) \mapsto t\tilde{\theta}_j(x).$$

Further, let $\{\phi_1, \ldots, \phi_N\}$ be a partition of unity subordinate to the covering $\mathcal{U}$.
First, we define a scale of cone Sobolev spaces $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$ on $X^\wedge = \mathbb{R}_+ \times X$ for real s, γ and ϱ; it will play the role of the spaces E^r, $r \in \mathbb{R}^m$, in the preceding section.
For $\gamma \in \mathbb{R}$ define a linear bijection $S_\gamma : C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{1+n})$ by

$$(S_\gamma u)(r, x) = e^{(\gamma - \frac{n+1}{2})r} u(e^{-r}, x), \qquad r \in \mathbb{R},\ x \in \mathbb{R}^n.$$

2.5 Definition. Let $\omega(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega \equiv 1$ near $t = 0$. Then $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$ denotes the closure of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\|u\|^2_{\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho} = \sum_{j=1}^N \|S_\gamma[(\omega\phi_j u) \circ (1 \times \chi_j)^{-1}]\|^2_{H^s(\mathbb{R}^{1+n})} + \|((1-\omega)\phi_j u) \circ \theta_j^{-1}\|^2_{H^{s,\varrho}(\mathbb{R}^{1+n})}.$$

Here, the functions on the right-hand side are extended by zero outside their natural domains. This yields a Hilbert space. The construction is independent of the choice of ω.

The proof of the following lemma is straightforward.

2.6 Lemma. *For each $\lambda > 0$ the mappings*

$$(\kappa_\lambda u)(t, x) = \lambda^{(n+1)/2} u(\lambda t, x), \quad u \in C_0^\infty(X^\wedge), \tag{2.5}$$

extend to continuous operators $\kappa_\lambda : \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho \to \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$. The operator-norm $\|\kappa_\lambda\|$ is locally bounded as a function of $(\lambda, s, \gamma, \varrho) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Furthermore, $\{\kappa_\lambda;\ \lambda > 0\}$ is a group action (cf. Definition 1.1) on each $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$, and is unitary on $\mathcal{K}^{0,0}(X^\wedge)^0$.

As a corollary, we obtain that the constant $M(s, \gamma, \varrho)$ associated to $\{\kappa_\lambda\}$ and $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$ via (1.1) is locally bounded as a function of $(s, \gamma, \varrho) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

2.7 Remark. From the well-known embedding properties of the usual (weighted) Sobolev spaces on $\mathbb{R}^{1+n}$ it is straightforward to verify that the embedding

$$\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho \hookrightarrow \mathcal{K}^{s',\gamma'}(X^\wedge)^{\varrho'}$$

is Hilbert-Schmidt if $s - s' > \frac{n+1}{2}$, $\gamma - \gamma' > 0$, and $\varrho - \varrho' > \frac{n+1}{2}$, i.e., (3) from Section 2.2 is valid with $r_0 = (\frac{n+1}{2}, 0, \frac{n+1}{2})$. Thus the scale $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$ satisfies conditions (1)–(4) from the beginning of Section 2.2. In particular, Lemma 2.3 and Corollary 2.4 hold.

3 The index formula

3.1 Green and smoothing Mellin operators

In [16] was introduced an algebra of pseudo-differential operators on an (open stretched) wedge $\mathbb{R}^q \times X^\wedge$, with X as in Section 2.3. In particular, this calculus allows a control of the asymptotics of solutions to elliptic equations. To deal with index theory, it is not necessary to handle these asymptotics. Thus we modify the material from [16] and develop a more general algebra. Proofs are omitted, since they are (simpler) variants of those in [16].

By $L^\mu(X)$ denote the space of pseudo-differential operators of order μ on the manifold X, and by $L^\mu(X;\mathbb{R})$ the parameter-dependent ones with parameter $\tau \in \mathbb{R}$. For real β set

$$\Gamma_\beta = \{z \in \mathbb{C};\ \mathrm{Re}\, z = \beta\}.$$

By means of the identification $\Gamma_\beta \to \mathbb{R} : \beta + i\tau \mapsto \tau$ we can define $L^\mu(X;\Gamma_\beta)$.
A function $\omega(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\omega \equiv 1$ near $t = 0$ is called *cut-off function.*
With a function $f \in L^\mu(X;\Gamma_{1/2-\gamma})$, $\gamma \in \mathbb{R}$, we associate a Mellin pseudo-differential operator

$$[\mathrm{op}_M^\gamma(f)u](t) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z} f(z)(\mathcal{M}u)(z)\, dz, \quad u \in C_0^\infty(X^\wedge).$$

Here $\mathcal{M}$ is the Mellin transform, defined for $u \in C_0^\infty(X^\wedge) = C_0^\infty(\mathbb{R}_+, C^\infty(X))$ by

$$(\mathcal{M}u)(z) = \int_0^\infty t^{z-1}u(t)\, dt.$$

For arbitrary $s, \varrho, \varrho' \in \mathbb{R}$ and cut-off functions $\omega, \tilde{\omega}$ we have continuous extensions

$$\omega \mathrm{op}_M^\gamma(f)\tilde{\omega} : \mathcal{K}^{s,\gamma+n/2}(X^\wedge)^\varrho \to \mathcal{K}^{s-\mu,\gamma+n/2}(X^\wedge)^{\varrho'}.$$

3.1 Definition. For $\varepsilon > 0$ set

$$S_\varepsilon = \{z \in \mathbb{C};\ \tfrac{n+1}{2} - \varepsilon < \mathrm{Re}\, z < \tfrac{n+1}{2} + \varepsilon\}.$$

Now, $M_\varepsilon^\mu(X)$, $\mu \in \mathbb{R}$, denotes the space of all functions $h \in \mathcal{A}(S_\varepsilon, L^\mu(X))$, i.e., h is holomorphic in $z \in S_\varepsilon$ with values in $L^\mu(X)$, such that

$$h(\beta + i\tau) \in L^\mu(X;\mathbb{R}_\tau) \text{ uniformly in } \beta \in \left]\tfrac{n+1}{2} - \varepsilon, \tfrac{n+1}{2} + \varepsilon\right[$$

.

$M_\varepsilon^\mu(X)$ becomes a Fréchet space if equipped with the system of semi-norms consisting of that from $\mathcal{A}(S_\varepsilon, L^\mu(X))$, and

$$\sup\left\{p(h(\beta + i\tau));\ \tfrac{n+1}{2} - \varepsilon < \beta < \tfrac{n+1}{2} + \varepsilon\right\},$$

where $p(\cdot)$ runs over a system of semi-norms of $L^\mu(X;\mathbb{R}_\tau)$.

3.2 Definition. For $\mu, m \in \mathbb{R}$, $\varepsilon > 0$, the space $M_\varepsilon^{\mu,m}(X)$ consists of all functions $h \in C^\infty(\mathbb{R}^q, M_\varepsilon^\mu(X))$ such that

$$\sup\left\{\langle y\rangle^{|\alpha|-m}\, p\left(\partial_y^\alpha h(y, \beta + i\tau)\right);\ \tfrac{n+1}{2} - \varepsilon < \beta < \tfrac{n+1}{2} + \varepsilon,\ y \in \mathbb{R}^q\right\} < \infty \qquad (3.6)$$

for all $\alpha \in \mathbb{N}_0^q$, and all semi-norms $p(\cdot)$ of $L^\mu(X;\mathbb{R}_\tau)$. On $M_\varepsilon^{\mu,m}(X)$ we define a Fréchet topology by means of the semi-norms of $C^\infty(\mathbb{R}^q, M_\varepsilon^\mu(X))$ and those from (3.6). Finally set $M_\varepsilon^{-\infty,m}(X) = \cap_{\mu\in\mathbb{R}} M_\varepsilon^{\mu,m}(X)$ and

$$M^{\mu,m}(X) = \cup_{\varepsilon>0} M_\varepsilon^{\mu,m}(X), \quad \mu \in \mathbb{R} \cup \{-\infty\}.$$

3.3 Definition. i) A *smoothing Mellin symbol* of order $(0, m)$ is an operator-valued function on $\mathbb{R}^q \times \mathbb{R}^q$ of the form

$$d(y,\eta) = \omega(t[\eta])\mathrm{op}_M^{-n/2}(h)(y)\tilde{\omega}(t[\eta]), \tag{3.7}$$

with cut-off functions $\omega, \tilde{\omega}$, and $h \in M^{-\infty,m}(X)$.

ii) For some fixed $L \in \mathbb{N}_0$ let $R_G^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q)$, $\nu, m \in \mathbb{R} \cup \{-\infty\}$, denote the space of all functions $\mathbf{g}(y,\eta) \in \cap_{s\in\mathbb{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,0}(X^\wedge)^0 \oplus \mathbb{C}^L, \mathcal{K}^{\infty,0}(X^\wedge)^0 \oplus \mathbb{C}^L)$ satisfying

$$\mathbf{g} \in \cap_{s\in\mathbb{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,0}(X^\wedge)^0 \oplus \mathbb{C}^L, \mathcal{K}^{\infty,\varepsilon}(X^\wedge)^\infty \oplus \mathbb{C}^L),$$

$$\mathbf{g}^* \in \cap_{s\in\mathbb{R}} S^{\nu,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,0}(X^\wedge)^0 \oplus \mathbb{C}^L, \mathcal{K}^{\infty,\varepsilon}(X^\wedge)^\infty \oplus \mathbb{C}^L),$$

for a certain $\varepsilon > 0$ (depending on $\mathbf{g}$). For a space E with group action $\{\kappa_\lambda\}$ we employ the action $\{\kappa_\lambda \oplus 1\}$ on $E \oplus \mathbb{C}^L$. Furthermore * means the pointwise formal adjoint with respect to the scalar product on $\mathcal{K}^{0,0}(X^\wedge)^0 \oplus \mathbb{C}^L$. These functions are called *Green symbols.*

iii) Let $R_{M+G}^{0,m}(\mathbb{R}^q \times \mathbb{R}^q)$ denote all symbols $\mathbf{m} + \mathbf{g}$, where, in block matrix notation,

$$\mathbf{m}(y,\eta) = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}(y,\eta) : \begin{array}{c} \mathcal{K}^{s,0}(X^\wedge)^0 \\ \oplus \\ \mathbb{C}^L \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{\infty,0}(X^\wedge)^0 \\ \oplus \\ \mathbb{C}^L \end{array}$$

with d as in (3.7), and $\mathbf{g} \in R_G^{0,m}(\mathbb{R}^q \times \mathbb{R}^q)$. In particular, we obtain that

$$R_{M+G}^{0,m}(\mathbb{R}^q \times \mathbb{R}^q) \subset \cap_{s\in\mathbb{R}} S^{0,m}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,0}(X^\wedge)^0 \oplus \mathbb{C}^L, \mathcal{K}^{\infty,0}(X^\wedge)^0 \oplus \mathbb{C}^L).$$

3.4 Remark. Let d be as in (3.7) with $h \in M_{\varepsilon_0}^{-\infty,m}(X)$. Then [16], Corollary 2.25 implies that for each $0 \le \varepsilon \le \varepsilon_0$

$$d(y,\eta) = \omega(t[\eta])\mathrm{op}_M^{\varepsilon-n/2}(h)(y)\tilde{\omega}(t[\eta]) \quad \forall\,(y,\eta) \in \mathbb{R}^{2q}$$

as operators on $\mathcal{K}^{s,\varepsilon}(X^\wedge)^\varrho$, hence

$$d \in \cap_{s,\varrho\in\mathbb{R}} S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\varepsilon}(X^\wedge)^\varrho, \mathcal{K}^{\infty,\varepsilon}(X^\wedge)^\infty).$$

Both symbol classes R_G and R_{M+G} form an algebra with respect to pointwise multiplication. The Green symbols form an ideal in the sense that

$$R_G^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q) \;\cdot\; R_{M+G}^{0,m'}(\mathbb{R}^q \times \mathbb{R}^q) \subset R_G^{\mu,m+m'}(\mathbb{R}^q \times \mathbb{R}^q);$$

the analogous statement holds if we interchange the factors on the left-hand side. Furthermore we have $\partial_\eta^\alpha \partial_y^\beta R_G^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q) \subset R_G^{\mu-|\alpha|,m-|\beta|}(\mathbb{R}^q \times \mathbb{R}^q)$, $\partial_y^\beta R_{M+G}^{0,m}(\mathbb{R}^q \times \mathbb{R}^q) \subset R_{M+G}^{0,m-|\beta|}(\mathbb{R}^q \times \mathbb{R}^q)$, and the important fact that

$$\partial_\eta^\alpha R_{M+G}^{0,m}(\mathbb{R}^q \times \mathbb{R}^q) \subset R_G^{-|\alpha|,m}(\mathbb{R}^q \times \mathbb{R}^q) \quad \forall |\alpha| \ge 1.$$

The corresponding pseudo-differential operators act continuously between the weighted wedge Sobolev spaces, cf. Sections 1 and 2.3. The composition of operators

$$\mathrm{op}(\mathbf{m}+\mathbf{g})\mathrm{op}(\mathbf{m}'+\mathbf{g}') \mapsto \mathrm{op}((\mathbf{m}+\mathbf{g})\#(\mathbf{m}'+\mathbf{g}'))$$

in the sense of Theorem 1.3 furnishes a continuous map

$$R^{0,m}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q) \times R^{0,m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q) \to R^{0,m+m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q),$$

with restrictions

$$R^{0,m}_{G}(\mathbb{R}^q \times \mathbb{R}^q) \times R^{0,m'}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q) \to R^{0,m+m'}_{G}(\mathbb{R}^q \times \mathbb{R}^q),$$

$$R^{0,m}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q) \times R^{0,m'}_{G}(\mathbb{R}^q \times \mathbb{R}^q) \to R^{0,m+m'}_{G}(\mathbb{R}^q \times \mathbb{R}^q).$$

Analogously the composition of Green operators yields a mapping

$$R^{\mu,m}_{G}(\mathbb{R}^q \times \mathbb{R}^q) \times R^{\mu',m'}_{G}(\mathbb{R}^q \times \mathbb{R}^q) \to R^{\mu+\mu',m+m'}_{G}(\mathbb{R}^q \times \mathbb{R}^q).$$

Now let $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in the sense of Definition 3.2.iii). The pseudo-differential operators we consider have symbols

$$\mathbf{1}+\mathbf{m}+\mathbf{g}, \quad \text{with} \quad \mathbf{m}+\mathbf{g} \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q).$$

In case of ellipticity, these operators are Fredholm. We will show an analytical index formula in Theorem 3.19.

3.5 Definition. Let $\mathbf{m}+\mathbf{g} \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q)$ with $\mathbf{m} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}$ and $d(y,\eta) = \omega(t[\eta])\, \mathrm{op}_M^{-n/2}(h)(y)\tilde{\omega}(t[\eta])$. The symbol $\mathbf{1}+\mathbf{m}+\mathbf{g}$ is called elliptic, if

i) $(1+h)^{-1} \in M^{0,0}(X)$,

ii) for large $|(y,\eta)|$

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})(y,\eta) : \begin{array}{c} \mathcal{K}^{0,0}(X^\wedge)^0 \\ \oplus \\ \mathbb{C}^L \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{0,0}(X^\wedge)^0 \\ \oplus \\ \mathbb{C}^L \end{array}$$

is invertible and the inverse is uniformly bounded in (y,η).

Then we find a parametrix to $\mathbf{1}+\mathbf{m}+\mathbf{g}$, i.e., an inverse under the Leibniz product modulo $R^{-\infty,-\infty}_G(\mathbb{R}^q \times \mathbb{R}^q)$. But we will only need the following

3.6 Proposition. (cf. [16], Proposition 3.10) *Let $\mathbf{1}+\mathbf{m}+\mathbf{g}$ be elliptic and $\phi \in C^\infty(\mathbb{R}^{2q})$, $\phi \equiv 1$ for large $|(y,\eta)|$, such that $\mathbf{1}+\mathbf{m}+\mathbf{g}$ is invertible in* $\mathrm{supp}\,\phi$. *Then there exist $\mathbf{m}_0+\mathbf{g}_0 \in R^{0,0}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q)$ and $\mathbf{g}_1, \mathbf{g}_2 \in R^{0,0}_G(\mathbb{R}^q \times \mathbb{R}^q)$ such that*

$$(\mathbf{1}+\mathbf{m}+\mathbf{g})(\mathbf{1}+\mathbf{m}_0+\mathbf{g}_0) = 1-(1-\phi)\mathbf{g}_1, \quad (\mathbf{1}+\mathbf{m}_0+\mathbf{g}_0)(\mathbf{1}+\mathbf{m}+\mathbf{g}) = 1-(1-\phi)\mathbf{g}_2.$$

3.2 Elliptic families and Chern characters

As indicated in the introduction, this paper employs techniques developed by FEDOSOV. This section gives a short review of the relevant material from [6].

3.7 Definition. i) Let $\mathfrak{A}$ be an associative algebra over $\mathbb{C}$ and $\mathfrak{J}$ a twosided ideal in $\mathfrak{A}$ with a trace, i.e., a linear map $\mathrm{Tr} : \mathfrak{J} \to \mathbb{C}$, satisfying

$$\mathrm{Tr}\,(ab) = \mathrm{Tr}\,(ba)$$

whenever a or b belongs to $\mathfrak{J}$. Then $\mathfrak{A}$ is called a *trace algebra* and $\mathfrak{J}$ a *trace ideal.*

ii) A collection $E = \{p_1,\, p_2,\, a,\, r\}$ of elements from $\mathfrak{A}$ is called an *elliptic collection* if

$$1)\ p_i^2 = p_i, \quad 2)\ ap_1 = p_2 a = a \text{ and } rp_2 = p_1 r = r, \quad 3)\ p_1 - ra,\, p_2 - ar \in \mathfrak{J}.$$

The *index* of E is defined by

$$\mathrm{ind}\, E = \mathrm{Tr}\,(p_1 - ra) - \mathrm{Tr}\,(p_2 - ar).$$

This notation is motivated by Proposition 3.13.

iii) If p_1, p_2 are idempotents in $\mathfrak{A}$ with $p_1 - p_2 \in \mathfrak{J}$, then $E = \{p_1,\, p_2,\, p_2p_1,\, p_1p_2\}$ is an elliptic collection which is shortly denoted by $E = \{p_1,\, p_2\}$. It is easily seen that in this case $\mathrm{ind}\, E = \mathrm{Tr}\,(p_1 - p_2)$.

We also will consider elliptic collections depending smoothly on a parameter $x \in \mathbb{R}^m$. Therefore we assume that $\mathfrak{A}$ and $\mathfrak{J}$ are equipped with suitable topologies such that $\mathfrak{J} \hookrightarrow \mathfrak{A}$ and Tr is a continuous functional on $\mathfrak{J}$.

A p-form $\Omega = \sum_{i_1 < \ldots < i_p} f_{i_1 \ldots i_p}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ is said to be an element of $A^p(\mathbb{R}^m, \mathfrak{A})$ (or $A^p(\mathbb{R}^m, \mathfrak{J})$, or $A_0^p(\mathbb{R}^m, \mathfrak{A})$) if the functions $f_{i_1 \ldots i_p}(x)$ belong to $C^\infty(\mathbb{R}^m, \mathfrak{A})$ (or $C^\infty(\mathbb{R}^m, \mathfrak{J})$, or $C_0^\infty(\mathbb{R}^m, \mathfrak{J})$). The graded commutator of a p-form Ω and a q-form $\tilde{\Omega}$ is defined as

$$[\Omega, \tilde{\Omega}] = \Omega \wedge \tilde{\Omega} - (-1)^{pq} \tilde{\Omega} \wedge \Omega.$$

3.8 Definition. A set $\mathbf{E} = \{p_1,\, p_2\}$ of functions $p_i \in C^\infty(\mathbb{R}^m, \mathfrak{A})$, is an *elliptic family* on $\mathbb{R}^m$ with values in $\mathfrak{A}$ if $p_i^2 = p_i$, and $p_1 - p_2 \in C_0^\infty(\mathbb{R}^m, \mathfrak{J})$. To p_i associate its curvature form $\Omega_i = (i/2\pi) p_i dp_i dp_i$ and then define the *Chern character* of $\mathbf{E}$ by

$$\mathrm{ch}\ \mathbf{E} = \mathrm{Tr}\,(p_1 - p_2) + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{Tr}\,(\Omega_1^k - \Omega_2^k).$$

This is a complex-valued inhomogeneous form of even degree with compact support. If $\mathbf{E}$ does not depend on a parameter, i.e., $\mathbb{R}^m = \mathbb{R}^0 := \{0\}$, we can view $\mathbf{E}$ as an elliptic collection, and then clearly $\mathrm{ind}\, \mathbf{E} = \mathrm{ch}\ \mathbf{E}$.

To a general elliptic family $\mathbf{E} = \{p_1,\, p_2,\, a,\, r\}$ (i.e., $p_i, a, r \in C^\infty(\mathbb{R}^m, \mathfrak{A})$, conditions 1), 2) of Definition 3.7.ii) hold pointwise and $p_1 - ra, p_2 - ar \in C_0^\infty(\mathbb{R}^m, \mathfrak{J})$) we associate

$$P_1(x) = \begin{pmatrix} p_1(x) - r(x)a(x) & r(x) \\ a(x)(p_1(x) - r(x)a(x)) & a(x)r(x) \end{pmatrix}, \quad P_2(x) = \begin{pmatrix} 0 & r(x) \\ 0 & p_2(x) \end{pmatrix}.$$

These are idempotents in the algebra of 2×2-matrices with entries from $\mathfrak{A}$. Here the trace ideal are those matrices with entries from $\mathfrak{J}$ and the trace is the sum of the traces of the diagonal elements. Then we define

$$\text{ch } \mathbf{E} = \text{ch } \{P_1, P_2\}.$$

In case $\mathbf{E}$ is independent of a parameter we again see that $\text{ind}\,\mathbf{E} = \text{ch } \mathbf{E}$.

3.9 Theorem. (cf. [6], (5.8)) *For an elliptic family* $\mathbf{E} = \{1, 1, a, r\}$ *with values in* $\mathfrak{A}$ *the Chern character is given by*

$$\text{ch } \mathbf{E} = \text{Tr}\,[a,r] - \sum_{k=1}^{\infty} \frac{(k-1)!}{(2\pi i)^k (2k-1)!} \text{Tr} \left\{ d(rda)^{2k-1} + \frac{1}{2}[(rda)^{2k-1}, rda] \right\} \tag{3.8}$$

modulo the exterior differential of a compactly supported, inhomogeneous form.

3.10 Definition. Let $\mathbb{R}^{2q} = \mathbb{R}^q_y \times \mathbb{R}^q_\eta$. Then $S(\mathbb{R}^q, \mathfrak{A})$ denotes the algebra of formal symbols

$$\bar{a} = \sum_{k=0}^{\infty} \lambda^k a_k, \qquad a_k \in C^\infty(\mathbb{R}^{2q}, \mathfrak{A}),$$

with multiplication

$$\bar{a} \circ \bar{b} = \sum_{k=0}^{\infty} \lambda^k \Big\{ \sum_{|\alpha|+l+m=k} \frac{1}{\alpha!} (\partial^\alpha_\eta a_l)(D^\alpha_y b_m) \Big\}.$$

The function a_0 is called *leading term* of $\bar{a}$. The trace ideal $S(\mathbb{R}^q, \mathfrak{J})$ consists of those symbols $\bar{a}$ with $a_k \in C^\infty_0(\mathbb{R}^{2q}, \mathfrak{J})$ for all $k \in \mathbb{N}_0$. On this ideal different traces are defined by

$$\text{Tr}_k\, \bar{a} = (2\pi)^{-q} \iint \text{tr } a_k(y,\eta)\, dy d\eta, \quad k \in \mathbb{N}_0,$$

where the 'tr' is the trace in $\mathfrak{J}$. For $M \in \mathbb{N}_0$ define the *truncated symbol* $\bar{a}|_M = \sum_{k=0}^M a_k$ (a function on $\mathbb{R}^{2q}$).

To an elliptic collection $\overline{E} = \{\bar{p}_1, \bar{p}_2, \bar{a}, \bar{r}\}$ we can associate different indices, depending on which of the traces Tr_k we use. They are denoted by ind_k, and simply ind if $k = q$.

3.11 Proposition. (cf. [6], Proposition 3.5). $\text{ind}_k \overline{E} = 0$ *for* $k \neq q$.

Obviously, $\overline{E}$ induces an elliptic family $\mathbf{E} = \{p_1, p_2, a, r\}$ on $\mathbb{R}^{2q}$ with values in $\mathfrak{A}$, consisting of the leading terms of the involved symbols. This family is called the *leading term* of $\overline{E}$.

3.12 Theorem. (cf. [6], (4.13)). *Let* $\overline{E}$ *be an elliptic collection in the algebra of symbols* $S(\mathbb{R}^q, \mathfrak{A})$ *and* $\mathbf{E}$ *its leading term. Then*

$$\text{ind}\,\overline{E} = \int_{\mathbb{R}^{2q}} \text{ch } \mathbf{E}, \tag{3.9}$$

where $\mathbb{R}^{2q}$ *is oriented by* $d\eta_1 \wedge dy_1 \wedge \ldots \wedge d\eta_q \wedge dy_q > 0$, *and on the right-hand side only the term of degree* $2q$ *of the inhomogeneous form* ch $\mathbf{E}$ *is integrated.*

3.3 The index of elliptic wedge pseudo-differential operators

First, we state a well-known connection between Fredholm and trace class operators.

3.13 Proposition. *Let H_0 be a Hilbert space and $T \in \mathcal{L}(H_0)$. If there is an operator $S \in \mathcal{L}(H_0)$ such that both $I - ST$ and $I - TS$ are of trace class, then T is a Fredholm operator with index*

$$\operatorname{ind} T = \operatorname{Tr}(I - ST) - \operatorname{Tr}(I - TS).$$

For abbreviation we set

$$H = \mathcal{K}^{0,0}(X^\wedge)^0 \oplus \mathbb{C}^L. \tag{3.10}$$

with a fixed $L \in \mathbb{N}_0$. Next, we introduce two trace algebras.

3.14 Definition. i) Let $\mathfrak{A} = \mathcal{L}(H)$ with trace ideal $\mathfrak{J} = \mathcal{L}^1(H)$ equipped with the usual trace.

ii) Let $\mathfrak{A}_\Psi$ be the algebra of pseudo-differential operators with symbols $a \in S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q; H, H)$ satisfying

$$\exists \varepsilon_0 > 0 \, \forall 0 \le \varepsilon \le \varepsilon_0 \, \forall s \ge 0 : \; a \in S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\varepsilon}(X^\wedge)^s \oplus \mathbb{C}^L, \mathcal{K}^{s,\varepsilon}(X^\wedge)^s \oplus \mathbb{C}^L).$$

Then, in particular, $\operatorname{op}(a) : \mathcal{W}^{0,0}(\mathbb{R}^q, H) \to \mathcal{W}^{0,0}(\mathbb{R}^q, H)$ continuously. By Theorem 2.3 there is an $N > q$ such that the embedding $\mathcal{W}^{s,\delta}(\mathbb{R}^q, \mathcal{K}^{r,\varepsilon}(X^\wedge)^\varrho \oplus \mathbb{C}^L) \hookrightarrow \mathcal{W}^{0,0}(\mathbb{R}^q, H)$ is of trace class for $s > N$, $\delta > q$, $\varepsilon > 0$, $r > n+1$ and $\varrho > n+1$. The trace ideal $\mathfrak{J}_\Psi$ of $\mathfrak{A}_\Psi$ consists of those operators where in addition

$$a \in S^{\mu,m}(\mathbb{R}^q \times \mathbb{R}^q; H, \mathcal{K}^{r,\varepsilon}(X^\wedge)^\varrho \oplus \mathbb{C}^L)$$

with certain $\mu < -N$, $m < -q$, $\varepsilon > 0$, $r > n+1$, $\varrho > n+1$, and the trace is defined (or given, cf. Corollary 2.4) by

$$\operatorname{Tr} \operatorname{op}(a) = (2\pi)^{-q} \iint \operatorname{Tr} a(y,\eta) \, dy d\eta;$$

here 'tr' is the trace on $\mathcal{L}^1(H)$.

3.15 Notation. In the following let $\mathbf{1} + \mathbf{m} + \mathbf{g}$, $\mathbf{1} + \mathbf{m}_0 + \mathbf{g}_0$, and ϕ as in Proposition 3.6. We use the abbreviation

$$a = \mathbf{1} + \mathbf{m} + \mathbf{g}, \qquad r_0 = \mathbf{1} + \mathbf{m}_0 + \mathbf{g}_0.$$

Then

$$a r_0 = \mathbf{1} - (1 - \phi)\mathbf{g}_1, \qquad r_0 a = \mathbf{1} - (1 - \phi)\mathbf{g}_2.$$

with suitable $\mathbf{g}_1, \mathbf{g}_2 \in R_G^{0,0}(\mathbb{R}^q \times \mathbb{R}^q)$. In particular, $r_0(y,\eta) = a(y,\eta)^{-1}$ for $(y,\eta) \in \mathbb{R}^{2q} \setminus \operatorname{supp}(1 - \phi)$, and

$$\mathbf{E} = \{1, 1, a, r_0\} \tag{3.11}$$

is an elliptic family on $\mathbb{R}^{2q}$ with values in $\mathfrak{A}$.

Now set

$$\bar{a} = a, \qquad \bar{r}_0 = r_0,$$

i.e., $\bar{a}, \bar{r}_0 \in S(\mathbb{R}^q, \mathfrak{A})$ are formal symbols consisting only of a principal symbol. Further, choose $\varphi \in C^\infty(\mathbb{R}^{2q})$ with $\varphi \equiv 0$ on the set $\{ar_0 \neq 1\}$ and $\varphi \equiv 1$ for large $|(y,\eta)|$. Then consider the formal symbol

$$\bar{r} = \bar{r}_0 + \bar{r}_0 \sum_{k=1}^{\infty} [\bar{\varphi}(\bar{1} - \bar{a}\bar{r}_0)]^k. \tag{3.12}$$

Now, $\bar{1} - \bar{a}\bar{r}_0 = (1 - ar_0) + \sum_{k=1}^{\infty} \lambda^k c_k$ with

$$c_k = \sum_{|\alpha|=k} \frac{1}{\alpha!}(\partial_\eta^\alpha a)(D_y^\alpha r_0) \ \in R_G^{-k,-k}(\mathbb{R}^q \times \mathbb{R}^q).$$

Especially, each coefficient of $\bar{1}-\bar{a}\bar{r}_0$ is a Green symbol. Together with the fact that $\varphi(1-ar_0) = 0$ this shows that $\bar{r} = r_0 + \sum_{k=1}^{\infty} \lambda^k r_k$ with

$$r_k \in R_G^{-k,-k}(\mathbb{R}^q \times \mathbb{R}^q), \quad k \geq 1.$$

Hereby, to the weights $\varepsilon_k > 0$ corresponding to r_k, cf. the definition of Green symbols, we find an $\varepsilon > 0$ such that $\varepsilon_k > \varepsilon$ for all $k \geq 1$. Furthermore, inserting (3.12) yields

$$\begin{aligned} \bar{1} - \bar{a}\bar{r} &= (\bar{1} - \bar{a}\bar{r}_0) - \sum_{k=1}^{\infty} [\bar{\varphi}(\bar{1} - \bar{a}\bar{r}_0)]^k + (\bar{1} - \bar{a}\bar{r}_0) \sum_{k=1}^{\infty} [\bar{\varphi}(\bar{1} - \bar{a}\bar{r}_0)]^k \\ &= (\bar{1} - \bar{\varphi})(\bar{1} - \bar{a}\bar{r}_0)\Big\{\bar{1} + \sum_{k=1}^{\infty} [\bar{\varphi}(\bar{1} - \bar{a}\bar{r}_0)]^k\Big\}, \end{aligned}$$

hence $\bar{1}-\bar{a}\bar{r} = \sum_{k=0}^{\infty} \lambda^k d_k$ with $d_k \in \cap_{s,r\in\mathbb{R}} C_0^\infty(\mathbb{R}^{2q}, \mathcal{L}(H, \mathcal{K}^{s,\varepsilon}(X^\wedge)^r) \oplus \mathbb{C}^L)$ with an appropriate $\varepsilon > 0$ independent of $k \in \mathbb{N}_0$. Analogous statements are true for $\bar{1} - \bar{r}\bar{a}$; hence

$$\overline{E} = \{\bar{1},\ \bar{1},\ \bar{a},\ \bar{r}\} \tag{3.13}$$

is an elliptic family in $S(\mathbb{R}^q, \mathfrak{A})$. The leading symbol of $\overline{E}$ is exactly $\mathbf{E}$, cf. (3.11). Finally, if we set

$$b_k = r_0 + \ldots + r_k, \quad k \in \mathbb{N}, \tag{3.14}$$

the (standard) construction of a parametrix shows that

$$1 - \mathrm{op}(b_k)\mathrm{op}(a) = \mathrm{op}(\mathbf{g}_{1,k}), \qquad 1 - \mathrm{op}(a)\mathrm{op}(b_k) = \mathrm{op}(\mathbf{g}_{2,k})$$

with $\mathbf{g}_{1,k}, \mathbf{g}_{2,k} \in R_G^{-k,-k}(\mathbb{R}^q \times \mathbb{R}^q)$; again the involved weights are independent of k in the above sense. In fact, $\bar{a} \circ \bar{r}_0$ corresponds to the asymptotic expansion of $a\#r_0$, cf. Theorem 1.3. Then Proposition 3.6 yields that $1 - a\#r_0 \in R_G^{-1,-1}(\mathbb{R}^q \times \mathbb{R}^q)$. Hence,

$$r_0 + r_0\# \sum_{j=1}^{k-1} (1 - a\#r_0)^{\#j}$$

is a Leibniz inverse of a modulo $R_G^{-k,-k}(\mathbb{R}^q \times \mathbb{R}^q)$. Since $(1-\varphi)\#(1-a\#r_0) \in R_G^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q)$, we obtain another inverse by

$$r_0 + r_0 \# \sum_{j=1}^{k-1} [\varphi \# (1 - a\#r_0)]^{\#j}.$$

Clearly in the expansion of this symbol we can omit all terms in $R_G^{-m,-m}(\mathbb{R}^q \times \mathbb{R}^q)$ for $m \geq k$, to obtain a third inverse of a, which is exactly b_k. Thus, for sufficiently large k,

$$E_\Psi = \{1,\ 1,\ \mathrm{op}(a),\ \mathrm{op}(b_k)\} \tag{3.15}$$

is an elliptic family in the algebra $\mathfrak{A}_\Psi$.

3.16 Definition. For $m \in \mathbb{N}$ and (appropriate) functions f, g defined on $\mathbb{R}^{2q}$ set $R_m(f,g) = \sum_{|\sigma|=m} \frac{m}{\sigma!} R_{m,\sigma}(f,g)$ with

$$R_{m,\sigma}(f,g)(y,\eta) = \int_0^1 (1-\theta)^{m-1} \iint e^{-ix\xi} \partial_\eta^\sigma f(y, \eta + \theta\xi) D_y^\sigma g(y+x, \eta)\, dx\,đ\xi\, d\theta,$$

cf. Theorem 1.3.

3.17 Theorem. (regularized trace). *Let H be as in (3.10), $M \in \mathbb{N}$ with $M > N$, and N as in Definition 3.14.ii). Then*

$$[\mathrm{op}(a), \mathrm{op}(b_M)] - \mathrm{op}([\bar{a}, \bar{r}]|_M) : \mathcal{W}^{0,0}(\mathbb{R}^q, H) \to \mathcal{W}^{0,0}(\mathbb{R}^q, H) \tag{3.16}$$

is a trace class operator with trace equal to 0. Here, $[\cdot,\cdot]$ denotes the commutator.

Proof: By definition of the product in $S(\mathbb{R}^q, \mathfrak{A})$ and Theorem 1.3 we get the identities

$$\begin{aligned} (\bar{a} \circ \bar{r})|_M &= \sum_{|\alpha|=0}^{M} \frac{1}{\alpha!} (\partial_\eta^\alpha a) D_y^\alpha \Big\{ \sum_{l=0}^{M-|\alpha|} r_l \Big\}, \\ a\#b_M &= \sum_{|\alpha|=0}^{M} \frac{1}{\alpha!} (\partial_\eta^\alpha a) D_y^\alpha \Big\{ \sum_{l=0}^{M} r_l \Big\} + R_{M+1}(a, b_M) \end{aligned}$$

and from this we see

$$(a\#b_M) - (\bar{a} \circ \bar{r})|_M = \sum_{|\alpha|=1}^{M} \frac{1}{\alpha!} (\partial_\eta^\alpha a) D_y^\alpha \Big\{ \sum_{l=M+1-|\alpha|}^{M} r_l \Big\} + R_{M+1}(a, b_M).$$

Analogously one can show that

$$(b_M \# a) - (\bar{r} \circ \bar{a})|_M = \sum_{|\alpha|=1}^{M} \frac{1}{\alpha!} \partial_\eta^\alpha \Big\{ \sum_{l=M+1-|\alpha|}^{M} r_l \Big\} (D_y^\alpha a) + R_{M+1}(b_M, a).$$

Now we can rewrite the operator in (3.16) as

$$\mathrm{op}\left((a\#b_M) - (\bar{a} \circ \bar{r})|_M\right) - \mathrm{op}\left((b_M \# a) - (\bar{r} \circ \bar{a})|_M\right) =: A_1 - A_2.$$

Hence, it is sufficient to show that both A_1 and A_2 are of trace class and $\mathrm{Tr}\, A_1 = \mathrm{Tr}\, A_2$. Therefore set $S^1_{\alpha,l} := (\partial^\alpha_\eta a)(D^\alpha_y r_l)$, $S^2_{\alpha,l} := (\partial^\alpha_\eta r_l)(D^\alpha_y a)$ for $1 \le |\alpha| \le M$, and $M+1-|\alpha| \le l \le M$. Then $S^1_{\alpha,l}, S^2_{\alpha,l} \in R_G^{-(M+1),-(M+1)}(\mathbb{R}^q \times \mathbb{R}^q)$, hence $\mathrm{op}(S^1_{\alpha,l})$ and $\mathrm{op}(S^2_{\alpha,l})$ are of trace class and

$$\mathrm{Tr}\,\mathrm{op}(S^1_{\alpha,l}) = \iint \mathrm{tr}\ \{(\partial^\alpha_\eta a)(D^\alpha_y r_l)\}(y,\eta)\, dy đ\eta.$$

Since $|\alpha|, l \ge 1$ the trace can be pulled in front of the integral and using integration by parts we can interchange ∂^α_η and D^α_y. Then by the invariance of the trace under cyclic permutations we obtain

$$\mathrm{Tr}\,\mathrm{op}(S^1_{\alpha,l}) = \iint \mathrm{tr}\ \{(\partial^\alpha_\eta r_l)(D^\alpha_y a)\}(y,\eta)\, dy đ\eta = \mathrm{Tr}\,\mathrm{op}(S^2_{\alpha,l}).$$

Finally, $\mathrm{Tr}\,\mathrm{op}(R_{M+1}(a, b_M)) = \mathrm{Tr}\,\mathrm{op}(R_{M+1}(b_M, a))$ by the following Lemma 3.18, and this finishes the proof. ∎

3.18 Lemma. *Let $m, k > N$. Then*

$$\mathrm{Tr}\,\mathrm{op}(R_{m,\sigma}(a, b_k)) = \mathrm{Tr}\,\mathrm{op}(R_{m,\sigma}(b_k, a))$$

as trace class operators $\mathcal{W}^{0,0}(\mathbb{R}^q, H) \to \mathcal{W}^{0,0}(\mathbb{R}^q, H)$.

Proof: Since both $R_{m,\sigma}(a, b_k))$ and $R_{m,\sigma}(b_k, a)$ are elements of $R_G^{-m,-m}(\mathbb{R}^q \times \mathbb{R}^q)$, Corollary 2.4 shows that the associated pseudo-differential operators are of trace class. Now let $\chi(x,\xi) \in \mathcal{S}(\mathbb{R}^{2q})$ with $\chi(0,0) = 1$ and set

$$f_\varepsilon(x,\xi,\theta,y,\eta) = \chi(\varepsilon x, \varepsilon\xi)\partial^\sigma_\eta a(y, \eta+\theta\xi) D^\sigma_y b_k(y+x, \eta).$$

Then, by definition of oscillatory integrals and Corollary 2.4 we have

$$\mathrm{Tr}\,\mathrm{op}(R_{m,\sigma}(a, b_k)) = \iint \mathrm{tr}\ \Big\{ \int_0^1 (1-\theta)^{m-1} \lim_{\varepsilon\to 0} \iint e^{-ix\xi} f_\varepsilon(x,\xi,\theta,y,\eta) dx\, đ\xi\, d\theta \Big\} dy\, đ\eta.$$

For each $l_0, l_1 \in \mathbb{N}$ we obtain after integration by parts

$$\begin{aligned} &\iint e^{-ix\xi} f_\varepsilon(x,\xi,\theta,y,\eta)\, dx\, đ\xi \\ &= \iint e^{-ix\xi} \langle\xi\rangle^{-2l_1} (1-\Delta_\xi)^{l_0} (1-\Delta_x)^{l_1} \{\langle x\rangle^{-2l_0} f_\varepsilon(x,\xi,\theta,y,\eta)\} dx\, đ\xi. \end{aligned}$$

Since $\partial^\sigma_\eta a \in R_G^{-m,0}(\mathbb{R}^q \times \mathbb{R}^q)$ and $D^\sigma_y b_k \in R^{0,-m}_{M+G}(\mathbb{R}^q \times \mathbb{R}^q)$ there exist appropriate reals $s > N$, $\gamma > 0$, and $\varrho > q$ such that $N > M(s,\gamma,\varrho)$ (the constant associated to $\{\kappa_\lambda\}$ and $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$ via (1.1)) and the latter integrand is a smooth function taking values in $\mathcal{L}(H, \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho \oplus \mathbb{C}^L)$, whose norm can be estimated from above by

$$\begin{aligned} &c\,\langle\xi\rangle^{-2l_1} \langle x\rangle^{-2l_0} \langle \eta+\theta\xi\rangle^{-m+M(s,\gamma,\varrho)} \langle y+x\rangle^{-m} \\ &\le\ c\,\langle\xi\rangle^{m+M(s,\gamma,\varrho)-2l_1} \langle x\rangle^{m-2l_0} \langle\eta\rangle^{-m+M(s,\gamma,\varrho)} \langle y\rangle^{-m}, \end{aligned}$$

where the constant c is independent of θ, ε. This allows us to write

$$\mathrm{Tr}\,\mathrm{op}(R_{m,\sigma}(a, b_k)) = \lim_{\varepsilon\to 0} \iint \int_0^1 (1-\theta)^{m-1} \mathrm{tr}\ \Big\{ \iint e^{-ix\xi} f_\varepsilon(x,\xi,\theta,y,\eta)\, dx\, đ\xi \Big\} d\theta\, dy\, đ\eta.$$

Now f_ε is, for fixed $\varepsilon > 0$, integrable as a function $\mathbb{R}^{2q}_{x,\xi} \times [0,1] \times \mathbb{R}^{2q}_{y,\eta} \to \mathcal{L}^1(H)$ and this justifies the following calculation. Using the transformations $y \mapsto y - x$ and $\eta \mapsto \eta - \theta\xi$ we can write

$$\iint \int_0^1 (1-\theta)^{m-1} \mathrm{tr} \iint e^{-ix\xi} f_\varepsilon(x,\xi,\theta,y,\eta)\, dx\, đ\xi\, d\theta\, dy\, đ\eta$$
$$= \iint \int_0^1 (1-\theta)^{m-1} \mathrm{tr} \iint e^{-ix\xi} \chi(\varepsilon x, \varepsilon\eta) \partial_\eta^\sigma a(y-x,\eta) D_y^\sigma b_k(y, \eta - \theta\xi)\, dy\, đ\eta\, d\theta\, dx\, đ\xi.$$

By means of an integration by parts we now can interchange the differentiations ∂_η^σ and D_y^σ. Thus, using the invariance of the trace under permutation and the transformations $x \mapsto -x$, $\xi \mapsto -\xi$, the latter expression equals

$$\iint \int_0^1 (1-\theta)^{m-1} \mathrm{tr} \iint e^{-ix\xi} \chi(-\varepsilon x, -\varepsilon\eta) \partial_\eta^\sigma b_k(y, \eta + \theta\xi) D_y^\sigma a(y+x,\eta)\, dx\, đ\xi\, d\theta\, dy\, đ\eta.$$

Taking the limit for $\varepsilon \to 0$ then yields

$$\mathrm{Tr}\, \mathrm{op}(R_{m,\sigma}(a, b_k)) = \iint \mathrm{Tr}\, R_{m,\sigma}(b_k, a)\, dy\, đ\eta.$$

But the right-hand side of this equation is just the trace of $\mathrm{op}(R_{m,\sigma}(b_k, a))$. ∎

3.19 Theorem. *The operator* $\mathrm{op}(a) : \mathcal{W}^{0,0}(\mathbb{R}^q, H) \to \mathcal{W}^{0,0}(\mathbb{R}^q, H)$ *is Fredholm and, for* $q \geq 2$, *its index is given by*

$$\mathrm{ind}\, \mathrm{op}(a) = -\frac{(q-1)!}{(2\pi i)^q (2q-1)!} \int_{\partial B} \mathrm{Tr}\, (a^{-1} da)^{2q-1}, \tag{3.17}$$

where B *is an open ball in* $\mathbb{R}^{2q}$ *centered at* 0 *such that* a^{-1} *exists on* $\mathbb{R}^{2q} \setminus B$. *The orientation on* ∂B *is that inherited from* $\mathbb{R}^{2q}$ *via Stokes' theorem, if* $\mathbb{R}^{2q}$ *is oriented by* $d\eta_1 \wedge dy_1 \wedge \ldots \wedge d\eta_q \wedge dy_q > 0$.

PROOF: For any integer $M > N$, by Proposition 3.13 and the definition of the index for elliptic collections we have

$$\mathrm{ind}\, \mathrm{op}(a) = \mathrm{ind}\, \{1,\ 1,\ \mathrm{op}(a),\ \mathrm{op}(b_M)\} = \mathrm{Tr}\, [\mathrm{op}(a), \mathrm{op}(b_M)].$$

The above Theorem 3.17 on the regularized trace shows that

$$\mathrm{Tr}\, [\mathrm{op}(a), \mathrm{op}(b_M)] = \mathrm{Tr}\, \mathrm{op}([\bar{a}, \bar{r}]|_M) = \sum_{k=0}^{M} \mathrm{ind}_k\, \{\bar{1},\ \bar{1},\ \bar{a},\ \bar{r}\}.$$

The latter identity is again just the definition. Now by Proposition 3.11 and Theorem 3.12

$$\sum_{k=0}^{M} \mathrm{ind}_k\, \{\bar{1},\ \bar{1},\ \bar{a},\ \bar{r}\} = \mathrm{ind}\, \{\bar{1},\ \bar{1},\ \bar{a},\ \bar{r}\} = \int_{\mathbb{R}^{2q}} \mathrm{ch}\, \{1,\ 1,\ a,\ r_0\}.$$

The latter Chern character is computed in Theorem 3.9, and from this we obtain

$$\mathrm{ind}\, \mathrm{op}(a) = -\frac{(q-1)!}{(2\pi i)^q (2q-1)!} \int_{\mathbb{R}^{2q}} \mathrm{Tr}\, \left\{ d(r_0 da)^{2q-1} + \frac{1}{2}[(r_0 da)^{2q-1}, (r_0 da)] \right\},$$

with $\mathbb{R}^{2q}$ being oriented by $d\eta_1 \wedge dy_1 \wedge \ldots \wedge d\eta_q \wedge dy_q > 0$. Furthermore,

$$\frac{1}{2}[(r_0 da)^{2q-1}, (r_0 da)] = (r_0 da)^{2q}.$$

Since $\partial_{\eta_i} a$ is pointwise trace class each coefficient of $(r_0 da)^{2q}$ is also trace class. But this implies that the trace of the commutator is equal to 0. Without loss of generality $r_0 = a^{-1}$ on $\mathbb{R}^{2q} \setminus B$, but then on this set $\mathrm{Tr}\,\{d(r_0 da)^{2q-1}\} = \mathrm{Tr}\,(a^{-1}da)^{2q} = 0$. Hence

$$\operatorname{ind}\operatorname{op}(a) = -\frac{(q-1)!}{(2\pi i)^q (2q-1)!} \int_B \mathrm{Tr}\,\{d(r_0 da)^{2q-1}\}. \tag{3.18}$$

In case of $q \geq 2$ each coefficient of $(r_0 da)^{2q-1}$ contains at least one factor $\partial_{\eta_i} a$ and thus is of trace class. Then one can interchange Tr and the exterior differentiation d, and the index formula follows from Stokes' theorem. ∎

3.20 Remark. For $q = 1$ the index of $\operatorname{op}(a)$ is also given by (3.18) from the proof of the latter theorem. But in general it is not possible to permute Tr and d, since $r_0 \partial_y a$ is not of trace class.

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