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GEOSPHERE LAMINATIONS IN FREE GROUPS

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Abstract. We construct *geosphere laminations* for free groups, which are codimension one analogues of geodesic laminations on surfaces. Other analogues that have been constructed by several authors are dimension-one instead of codimension-one.

Our main result is that the space of such laminations is compact. This in turn is based on the result that crossing, in the sense of Scott-Swarup, is an open condition. Our construction is based on Hatcher's normal form for spheres in the model manifold.

1. Introduction

Geodesic laminations (and measured laminations) on surfaces have proved to be very fruitful in three-manifold topology, Teichmüller theory and related areas. In this paper, we construct analogously *geosphere laminations* for free groups. They have the same relation to (disjoint unions of) embedded spheres in the connected sum $M = \sharp_n S^2 \times S^1$ of n copies of $S^2 \times S^1$ as geodesic laminations on surfaces have to (disjoint unions of) simple closed curves on surfaces. The manifold M has fundamental group the free group on n generators, and is a natural model for the study of free groups.

Laminations for groups (including free groups) have been constructed and studied in various contexts. In [2], laminations in the free group context have been defined in three different approaches, algebraic laminations, symbolic laminations and laminary languages. The set of each of these three objects naturally come with a topology and an action of the group $Out(F_n)$ of outer automorphisms of the free group F_n. These three approaches turn out to be equivalent. In [3], dual lamination for any isometric very small F_n-action on an R-tree is defined. In this paper, an $Out(F_n)$-equivariant map from the boundary of the outer space to the space of laminations is obtained. This map generalizes the corresponding basic construction for surfaces. In [1], laminations for free groups are defined and studied using graphs as a model for free groups. However, these laminations are one dimensional objects, corresponding to geodesics. We study here objects of codimension one, which correspond to splittings of free groups. In the case of surfaces, dimension one and codimension one coincide.

Our main result is a compactness theorem for the space of (non-trivial) geosphere laminations. We also show that embedded spheres in M are geosphere laminations. Hence sequences of spheres, in particular under iterations of an outer automorphism of the free group, have subsequences converging to geosphere laminations. It is such

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limiting constructions that make geodesic laminations for surfaces a very useful construction.

Our construction is based on the normal form for disjoint unions of spheres in M due to Hatcher. The normal form is relative to a decomposition of M with respect to a maximal collection of disjointly embedded spheres in M. This is in many respects analogous to a normal form with respect to an ideal triangulation of a punctured surface. In particular, isotopy for spheres in normal form implies normal isotopy, i.e., the normal form is unique.

As in the case of normal curves on surfaces and normal surfaces in three-manifolds, we can associate the number of pieces of each type to a collection of spheres in Hatcher's normal form. However, these numbers do not determine the (collection of) spheres up to isotopy. We instead proceed by considering lifts of normal spheres to the universal cover $\widetilde{M}$ of M.

In the universal cover $\widetilde{M}$, a normal sphere is determined by a *finite* subtree τ of a tree T associated to $\widetilde{M}$ together with some additional data. We construct geospheres in $\widetilde{M}$ by dropping the finiteness condition. We construct an appropriate topology on the space of geospheres and show that the space is locally compact and totally disconnected.

The lift of a normal sphere in M to its universal cover satisfies an additional condition, namely it is disjoint from all its translates. This can be reformulated in terms of the notion of *crossing* of spheres in $\widetilde{M}$, following Scott-Swarup, which depends on the corresponding partitions of ends of $\widetilde{M}$. We show that there is an appropriate notion of crossing for geospheres, which is defined in terms of the appropriate partition of ends (into three sets in this case).

Our main technical result is that crossing is an open condition. We recall that this is the case for crossing of geodesics in hyperbolic space, and that this plays a central role in the study of geodesic laminations. The proof of compactness of the space of geosphere laminations uses the result that crossing is an open condition.

The construction based on normal forms is not intrinsic, as it depends on the maximal collection of spheres with respect to which M is decomposed. However, we show that geospheres can be described in terms of their associated partitions. This gives an intrinsic definition.

We end with a list of problems and questions in Section 10.

Contents

2. Preliminaries

2.1. **The model 3-manifold.** Consider the 3-manifold $M = \sharp_k S^2 \times S^1$, i.e., the connected sum of k copies of $S^2 \times S^1$. A description of M can be given as follows: Consider the 3-sphere S^3 and let $A_i, B_i, 1 \leq i \leq k$, be a collection of $2k$ disjoint embedded balls in S^3. Let P be the complement of the union of the interiors of these balls and let S_i (respectively, T_i) denote the boundary of A_i (respectively, B_i). Then, M is obtained from P by gluing together S_i and T_i with an orientation reversing diffeomorphism φ_i for each $i, 1 \leq i \leq k$. The image of S_i (hence T_i) in M will be denoted Σ_i. The fundamental group $\pi_1(M)$ of M, which is a free group of rank k, acts freely on the universal cover $\widetilde{M}$ of M by deck transformations.

Definition 2.1. A smooth, embedded 2-sphere in M is said to be essential if it does not bound a 3-ball in M.

Definition 2.2. A system of 2-spheres in M is defined to be a finite collection of disjointly embedded smooth essential 2-spheres $\Sigma_i \subset M$ such that no two spheres in this collection are isotopic.

Let $\Sigma = \cup_j \Sigma_j$ be a maximal system of 2-sphere in M. We shall call the spheres Σ_j as well as their lifts to $\widetilde{M}$ *standard spheres.* Let M^* be obtained by *splitting* M along Σ, i.e., M^* is obtained from $M - \Sigma$ by completing with respect to the restriction of a Riemannian metric on M. Then, M^* is a finite collection of 3-punctured 3-spheres P_k, whose boundary components correspond to the spheres Σ_i. Here, a 3-punctured 3-sphere is the complement of the interiors of three disjointly embedded 3-balls in a 3-sphere. Let $\widetilde{M}^*$ be obtained similarly by splitting $\widetilde{M}$ along standard spheres. Then $\widetilde{M}^*$ is the disjoint union of lifts $\widetilde{P}_k$ of the components P_k of M^*.

There are natural inclusion maps $M^* \to M$ and $\widetilde{M}^* \to \widetilde{M}$. These maps are one-to-one on the union of the interiors of (the components of) M^* and on each boundary component. Boundary components are identified in pairs under these maps. Further, the map from $\widetilde{M}^* \to M$ is injective on each component as identifications are between boundaries of different components. We identify each component of $\widetilde{M}^*$ with its image in $\widetilde{M}$.

2.2. **Construction of the tree T.** We recall some constructions from [6].

We associate a tree T to $\widetilde{M}$ corresponding to the decomposition of M by Σ. Let $\widetilde{\Sigma}$ be the pre-image of Σ in $\widetilde{M}$. The vertices of the tree T are of two types, with one vertex corresponding to each component of $\widetilde{M}^*$ and one vertex for each component of $\widetilde{\Sigma}$. An edge of T joins a pair of vertices if one of the vertices corresponds to a component X of $\widetilde{M}^*$ and the other vertex corresponds to a component of $\widetilde{\Sigma}$ that is in the image of the boundary of X. Thus, we have a Y-shaped subtree corresponding to each component of $\widetilde{M}^*$. The end points of different Y's that correspond to the same sphere in $\widetilde{\Sigma}$ are identified. We pick an embedding of T in $\widetilde{M}$ respecting the correspondences. The tree T has bivalent and trivalent vertices with bivalent vertices corresponding to components of $\widetilde{\Sigma}$. We call each sphere Σ_i a standard sphere in M and each component of $\widetilde{\Sigma}$ a standard sphere in $\widetilde{M}$. We call a vertex of T which corresponds to a standard sphere a standard vertex.

Let $\tau = \tau_1 \subset \tau_2 \subset \dots$ be an exhaustion of T by finite subtrees of T such that all the terminal vertices of each τ_i are bivalent in T. Let K_{τ_i} be the union of closures of components $\widetilde{P}$ of $\widetilde{M^*}$ which corresponds to vertices in τ_i which are trivalent in T. Then, one can easily see that K_{τ_i} is a compact, simply-connected space homeomorphic to a space of the form $S^3 - \cup_{j=1}^{n} int(D_j)$ with D_j disjoint embedded balls in S^3.

2.3. **Ends of $\widetilde{M}$.** We recall the notion of ends of a topological space. An end of a topological space is a point of the so called Freudenthal compactification of the space. It can be viewed as a way to approach infinity within the space.

Namely, let X be a topological space. For a compact set $K \subset X$, let $C(K)$ denote the set of components of $X - K$. For L compact with $K \subset L$, we have a natural map $C(L) \to C(K)$. Thus, as compact subsets of X define a directed system under inclusion, we can define the set of ends $E(X)$ as the inverse limit of the sets $C(K)$. Further, we can compute the inverse limit with respect to any exhaustion by compact sets.

It is easy to see that a proper map $f : X \to Y$ induces a map $E(f) : E(X) \to E(Y)$ and that if $f : X \to Y$ and $g : Y \to Z$ are proper maps, then $E(g \circ f) = E(g) \circ E(f)$. In particular, the real line $\mathbb{R}$ has two ends which can be regarded as ∞ and $-\infty$. Hence, a proper map $c : \mathbb{R} \to X$ gives a pair of ends c_- and c_+ of X.

The space $\widetilde{M}$ is non-compact and it has infinitely many ends. We denote the set of ends of $\widetilde{M}$ by $E(\widetilde{M})$.

Consider a proper map $c : \mathbb{R} \to \widetilde{M}$. As $\widetilde{M}$ is a union of the simply-connected compact sets K_τ, where τ is a finite subtree of T, the following lemma is straightforward.

Lemma 2.3. *There is a one-one correspondence between proper homotopy classes of maps $c : \mathbb{R} \to \widetilde{M}$ and pairs $(c_-, c_+) \in E(\widetilde{M}) \times E(\widetilde{M})$ with $c_+ \neq c_-$.*

2.4. **Topology on the set of ends of $\widetilde{M}$.** We define a topology on $E(\widetilde{M})$. If K is any compact subset of $\widetilde{M}$, then $\widetilde{M} - K$ has finitely many components. Consider the collection of sets

$$\mathcal{U} = \{E(W) : W \text{ a component of } M - K \text{ for a compact set } K\}.$$

It is easy to see that $\mathcal{U}$ forms a basis for a topology on $E(\widetilde{M})$. The set $E(\widetilde{M})$ is homeomorphic to a Cantor set, in particular compact. Note that the set $E(T)$ of ends of T can be identified with the set $E(\widetilde{M})$.

Given an oriented essential embedded sphere $S \subset \widetilde{M}$, we get a partition of the ends of $E(\widetilde{M})$. Namely, as $H_1(M)$ is trivial, it follows by Alexander duality that $\widetilde{M} - S$ has two components, say V^+ and V^-, with V^+ on the positive side of S according to the given orientations on S and $\widetilde{M}$. The sets $E^{\pm}(S)$ are the sets of ends $E(V^{\pm})$ of these components. The sets $E^{\pm}(S)$ are open in $E(\widetilde{M})$. As the sets $E^{\pm}(S)$ give partition of $E(\widetilde{M})$, both E^+ and E^- are closed subsets of $E(\widetilde{M})$. As $E(\widetilde{M})$ is compact, both E^+ and E^- are compact subsets of $E(\widetilde{M})$.

If S' is an embedded sphere in $\widetilde{M}$, homologous to S, then both S and S' give the same partition of the set of ends of $\widetilde{M}$. Conversely, if S and S' are two embedded spheres in $\widetilde{M}$ such that they give the same partition (E^+, E^-) of the set $E(\widetilde{M})$ of ends of $\widetilde{M}$, then S and S' are homologous in $\widetilde{M}$.

2.5. **Crossings of spheres in $\widetilde{M}$.** Let A and B be two homology classes in $H_2(\widetilde{M})$ represented by embedded spheres in $\widetilde{M}$. A homology classes A of embedded spheres S in $\widetilde{M}$ is completely determined by a partition of $E(\widetilde{M})$ into two open subsets of $E(\widetilde{M})$. If S gives partition of $E(\widetilde{M})$ into two open subsets $E^+(S)$ and $E^-(S)$ of $E(\widetilde{M})$, then we can write $E^+(A) = E^+(S)$ and $E^-(A) = E^-(S)$.

Definition 2.4. We say that A and B *cross* if we have

$$E^{\varepsilon}(A) \cap E^{\eta}(B) \neq \phi$$

for each of the four sets obtained by choosing signs ε and η in $\{+, -\}$.

Suppose A and B do not cross, then for some choice of sign $E^{\varepsilon}(A) \supset E^{\eta}(B)$. It follows that $E^{\bar{\varepsilon}}(A) \subset E^{\bar{\eta}}(B)$, where $\bar{\varepsilon}$ and $\bar{\eta}$ denote the opposite signs. Further, if $A \neq B$, then the inequalities are strict.

Definition 2.5. We say that B is on the positive side of A if $E^+(A) \supset E^{\eta}(B)$ for some sign η. Otherwise, we say that B is on the negative side of A. In general, we say that B is on the ε-side of A for the appropriate sign ε.

If A and B are two homology classes in $H_2(\widetilde{M})$ represented by embedded spheres in $\widetilde{M}$. Then, A and B can be represented by disjoint embedded spheres in $\widetilde{M}$ if and only if A and B do not cross (for the proof, see [5]).

Group theoretically, embedded spheres in M correspond to splittings of the fundamental group of M. Scott and Swarup [8] introduced an algebraic analogue, called the *algebraic intersection number*, for a pair of *splittings* of a group. This is based on the associated partition of the ends of a group. Given a pair of embedded spheres in M, we can consider their *geometric intersection number* as well as the *algebraic intersection number* of Scott and Swarup for the corresponding splittings. In [4], it is shown that for embedded spheres, these two intersection numbers coincide.

3. Normal form

We recall the notion of *normal sphere systems* from [6].

3.1. **Normal form for sphere systems.** Let $\mathbf{\Sigma} = \cup_j \Sigma_j$ be a maximal system of 2-sphere in M. We recall that splitting M along Σ gives a manifold M^*, which is a finite collection of 3-punctured 3-spheres. Here, a 3-punctured 3-sphere is the complement of the interiors of three disjointly embedded 3-balls in a 3-sphere.

Definition 3.1. A system of 2-spheres $\mathbf{S} = \cup_i S_i$ in M is said to be in *normal form with respect to* $\mathbf{\Sigma}$ if each component S_i either coincides with a sphere Σ_j or meets $\mathbf{\Sigma}$ transversely in a non-empty finite collection of circles splitting S_i into components (which we call pieces) $p \subset P$, P a component of M^*, such that the following two conditions hold:

(1) Each piece p in P meets each component of ∂P in at most one circle.
(2) No piece in P is a disk which is isotopic, fixing its boundary, to a disk in ∂P.

Thus, each piece $p \subset P$ is a disk, a cylinder or a pair of pants. A disk piece has its boundary on one component of ∂P and separates the other two components of ∂P for some component P of M^*. A cylinder piece (tube) joins any two boundary components of ∂P and a pants piece joins all three boundary spheres of P.

Recall the following result from [6].

Proposition 3.2 (Hatcher)**.** *Every system* $\mathbf{S} \subset M$ *can be isotoped to be in normal form with respect to* $\mathbf{\Sigma}$*. In particular, every embedded sphere* S *which does not bound a ball in* M *can be isotoped to be in normal form with respect to* $\mathbf{\Sigma}$.

Similarly, we can define sphere systems in *normal form* with respect to the pre-image $\widetilde{\mathbf{\Sigma}}$ of $\mathbf{\Sigma}$ in $\widetilde{M}$.

3.2. **A description of an embedded sphere in $\widetilde{M}$ using its normal form.** Given a sphere S in normal form with respect to $\widetilde{\mathbf{\Sigma}}$ in $\widetilde{M}$, we associate a tree τ corresponding to the decomposition of S into pieces. Namely, we consider a tree τ with two kinds of vertices, one for each piece $p \subset \widetilde{P}$ of S and one for each component C of $S \cap \widetilde{\Sigma_i}$ for standard spheres $\widetilde{\Sigma}_i$. Edges join vertices corresponding to pieces p to those corresponding to $C \subset \partial P$. Here we identify P with its image in $\widetilde{M}$.

In [6], it is shown that τ is a tree. Moreover, the inclusion $S \hookrightarrow \widetilde{M}$ induces a natural inclusion map $\tau \hookrightarrow T$. Hence we can view τ as a subtree of T. It is easy to see that each component $\widetilde{P}$ of $\widetilde{M^*}$ contains at most one piece of S. If S is a standard sphere (or can be isotoped to standard sphere), then the associated tree τ is single vertex in T corresponding to that standard sphere.

Let $N(\tau)$ be the subgraph of T consisting of points with distance at most 1 from τ. Then, $N(\tau)$ is a tree, which is the union of τ with the following two kinds of edges:

(1) For each terminal vertex v of τ, we have a pair of edges $e_1(v) \notin \tau$ and $e_2(v) \notin \tau$ with v as an end-vertex. Let v_1 and v_2 be the other end vertices of e_1 and e_2, respectively.
(2) A non-standard bivalent vertex of τ is a bivalent vertex of τ which is not a standard vertex in T. For each non-standard bivalent vertex w of τ, we have an edge $e(w) \notin \tau$ with w as an end-vertex. Let w_1 be its other end vertex.

Fix an orientation of S. As S separates $\widetilde{M}$, the orientation on S determines the positive and negative sides of S in $\widetilde{M}$. A terminal vertex v of τ corresponds to a disc piece in a thrice-punctured 3-sphere $P(v)$, which separates the two other boundary components of $P(v)$. The standard sphere corresponding to one of v_1 and v_2 is on the positive side of S and other standard sphere is on the negative side. The vertices v_1 and v_2 are end vertices of e_1 and e_2 respectively. So, we can assign positive or negative signs to these edges accordingly. We denote the positive edge by $e_+(v)$ and denote the other edge (which is on the negative side) by $e_-(v)$. We denote the standard spheres corresponding to v_1 and v_2 by $\widetilde{\Sigma}(v_1) = \widetilde{\Sigma}(e_1)$ and $\widetilde{\Sigma}(v_2) = \widetilde{\Sigma}(e_2)$, respectively.

A non-standard bivalent vertex w of τ corresponds to an annulus piece (cylinder piece) in $P(w)$. The boundary component of $P(w)$ not intersecting the annulus is on either the positive or the negative side of S. For a non-standard bivalent vertex w of τ, we can associate a sign $\epsilon(w)$ so that $\widetilde{\Sigma}(w_1) = \widetilde{\Sigma}(e(w))$ is on the $\epsilon(w)$-side of S.

Thus, we can associate a triple (τ, ϵ, e_+), with τ a subtree of T, to a normal sphere in $\widetilde{M}$. In [5], it is shown that the triple (τ, ϵ, e_+) determines the normal sphere S and the partition $(E^+(S), E^-(S))$ of $E(\widetilde{M})$ given by S.

4. Geospheres

To construct geosphere laminations in M, we first need the analogue of (not necessarily closed) geodesics in M. We first construct the analogue of geodesics in $\widetilde{M}$, which we call *geospheres*. We then consider when two such geospheres cross, and deduce basic properties of crossing. This allows us to study the appropriate notion of geospheres embedded in M. Our main technical lemma says that crossing is an open condition. This allows us to construct limiting laminations and prove a compactness theorem for geosphere laminations in M.

4.1. Geospheres and trees.

In Section 3, we have seen that a normal sphere in $\widetilde{M}$ is determined by a triple (τ, ϵ, e_+), with τ either a finite subtree of T with each terminal vertex of τ a trivalent vertex of T or τ a standard vertex, ϵ an assignment of sign to each non-standard bivalent vertex of τ and e_+ an assignment to each univalent vertex v of τ an edge containing v and not contained in τ.

Geospheres are generalizations of such spheres where we drop the condition that τ is finite.

Definition 4.1. A geosphere σ in $\widetilde{M}$ is a triple $\sigma = (\tau, \epsilon, e_+)$ with

- τ a subtree of T such that either τ is a bivalent vertex of T (and has no edges) or τ has at least one edge and each univalent (terminal) vertex of τ is a trivalent vertex in T.
- If $B(\tau)$ is the set of non-standard bivalent vertices of τ, ϵ is a function $\epsilon : B(\tau) \to \{+, -\}$.
- If τ is not a single bivalent vertex, then if $C(\tau)$ is the set of terminal vertices of τ, e_+ associates to each vertex in $C(\tau)$ an edge containing v and not contained in τ.

Let $GS(\widetilde{M})$ be the set of such geospheres in $\widetilde{M}$. To construct a topology on $GS(\widetilde{M})$, we consider restrictions to compact subtrees $\kappa \subset T$ such that each of its terminal vertices is a trivalent vertex in T. We call a tree containing no edge as a degenerate tree. We define for a non-degenerate tree κ, $N(\kappa)$ to be the set of points of distance at most 1 from κ. For a degenerate tree κ, we define $N(\kappa) = \kappa$.

Henceforth, we consider only subtrees κ of T such that all univalent vertices of κ are trivalent in T or κ is a degenerate tree.

Definition 4.2. If $\sigma = (\tau, \epsilon, e_+)$ is a geosphere, then the restriction $res_\kappa(\sigma)$ of σ to κ is the triple $\sigma|_\kappa = (\tau \cap N(\kappa), \epsilon|_{B(\tau)\cap\kappa}, e_+|_{C(\tau)\cap\kappa})$.

Note that the valence of a vertex v of τ such that $v \in \kappa$ is determined by $\tau \cap N(\kappa)$. Further, for univalent vertices v of $\tau \cap \kappa$, the edges $e_+(v)$ (and $e_-(v)$) are in $N(\tau)$. Thus, we can view $res|_\kappa$ as a map from $GS(\widetilde{M})$ to the set $GS(\kappa)$ defined as below:

Definition 4.3. For a subtree $\kappa \subset T$, we define $GS(\kappa)$ to be the set of triples $\sigma = (\tau, \epsilon, e_+)$ with

- τ a subtree of $N(\kappa)$ or the empty graph.
- If $B(\tau)$ is the set of vertices $\tau \cap \kappa$ which are non-standard bivalent vertices in τ, ϵ is a function $\epsilon : B(\tau) \to \{+, -\}$.
- If $C(\tau)$ is the set of vertices of $\tau \cap \kappa$ which are univalent in τ and trivalent in T, then e_+ associates to each vertex v in $C(\tau)$ an edge containing v and not contained in τ.

We remark that $GS(\kappa)$ is not a subset of $GS(\widetilde{M})$, as an element of $GS(\kappa)$ can correspond to the empty graph, and terminal vertices of the tree τ corresponding to an element of $GS(\kappa)$ may be bivalent in T.

Note that if κ is a finite subtree of T, then the set $GS(\kappa)$ is finite. We say that an element $\sigma = (\tau, \epsilon, e_+)$ of $GS(\kappa)$ is *non-trivial* if τ is non-empty.

Suppose κ' is a subtree of T such that $\kappa' \supset \kappa$, then we can similarly define a restriction map $res_{\kappa,\kappa'} : GS(\kappa') \to GS(\kappa)$. Further, $res_\kappa = res_{\kappa,\kappa'} \circ res_{\kappa'}$. In particular, we can denote without ambiguity the map $res_{\kappa,\kappa'}$ as simply res_κ.

4.2. **The topology on Geospheres.** We define a topology on $GS(\widetilde{M})$ using the restriction maps. Namely, for each subtree κ of T and each $\sigma_0 \in GS(\kappa)$, consider the set

$$U(\kappa, \sigma_0) = \{\sigma \in GS(\widetilde{M}) : res_\kappa(\sigma) = \sigma_0\}$$

Lemma 4.4. *The sets $U(\kappa, \sigma_0)$ for finite subtrees κ of T form a basis for a topology on $GS(\widetilde{M})$.*

Proof. Showing that the sets $U(\kappa, \sigma_0)$ form a basis for a topology on $GS(\widetilde{M})$ is equivalent to showing that if $U(\kappa^i, \sigma_0^i)$, $1 \le i \le n$ is a finite collection of basic open sets and $\sigma \in \cap_i U(\kappa^i, \sigma_0^i)$, then there is a basic open set containing σ and contained in each of the sets $U(\kappa^i, \sigma_0^i)$.

To show this, let κ be the finite subtree of T spanned by the subtrees κ^i, and let $\sigma_0 = \sigma|_\kappa$. Note that as $\sigma \in U(\kappa^i, \sigma_0^i)$, $res_{\kappa^i}(\sigma) = \sigma_0^i$. Hence, if $\sigma' \in U(\kappa, \sigma|_\kappa)$, as $\kappa \supset \kappa^i$, $res_{\kappa^i}(\sigma') = res_{\kappa^i}(\sigma) = \sigma_0^i$. Thus, $U(\kappa, \sigma|_\kappa) \subset U(\kappa^i, \sigma_0^i)$, for each i as required. $\square$

Lemma 4.5. *Each basic open set $U(\kappa, \sigma_0)$ is closed.*

Proof. Observe that the complement of $U(\kappa, \sigma_0)$ is the union of basic open sets

$$U(\kappa, \sigma_0)^c = \bigcup_{\sigma \in GS(\kappa),\ \sigma \neq \sigma_0} U(\kappa, \sigma).$$

□

Henceforth, consider $GS(\widetilde{M})$ with the topology whose basis is given by the sets $U(\kappa, \sigma_0)$ as above. By construction, $GS(\widetilde{M})$ is second countable. If $\kappa = \kappa_1 \subset \kappa_2 \subset \dots$ is an exhaustion of T by finite subtrees of T, then for each i, the collection $\{U(\kappa_i, \sigma) : \sigma \in GS(\kappa_i)\}$ is finite. Hence, one can easily see that the collection $\cup_i \{U(\kappa_i, \sigma) : \sigma \in GS(\kappa_i)\}_i$ gives a countable basis for the topology on $GS(\widetilde{M})$.

If $\kappa \subset T$ is a finite tree and σ_1, σ_2 are elements of $GS(\kappa)$ such that $\sigma_1 \neq \sigma_2$, then $U(\kappa, \sigma_1) \cap U(\kappa, \sigma_2) = \phi$ and $GS(\widetilde{M}) = \amalg U(\kappa, \sigma_i)$, where $\sigma_i \in GS(\kappa)$.

We see that the space $GS(\widetilde{M})$ is Hausdorff, in fact totally disconnected.

Lemma 4.6. *The space $GS(\widetilde{M})$ is totally disconnected.*

Proof. Let $\sigma^i = (\tau^i, \epsilon^i, e^i_+)$, $i = 1, 2$, be two distinct points in $GS(\widetilde{M})$. It is easy to see that for some finite tree κ, $res_\kappa(\sigma^1) \neq res_\kappa(\sigma^2)$. As $GS(\kappa)$ is a finite set, it follows that we can partition $GS(\kappa)$ into finite sets S_1 and S_2 with $res_\kappa(\sigma^i) \in S_i$, for $i = 1, 2$.

It now follows from Lemma 4.5 that $GS(\widetilde{M})$ is totally disconnected. □

4.3. **A compactness theorem for geospheres.** The main result we need about the topology is the following compactness theorem. This is the analogue of the fact that the set of geodesics in hyperbolic space (more generally, in any Riemannian manifold) that intersect a fixed compact set is compact.

Theorem 4.7. *For a fixed finite subtree $\kappa \subset T$, the set of all geospheres whose restriction to κ is non-trivial is compact.*

Proof. Let A be the set of all geospheres whose restriction to κ is non-trivial. As $GS(\widetilde{M})$ is second countable and Hausdorff, it is metrizable. Hence, it suffices to show that every sequence in the given subspace A has a convergent subsequence in A.

Let $\kappa = \kappa_1 \subset \kappa_2 \subset \dots$ be an exhaustion of T by finite subtrees of T. Let σ_i be a sequence of geospheres in $\widetilde{M}$ so that the restriction of each σ_i to κ is non-trivial. We construct a convergent subsequence of σ_i.

Firstly, for each i, $res_{\kappa_1}(\sigma_i) \in GS(\kappa_1)$ and $GS(\kappa)$ is a finite set. Hence, on passing to a subsequence (which we continue to denote by σ_i), we can assume that $res_{\kappa_1}(\sigma_i)$ is constant. Similarly, passing to a further subsequence, we can assume that $res_{\kappa_2}(\sigma_i)$ is constant. Iterating this and passing to a diagonal subsequence, we obtain a sequence, which we also denote σ_i, so that the restriction of σ_i to each of the sets κ_i is *eventually* constant. More concretely, we can assume that for $j, k \geq i$, $res_{\kappa_i}(\sigma_j) = res_{\kappa_i}(\sigma_k)$.

We claim that the subsequence σ_i constructed as above has a limit $\sigma = (\tau, \epsilon, e_+)$. Namely, to determine whether an edge e is in τ, consider i large enough that $e \in \kappa_i$. Then, as $res_{\kappa_i}(\sigma_j) = res_{\kappa_i}(\sigma_i)$ for $j \geq i$ (taking $k = i$), either $e \in \tau_j$ for all j large enough or $e \notin \tau_j$ for all $j \geq i$, where τ_j is the tree corresponding to σ_j. In the former case, we declare $e \in \tau$ and in the latter case $e \notin \tau$. We can see $\tau_1 \subset \tau_2 \subset \dots$ is an exhaustion of τ. We similarly can decide what vertices are in τ and also the values of the functions ϵ and e_+.

As the restriction of each σ_i is non-empty, the limiting subgraph τ is non-empty.

One can show that τ is connected. Namely, is v and w are vertices of τ, for j sufficiently large, v and w are contained in the tree τ_j, hence there is a unique reduced path λ between them. It follows that $\lambda \subset \tau$ by the definition of τ.

Thus, $\sigma \in GS(\widetilde{M})$. Finally, as κ_i is an exhaustion of T by compact subtrees, any compact subtree κ' is contained in κ_j for some j. Hence, for $k > j$, $res_{\kappa'}(\sigma_k) = res_{\kappa'}(\sigma)$. By the definition of the topology on $GS(\widetilde{M})$, we see that $\sigma_i \to \sigma$.

□

As a corollary, we see that $GS(\widetilde{M})$ is locally compact. In fact, every geosphere σ is contained in a compact open subset of $GS(\widetilde{M})$.

Proposition 4.8. *Any geosphere σ is contained in a compact open subset U of $GS(\widetilde{M})$.*

Proof. It is easy to see that there is a finite tree κ such that $res_\kappa(\sigma)$ is non-trivial. Let

$$U = \{\sigma' \in GS(\widetilde{M}) : res_\kappa(\sigma) = res_\kappa(\sigma')\}$$

By Theorem 4.7, U is compact. The set U is open by definition of the topology on $GS(\widetilde{M})$. □

In Section 3, we have seen that a normal sphere S in $\widetilde{M}$ is determined by triple $\sigma = (\tau, \epsilon, e_+)$, with τ is a finite subtree of T, ϵ is an assignment of sign to each non-standard bivalent vertex of τ and e_+ an assignment to each univalent vertex of τ an edge containing v and not contained in τ. Hence, normal spheres $\widetilde{M}$ are geospheres.

Let $S(\widetilde{M})$ be the set of all normal spheres in $\widetilde{M}$, i.e., $S(\widetilde{M})$ is the set of all geospheres $\sigma = (\tau, \epsilon, e_+)$, where τ is a finite subtree of T.

Proposition 4.9. *The set $S(\widetilde{M})$ is the set of isolated points of $GS(\widetilde{M})$ and is dense in $GS(\widetilde{M})$.*

Proof. Let $\sigma^0 = (\tau^0, \epsilon^0, e_+^0)$ be a normal sphere in $\widetilde{M}$. We see that $res_{\tau^0}(\sigma^0) = \sigma^0$. Consider $U(\tau^0, \sigma^0)$. If $\sigma' = (\tau', \epsilon', e'_+) \in U(\tau^0, \sigma^0)$, then $res_{\tau^0}(\sigma') = \sigma^0$. By definition of res,

$$res_{\tau^0}(\sigma') = (\tau' \cap N(\tau^0), \epsilon'_{|_{B(\tau')\cap\tau^0}}, e'_+|_{C(\tau')\cap\tau^0}) = (\tau^0, \epsilon^0, e_+^0).$$

As $\tau' \cap N(\tau^0) = \tau^0$, we have $\tau' = \tau^0$ and $\epsilon' = \epsilon^0, e'_+ = e_+^0$. Thus, $\sigma' = \sigma^0$. This implies $U(\tau^0, \sigma^0) = \{\sigma^0\}$ and hence, σ_0 is an isolated point in $GS(\widetilde{M})$.

Next, let $\sigma = (\tau, \epsilon, e_+)$ be a geosphere in $\widetilde{M}$, where τ is a subtree of T with each univalent vertex of τ a trivalent vertex in T. We call such a geosphere as a non-degenerate geosphere. We show that σ is a limit of spheres. Namely, we consider an exhaustion of T by finite subtrees κ_i. It is easy to construct spheres σ_i with $res_{\kappa_i}(\sigma_i) = res_{\kappa_i}(\sigma)$. We claim that σ_i converges to σ. Namely, given a basic open set $U(\kappa, \sigma_0)$ containing σ, there is an integer N such that for $i > n$ we have $\kappa \subset \kappa_i$. This implies that $res_\kappa(\sigma_i) = res_\kappa(\sigma) = \sigma_0$. Thus, for $i > n$ we have $\sigma_i \in U(\kappa, \sigma_0)$. As such a relation holds for all basic open sets containing σ, it follows that σ_i converges to σ.

Thus, every geosphere $\sigma \notin S(\widetilde{M})$, is the limit of a sequence of points of $S(\widetilde{M})$ and hence, it is not an isolated point in $GS(\widetilde{M})$. This shows that the set $S(\widetilde{M})$ is the set of isolated points of $GS(\widetilde{M})$ and is dense in $GS(\widetilde{M})$. □

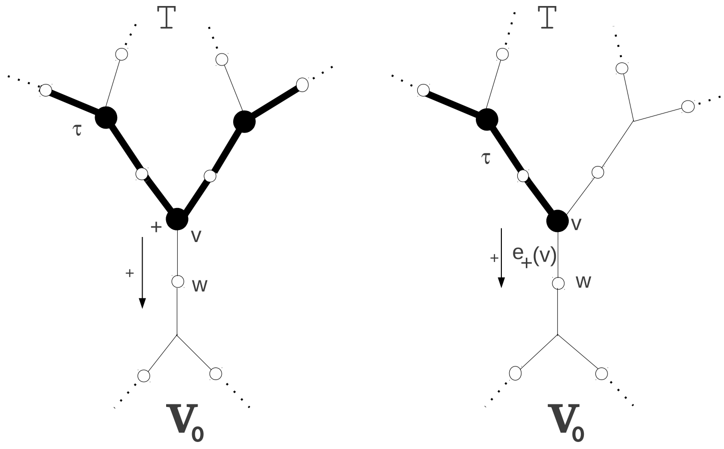

FIGURE 1. Bivalent and Univalent vertices

5. Crossing of geospheres

5.1. **Partitions and Crossing.** As in the case of spheres, we can associate to a geosphere a partition of the ends of $\widetilde{M}$, which can be identified with the set of ends $E(T)$. However, in the case of a geosphere $\sigma = (\tau, \epsilon, e_+)$, we get a partition into three sets

$$E(T) = E^\infty(\sigma) \amalg E^+(\sigma) \amalg E^-(\sigma)$$

with $E^\infty(\sigma)$ closed and $E^\pm(\sigma)$ open.

The set $E^\infty(\sigma)$ is defined to be the set of ends of τ. It is easy to see that, as τ is a subtree of T, τ is closed. Hence, $E^\infty(\sigma)$ is closed in $E(T)$. Observe that $E^\infty(\sigma)$ can also be interpreted as the set of ends of $N(\tau)$.

The complement $V(\sigma) = T - N(\tau)$ of $N(\tau)$ is an open set. We shall partition the components of $V(\sigma)$ into sets $V^+(\sigma)$ and $V^-(\sigma)$ using the data for σ, in analogy with the case of spheres. We shall define $E^\pm(\sigma)$ as the set of ends of $V^\pm(\sigma)$.

Let V_0 be a component of $T - N(\tau)$. Then, as τ is a tree, the closure of V_0 contains exactly one vertex w of $N(\tau)$, which in turn is at a distance 1 from a unique vertex v of τ which is either bivalent or univalent (see figure 1). If v is bivalent, we say that V_0 is positive (and w is on the positive side of v) if $\epsilon(v) = +$ and say that V_0 is negative otherwise. If v is univalent, we say that V_0 is positive (and w is on the positive side of v) if the edge $e_+(v)$ joins v to w and say that V_0 is negative otherwise.

By the above rule, each component of $V(\sigma)$ is assigned a sign. We define $V^+(\sigma)$ to be the union of the positive components and $V^-(\sigma)$ the union of negative components. We define $E^\pm(\sigma)$ as the set of ends of $V^\pm(\sigma)$.

Given two geospheres σ_1 and σ_2, we can define when they cross.

Definition 5.1. The geospheres $\sigma^i = (\tau^i, \epsilon^i, e^i_+)$, $i = 1, 2$ cross if either each of the four sets

$$E^\pm(\sigma^1) \cap (E^\pm(\sigma^2) \cup E^\infty(\sigma^2))$$

is non-empty or if each of the four sets

$$E^{\pm}(\sigma^2) \cap (E^{\pm}(\sigma^1) \cup E^{\infty}(\sigma^1))$$

is non-empty.

We remark that it is necessary to consider both the above collections of four sets separately.

The above definition is motivated by the observation that if, for instance, σ^2 is on the positive side of σ^1, then all ends (in fact points) on either the negative side of σ^2 or the positive side of σ^2 (the side away from σ^1) are on the positive side of σ^1. Hence, one of the intersections $E^-(\sigma^1) \cap (E^{\pm}(\sigma^2) \cup E^{\infty}(\sigma^2))$ is empty.

Lemma 5.2. *Let $\sigma^i = (\tau^i, \epsilon^i, e^i_+)$, $i = 1, 2$ be geospheres. If $\tau^1 \cap \tau^2 = \phi$, then σ^1 and σ^2 do not cross.*

Proof. As τ^1 and τ^2 are subtrees of T and $\tau^1 \cap \tau^2 = \phi$, for some component V^1_0 of $T - N(\tau^1)$, τ^2 is contained in V^1_0. Let v^1 be the point in τ^1 that is unit distance from V^1_0. Without loss of generality assume V^1_0 is positive.

As τ^2 is contained in the closure of V^1_0, $E^{\infty}(\sigma^2)$ is contained in the ends of V^1_0, and hence is contained in $E^+(\sigma^1)$. Further, as τ^1 is a tree, τ^1 is contained in a component V^2_0 of $T - N(\tau^2)$ and all other components of $T - N(\tau^2)$ are contained in V^1_0. Hence, if V^2_0 is positive, then $E^-(\sigma^2)$ is contained in the ends of V^1_0, and hence is contained in $E^+(\sigma^1)$.

Thus, as V^1_0 and V^2_0 are positive, the intersection $E^-(\sigma^1) \cap (E^-(\sigma^2) \cup E^{\infty}(\sigma^2))$ is empty. Considering other cases similarly, we see that in each case, at least one of the intersections $E^{\pm}(\sigma^1) \cap (E^{\pm}(\sigma^2) \cup E^{\infty}(\sigma^2))$ is empty.

Reversing the roles of τ^1 and τ^2, we see that one of the four intersections $E^{\pm}(\sigma^2) \cap (E^{\pm}(\sigma^1) \cup E^{\infty}(\sigma^1))$ is also empty. Thus, σ^1 and σ^2 do not cross. ☐

5.2. **Stability of Crossings.** Our main technical result is that crossing is an open condition.

Lemma 5.3. *Suppose $\sigma^i = (\tau^i, \epsilon^i, e^i_+)$, $i = 1, 2$ cross, then there are open sets U^i, $i = 1, 2$, with $\sigma^i \in U^i$ so that if $s^i \in U^i$ for $i = 1, 2$, then s^1 crosses s^2.*

Proof. Without loss of generality, we assume that each of the four intersections

$$E^{\pm}(\sigma^1) \cap (E^{\pm}(\sigma^2) \cup E^{\infty}(\sigma^2))$$

is non-empty. We shall construct open sets U^i containing σ^i so that for $s^i \in U^i$,

$$E^+(s^1) \cap (E^+(s^2) \cup E^{\infty}(s^2)) \neq \phi$$

We can similarly construct open sets for which each of the other three intersections is non-empty. The intersections of the four pairs of open sets thus constructed give the required neighbourhoods of σ^i.

We first make some observations. Suppose $\xi \in E^+(\sigma^1)$ is an end. Then, there is a component V_0 of $T - N(\tau^1)$ so that $\xi \in E(V_0)$. The intersection of the closure of V_0 with $N(\tau^1)$ is a vertex w, which is unit distance from a unique vertex v of τ^1. Further, the vertex is bivalent or univalent, with w on the positive side of v (see figure 1).

Let κ be a finite tree containing v. Then, if $(\tau^0, \epsilon^0, e^0_+)$ is another geosphere with $res_\kappa(\sigma^0) = res_\kappa(\sigma^1)$, then as $N(\kappa) \cap \tau^0 = N(\kappa) \cap \tau^1$, w is a vertex of $N(\tau^0) - \tau^0$ and v is in τ^0. As $\epsilon^0 = \epsilon^1$ and $e^0_+ = e^+_1$, w is on the positive side of v with respect

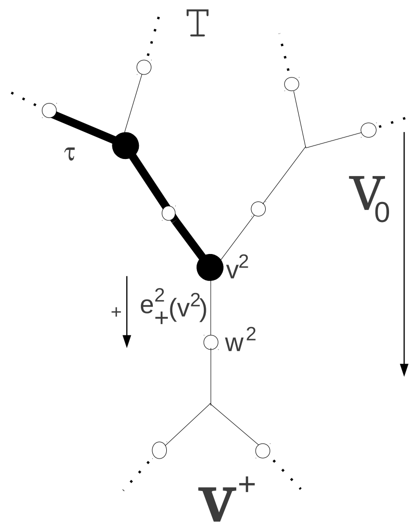

FIGURE 2

to σ^0. It follows, as τ^0 is connected, that V_0 is a component of $T - N(\tau^0)$ which is positive.

Suppose now that ξ is an end in $E^+(\sigma^1) \cap (E^+(\sigma^2) \cup E^\infty(\sigma^2))$. We consider two cases. Firstly, if $\xi \in E^+(\sigma^1) \cap E^+(\sigma^2)$, then as above we have positive components V_0^i of $T - N(\tau^i)$ containing ξ, for $i = 1, 2$, and corresponding vertices v^i and w^i. Let κ be a finite tree containing v^1 and v^2 and let $U^i = U(\kappa, res_\kappa(\sigma^i))$.

Suppose $s^i = (t^i, \epsilon^i, e^i_+) \in U^i$, $i = 1, 2$, then, as above, V_0^i is a component of $T - N(t^i)$ and is positive. Hence, $\xi \in E^+(s^i)$ for $i = 1, 2$, i.e., $\xi \in E^+(s^1) \cap E^+(s^2) \subset E^+(s^1) \cap (E^+(s^2) \cup E^\infty(s^2))$.

Next, consider the case when $\xi \in E^+(\sigma^1) \cap E^\infty(\sigma^2)$. Let V_0 be the component of $T - N(\sigma^1)$ that has ξ as an end and let v and w be as above. As $\xi \in E^\infty(\sigma^2)$, the intersection $\tau^2 \cap V_0$ is infinite.

Note that as σ^1 and σ^2 cross, we cannot have $\tau^1 \cap \tau^2 = \phi$, as this would imply that one of the intersections $E^-(\sigma^1) \cap (E^\pm(\sigma^2) \cup E^\infty(\sigma^2))$ is empty. As τ^2 is connected and $\tau^1 \cap \tau^2 \neq \phi \neq V_0 \cap \tau^2$, it follows that v and w are vertices of τ^2.

Let κ be a finite tree containing v and w and let $U^i = U(\kappa, res_\kappa(\sigma^i))$ and s^i be as before. As in the first case, if $s^1 \in U^1$, then V_0 is a positive component of $T - N(t^1)$. To complete the proof, we show that if $s^2 \in U^2$, then the set of ends of V_0 contains either a point of $E^\infty(s_2)$ or a point of $E^+(s_2)$.

To see this, observe that as $\tau^2 \cap V_0$ is infinite and $t^2 \cap N(\kappa) = \tau^2 \cap N(\kappa)$, with κ a tree containing w, $t^2 \cap V_0$ is non-empty. Suppose $t^2 \cap V_0$ is infinite, then an end of $t^2 \cap V_0$ lies in $V_0 \cap E^\infty(s^2)$, as claimed.

On the other hand, if $t^2 \cap V_0$ is finite, it has a terminal vertex v^2 distinct from w. The other vertex w^2 of $e^2_+(v^2)$ is in the closure of a component of V^+ of $T - N(t^2)$, with $E(V^+) \subset E^+(s^2)$ (see figure 2).

By construction $V^+ \subset V_0$. An end of V^+ gives an element $E^+(s_2)$ which is an end of V_0, hence in $E^+(s^1)$.

Thus, we have shown that in all cases $E^+(s^1)\cap(E^+(s_2)\cup E^\infty(s_2))$ is non-empty for $s^i \in U^i$.

$\square$

6. Geosphere laminations in M

We are now in a position to define geosphere laminations in M, which are the analogues of (embedded) geodesic laminations in a surface. Recall that the group $\pi_1(M)$ acts on $\widetilde{M}$ by deck transformations. Geosphere laminations are the natural completion of the inverse image in $\widetilde{M}$ of a sphere (or a collection of spheres) in M.

Definition 6.1. A subset $X \subset GS(\widetilde{M})$ is said to be *embedded in* M if for $\sigma_1, \sigma_2 \in X$, σ_1 does not cross σ_2.

Definition 6.2. A geosphere lamination in M is a subset $\Gamma \subset GS(\widetilde{M})$ such that

(1) Γ is closed in $GS(\widetilde{M})$.
(2) Γ is invariant under the action of $\pi_1(M)$.
(3) Γ is embedded in M.

We denote the set of geosphere laminations in M by $L(M)$.

Definition 6.3. Let Γ be geosphere lamination in M. A geosphere $\sigma \in \Gamma$ is called a leaf of Γ.

Definition 6.4. A subset Γ' of a geosphere lamination Γ is said to be a sublamination of Γ if Γ' itself is a geosphere lamination.

Definition 6.5. A geosphere lamination Γ is said to be maximal if Γ is not a proper sublamination of any geosphere lamination in M.

Definition 6.6. A geosphere lamination Γ is said to be minimal if no proper subset of Γ is a sublamination of Γ.

We shall see that this contains all collections of disjoint, non-parallel spheres in M, and that the space of non-trivial geosphere laminations is compact. This allows us to consider limits of spheres in M.

We first observe that the condition that Γ is closed is easy to achieve.

Lemma 6.7. *Suppose $X \subset GS(\widetilde{M})$ is embedded in M, then so is its closure $\bar{X}$.*

Proof. Suppose σ_1 and σ_2 are geospheres in $\bar{X}$ that cross. By Lemma 5.3, there are open sets U_i with $\sigma_i \in U_i$ so that if $s_i \in U_i, i = 1, 2$, then s_1 and s_2 cross. As $\sigma_i \in \bar{X}$, there are elements $s_i \in X \cap U_i$, which thus cross. But, this contradicts the hypothesis that X is embedded in M. Thus, $\bar{X}$ is embedded in M. $\square$

It is clear that the closure of a $\pi_1(M)$-invariant set in $GS(\widetilde{M})$ is $\pi_1(M)$-invariant. Thus, if X is not closed but satisfies the other two conditions for being a geosphere lamination, then its closure is a geosphere lamination.

6.1. **Topology on $L(M)$.** We shall make the set $L(M)$ of geosphere laminations in M into a topological space by defining a topology on $L(M)$. To do this, we first define a topology on the set of *closed* subsets of $GS(\widetilde{M})$, which we denote by $C(\widetilde{M})$.

The topology we construct is analogous to the Hausdorff topology. Namely, if $\Gamma \subset GS(\widetilde{M})$ is closed and κ is a finite subtree of T, consider the image $res_\kappa(\Gamma)$ of Γ under the restriction map. For $S \subset GS(\kappa)$, consider the set

$$\mathcal{U}(\kappa, S) = \{\Gamma \in C(\widetilde{M}) : res_\kappa(\Gamma) = S\}.$$

Lemma 6.8. *The sets $\mathcal{U}(\kappa, S)$ for finite subtrees κ of T form a basis for a topology on $C(\widetilde{M})$.*

Proof. Showing that the sets $\mathcal{U}(\kappa, S)$ form a basis for a topology on $C(\widetilde{M})$ is equivalent to showing that if $\mathcal{U}(\kappa^i, S^i)$, $1 \leq i \leq n$ is a finite collection of basic open sets and $\Gamma \in \cap_i \mathcal{U}(\kappa^i, S^i)$, then there is a basic open set containing Γ and contained in each of the sets $\mathcal{U}(\kappa^i, S^i)$.

To show this, let κ be the finite subtree of T spanned by the subtrees κ^i, and let $S_0 = res_\kappa(\Gamma)$. Note that as $\Gamma \in \mathcal{U}(\kappa^i, S^i)$, $res_{\kappa^i}(\Gamma) = S^i$. Hence, if $\Gamma' \in \mathcal{U}(\kappa, S_0)$, as $\kappa \supset \kappa^i$, $res_{\kappa^i}(\Gamma') = res_{\kappa^i}(\Gamma) = S^i$, for each i. Thus, $\mathcal{U}(\kappa, S_0) \subset \mathcal{U}(\kappa^i, \sigma_0^i)$, for each i as required. □

Thus, the sets $\mathcal{U}(\kappa, S)$ form the basis for a topology, which we take to be the topology on $C(\widetilde{M})$. Note that as $GS(\kappa)$ is finite, so is the collection of subsets of $GS(\kappa)$.

If $\kappa \in T$ is a finite tree and S_1 and S_2 are subsets of $GS(\kappa)$ such that $S_1 \neq S_2$, then $\mathcal{U}(\kappa, S_1) \cap \mathcal{U}(\kappa, S_2) = \phi$ and $C(\widetilde{M}) = \amalg\, \mathcal{U}(\kappa, S_i)$, where S_i is a subset of $GS(\kappa)$.

We can easily see that $C(\widetilde{M})$ is second countable. We see that the topology is Hausdorff, in fact totally disconnected. This is based on the following lemma.

Lemma 6.9. *If $\Gamma_1, \Gamma_2 \subset GS(\widetilde{M})$ are closed sets with $\Gamma_1 \neq \Gamma_2$, then for some finite subtree κ of T, $res_\kappa(\Gamma_1) \neq res_\kappa(\Gamma_2)$.*

Proof. As $\Gamma_1 \neq \Gamma_2$, without loss of generality, there is a point $\sigma \in \Gamma_1 \setminus \Gamma_2$. As Γ_2 is closed subset of $GS(\widetilde{M})$, there is a basic open set $\mathcal{U} = \mathcal{U}(\kappa, \sigma_0)$ with $\sigma \in U$ but $\mathcal{U} \cap \Gamma_2 = \phi$. But this means that $res_\kappa(\sigma) \in res_\kappa(\Gamma_1) \setminus res_\kappa(\Gamma_2)$. Hence, $res_\kappa(\Gamma_1) \neq res_\kappa(\Gamma_2)$. □

It is easy to deduce that the topology on $C(\widetilde{M})$ is totally disconnected. The proof is analogous to Lemma 4.6.

Lemma 6.10. *Given $\Gamma_1, \Gamma_2 \in C(\widetilde{M})$, there are disjoint open sets $\mathcal{U}_1, \mathcal{U}_2 \subset C(\widetilde{M})$ with $\Gamma_i \subset \mathcal{U}_i$ so that $\mathcal{U}_1 \cup \mathcal{U}_2 = C(\widetilde{M})$.*

□

We can consider $S(\widetilde{M})$ as a subset of $C(\widetilde{M})$. If $\sigma = (\tau, \epsilon, e_+) \in S(\widetilde{M})$, then $\{\sigma\} \in C(\widetilde{M})$ and $res_\tau(\sigma) = \sigma \in GS(\tau)$. One can easily see that in fact $\mathcal{U}(\tau, \{\sigma\}) = \{\{\sigma\}\}$.

The topology on $C(\widetilde{M})$ restricts to one on $L(M)$. To study the restriction, the following lemma is useful.

Lemma 6.11. *The subspace $L(M) \subset C(\widetilde{M})$ is closed.*

Proof. As the topology on $C(\widetilde{M})$ is second countable and Hausdorff, it suffices to show that if $\Gamma_0 \in C(\widetilde{M})$ is the limit of a sequence $\Gamma_i \in L(M)$, then $\Gamma_0 \in L(M)$.

Firstly, as $C(\widetilde{M})$ is Hausdorff, limits are well-defined. Hence, if $g \in \pi_1(M)$, as $g\Gamma_i = \Gamma_i$ and $g\Gamma_i \to g\Gamma_0$ (as the deck transformation g is a homeomorphism), $g\Gamma_0 = \Gamma_0$. Thus, Γ_0 is $\pi_1(M)$-invariant. Further, Γ_0 is closed as it is an element of $C(\widetilde{M})$. Thus, to complete the proof it suffices to show that Γ_0 is embedded in M.

Suppose Γ_0 is not embedded in M, then there are elements σ_1, σ_2 in Γ_0 that cross. By Lemma 5.3, there are open sets $\mathcal{U}_i$ with $\sigma_i \in \mathcal{U}_i$ so that if $s_i \in \mathcal{U}_i$, then s_1 and s_2 cross. By the definition of the topology on $GS(\widetilde{M})$, for some finite tree κ, $\mathcal{U}_i$ contains the open set $\mathcal{U}(\kappa, res_\kappa(\sigma_i))$. As $\Gamma_i \to \Gamma_0$, for i sufficiently large, $res_\kappa(\Gamma_i) = res_\kappa(\Gamma_0)$, in particular, there are elements $s_i \in \Gamma_i$ with $s_i \in \mathcal{U}_i$. It follows that s_1 and s_2 cross, contradicting the hypothesis that $\Gamma_i \in L(M)$. ☐

6.2. **Geosphere laminations from spheres.** We see that (collections of) spheres in M have associated geosphere laminations.

Suppose that Σ' is a collection of disjoint, non-parallel spheres in M which are in normal form with respect to Σ. Let $\widetilde{\Sigma}'$ be the collection of lifts of the spheres in Σ', i.e., the inverse image of Σ' under the covering map $\widetilde{M} \to M$. Each element of $\widetilde{\Sigma}'$ is a sphere, and hence, gives a geosphere. Thus, $\widetilde{\Sigma}'$ can be viewed as a subset of $GS(\widetilde{M})$.

It is immediate that the set $\widetilde{\Sigma}'$ is $\pi_1(M)$-invariant. The set $\widetilde{\Sigma}'$ is embedded in M as it is a union of disjoint spheres. To see that $\widetilde{\Sigma}'$ gives an element in $L(M)$), it only remains to show that the set $\widetilde{\Sigma}'$ is a closed subset of $GS(\widetilde{M})$.

Lemma 6.12. *The set $\widetilde{\Sigma}'$ is closed in $GS(\widetilde{M})$, hence a lamination.*

Proof. The tree τ corresponding to each element $\sigma \in \widetilde{\Sigma}'$ is finite, with diameter determined by the corresponding sphere in M. Hence, there is an integer $D > 0$ such that the trees τ corresponding to elements $\sigma \in \widetilde{\Sigma}'$ have diameter at most D.

Suppose now σ_0 is in the closure of $\widetilde{\Sigma}'$, with τ_0 the tree corresponding to σ_0. Let v be a vertex of τ_0 and let κ be the tree consisting of all points of distance at most D from v.

As σ_0 is in the closure of $\widetilde{\Sigma}'$, $res_\kappa(\sigma_0) = res_\kappa(\sigma)$ for some $\sigma \in \widetilde{\Sigma}'$. If τ is the tree corresponding to σ, then $v \in \tau$ and τ has diameter at most D. It follows that $\tau \subset \kappa$, and hence, $\tau = \tau \cap N(\kappa)$ and is contained in the interior of $N(\kappa)$. As $\tau_0 \cap N(\kappa) = \tau \cap N(\kappa)$, $\tau_0 \cap N(\kappa)$ is contained in the interior of $N(\kappa)$. Hence, as τ_0 is connected, $\tau_0 = \tau_0 \cap N(\kappa) = \tau \cap N(\kappa) = \tau$. As $res_\kappa(\sigma_0) = res_\kappa(\sigma)$, it follows that $\sigma_0 = \sigma$, hence $\sigma_0 \in \widetilde{\Sigma}$. Thus, any element of the closure of $\widetilde{\Sigma}'$ is in $\widetilde{\Sigma}'$, showing that $\widetilde{\Sigma}'$ is closed. ☐

Thus, given any embedded sphere S in normal form with respect to Σ in M, we have a geosphere lamination associated to it, namely, the inverse image of S in $\widetilde{M}$ under the covering map. So, we can regard S as a geosphere lamination in M. Let $S_0(M)$ be the set of isotopy classes spheres in M. Then, $S_0(M)$ can be considered as a subset of $L(M)$.

Proposition 6.13. *The geosphere lamination Σ' is an isolated point in $L(M)$.*

Proof. Fix a lift $\widetilde{\Sigma'_i}$ of Σ'_i to $\widetilde{M}$. Note that $\widetilde{\Sigma'_i}$ is a normal sphere in $\widetilde{M}$. Let $\widetilde{\Sigma'_i} = (\tau^i, \epsilon^i, e^i_+)$. Let κ be a subtree of T spanned by trees τ^i. Let $\mathcal{S} = \{\widetilde{\Sigma'_i}\}_i$. The set $res_\kappa(\mathcal{S})$ is $\mathcal{S}$. Then, one can easily see that $\mathcal{U}(\kappa, res_\kappa(\mathcal{S}) \cup \{\phi\}) = \Sigma'$. This shows that Σ' is an isolated point in $L(M)$. ☐

7. Compactness for geosphere laminations

7.1. **The Compactness Theorem.** Our main result concerning geosphere laminations is the following compactness theorem.

Theorem 7.1. *The spaces $L(M)$ and $C(\widetilde{M})$ are compact.*

Proof. First observe that as $L(M)$ is a closed subset of $C(\widetilde{M})$, it suffices to show that $C(\widetilde{M})$ is compact. Further, as $C(\widetilde{M})$ is second countable and Hausdorff, it suffices to show that any sequence $\Gamma_i \in C(\widetilde{M})$ has a convergent subsequence.

As in the proof of Theorem 4.7, let κ_i be an exhaustion of T by finite subtrees. Observe that $res_{\kappa_1}(\Gamma_i) \in GS(\kappa_i)$ is contained in a finite set, namely the set of subsets of $GS(\kappa_i)$. Hence, passing to a subsequence, we can assume that this is constant. Similarly, passing to a further subsequence, we can assume that $res_{\kappa_j}(\Gamma_i)$ is constant for each successive integer j. Iterating this and passing to a diagonal subsequence, we obtain a sequence, which we also denote Γ_i, so that the restriction of Γ_i to each of the sets κ_j is *eventually* constant. More concretely, we can assume that for $j, k \geq i$, $res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_k) = res_{\kappa_i}(\Gamma_i)$.

We claim that the subsequence Γ_i constructed as above has a limit Γ_0. Let $X_i = \{\sigma \in GS(\widetilde{M}) : res_{\kappa_i}(\sigma) \in res_{\kappa_i}(\Gamma_i)\}$. It is immediate that $\Gamma_i \subset X_i$. We let $\Gamma_0 = \cap_i X_i$.

We claim that $\Gamma_i \to \Gamma_0$. As the finite trees κ_i form an exhaustion, it suffices to show that for j sufficiently large, $res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_0)$. We show this for $j \geq i$.

Observe that for $j \geq i$, $X_j \subset X_i$. This is because if $\sigma \in X_j$, by definition there is a geosphere $\sigma' \in \Gamma_j$ with $res_{\kappa_j}(\sigma) = res_{\kappa_j}(\sigma')$. As $\kappa_i \subset \kappa_j$, it follows that $res_{\kappa_i}(\sigma) = res_{\kappa_i}(\sigma')$ and hence $res_{\kappa_i}(\sigma) \in res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_i)$, which implies that $\sigma \in X_i$. As $\sigma \in X_j$ was arbitrary, $X_j \subset X_i$.

Next, note that $res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_i)$ for $j \geq i$. Hence, we are reduced to showing that $res_{\kappa_i}(\Gamma_i) = res_{\kappa_i}(\Gamma_0)$. Firstly, as $\Gamma_0 \subset X_i$ and for $\sigma \in X_i$, $res_{\kappa_i}(\sigma) \in res_{\kappa_i}(\Gamma_i)$, we have $res_{\kappa_i}(\Gamma_0) \subset res_{\kappa_i}(\Gamma_i)$.

Conversely, suppose $\sigma_0 \in res_{\kappa_i}(\Gamma_i)$, and without loss of generality, σ_0 is not degenerate. Then, as $res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_i)$ for $j \geq i$ and $\Gamma_j \subset X_j$, $\sigma_0 \in res_{\kappa_i}(X_j)$. Hence, for $j \geq i$, there is an element $\sigma_j \in X_j$ with $res_{\kappa_i}(\sigma_j) = \sigma_0$.

By the compactness theorem, Theorem 4.7, there is a subsequence σ_{n_j} that converges to a geosphere σ. By construction $res_{\kappa_i}(\sigma) = \sigma_0$. We finish the proof by showing that $\sigma \in \Gamma_0$, hence $\sigma_0 \in res_{\kappa_i}(\Gamma_0)$.

Assume without loss of generality that $n_j \geq j$ for all j. Hence, if $j \geq i$ is fixed, for $k \geq j$, $\sigma_{n_k} \in X_{n_k} \subset X_j$. As X_j is closed and $\sigma_{n_k} \to \sigma$, $\sigma \in X_j$. As $j \geq i$ was arbitrary, $\sigma \in \cap_j X_j = \Gamma_0$. Thus, $\sigma_0 = res_{\kappa_i}(\sigma) \in res_{\kappa_i}(\Gamma_0)$.

□

7.2. **Limit laminations.** Thus, we can extract limits of geosphere laminations, in particular those of collections of spheres. For this construction to be useful, one would like the limit to be non-trivial. This turns out to be automatic for geosphere laminations embedded in M.

Proposition 7.2. *The empty subset $\phi \in L(M)$ is an isolated point.*

Proof. As $\pi_1(M)$ acts cocompactly on T, there is a finite tree κ such that the translates of κ cover T. Let $\mathcal{U}$ be the open set in $C(\widetilde{M})$ given by $\mathcal{U} = \{\Gamma \in C(\widetilde{M}) :$

$res_\kappa(\sigma) = \phi\}$. Clearly, $\phi \in \mathcal{U}$ for the empty lamination ϕ. We shall show that if $\Gamma \in L(M)$ and $\Gamma \neq \phi$, then $\Gamma \notin \mathcal{U}$.

Suppose $\Gamma \in L(M)$ is non-trivial. Let $\sigma \in \Gamma$ be a geosphere. Let v be a vertex in the tree τ corresponding to σ. Then, as the translates of κ cover T, $v \in g\kappa$ for some $g \in \pi_1(M)$. Hence, $g^{-1}v \in \kappa$, which implies that $g^{-1}\tau \cap \kappa \neq \phi$.

It follows that $res_\kappa(g^{-1}\Gamma) \neq \phi$. But, as $\Gamma \in L(M)$, $g^{-1}\Gamma = \Gamma$ and hence, $res_\kappa(\Gamma) \neq \phi$, i.e., $\Gamma \notin \mathcal{U}$ as claimed. ☐

An important example is that associated to an outer automorphism φ of the free group and a sphere S. Namely, the sphere S is an element of $L(M)$ by Lemma 6.12. Hence, we obtain a sequence of laminations $\varphi^k(S)$. A convergent subsequence of this gives a limiting lamination.

7.3. **Laminations that are not limits of spheres.** It is natural to ask whether laminations corresponding to spheres in M are the only isolated points of $L(M)$. The analogous result holds for geodesic laminations, namely the only isolated geodesic laminations are unions of simple closed curves. However, we see that there are isolated geosphere laminations which are not unions of spheres.

Consider the geosphere lamination $\Gamma_0 = \{\sigma_o\}$, where $\sigma_o = (\tau_0, \epsilon_0, e_{o+})$ is geosphere such that $\tau_0 = T$. Observe that τ_0 has no terminal or non-standard bivalent vertices.

Proposition 7.3. *The geosphere lamination Γ_0 is an isolated point in $L(\widetilde{M})$.*

Proof. Consider the projection of the tree $T \subset \widetilde{M}$ under covering map in M. Then, the projection is a graph $\mathcal{G}$. Choose a maximal tree $\mathcal{T}$ in $\mathcal{G}$ and fix a lift κ of $\mathcal{T}$ in the tree T.

The $res_\kappa(\Gamma_0)$ contains only one element $\sigma^0 = (\tau^0, \epsilon^0, e^0_+)$, where $\tau^0 = N(\kappa)$. Then, τ^0 has no terminal or non-standard bivalent vertex. We shall show that $\mathcal{U}(\kappa, \Gamma_0) = \{\Gamma_0\}$.

If $\Gamma \neq \Gamma_0$ is a geosphere lamination, then Γ has a leaf $\sigma = (\tau, \epsilon, e_+)$ (geosphere) such that either τ is a bivalent vertex or contains at least one vertex which is a non-standard bivalent vertex or a terminal vertex. Suppose τ has a non-standard bivalent vertex. Then, one can easily see that there exists a translate $g\sigma$ of σ such that if we consider $res_\kappa(g\sigma) = (\tau', \epsilon', e'_+)$, then τ' has a non-standard bivalent vertex. A similar argument holds in the other cases. This shows that $res_\kappa(\Gamma_0) \neq res_\kappa(\Gamma)$, for all Γ. Therefore, $res_\kappa(\Gamma_0) = \{\Gamma_0\}$. Hence, Γ_0 is an isolated point of $L(\widetilde{M})$.

☐

Using the above argument, we have the following proposition:

Proposition 7.4. *A geosphere lamination Γ such that no leaf of Γ has a terminal vertex, is not a limit of spheres.*

8. Geospheres and partitions

The definition of geospheres *a priori* depends on the choice of standard spheres for M. However, we show that geospheres can be defined intrinsically by showing that they are determined by the partition of the space of ends.

As we have seen, every geosphere $\sigma = (\tau, \epsilon, e_+)$ gives a partition of the set of $E(\widetilde{M})$ of ends of $\widetilde{M}$ in to three sets $E^+(\sigma), E^-(\sigma)$ and $E^\infty(\sigma)$. If τ is a finite tree, then $E^{\infty}(\sigma) = \phi$. If $\tau = T$, then $E^{\infty} = E(\widetilde{M})$ and $E^{+}(\sigma) = E^{-}(\sigma) = \phi$. In general, we get a partition with $E^{\pm}(\Sigma)$ open sets and $E^{\infty}(\Sigma)$ a closed set.

We show that any such partition corresponds to a geosphere.

Theorem 8.1. *Given a partition $E(\widetilde{M}) = E^{+} \cup E^{-} \cup E^{\infty}$ of the ends of M (hence of T) into disjoint sets so that $E^{\pm}$ are open (and hence E^{∞} is closed) so that either E^{∞} has at least two points or both E^{+} and E^{-} are non-empty, there is a geosphere $\sigma = (\tau, \epsilon, e_{+})$ so that $E^{\pm}(\Sigma) = E^{\pm}$ and $E^{\infty}(\Sigma) = E^{\infty}$*

We prove this by constructing the appropriate geosphere.

8.1. **Geospheres from partitions.** Consider a partition of $E(\widetilde{M})$ satisfying the hypothesis, say $A = (E^{+}, E^{-}, E^{\infty}) = (E^{+}(A), E^{-}(A), E^{\infty}(A))$. We note that it makes sense to talk of partitions crossing (as in the Definition 5.1).

Firstly, we associate a subgraph τ of T to A motivated by the following lemma (see also [4]).

Lemma 8.2. *A non-degenerate geosphere $\sigma = (\tau, \epsilon, e_{+})$ crosses a standard sphere S if and only if τ contains the corresponding vertex v.*

Proof. If τ does not contain v, it is either on the positive or the negative side of v. As in Lemma 5.2, we see that S does not cross σ.

Conversely, suppose τ contains v. As v is a standard vertex, v is not a terminal vertex of τ. Hence, if $W^{\pm}$, are the complementary components of v corresponding to a chosen orientation on S, then τ has non-empty intersection with $W^{\pm}$.

It follows that $\tau \cap W^{\pm}$ is either infinite or has a terminal vertex. In the first case, $E^{\pm}(S) \cap E^{\infty}(\sigma)$ is non-empty. In the second case, as terminal vertices are adjacent to edges on both the positive and negative sides, as in the proof of Lemma 5.3, $E^{\pm}(S)$ intersects both $E^{+}(\sigma)$ and $E^{-}(\sigma)$. In either case, S crosses σ. $\square$

We construct τ as follows. If A crosses a standard sphere $\widetilde{\Sigma_i}$, then τ contains the bivalent vertex v_i corresponding to $\widetilde{\Sigma_i}$ and the edges e_1^i and e_2^i containing that vertex v_i. The other end vertex v_j^i of the edge e_j^i, for $j = 1, 2$, is a trivalent vertex in T which corresponds to a component of $\widetilde{M} - \widetilde{\Sigma}$. Each v_j^i may be a univalent, bivalent or trivalent vertex in τ. If A does not cross some standard sphere in $\widetilde{M}$, then τ does not contain the standard vertex corresponding to this standard sphere and hence, it does not contain the edges containing this standard vertex.

Lemma 8.3. *If the partition A does not cross any standard sphere in $\widetilde{M}$, then $E^{\infty}(A) = \phi$ and there exists a standard sphere Σ_0 such that $E^{\pm} = E^{\pm}(\Sigma_0)$.*

Proof. Firstly we shall show that $E^{\infty}(A) = \phi$. Suppose $E^{\infty}(A) \neq \phi$. Let $P \in E^{\infty}$. Suppose E^{∞} has another point Q, we consider the geodesic $\gamma \subset T$ from P to Q. Given any edge e of γ, if $\Sigma(e)$ is the standard sphere corresponding to the standard vertex of e oriented appropriately, then $P \in E^{-}(\Sigma(e))$ and $Q \in E^{+}(\Sigma(e))$. Hence, $\Sigma(e)$ crosses the given partition A. This is a contradiction to the hypothesis as A does not cross any standard sphere.

On the other hand, if P is the only point in $E^{\infty}(A)$, then there are points $Q^{\pm} \in E^{\pm}(A)$. Let α be the geodesic from Q^{-} to Q^{+} and let γ be the unique geodesic ray from a point of α to P with the property that its interior is disjoint from α. Given any edge e of γ, if $\Sigma(e)$ is the standard sphere corresponding to the standard vertex of e oriented appropriately, then $P \in E^{-}(\Sigma(e))$ and $Q^{\pm} \in E^{+}(\Sigma(e))$. Hence, $\Sigma(e)$

crosses the given partition A. This is a contradiction to the hypothesis. Therefore, $E^\infty(A) = \phi$.

Now, by hypothesis, if v is a standard bivalent vertex of T, the standard sphere $\Sigma(v)$ corresponding to v does not cross A. Hence, after choosing orientations appropriately, either $E^+(\Sigma(v)) \subset E^+(A)$ or $E^-(\Sigma(V)) \subset E^-(A)$. If $\Sigma(v) = \Sigma_0$ satisfies both the conditions, then $E^\pm(A) = E^\pm(\Sigma_0)$.

Suppose no $\Sigma(v)$ satisfies both the above conditions, we get a partition of bivalent vertices of T as

$$V^+ = \{v : E^+(\Sigma(v)) \subset E^+(A)\}$$

and

$$V^- = \{v : E^-(\Sigma(v)) \subset E^-(A)\}.$$

Let $X^\pm$ is the union of all the edges e in T such that the bivalent vertex of e lies in $V^\pm$. Then, $X^\pm$ are closed and $T = X^+ \cup X^-$. Hence, $X^+ \cap X^- \neq \phi$. By construction, $X^+ \cap X^-$ consists of trivalent vertices of T. Let $w \in X^+ \cap X^-$ and let v_1, v_2 and v_3 be bivalent vertices adjacent to w. Note that at least one $v_i \in X^+$ and at least one $v_j \in X^-$. Without loss of generality, suppose $v_1, v_2 \in X^+$ and $v_3 \in X^-$. Let $N(w)$ denote the set of all the points in T distance at most 1 from w. Then, $T - N(w)$ has three components V_1, V_2 and V_3 whose closures contain the vertices v_1, v_2 and v_3, respectively. It is easy to see that $E(V_1) \subset E^+, E(V_2) \subset E^+$ and $E(V_3) \subset E^-$. It follows that $E^+(\Sigma(v_3)) = E^+(\Sigma(v_1)) \cup E^+(\Sigma(v_2))$. This implies $E^+(\Sigma(v_3)) \subset E^+$. As $v_3 \in X^-$, $E^-(\Sigma(v_3)) \subset E^-$. But then, $v_3 \in V^+ \cap V^-$. This is a contradiction as V^+ and V^- are disjoint. Hence, there must exist a standard sphere Σ_0 such that $E^\pm(A) = E^\pm(\Sigma_0)$. □

If A does not cross any standard sphere, the tree τ associated to A is a standard vertex corresponding to the standard sphere representing A. Note that any edge e in T has a unique end vertex which is a standard bivalent vertex in T.

We make the following observations :

If the partition $A = (E^+, E^-, E^\infty)$ of $E(\widetilde{M})$ crosses a sphere $S = (E^+(S), E^-(S))$ in $\widetilde{M}$, where $(E^+(S), E^-(S))$ is a partition of $E(\widetilde{M})$ given by S, then all the four intersections $E^\pm(S) \cap (E^\pm \cup E^\infty)$ are non-empty. For, if $E^\varepsilon(S) \cap (E^\eta(A) \cup E^\infty(A)) = \phi$, for some sign ε and η, then $E^\eta(A) \subset E^{\bar{\varepsilon}}(S)$ and hence, $E^\eta(A) \cap E^\varepsilon(S) = \phi$. This is a contradiction to the fact the partition A crosses S.

Lemma 8.4. *The graph τ associated to the partition A is connected, and hence is a subtree of T.*

Proof. Suppose S, S' and S'' are standard spheres in $\widetilde{M}$ such that the standard bivalent vertex v' in T corresponding to S' lies on the geodesic in T joining the standard bivalent vertices v and v'' in T corresponding to S and S'', respectively (the geodesic is the dark line in Figure 3). By giving appropriate orientations to S, S' and S'', we can assume that $E^+(S'') \subset E^+(S') \subset E^+(S)$ and $E^-(S) \subset E^-(S') \subset E^-(S'')$. Now, if A crosses S and S'', it follows that each of the four intersections $E^\pm(S) \cap (E^+(A) \cap E^\infty(A))$ is non-empty, hence A crosses S'. This shows that the geodesic in T joining v and v'' in T is completely contained in τ. From this, one easily see that τ is connected and hence a subtree of T. □

Note that the terminal vertices of τ are trivalent vertices in T.

If A does not cross a sphere $S = (E^+(S), E^-(S))$ in $\widetilde{M}$, where $(E^+(S), E^-(S))$ is a partition of $E(\widetilde{M})$ given by S, then $E^\varepsilon(S) \cap (E^\eta(A) \cup E^\infty(A)) = \phi$, for some

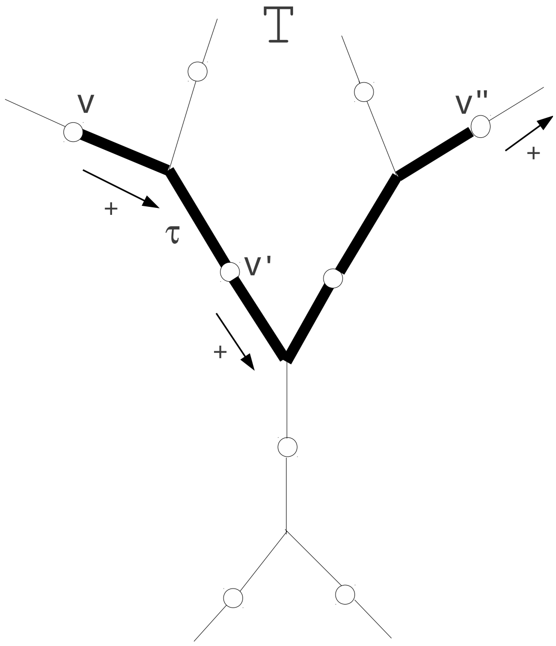

FIGURE 3

sign ε and η obtained by choosing signs ε and η in $\{+,-\}$. Then, $E^{\varepsilon}(S) \subset E^{\bar{\eta}}(A)$ and $(E^{\eta}(A) \cup E^{\infty}(A)) \subset E^{\bar{\varepsilon}}(S)$. In this case, we say S is on the $\bar{\eta}$-side of A and A is on $\bar{\varepsilon}$-side of S.

Note that the tree τ may or may not have terminal vertices. Suppose v is a vertex of τ adjacent to a single edge $e_0 \in \tau$, i.e., a terminal vertex of τ. Let $v_0 \in \tau$ be the other end vertex of e_0 and Σ_0 be the standard sphere in $\widetilde{M}$ corresponding to v_0. Then, A crosses Σ_0. Let the other edges adjacent to v in T be e_1 and e_2 with other end vertices v_1 and v_2, respectively (see figure 4). Consider the standard spheres $\widetilde{\Sigma}_i = \widetilde{\Sigma}(v_i)$ corresponding to vertices v_i, with orientations chosen so that for $i = 1, 2$, the set $E^+(\widetilde{\Sigma}_i)$ is the set of ends of the component of $\widetilde{M} - \widetilde{\Sigma}_i$ that does not contain $\widetilde{\Sigma}_0$. We can orient $\widetilde{\Sigma}_0$ so that $E^+(\widetilde{\Sigma}_0) = E^+(\widetilde{\Sigma}_1) \cup E^+(\widetilde{\Sigma}_2)$.

Lemma 8.5. *For some sign ε, $E^{\varepsilon}(A) \supset E^+(\widetilde{\Sigma}_1)$ and $E^{\bar{\varepsilon}}(A) \supset E^+(\widetilde{\Sigma}_2)$.*

Proof. First note that for each i, $i = 1, 2$, $E^+(\widetilde{\Sigma}_i) \cap E^{\infty}(A) = \phi$. For, if $E^+(\widetilde{\Sigma}_i) \cap E^{\infty}(A) \neq \phi$, then $E^+(\widetilde{\Sigma}_i) \cap (E^{\pm}(A) \cup E^{\infty}(A)) \neq \phi$. As $E^-(\widetilde{\Sigma}_0) \subset E^-(\widetilde{\Sigma}_i)$ and A crosses $\widetilde{\Sigma}_0$, we have $E^-(\widetilde{\Sigma}_i) \cap (E^{\pm}(A) \cup E^{\infty}(A)) \neq \phi$. This implies that A crosses $\widetilde{\Sigma}_i$, which is a contradiction. Thus, $E^+(\widetilde{\Sigma}_i) \cap E^{\infty}(A) = \phi$.

As A does not cross the spheres $\widetilde{\Sigma}_i$, for appropriate signs ε_i, $(E^{\varepsilon_i}(A) \cup E^{\infty}(A)) \cap E^+(\widetilde{\Sigma}_i) = \phi$. Then, we have $E^+(\widetilde{\Sigma}_i) \subset E^{\bar{\varepsilon_i}}(A)$, for $i = 1, 2$. Finally, if $\varepsilon_1 = \varepsilon_2 = \varepsilon$, then $E^{\bar{\varepsilon}}(A) \supset E^+(\widetilde{\Sigma}_0)$ as $E^+(\widetilde{\Sigma}_0) = E^+(\widetilde{\Sigma}_1) \cup E^+(\widetilde{\Sigma}_2)$. As $E^{\infty}(A) \cap E^+(\widetilde{\Sigma}_0) = \phi$,

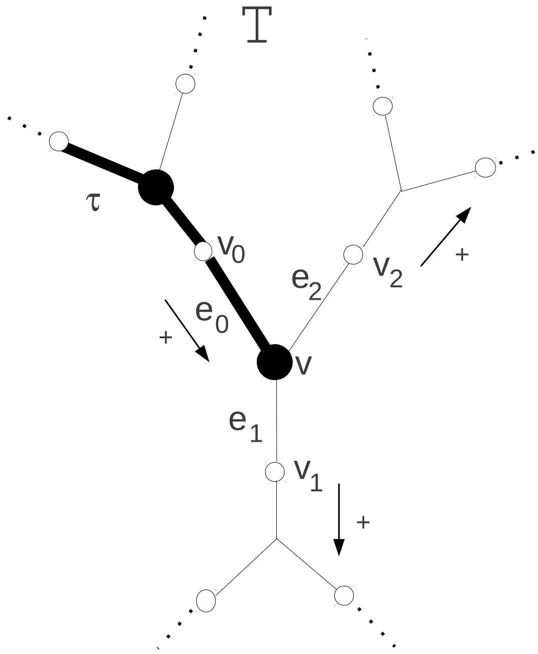

FIGURE 4

we get $E^+(\widetilde{\Sigma}_0) \cap (E^\varepsilon(A) \cup E^\infty(A)) = \phi$, contradicting the hypothesis that A crosses $\widetilde{\Sigma}_0$. Therefore, $\varepsilon_1 \neq \varepsilon_2$. Hence the result. □

Thus, one of the spheres $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ is on the positive side of A and the other on the negative side. In the case of a vertex v of valence 2 of τ, either it is a bivalent vertex (standard vertex) of T or there is an edge e_v of T adjacent to v which is not in τ. The standard sphere $\widetilde{\Sigma}(e_v)$ corresponding to the other end vertex of the edge e_v is either on the positive side of A or on the negative side.

Let $N(\tau)$ be the subgraph of T consisting of points with distance at most 1 from τ. Then, $N(\tau)$ is a tree, which is the union of τ with the following two kinds of edges:

(1) For each terminal vertex v of τ, we have a pair of edges $e_1(v) \notin \tau$ and $e_2(v) \notin \tau$ with v as an end-vertex. Let v_1 and v_2 be the other end vertices of e_1 and e_2, respectively.
(2) For each non-standard bivalent vertex w of τ, we have an edge $e(w) \notin \tau$ with w as an end-vertex. Let w_1 be its other end vertex.

By Lemma 8.5, for a terminal vertex v, the sphere corresponding to one of v_1 and v_2 is on the positive side of τ (positive side of A). The vertices v_1 and v_2 are end vertices of e_1 and e_2 respectively. So, we can assign positive or negative signs to these edges accordingly. We denote this by $e_+(v)$ and denote the other edge (which is on the negative side) by $e_-(v)$. We denote the standard spheres corresponding to v_1 and v_2 by $\widetilde{\Sigma}(v_1) = \widetilde{\Sigma}(e_1)$ and $\widetilde{\Sigma}(v_2) = \widetilde{\Sigma}(e_2)$, respectively. For a non-standard bivalent vertex w of τ, we can associate a sign $\epsilon(w)$ so that $\widetilde{\Sigma}(w_1) = \widetilde{\Sigma}(e(w))$ is on the $\epsilon(w)$-side of A. Thus, we have a triple $\sigma = (\tau, \epsilon, e_+)$ which is geosphere in $\widetilde{M}$.

8.2. **Partitions correspond to Geospheres.** Now we shall show that σ gives the partition A of $E(\widetilde{M})$.

Lemma 8.6. *The partition $(E^+(\sigma), E^-(\sigma), E^\infty(\sigma))$ of $E(\widetilde{M})$ given by the geosphere σ is the same as the partition A of $E(\widetilde{M})$.*

Proof. Let $P \in E^+(A)$. As $E^+(A)$ is open in the space of ends of T, there is a finite connected tree $\kappa \subset T$ and a component V of $T - \kappa$ so that $P \in E(V) \subset E^+(A)$. We shall show that no edge of V is contained in τ. Let e be an edge of T contained in $V = T - \kappa$. Then, as κ is connected, some component W of $T - e$ is disjoint from κ, and hence contained in V. Suppose v is the end vertex of e such that v is a standard bivalent vertex in T. Let $\Sigma(v)$ be the standard sphere corresponding to v, then it follows that for some sign ε, $E^\varepsilon(\Sigma(v)) \subset E(V) \subset E^+(A)$, and hence, $\Sigma(v)$ does not cross A. This implies v is not in τ. It follows that e is not in τ. Thus, no edge of V is in τ, as required.

Let W_0 be the component of $T - \tau$ that contains V. Then, the closure of W_0 intersects τ in a single vertex, which is either a terminal vertex or a non-standard bivalent vertex. In either case, $E(W_0) \subset E^+(\sigma)$ by construction of the partition associated to a geosphere. Then, as $P \in E(V) \subset E(W_0)$, $P \in E^+(\sigma)$. Thus, $E^\pm \subset E^\pm(\sigma)$.

We next show that $E^\infty(A) \subset E^\infty(\sigma)$. Let $P \in E^\infty(A)$. Suppose $E^\infty(A)$ has another point Q, we consider the geodesic $\gamma \subset T$ from P to Q. Given any edge e of γ, if $\Sigma(e)$ is the standard sphere corresponding to the standard vertex of e oriented appropriately, then $P \in E^-(\Sigma(e))$ and $Q \in E^+(\Sigma(e))$. Hence, $\Sigma(e)$ crosses the given partition A, so $v \in \tau$ and hence, $e \in \tau$. Thus, $\gamma \subset \tau$ and hence $P \in E^\infty(\tau)$.

On the other hand, if P is the only point in $E^\infty(A)$, then there are points $Q^\pm \in E^\pm(A)$. Let α be the geodesic from Q^- to Q^+ and let γ be the unique geodesic ray from a point of α to P with the property that its interior is disjoint from α. Given any edge e of γ, if $\Sigma(e)$ is the standard sphere corresponding to the standard vertex of e oriented appropriately, then $P \in E^-(\Sigma(e))$ and $Q^\pm \in E^+(\Sigma(e))$. Hence, $\Sigma(e)$ crosses the given partition A, so $e \in \tau$. Thus, $\gamma \subset \tau$ and hence, $P \in E^\infty(\tau)$.

This shows that $E^\infty(A) \subset E^\infty(\sigma)$. Thus, as $(E^+(\sigma), E^-(\sigma), E^\infty(\sigma))$ and A form partitions of $E(\widetilde{M})$, both are the same. ☐

This completes the existence part of the proof of Theorem 8.1. In the degenerate case, which corresponds to A not crossing any standard sphere, uniqueness follows as different standard spheres clearly give different partitions. In the non-degenerate case, it follows from Lemma 8.2 that if $\sigma = (\tau, \epsilon, e_+)$ corresponds to A, then τ is determined by A. Further, ϵ and e_+ are also determined by A, as they are determined by whether the set of ends of a given component of $T - \tau$ is contained in $E^+(A)$ or in $E^-(A)$. Thus A uniquely determines σ. ☐

9. The Topology in terms of Partitions

In Section 8, we saw that a geosphere σ can be defined as the triple (E^+, E^-, E^∞), where the sets $E^\pm \subset E(\widetilde{M})$ are open, E^∞ is closed and these sets form a partition of $E(\widetilde{M})$. Now, we define an appropriate topology on the set $\mathcal{GS}(\widetilde{M})$ of geospheres, viewed as partitions, and accordingly define topology on the set $\mathcal{L}(M)$ of geosphere laminations where geospheres are partitions. We shall show that the spaces $L(\widetilde{M})$ and $\mathcal{L}(\widetilde{M})$ are homeomorphic.

9.1. **Topology on the set $\mathcal{GS}(\widetilde{M})$.** Let $\sigma_i = (E_i^+, E_i^-, E_i^\infty)$, $i = 1, 2$, be two geospheres in $\mathcal{GS}(\widetilde{M})$. Let U be a basic open subset of the space $E(\widetilde{M})$ of the ends of $\widetilde{M}$.

Definition 9.1. We say σ_1 is U-equivalent to σ_2 if

(1) $E_1^\pm \cap U$ is non-empty if and only if $E_2^\pm \cap U$ is non-empty and,
(2) $(E_1^\pm \cup E_1^\infty) \cap U$ is non-empty if and only if $(E_2^\pm \cup E_2^\infty) \cap U$ is non-empty.

This is clearly an equivalence relation for each open set U, which we shall call U-equivalence. Let $U(\sigma, U) \subset \mathcal{GS}(\widetilde{M})$ be the U-equivalence class of σ. We consider the topology on geospheres with sub-basis the set of all U-equivalent subsets $\mathcal{U}(\sigma, U)$ of $\mathcal{GS}(\widetilde{M})$, for all $\sigma \in \mathcal{GS}(\widetilde{M})$ and for all basic open subsets U of $E(\widetilde{M})$. Thus, a basic open subset of $\mathcal{GS}(\widetilde{M})$ is a finite intersection of U-equivalent subsets of $\mathcal{GS}(\widetilde{M})$.

Now, we shall show that the space $\mathcal{GS}(\widetilde{M})$ with the topology defined above is homeomorphic to the space $GS(\widetilde{M})$. The identity map gives a bijective correspondence $\theta : \mathcal{GS}(\widetilde{M}) \to GS(\widetilde{M})$. We shall show that this correspondence is a homeomorphism.

Theorem 9.2. *The space $\mathcal{GS}(\widetilde{M})$ is homeomorphic to the space $GS(\widetilde{M})$.*

Proof. Firstly, we shall show that θ is continuous. Let $\sigma_n \in GS(\widetilde{M})$ and let κ be a finite subtree of T. Consider the basic open set $\mathcal{U}(\sigma_n, \kappa)$. Let σ_p be the geosphere in $\mathcal{GS}(\widetilde{M})$ corresponding to σ_n. We shall show that there exists a basic neighborhood of σ_p in $\mathcal{GS}(\widetilde{M})$ whose image under θ is contained in $\mathcal{U}(\sigma_n, \kappa)$.

Let $res_\kappa(\sigma_n) = (\tau, \epsilon, e_+)$. Note that $\tau \subset N(\kappa)$. Consider $T \backslash \tau$. It has only finitely many components W_i. Let $U_i = E(W_i) \subset E(\widetilde{M})$. Consider the basic neighborhood $\cap_i \mathcal{U}(\sigma_p, U_i)$ of σ_p in $\mathcal{GS}(\widetilde{M})$. We claim that for every geosphere $\sigma'_p \in \cap_i \mathcal{U}(\sigma_p, U_i)$, the corresponding geosphere σ'_n has the same restriction to κ as the restriction to κ of σ_n. Let $res_\kappa(\sigma'_n) = (\tau', \epsilon', e'_+)$.

Let w be any vertex of $N(\kappa) \setminus \tau$, then w lies in a component, say W_{i_0} of $T \setminus \tau$. For this component, $E(W_{i_0}) = U_{i_0}$ is contained either in $E^+(\sigma_n)$ or in $E^-(\sigma_n)$. Without loss of generality, we assume $U_{i_0} \subset E^+(\sigma_n) = E^+(\sigma_p)$. In this case, we say w (and its adjacent edges) are on the positive side of τ. As $E^+(\sigma_p) \cap U_{i_0}$ is non-empty and $(E^-(\sigma_p) \cup E^\infty(\sigma_p)) \cap U_{i_0}$ is empty, $E^+(\sigma'_p) \cap U_{i_0}$ is non-empty and $(E^-(\sigma'_p) \cup E^\infty(\sigma'_p)) \cap U_{i_0}$ is empty. This shows that $U_{i_0} \subset E^+(\sigma'_p) = E^+(\sigma'_n)$ and $\tau' \cap W_{i_0}$ is empty. This imples $\tau' \subset \tau$ and w lies on the positive side of τ'. Similary, we can show that $\tau \subset \tau'$. Hence, $\tau = \tau'$. The above arguments also show that $\epsilon' = \epsilon$ and $e'_+ = e_+$. Thus, $res_\kappa(\sigma'_n) = res_\kappa(\sigma_n)$. Therefore, image under θ of $\cap_i \mathcal{U}(\sigma_p, U_i)$ is contained $\mathcal{U}(\sigma_n, \kappa)$. This shows that the map θ is continuous.

Now, we shall show that the map θ^{-1} is continuous. Let $\sigma_p \in U = \cap_{i=1}^k \mathcal{U}(\sigma_p^i, U_i)$. For each basic open set U_i, there exists a finite subtree κ_i of T such that for some component W^i of $T - \kappa_i$, $E(W^i) = U_i$. Further, if (the open set) $E^\pm(\sigma_p) \cap U_i$

is non-empty, there is a finite tree $\kappa_i^\pm$ and a component $W_i^\pm$ of $T - \kappa_i^\pm$ so that $E(W_i^\pm) \subset E^\pm(\sigma_p) \cap U_i$.

Let κ be a finite subtree of T containing the subtrees κ_i and $\kappa_i^\pm$ in its interior. Consider the basic open set $\mathcal{U}(\sigma_n, \kappa)$. Note that $\mathcal{U}(\sigma_n, \kappa) \subset \mathcal{U}(\sigma_n, \kappa_i)$ and $\mathcal{U}(\sigma_n, \kappa) \subset \mathcal{U}(\sigma_n, \kappa_i^\pm)$ whenever $E^\pm(\sigma_n) \cap U_i \neq \phi$.

Suppose $\sigma' \in U(\sigma_n, \kappa)$, then if $E^\pm(\sigma_n) \cap U_i \neq \phi$, then as $res_{\kappa_i^\pm}(\sigma') = res_{\kappa_i^\pm}(\sigma)$, it follows that $E(W_i^\pm) \subset E^\pm(\sigma') \cap U_i$ and hence $E^\pm(\sigma') \cap U_i \neq \phi$.

Next, if $(E^\pm(\sigma_p) \cup E^\infty(\sigma_p)) \cap U_i \neq \phi$, either $E^\pm(\sigma_p) \cap U_i \neq \phi$ or $E^\infty(\sigma_p) \cap U_i \neq \phi$. In the first case the claim follows as above. Thus, we may assume that $E^\infty(\sigma_p) \cap U_i \neq \phi$, and hence $\tau \cap E(W^i)$ is an infinite tree. It follows that, if τ' is the tree corresponding to $\sigma' \in U(\sigma_n, \kappa)$, then $\tau' \cap E(W^i)$ is non-trivial. Hence, either $\tau' \cap E(W^i)$ is infinite, in which case $E^\infty(\sigma') \cap U_i \neq \phi$, or $\tau' \cap E(W^i)$ has a terminal vertex, in which case $E^\pm(\sigma') \cap U_i \neq \phi$ as each terminal vertex is adjacent to both positive and negative vertices. This shows that $\sigma'_p \in U$.

Thus, θ is homeomorphism.

□

9.2. **Topology on the set $\mathcal{C}(\widetilde{M})$ and $\mathcal{L}(M)$.** Let $\mathcal{C}(\widetilde{M})$ be the collection of all closed subsets of $\mathcal{GS}(\widetilde{M})$. Let U be basic open subset of $E(\widetilde{M})$. Given two elements F_1 and F_2 of $\mathcal{C}(\widetilde{M})$, we say F_1 is U-equivalent to F_2, if each geosphere of F_1 is U-equivalent to some geosphere of F_2 and vice versa. Then, the U-equivalent subsets of $\mathcal{C}(\widetilde{M})$ form a sub-basis for a topology on $\mathcal{C}(\widetilde{M})$. Using arguments similar to those in Section 9.1, one can see that the space $\mathcal{C}(\widetilde{M})$ is homeomorphic to $C(\widetilde{M})$. One can define geosphere laminations where geospheres are the partitions in the same way as the geosphere lamination defined in the Section 6. We can restrict the topology on $\mathcal{C}(\widetilde{M})$ to the set $\mathcal{L}(M)$ of geosphere laminations with geospheres as partitions. One can easily conclude that the space $\mathcal{L}(M)$ is homeomorphic to the space $L(M)$.

Remark 9.3. From this definition of the topology, it is immediate that crossing is an open condition. Indeed half of the proof of the equivalence of the two topologies is essentially the same as the proof that crossing is an open condition.

10. Some Questions

We end with a list of questions and problems regarding geosphere laminations, the solutions to some of which should be straightforward adaptations of results for geodesic laminations while others require new insights.

(1) Given any outer automorphism φ of the free group, show that there is a lamination that is a limit of spheres and is invariant under φ.
(2) Characterise geosphere laminations that are limits of spheres.
(3) Show that an invariant lamination of a totally irreducible automorphism of a free group is *filling*, i.e., intersects every essential sphere.
(4) Is there a lamination that is filling, minimal and a limit of spheres.

    Remark 10.1. This may be obtained either by 3 or some iterated construction. One may be able to deduce infinite diameter of the sphere complex from this.

(5) Construct measured geosphere laminations and show that space is compact.

(6) Which geosphere laminations admit transverse measures. There is a corresponding characterisation for geodesic laminations.
(7) Do we have a *Cauchy inequality* in the sense of Luo-Stong (see [7]) for embedded spheres in M, (we should, but it will also be interesting if we show this false) allowing a completion of the space of spheres with intersection number to 'measured laminations'.
(8) If 7 is true, are the limiting objects geosphere laminations with transversal measure.

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