Algebraic equivalence of real divisors

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1 Introduction and results

Let X be a smooth projective variety over $\mathbb{R}$ of dimension n (that is, an integral smooth projective scheme over $\mathbb{R}$ of dimension n). We endow the set $X(\mathbb{R})$ of $\mathbb{R}$-rational points of X, regarded here as a subset of X, with the topology induced by the usual metric topology on $\mathbb{R}$ and assume that $X(\mathbb{R})$ is nonempty. Thus $X(\mathbb{R})$ is a closed n-dimensional manifold of class $\mathcal{C}^\infty$. The number of connected components of $X(\mathbb{R})$ will be denoted by $c(X)$. Let, as usual, $\mathrm{Div}(X)$ denote the group of Weil divisors on X. There is a unique group homomorphism

$$\mathrm{cl}_{\mathbb{R}} : \mathrm{Div}(X) \to H_{n-1}(X(\mathbb{R}), \mathbb{Z}/2)$$

such that for any closed subvariety V of X of dimension $n-1$ the homology class $\mathrm{cl}_{\mathbb{R}}(V)$ is determined by $V(\mathbb{R})$ if $\dim V(\mathbb{R}) = n-1$ (cf. [5] or [3, Section 11.3] for the definition of this homology class) and $\mathrm{cl}_{\mathbb{R}}(V) = 0$ if $\dim V(\mathbb{R}) \leq n-2$. Of considerable interest in real algebraic geometry is the image

$$H^{\mathrm{alg}}_{n-1}(X(\mathbb{R}), \mathbb{Z}/2) = \mathrm{cl}_{\mathbb{R}}(\mathrm{Div}(X))$$

of $\mathrm{Div}(X)$ via $\mathrm{cl}_{\mathbb{R}}$, cf. [4] for a survey of results and references to the literature. This note is concerned with the subgroup

$$D(X) = \mathrm{cl}_{\mathbb{R}}(\mathrm{Div}_a(X))$$

of $H^{\mathrm{alg}}_{n-1}(X(\mathbb{R}), \mathbb{Z}/2)$, where $\mathrm{Div}_a(X)$ is the subgroup of $\mathrm{Div}(X)$ consisting of all divisors algebraically equivalent to 0, cf. [7, Chapter 10] for the theory of algebraic equivalence. It

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has been noticed first by R. Silhol [14] that $D(X)$ plays a nontrivial role in real algebraic geometry. More recent results are contained in [1, 12]. We prove here that the dimension

$$d(X) = \dim_{\mathbb{Z}/2} D(X)$$

of the $\mathbb{Z}/2$-vector space $D(X)$ is a birational invariant (it is a trivial observation that $\dim_{\mathbb{Z}/2} H^{\mathrm{alg}}_{n-1}(X(\mathbb{R}), \mathbb{Z}/2)$ is not a birational invariant for $n \geq 2$).

Theorem 1.1 *Let X and Y be smooth projective varieties over $\mathbb{R}$ with $X(\mathbb{R})$ and $Y(\mathbb{R})$ nonempty. If X and Y are birationally equivalent over $\mathbb{R}$, then $d(X) = d(Y)$.*

The behavior of $d(-)$ will now be illustrated by several examples.

Example 1.2. Suppose $n = 1$, that is, X is a curve. Then

$$D(X) = \tilde{H}_0(X(\mathbb{R}), \mathbb{Z}/2) \text{ and } d(X) = c(X) - 1,$$

where $\tilde{H}_0(X(\mathbb{R}), \mathbb{Z}/2)$ is the reduced 0th homology group (in other words, $\tilde{H}_0(X(\mathbb{R}), \mathbb{Z}/2)$ consists precisely of these homology classes in $H_0(X(\mathbb{R}), \mathbb{Z}/2)$ that are represented by an even number of points). One argues as follows.

If p_0 and p_1 are distinct points of $X(\mathbb{R})$, then the divisor $p_1 - p_0$ is in $\mathrm{Div}_a(X)$ and hence $\mathrm{cl}_{\mathbb{R}}(p_1 - p_0)$ is in $D(X)$. Since $\mathrm{cl}_{\mathbb{R}}(p_1 - p_0)$ is represented by $\{p_0, p_1\}$, it follows that any homology class in $H_0(X(\mathbb{R}), \mathbb{Z}/2)$ represented by an even number of points belongs to $D(X)$. Hence $\tilde{H}_0(X(\mathbb{R}), \mathbb{Z}/2) \subseteq \mathrm{Div}(X)$.

On the other hand, if D is in $\mathrm{Div}_a(X)$, then $\deg D = 0$ [7, p. 180, Proposition 10.2]. Writing D in the form $D = D_1 + D_2$, where D_1 (resp. D_2) is a formal linear combination of $\mathbb{R}$-rational (resp. $\mathbb{C}$-rational) points, we obtain $\deg D_1 = 0$. Hence $\mathrm{cl}_{\mathbb{R}}(D_1)$ is represented by an even number of points of $X(\mathbb{R})$. It follows that $D(X) \subseteq \tilde{H}_0(X(\mathbb{R}), \mathbb{Z}/2)$.

We therefore have $D(X) = \tilde{H}_0(X(\mathbb{R}), \mathbb{Z}/2)$. The equality $d(X) = c(X) - 1$ is obvious since $\dim_{\mathbb{Z}/2} \tilde{H}_0(X(\mathbb{R}), \mathbb{Z}/2) = c(X) - 1$.

Example 1.3. If X is an Abelian variety over $\mathbb{R}$, then $2^{d(X)} = c(X)$ [12, Theorem 2.3]. If $X^\vee$ is the dual Abelian variety of X, then $c(X^\vee) = c(X)$ [10, Proposition 1.2] and hence

$d(X^\vee) = d(X)$. Recall that given an integer s satisfying $0 \le s \le n$, there exists an Abelian variety Y over $\mathbb{R}$ of dimension n and with $c(Y) = 2^s$ [15, p. 90].

Example 1.4. If $\dim X = 1$ and J is the Jacobian variety of X, then $c(J) = 2^{c(X)-1}$ [10, Proposition 3.2], and hence $d(X) = d(J)$ by virtue of Examples 1.2 and 1.3.

The invariant $d(-)$ is subtler than the examples above might suggest. For instance, $c(X) = 1$ (resp. $c(X) \ge 2$) does not imply $d(X) = 0$ (resp. $d(X) \ge 1$) and Example 1.4 has no obvious generalization in dimensions greater than 1.

Example 1.5. Fix a positive integer r. Given nonnegative integers k and ℓ satisfying $k + \ell \ge 1$ and $k \le r$, there exists a smooth projective surface $X^{k,\ell}$ over $\mathbb{R}$ such that

$$c(X^{k,\ell}) = k + \ell,$$

$X^{k,\ell}(\mathbb{R})$ has k (resp. ℓ) connected components diffeomorphic to the real torus $\mathbb{R}^2/\mathbb{Z}^2$ (resp. the real projective plane $\mathbb{P}^2_{\mathbb{R}}(\mathbb{R})$), and

$$d(X^{k,\ell}) = k,\ \dim_{\mathbb{Z}/2} H_1^{\mathrm{alg}}(X^{k,\ell}(\mathbb{R}), \mathbb{Z}/2) = k + \ell$$

(note $\dim_{\mathbb{Z}/2} H_1(X^{k,\ell}(\mathbb{R}), \mathbb{Z}/2) = 2k + \ell$). Furthermore, for all k and ℓ, the surfaces $X^{k,\ell}$ share the same Albanese variety and hence also the same Picard variety; if A denotes the Albanese variety and P denotes the Picard variety, then A is isomorphic to P over $\mathbb{R}$ and

$$d(A) = d(P) = r.$$

In addition, the surfaces $X^{k,\ell}_{\mathbb{C}} = X^{k,\ell} \times_{\mathbb{R}} \mathbb{C}$ are pairwise birationally equivalent over $\mathbb{C}$. It is explained in Section 3, Example 3.3 how to construct $X^{k,\ell}$ with such properties.

If X is as at the beginning of the section and $\dim X = 2$, the intersection product

$$H_1(X(\mathbb{R}), \mathbb{Z}/2) \times H_1(X(\mathbb{R}), \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

$$(\alpha, \beta) \longrightarrow \alpha \cdot \beta$$

makes of $H_1(X(\mathbb{R}), \mathbb{Z}/2)$ an inner product space, cf. [13]. For any $\mathbb{Z}/2$-vector subspace N of $H_1(X(\mathbb{R}), \mathbb{Z}/2)$, let $N^\perp$ denote the orthogonal complement of N:

$$N^\perp = \{\alpha \in H_1(X(\mathbb{R}), \mathbb{Z}/2) \mid \alpha \cdot \beta = 0 \text{ for all } \beta \text{ in } N\}.$$

Proposition 1.6 *Let X be a smooth projective surface over $\mathbb{R}$ with $X(\mathbb{R})$ nonempty. If $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ is a ruled surface defined over $\mathbb{C}$, then*

$$H_1^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = D(X)^{\perp}.$$

We use here the definition of ruled surfaces (defined over $\mathbb{C}$) given in [2]; sometimes such surfaces are called birationally ruled. In Proposition 1.6 the groups $D(X)$ and $H_1^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ determine each other since, obviously,

$$D(X) = H_1^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)^{\perp}.$$

Let us mention that the surface $X^{k,\ell}$ in Example 1.5 can be chosen in such a way that $X_{\mathbb{C}}^{k,\ell}$ is ruled, cf. Example 3.3.

Theorem 1.1 and Proposition 1.6 are proved in Section 2 and Section 3, respectively. A method for computing the groups that appear in Proposition 1.6 is also described in Section 3.

2 Proof of Theorem 1.1

Let X be as at the beginning of Section 1. It will be more convenient to work with

$$H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = P^{-1}(H_{n-1}^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)),$$
$$D^1(X) = P^{-1}(D(X)),$$

where $P : H^1(X(\mathbb{R}), \mathbb{Z}/2) \to H_{n-1}(X(\mathbb{R}), \mathbb{Z}/2)$ is the Poincaré duality isomorphism. Clearly, $d(X) = \dim_{\mathbb{Z}/2} D^1(X)$.

Given an invertible sheaf $\mathcal{L}$ on X, we denote by $\mathcal{L}(\mathbb{R})$ the topological real line bundle on $X(\mathbb{R})$ determined by $\mathcal{L}$ in the usual way. The correspondence which assigns to $\mathcal{L}$ the first Stiefel-Whitney class $w_1(\mathcal{L}(\mathbb{R}))$ of $\mathcal{L}(\mathbb{R})$ gives rise to a canonical homomorphism

$$w_1 : \mathrm{Pic}(X) \to H^1(X(\mathbb{R}), \mathbb{Z}/2)$$

defined on the Picard group of isomorphism classes of invertible sheaves on X. If $\mathcal{O}(D)$ is the invertible sheaf associated with a divisor D on X, then $P(w_1(\mathcal{O}(D)(\mathbb{R}))) = \mathrm{cl}_{\mathbb{R}}(\mathrm{D})$ [5, p. 498]. Since every invertible sheaf on X is isomorphic to $\mathcal{O}(D)$ for some D, we have

$$H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = w_1(\mathrm{Pic}(X)),$$
$$D^1(X) = w_1(\mathrm{Pic}^0(X)),$$

where $\mathrm{Pic}^0(X)$ is the subgroup of $\mathrm{Pic}(X)$ consisting of the isomorphism classes of invertible sheaves of the form $\mathcal{O}(D)$ for D in $\mathrm{Div}_a(X)$.

Given a morphism $f : X \to Y$ over $\mathbb{R}$ into a smooth projective variety Y over $\mathbb{R}$, we denote by $f(\mathbb{R}) : X(\mathbb{R}) \to Y(\mathbb{R})$ the restriction of f. Thus $f(\mathbb{R})$ is a $\mathcal{C}^\infty$ map. One readily checks that the diagram

$$\begin{array}{ccc} \mathrm{Pic}(Y) & \xrightarrow{f^*} & \mathrm{Pic}(X) \\ {\scriptstyle w_1}\downarrow & & \downarrow{\scriptstyle w_1} \\ H^1(Y(\mathbb{R}), \mathbb{Z}/2) & \xrightarrow{f(\mathbb{R})^*} & H^1(X(\mathbb{R}), \mathbb{Z}/2), \end{array}$$

where f^* and $f(\mathbb{R})^*$ are the induced homomorphisms, is commutative. Hence

$$f(\mathbb{R})^*(H^1_{\mathrm{alg}}(Y(\mathbb{R}), \mathbb{Z}/2)) \subseteq H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2).$$

Since $f^*(\mathrm{Pic}^0(Y)) \subseteq \mathrm{Pic}^0(X)$, we also obtain

$$f(\mathbb{R})^*(D^1(Y)) \subseteq D^1(X).$$

Let us recall certain facts, which are special cases of a much more general theory [9] (note that $X(\mathbb{R})$ nonempty is assumed). The homomorphism $p^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbb{C}})$, induced by the canonical projection $p : X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C} \to X$, is injective and $p^*(\mathrm{Pic}(X)) = \mathrm{Pic}(X_{\mathbb{C}})^G$, where $\mathrm{Pic}(X_{\mathbb{C}})^G$ is the subgroup of $\mathrm{Pic}(X_{\mathbb{C}})$ of all elements fixed under the usual action of the Galois group $G = \mathrm{Gal}(\mathbb{C}|\mathbb{R})$ on $\mathrm{Pic}(X_{\mathbb{C}})$. Furthermore, the action of G on $\mathrm{Pic}(X_{\mathbb{C}})$ restricts to the action on $\mathrm{Pic}^0(X_{\mathbb{C}})$ and $p^*(\mathrm{Pic}^0(X)) = \mathrm{Pic}^0(X_{\mathbb{C}})^G$. The group $\mathrm{Pic}(X)$ (resp. $\mathrm{Pic}^0(X)$) is often identified via p^* with $\mathrm{Pic}(X_{\mathbb{C}})^G$ (resp. $\mathrm{Pic}^0(X_{\mathbb{C}})^G$).

Lemma 2.1 *Let A be the Albanese variety of X and let $\alpha : X \to A$ be the Albanese morphism corresponding to some point in $X(\mathbb{R})$. Then*

$$\alpha(\mathbb{R})^*(D^1(A)) = D^1(X).$$

In particular, $d(X) \leq d(A)$.

Proof. It is well-known that the induced homomorphism $\alpha^* : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$ is an isomorphism. (The reader accustomed to working with varieties over $\mathbb{C}$ may first extend the base field to $\mathbb{C} : \alpha_{\mathbb{C}} : X_{\mathbb{C}} \to A_{\mathbb{C}}$. Then $\alpha_{\mathbb{C}}^* : \mathrm{Pic}^0(A_{\mathbb{C}}) \to \mathrm{Pic}^0(X_{\mathbb{C}})$ is an isomorphism [8, p. 332, p. 462], which is obviously G-invariant. Hence $\alpha_{\mathbb{C}}^* : \mathrm{Pic}^0(A_{\mathbb{C}})^G \to \mathrm{Pic}^0(X_{\mathbb{C}})^G$ is an isomorphism corresponding to $\alpha^* : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$ via the identification described immediately before the proof.) In the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^0(A) & \xrightarrow{\alpha^*} & \mathrm{Pic}^0(X) \\ {\scriptstyle w_1}\downarrow & & \downarrow{\scriptstyle w_1} \\ D^1(A) & \xrightarrow{\alpha(\mathbb{R})^*} & D^1(X) \end{array}$$

α^* is an isomorphism and the vertical arrows are surjective. Hence $\alpha(\mathbb{R})^* : D^1(A) \to D^1(X)$ is also surjective, as required. The inequality $d(X) \leq d(A)$ is now obvious. □

If $\dim X = 1$ in Lemma 2.1, then $\alpha(\mathbb{R})^* : D^1(A) \to D^1(X)$ is an isomorphism by virtue of Example 1.4. This, however, is no longer true in higher dimensions, cf. Examples 1.5 and 3.3.

Proof of Theorem 1.1. Since X and Y are birationally equivalent over $\mathbb{R}$, it follows from Hironaka's theorem on resolution of points of indeterminacy [11] that there exist a smooth projective variety V over $\mathbb{R}$ and two birational morphisms $f : V \to X$ and $g : V \to Y$ over $\mathbb{R}$.

We claim that $f(\mathbb{R})^*(D^1(X)) = D^1(V)$. Indeed, let A be the Albanese variety of X and let $\alpha : X \to A$ be the Albanese morphism corresponding to $f(p)$, where p is a point in $V(\mathbb{R})$. Then A is also the Albanese variety of V and $\alpha \circ f : V \to A$ is the Albanese morphism corresponding to p. By Lemma 2.1, we have $\alpha(\mathbb{R})^*(D^1(A)) = D^1(X)$ and

$((\alpha \circ f)(\mathbb{R}))^*(D^1(A)) = D^1(V)$. Hence in view of $((\alpha \circ f)(\mathbb{R}))^* = (\alpha(\mathbb{R}) \circ f(\mathbb{R}))^* = f(\mathbb{R})^* \circ \alpha(\mathbb{R})^*$, we obtain

$$f(\mathbb{R})^*(D^1(X)) = f(\mathbb{R})^*(\alpha(\mathbb{R})^*(D^1(A))) = ((\alpha \circ f)(\mathbb{R}))^*(D^1(A)) = D^1(V),$$

as claimed.

Note that the homomorphism

$$f(\mathbb{R})^* : H^1(X(\mathbb{R}), \mathbb{R}/2) \to H^1(V(\mathbb{R}), \mathbb{Z}/2)$$

is injective, the map $f(\mathbb{R}) : V(\mathbb{R}) \to X(\mathbb{R})$ having topological degree modulo 2 equal to 1. Hence the claim implies $d(X) = d(V)$.

By summetry, we also get $d(Y) = d(V)$. Thus $d(X) = d(Y)$ and the proof is complete.□

3 Conic fibrations

Let X be a smooth projective surface over $\mathbb{R}$ with $X(\mathbb{R})$ nonempty. Recall that

$$H^1(X(\mathbb{R}), \mathbb{Z}/2) \times H^1(X(\mathbb{R}), \mathbb{Z}/2) \to \mathbb{Z}/2$$
$$(u, v) \to < u \cup v, [X(\mathbb{R})] >$$

is a dual pairing [6, p. 300, Proposition 8.13] (standard notation is used: $\cup$ and $< -, - >$ denote the cup product and scalar product, respectively, and $[X(\mathbb{R})]$ stands for the fundamental class of $X(\mathbb{R})$ in $H_2(X(\mathbb{R}), \mathbb{Z}/2)$). Of course, this dual pairing corresponds via Poincaré duality to the intersection product considered in Section 1. More precisely, the Poincaré duality $H^1(X(\mathbb{R}), \mathbb{Z}/2) \to H_1(X(\mathbb{R}), \mathbb{Z}/2)$ is an isomorphism of inner product spaces. For any $\mathbb{Z}/2$-vector subspace N of $H^1(X(\mathbb{R}), \mathbb{Z}/2)$, let $N^\perp$ denote its orthogonal complement:

$$N^\perp = \{u \in H^1(X(\mathbb{R}), \mathbb{Z}/2) \mid < u \cup v, [X(\mathbb{R})] >= 0 \text{ for all } v \text{ in } N\}.$$

By [12, Theorem 2.1], we have

$$H^1_{\text{alg}}(X(\mathbb{R}), \mathbb{Z}/2) \subseteq D^1(X)^\perp. \tag{3.1}$$

It is interesting to distinguish a class of surfaces for which (3.1) is an equality.

Let $\pi : X \to B$ be a *conic fibration* defined over $\mathbb{R}$. This means that π is a morphism over $\mathbb{R}$ from a smooth projective surface X over $\mathbb{R}$ into a smooth projective curve B over $\mathbb{R}$ such that the generic fiber of π is isomorphic to a plane conic over the function field of B (equivalently, the generic fiber of π is an integral smooth curve of genus 0). If $X(\mathbb{R})$ is nonempty (hence $B(\mathbb{R})$ is also nonempty), then a detailed topological description of the map $\pi(\mathbb{R}) : X(\mathbb{R}) \to B(\mathbb{R})$ is available [15, Chapter V]. In particular, one fully understands the induced homomorphism

$$\pi(\mathbb{R})^* : H^1(B(\mathbb{R}), \mathbb{Z}/2) \to H^1(X(\mathbb{R}), \mathbb{Z}/2),$$

which, as we shall see below, completely determines $D^1(X)$ and $H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$.

Denote by $H^1_0(B(\mathbb{R}), \mathbb{Z}/2)$ the subgroup of $H^1(B(\mathbb{R}), \mathbb{Z}/2)$ that consists of all cohomology classes z such that the homology class Poincaré dual to z belongs to $\tilde{H}_0(B(\mathbb{R}), \mathbb{Z}/2)$. By Example 1.2, $D^1(B) = H^1_0(B(\mathbb{R}), \mathbb{Z}/2)$.

Proposition 3.2 *Let $\pi : X \to B$ be a conic fibration defined over $\mathbb{R}$, with $X(\mathbb{R})$ nonempty. Let A be the Albanese variety of X. Then*

(i) $D^1(X) = \pi(\mathbb{R})^*(D^1(B)) = \pi(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2))$,

(ii) $d(X) \leq d(A) = d(B) = c(B) - 1$,

(iii) $H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = D^1(X)^{\perp} = (\pi(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)))^{\perp}$,

(iv) $D^1(X) \neq H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ *if the map $\pi(\mathbb{R}) : X(\mathbb{R}) \to B(\mathbb{R})$ is surjective.*

Proof. The morphism $\pi_{\mathbb{C}} : X_{\mathbb{C}} \to B_{\mathbb{C}}$ over $\mathbb{C}$, obtained by extending the base field to $\mathbb{C}$, is a trivial $\mathbb{P}^1_{\mathbb{C}}$-bundle over a nonempty Zariski open subset of $B_{\mathbb{C}}$ [2, Theorem III.4]. It therefore readily follows that the Albanese variety A of X is also the Jacobian variety of B. Hence we have

$$d(A) = d(B) = c(B) - 1$$

in view of Examples 1.2 and 1.4. Furthermore, if x_0 is a point in $X(\mathbb{R})$ and $\beta : B \to A$ is the Abel-Jacobi (that is, Albanese) morphism corresponding to $\pi(x_0)$, then $\beta \circ \pi : X \to A$ is the Albanese morphism corresponding to x_0. By Lemma 2.1, $\beta(\mathbb{R})^*(D^1(A)) = D^1(B)$ and $((\beta \circ \pi)(\mathbb{R}))^*(D^1(A)) = D^1(X)$. Since $((\beta \circ \pi)(\mathbb{R}))^* = (\beta(\mathbb{R}) \circ \pi(\mathbb{R}))^* = \pi(\mathbb{R})^* \circ \beta(\mathbb{R})^*$, we obtain

$$D^1(X) = \pi(\mathbb{R})^*(D^1(B)).$$

By virtue of Example 1.2,

$$D^1(B) = H^1_0(B(\mathbb{R}), \mathbb{Z}/2).$$

Hence (i) and (ii) are proved.

In order to prove (iii) and (iv) we argue as follows. Suppose that there is a commutative diagram of morphisms over $\mathbb{R}$:

(∗)
$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\rho"] \\ & B & \end{tikzcd}$$

where $p : X \to Y$ is the blowup at a closed point y of Y. Then $\rho : Y \to B$ is also a conic fibration defined over $\mathbb{R}$ and $Y(\mathbb{R})$ is nonempty. We shall consider two cases, depending on whether the point y is $\mathbb{R}$-rational or $\mathbb{C}$-rational.

If y is an $\mathbb{R}$-rational point, then

$$H^1(X(\mathbb{R}), \mathbb{Z}/2) = p(\mathbb{R})^*(H^1(Y(\mathbb{R}), \mathbb{Z}/2)) \oplus (\mathbb{Z}/2)e,$$

$$H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = p(\mathbb{R})^*(H^1_{\mathrm{alg}}(Y(\mathbb{R}), \mathbb{Z}/2)) \oplus (\mathbb{Z}/2)e,$$

where e is the cohomology class in $H^1(X(\mathbb{R}), \mathbb{Z}/2)$ Poincaré dual to the homology class in $H_1(X(\mathbb{R}), \mathbb{Z}/2)$ represented by $p(\mathbb{R})^{-1}(y)$. Note that e is in $(\pi(\mathbb{R})^*(H^1(B(\mathbb{R}), \mathbb{Z}/2)))^\perp$ and therefore also in $(\pi(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)))^\perp$. Keeping in mind $p(\mathbb{R})_*([X(\mathbb{R})]) = [Y(\mathbb{R})]$, one readily checks that the homomorphism

$$p(\mathbb{R})^* : H^1(Y(\mathbb{R}), \mathbb{Z}/2) \to H^1(X(\mathbb{R}), \mathbb{Z}/2)$$

preserves the inner product. Since $\pi(\mathbb{R})^* = (\rho(\mathbb{R}) \circ p(\mathbb{R}))^* = p(\mathbb{R})^* \circ p(\mathbb{R})^*$, we get

$$(\pi(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)))^\perp = p(\mathbb{R})^*((\rho(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)))^\perp) \oplus (\mathbb{Z}/2)e.$$

If y is a $\mathbb{C}$-rational point, then p is an isomorphism (over $\mathbb{R}$) of a neighborhood of $X(\mathbb{R})$ onto a neighborhood of $Y(\mathbb{R})$, and hence

$$p(\mathbb{R})^* : H^1(Y(\mathbb{R}), \mathbb{Z}/2) \to H^1(X(\mathbb{R}), \mathbb{Z}/2)$$

is an isomorphism of inner product spaces satisfying

$$H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = p(\mathbb{R})^*(H^1_{\mathrm{alg}}(Y(\mathbb{R}), \mathbb{Z}/2))$$
$$(\pi(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)))^{\perp} = p(\mathbb{R})^*((\rho(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)))^{\perp}).$$

In conclusion, in either case, if

$$H^1_{\mathrm{alg}}(Y(\mathbb{R}), \mathbb{Z}/2) = (\rho(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)))^{\perp},$$

then

$$H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = (\pi(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)))^{\perp}.$$

Thus, since (i) is already proved, we get that (iii) and (iv) hold for $\pi : X \to B$ whenever they hold for $\rho : Y \to B$.

It is well-known that a diagram like $(*)$ exists if and only if some fiber of π over a closed point of B contains an exceptional curve of the first kind (according to a suitable definition for varieties over $\mathbb{R}$ [15, p. 51, Definition 6.3]), and then $p : X \to Y$ is obtained by contracting this exceptional curve [15, p. 50, Proposition 6.2]. Thus for the proof of (iii) and (iv) we may assume that $\pi : X \to B$ is B-minimal, that is, no diagram of the form $(*)$ exists (or equivalently, no fiber of π over a closed point of B contains an exceptional curve of the first kind).

Assuming that $\pi : X \to B$ is B-minimal, the map $\pi(\mathbb{R}) : X(\mathbb{R}) \to B(\mathbb{R})$ has the following properties (cf. [15, p. 105, Theorem 3.4 and its proof]): Each connected component M of $X(\mathbb{R})$ is homeomorphic either to a 2-sphere or a 2-torus or a Klein bottle. Denote by $B(\mathbb{R})_M$ the connected component of $B(\mathbb{R})$ containing $\pi(\mathbb{R})(M)$. If M is homeomorphic to a 2-sphere, then $\pi(\mathbb{R})(\pi(\mathbb{R})^{-1}(B(\mathbb{R})_M)) \neq B(\mathbb{R})_M$. If M is not homeomorphic to a 2-sphere, then $M = \pi(\mathbb{R})^{-1}(B(\mathbb{R})_M)$ and $\pi(\mathbb{R}) : M \to B(\mathbb{R})_M$ is a $\mathcal{C}^\infty$ circle bundle over $B(\mathbb{R})_M$. In particular, the map $\pi(\mathbb{R}) : X(\mathbb{R}) \to B(\mathbb{R})$ is surjective if and only if it is a $\mathcal{C}^\infty$ circle bundle over $B(\mathbb{R})$, in which case there is a $\mathcal{C}^\infty$ section $s : B(\mathbb{R}) \to X(\mathbb{R})$ of the bundle.

It follows from these properties that

$$(\pi(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)))^\perp = \pi(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)) \oplus (\mathbb{Z}/2)u,$$

where $u = 0$ if $\pi(\mathbb{R}) : X(\mathbb{R}) \to B(\mathbb{R})$ is not surjective, while otherwise u is the cohomology class in $H^1(X(\mathbb{R}), \mathbb{Z}/2)$ Poincaré dual to the homology class in $H_1(X(\mathbb{R}), \mathbb{Z}/2)$ represented by $s(B(\mathbb{R}))$. Since $H^1_{\mathrm{alg}}(B(\mathbb{R}), \mathbb{Z}/2) = H^1(B(\mathbb{R}), \mathbb{Z}/2)$, we have

$$\pi(\mathbb{R})^*(H^1(B(\mathbb{R}), \mathbb{Z}/2)) \subseteq H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2).$$

Supposing for a moment that u is in $H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$, we obtain

$$(\pi(\mathbb{R})^*(H^1_0(B(\mathbb{R}), \mathbb{Z}/2)))^\perp \subseteq H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2).$$

In view of (3.1) and (already proved)(i), we conclude that (iii) and (iv) hold.

Thus in order to complete the proof it sufficies to show that u belongs to $H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$. This is obvious if $\pi(\mathbb{R}) : X(\mathbb{R}) \to B(\mathbb{R})$ is not surjective since then $u = 0$.

Suppose $\pi(\mathbb{R}) : X(\mathbb{R}) \to B(\mathbb{R})$ is surjective. We just have to demonstrate that $\pi : X \to B$ has a section (which is a morphism over $\mathbb{R}$). By definition, the generic fiber of π is isomorphic to a plane conic C over the function field $\mathbb{R}(B)$ of B. It is known that in suitably chosen coordinates C has an affine equation of the form

$$x_1^2 + x_2^2 = h,$$

where h is in $\mathbb{R}(B)$ and has no poles on $B(\mathbb{R})$ [15, p. 102, Corollary 2.7]. Let $h(\mathbb{R}) : B(\mathbb{R}) \to \mathbb{R}$ be the function determined by h. Since $\pi(\mathbb{R}) : X(\mathbb{R}) \to B(\mathbb{R})$ is surjective, it follows that $h(\mathbb{R})(b) \geq 0$ for all b in $B(\mathbb{R})$. Hence h is a sum of 2 squares in $\mathbb{R}(B)$ [3, Theorem 6.4.18]. In other words, C has an $\mathbb{R}(B)$-rational point. Consequently, $\pi : X \to B$ has a rational section (defined over $\mathbb{R}$), which automatically is a morphism. The proof is complete. $\square$

Example 3.3. We shall use Proposition 3.2 and its proof. Let r be a positive integer and let B be any smooth projective curve over $\mathbb{R}$ with $c(B) = r + 1$. Let k and ℓ be nonnegative integers satisfying $k+\ell \geq 1$ and $k \leq r$. By [15, p. 105, Lemma 3.2], there exists a B-minimal conic fibration $\rho^{k,\ell} : Y^{k,\ell} \to B$, defined over $\mathbb{R}$, such that $c(Y^{k,\ell}) = k + \ell$ and $Y^{k,\ell}(\mathbb{R})$ has k (resp. ℓ connected comoponents diffeomorphic to a 2-torus (resp. 2-sphere). Let $p : X^{k,\ell} \to Y^{k,\ell}$ be the blowup at ℓ points $y_1, \ldots, y_\ell$ such that the intersection of $\{y_1, \ldots, y_\ell\}$ with each connect component of $Y^{k,\ell}(\mathbb{R})$ diffeomorphic to a 2-sphere consists precisely of one point. Then $c(X^{k,\ell}) = k+\ell$ and $X^{k,\ell}(\mathbb{R})$ has k (resp. ℓ) connected components diffeomorphic to a 2-torus (resp. $\mathbb{P}^2_{\mathbb{R}}(\mathbb{R})$). It follows from the proof of Proposition 3.2 that

$$d(X^{k,\ell}) = k, \ \dim_{\mathbb{Z}/2} H^1_{\mathrm{alg}}(X^{k,\ell}(\mathbb{R}), \mathbb{Z}/2) = k + \ell.$$

Furthermore, the Albanese variety A of $X^{k,\ell}$ coincides with the Jacobian variety of B, and hence the Picard variety P of $X^{k,\ell}$ is isomorphic to A over $\mathbb{R}$. We have

$$d(A) = d(P) = d(B) = c(B) - 1 = r.$$

In addition, $X^{k,\ell}_{\mathbb{C}}$ is birationally equivalent over $\mathbb{C}$ to $B_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$.

Proof of Proposition 1.6. If the surface $X_{\mathbb{C}}$ over $\mathbb{C}$ is rational, then $D(X) = 0$ since $\mathrm{Pic}^0(X) = 0$. On the other hand, $H^1_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = H_1(X(\mathbb{R}), \mathbb{Z}/2)$ [15, p. 67, Theorem 3.4]. Hence the conclusion holds in this case.

If $X_{\mathbb{C}}$ is not rational, then there exist a smooth projective curve B over $\mathbb{R}$ and a conic fibration $\pi : X \to B$ defined over $\mathbb{R}$ [15, p. 98, Proposition 1.8]. Now the conclusion follows from Proposition 3.2. $\square$

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