Rankin-Cohen Type Differential Operators for Siegel Modular Forms

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Abstract

Let $\mathbb{H}_n$ be the Siegel upper half space and let F and G be automorphic forms on $\mathbb{H}_n$ of weights k and l, respectively. We give explicit examples of differential operators D acting on functions on $\mathbb{H}_n \times \mathbb{H}_n$ such that the restriction of

$$D(F(Z_1)G(Z_2))$$

to $Z = Z_1 = Z_2$ is again an automorphic form of weight $k + l + v$ on $\mathbb{H}_n$. Since the elliptic case, *i.e.* $n = 1$, has already been studied some time ago by R. Rankin and H. Cohen we call such differential operators Rankin-Cohen type operators.

We also discuss a generalisation of Rankin-Cohen type operators to vector valued differential operators.

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1 Introduction

In this paper we are concerned with the explicit construction of bilinear differential operators for Siegel modular forms mapping $M(\Gamma)_k \times M(\Gamma)_l$ to $M(\Gamma)_{k+l+v}$ for all even non-negative integers v and Γ some discrete subgroup of $\mathrm{Sp}(2n, \mathbb{R})$ with finite co-volume. It has been shown in ref. [4] that if the weights k and l are sufficiently large then there is a one-to-one correspondence between such bilinear differential operators and certain invariant pluri-harmonic polynomials (for more details see §2). Therefore, one possibility for constructing such differential operators is to construct the corresponding invariant pluri-harmonic polynomials. As we will choose exactly this way for our construction one may view our paper also as an attempt to describe certain spaces of invariant pluri-harmonic polynomials explicitly.

Some results concerning the explicit construction of bilinear differential operators for Siegel modular forms –which we will call Rankin-Cohen type operators following ref. [11]– are already known in the literature: the genus one case has been considered R. Rankin [8] and H. Cohen [2] and, more recently, the genus two case by Y. Choie and the first author [1]. Both of these results are special cases of the construction presented in this paper. We are, however, not able to give closed explicit formulas for all Rankin-Cohen type operators, *i.e.* for general values of n and v. Instead we derive a system of recursion equations which is indeed very simple to solve for any *numerical* values of n and v. In several special cases, however, we obtain explicit closed formulas. In addition we show that the image of the Rankin-Cohen type operators is, for $v > 0$, contained in the space of cusp forms. Finally, we discuss vector valued generalisations of the Rankin-Cohen type operators, *i.e.* bilinear differential operators which map two Siegel modular forms to a vector valued modular Siegel modular form that transforms under a certain representation ρ of $\mathrm{GL}(n, \mathbb{C})$.

This paper is organised as follows. In section 2, after reviewing some basic definitions and standard notations, we recall the one-to-one correspondence between invariant pluri-harmonic polynomials and covariant differential operators. In §3 we derive our main result in form of a set of recursion relations which allow to determine certain invariant pluri-harmonic polynomials $Q_{n,v}$ (Theorem 3.4). We also show that, for $v > 0$, the bilinear differential operators associated to these polynomials map two Siegel modular forms to a Siegel cusp form. Section 4 contains the proof of a uniqueness result in §2 as well as the proofs of three lemmas and our main theorem in §3 and the result about cusp forms. We then discuss in §5 several special cases of our main result explicitly: the cases $v = 2, 4$ for general n and the cases $n = 1, 2$ for general v. Finally, we generalise Rankin-Cohen type differential operators to vector valued differential operators in section 6. Here the case $n = 2$ is treated explicitly and in some detail. We conclude in section 7 with some remarks and open questions.

2 Siegel modular forms, pluri-harmonic polynomials and differential operators

In this section we recall some standard notations and review a result of the second author on the relation between invariant pluri-harmonic polynomials and differential operators (Theorem 2.3).

Let $\mathbb{H}_n$ be the space of complex symmetric $n\times n$ matrices with positive definite imaginary part and define an action of $\mathrm{Sp}(2n,\mathbb{R})$ on functions $f:\mathbb{H}_n\to\mathbb{C}$ by

$$(f|_M^k)(Z)=f(MZ)\det(CZ+D)^{-k}\qquad (M\in\mathrm{Sp}(2n,\mathbb{R}))$$

where $Z\in\mathbb{H}_n$, $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}$ with $n\times n$ matrices A,B,C,D, and, where $MZ=(AZ+B)(CZ+D)^{-1}$.

We define Siegel modular forms on some discrete subgroup $\Gamma\subset\mathrm{Sp}(2n,\mathbb{R})$ of finite co-volume (finite co-volume means that the volume of $\mathrm{Sp}(2n,\mathbb{R})/\Gamma$ is finite).

Definition 2.1 *A holomorphic function $f:\mathbb{H}_n\to\mathbb{C}$ is called a Siegel modular form of non-negative weight k on Γ if*

$$(f|_M^k)(Z)=f(Z)\qquad\text{for all } M\in\Gamma$$

(and $f(Z)$ is bounded at the cusps for $n=1$).

We denote the space of all Siegel modular forms of weight k on Γ by $M(\Gamma)_k$.

We review some notations concerning pluri-harmonic polynomials.

Definition 2.2 *Let P be a polynomial in the matrix variable $X=(x_{r,s})\in M_{n,d}$ and define*

$$\Delta_{i,j}(X)=\sum_{\nu=1}^{d}\frac{\partial^2}{\partial x_{i,\nu}\partial x_{j,\nu}}\qquad(1\le i,j\le n).$$

Then the polynomial P is called harmonic if $\sum_{i=1}^{n}\Delta_{i,i}(X)P=0$ and P is called pluri-harmonic if $\Delta_{i,j}(X)P=0$ for all $1\le i,j\le n$.

The group $\mathrm{GL}(n,\mathbb{C})\times O(d)$ acts on such polynomials by $P(X)\to P(A^tXB)$ ($A\in\mathrm{GL}(n,\mathbb{C})$, $B\in O(d)$). We will be interested in pluri-harmonic polynomials which are invariant under a subgroup of $O(d)$ and transform under a certain representation of $\mathrm{GL}(n,\mathbb{C})$.

A polynomial P of a matrix variable $X\in M_{n,d}$ is called homogeneous of weight v if $P(A^tX)=\det(A)^v\,P(X)$ for all $A\in\mathrm{GL}(n,\mathbb{C})$. It was already pointed out in ref. [6] that a polynomial $P(X)$ is pluri-harmonic if and only if $P(AX)$ is harmonic for all $A\in\mathrm{GL}(n,\mathbb{C})$. Therefore, a homogeneous polynomial of some weight v is pluri-harmonic if and only if it is harmonic.

Let d_i $(1\le i\le r)$ be natural numbers such that $d_i\ge n$ and $d_1+\ldots+d_r=d$. Define an embedding of $K=O(d_1)\times\ldots\times O(d_r)$ into $O(d)$ by

$$(B_1,\ldots,B_r)\to\begin{pmatrix}B_1&0&\ldots&\ldots&0\\0&B_2&0&\ldots&0\\\vdots&0&\ddots&&\vdots\\\vdots&\vdots&&\ddots&0\\0&0&&0&B_r\end{pmatrix}.$$

A polynomial P of a matrix variable $X\in M_{n,d}$ which is invariant under the action of K is called K-invariant. If we write $X=(X_1,\ldots,X_r)$ with $X_i\in M_{n,d_i}$ $(1\le i\le r)$ then, by virtue of H. Weyl, for each K-invariant polynomial P there exists a polynomial Q such that $P(X) = Q(X_1X_1^t, \ldots, X_rX_r^t)$. (Here we have used the assumption $d_i \geq n$ $(1 \leq i \leq r)$.) Following ref. [4] we call Q the associated polynomial (map) of P. Note that $\mathrm{GL}(n, \mathbb{C})$ acts on associated polynomials Q by mapping $Q(R_1, \ldots, R_r)$ to $Q(A^tR_1A, \ldots, A^tR_rA)$ $(A \in \mathrm{GL}(n, \mathbb{C}))$.

Let us introduce some notations for the certain spaces of polynomials. Denote the space of all homogeneous polynomials P of weight v which are K-invariant by $\mathcal{P}_{n,v}(d_1, \ldots, d_r)$. Furthermore, let $\mathcal{Q}_{n,v}(r)$ be the space of all polynomials Q in (the coefficients of) symmetric matrices $R_1, \ldots, R_r \in M_{n,n}$ such that $Q(A^tR_1A, \ldots, A^tR_rA) = \det(A)^vQ(R_1, \ldots, R_r)$ for all $A \in \mathrm{GL}(n, \mathbb{C})$. Finally, let $\mathcal{H}_{n,v}(d_1, \ldots, d_r)$ be the subspace of $\mathcal{Q}_{n,v}(r)$ consisting of elements Q such that $Q(X_1X_1^t, \ldots, X_rX_r^t)$ is pluri-harmonic for $X = (X_1, \ldots, X_r)$ where $X_i \in M_{n,d_i}$ $(1 \leq i \leq r)$. When $d_i \geq n$ then $\mathcal{H}_{n,v}(d_1, \ldots, d_r)$ is the space of polynomials associated to the invariant pluri-harmonic polynomials in $\mathcal{P}_{n,v}(d_1, \ldots, d_r)$.

Before we specialise to the case $r = 2$ on which we will concentrate in the following sections let us recall the connection between pluri-harmonic polynomials and differential operators.

For $Z = (z_{r,s}) \in \mathbb{H}_n$ we define ∂_Z to be the $n \times n$ matrix with components

$$\frac{1}{2}(1 + \delta_{r,s})\frac{\partial}{\partial z_{r,s}}$$

Let Q be a polynomial in $\mathcal{Q}_n(r)$ and set $D = Q(\partial_{Z_1}, \ldots, \partial_{Z_r})$. For fixed, even $d_i = 2k_i$ $(1 \leq i \leq r)$ and fixed non-negative v, *i.e.* $v \in \mathbb{N}_0$, we say that D satisfies the 'commutation relation' if the following condition is satisfied.

For all holomorphic functions $F(Z_1, \ldots, Z_r) : \mathbb{H}_n^r \to \mathbb{C}$ and all $\gamma \in \mathrm{Sp}(2n, \mathbb{R})$ one has

$$D\left(F(Z_1, \ldots, Z_r)\prod_{i=1}^{r}\det(CZ_i + D)^{-k_i}\right)|_{Z=Z_1=\cdots=Z_r} = DF(\gamma Z, \ldots, \gamma Z)\det(CZ + D)^{-k-v}$$

where $k = k_1 + \ldots + k_r$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $n \times n$ matrices A, B, C and D.

Using this condition we can now state (a special case of) Theorem 2 of ref. [4].

Theorem 2.3 *[4] For fixed integers $k_1, \ldots, k_r$ with $2k_i \geq n$ and a non-negative integer v the above commutation relation is satisfied if and only if $Q \in \mathcal{H}_{n,v}(2k_1, \ldots, 2k_r)$. In particular, let $F_i : \mathbb{H}_n \to \mathbb{C}$ $(1 \leq i \leq r)$ be Siegel modular forms on some discrete subgroup $\Gamma \subset \mathrm{Sp}(2n, \mathbb{R})$ of finite co-volume with weights $2k_i \geq n$, respectively. Let Q be in $\mathcal{H}_{n,v}(2k_1, \ldots, 2k_r)$ and set $D = Q(\partial_{Z_1}, \ldots, \partial_{Z_r})$. Then the restriction of*

$$D\left(F_1(Z_1)\cdots F_n(Z_r)\right)$$

to $Z = Z_1 = \ldots = Z_r$ is a Siegel modular form of weight $k_1 + \ldots + k_r + v$ on Γ.

From this theorem it is obvious that every element of $\mathcal{H}_{n,v}(d_1, \ldots, d_r)$ defines a r-linear covariant differential operator and vice versa. Here a differential operator D is called covariant if D satisfies the commutation relation above. In the rest of this paper we will be interested in the case of bilinear differential operators, *i.e.* the case $r = 2$.

Finally, let us recall a result on the dimension of $\mathcal{H}_{n,v}(d_1, d_2)$ which was already stated in ref. [4].

Proposition 2.4 *[4] For $d_1 \geq n$ and $d_2 \geq n$ one has $\dim(\mathcal{H}_{n,v}(d_1, d_2)) = 1$ for all non-negative, even v.*

Proof. The proof for the case $d_1, d_2 \geq 2n$ is already contained in ref. [4]; we give the complete proof in section 4.

In the next section we will, for $d_1, d_2 \geq n$, give a (quite explicit) description of $\mathcal{H}_{n,v}(d_1, d_2)$.

3 Explicit description of $\mathcal{H}_{n,v}(d_1, d_2)$

In this section we will obtain a description of $\mathcal{H}_{n,v}(d_1, d_2)$ for $d_1 \geq n$ and $d_2 \geq n$. As $\mathcal{H}_{n,v}(d_1, d_2)$ is one dimensional in this case we only have to find a non-zero element in this space. To find such an element we study the structure of $\mathcal{Q}_n(2)$ first. More precisely, we show that $\mathcal{Q}_n(2)$ is freely generated by certain homogeneous polynomials P_α $(0 \leq \alpha \leq n)$ of weight 2 (Lemma 3.1). Using this explicit description of $\mathcal{Q}_n(2)$ we can write an element $Q_{n,v} \in \mathcal{H}_{n,v}(d_1, d_2)$ as a linear combination of monomials in the P_α's. We then calculate the Laplacian of $Q_{n,v}$ and find a set of recursion relations for the coefficients in the linear combination expressing $Q_{n,v}$ in terms of the P_α's (Theorem 3.4). These recursion relations allow to solve uniquely (up to multiplication by a scalar) for the coefficients. In the case of general n and v, however, we do not obtain explicit closed formulas for the coefficients; several special cases where we find such formulas are discussed in §5.

In order to present our line of arguments in a transparent way we will postpone the proofs of various lemmas and our main theorem to the next section.

We can give $\mathcal{Q}_n(2)$ (and therefore also each $\mathcal{P}_{n,v}(d_1, d_2)$) explicitly as follows.

Firstly, it is clear that $\mathcal{Q}_{n,v}(2) = 0$ if v is odd. Indeed, any $Q(R, R') \in \mathcal{Q}_{n,v}(2)$ is determined by its values for $R = \mathbb{1}_n$ and diagonal matrices R'. Taking $A = (a_{i,j})$ as $a_{1,1} = -1$, $a_{i,i} = 1$ for $i \neq 1$ and $a_{i,j} = 0$ for all $i \neq j$, we get $\det(A) = -1$ and $A^t R A = R$, $A^t R' A = R'$ for the above R and R'. Hence, if v is odd, then $Q(R, R') = (-1)^v Q(R, R')$ and $Q = 0$.

Secondly, we give some typical elements of $\mathcal{Q}_{n,2}(2)$. For $0 \leq \alpha \leq n$ denote by P_α the polynomial in the matrix variables $R \in M_{n,n}$ and $R' \in M_{n,n}$ defined by

$$\det(R + \lambda R') = \sum_{\alpha=0}^{n} P_\alpha(R, R')\ \lambda^\alpha.$$

Then, each P_α obviously belongs to $\mathcal{Q}_{n,2}(2)$ and, therefore, all polynomials in the P_α belong to $\mathcal{Q}_n(2)$.

Thirdly, we show that the converse also holds true so that we obtain an explicit description of $\mathcal{Q}_n(2)$.

Lemma 3.1 *The ring $\mathcal{Q}_n(2)$ is generated by the algebraically independent polynomials $P_0, \ldots, P_n$.*

Now let $Q_{n,2v}$ be an element of $\mathcal{H}_{n,2v}(d_1, d_2) \subset \mathcal{Q}_{n,2v}(2)$. Then, by the last lemma, $Q_{n,2v}$ can be written in the form

$$Q_{n,2v} = \sum_{a \in I_{n,2v}} C(a) \ \prod_{\alpha=0}^{n} P_\alpha^{a_\alpha}$$

where $I_{n,2v} = \{a = (a_0, \ldots, a_n) \in \mathbb{N}_0^{n+1} \mid \sum_{\alpha=0}^{n} a_\alpha = v\}$.

The definition of $\mathcal{H}_{n,2v}(d_1, d_2)$ in the last section was rather indirect as $\mathcal{H}_{n,2v}(d_1, d_2)$ was defined of the space of polynomials $Q \in \mathcal{Q}_{n,2v}(2)$ such that $Q(X_1X_1^t, X_2X_2^t)$ is pluri-harmonic for $X = (X_1, X_2)$ with $X_i \in M_{n,d_i}$ $(1 \leq i \leq 2)$. One can, however, also give a direct definition as follows.

Define two differential operators $L_{i,j}^{(d_1)}$ and $L'^{(d_2)}_{i,j}$ acting on elements of $\mathcal{Q}_{n,v}(2)$ by

$$\begin{aligned} L_{i,j}^{(d_1)} &= d_1(1+\delta_{ij})D_{ij} + 4R_{i,j}D_{i,i}D_{j,j} + \\ &\quad \sum_{m'\neq j, m\neq i} R_{m',m}D_{m',j}D_{m,i} + 2\sum_{m'\neq j} R_{m',i}D_{m',j}D_{i,i} + 2\sum_{m\neq i} R_{m,j}D_{j,j}D_{m,i}, \\ L'^{(d_2)}_{i,j} &= d_2(1+\delta_{ij})D'_{ij} + 4R'_{ij}D'_{ii}D'_{jj} + \\ &\quad \sum_{m'\neq j, m\neq i} R'_{m',m}D'_{m',j}D'_{m,i} + 2\sum_{m'\neq j} R'_{m',i}D'_{m',j}D'_{i,i} + 2\sum_{m\neq i} R'_{m,j}D'_{j,j}D'_{m,i} \end{aligned}$$

where we have used $R = (R_{i,j}), R' = (R'_{i,j}) \in M_{n,n}$ for the two matrix variables and $D_{i,j}$ and $D'_{i,j}$ for the differential operators

$$D_{i,j} = \frac{\partial}{\partial R_{ij}}, \qquad D'_{i,j} = \frac{\partial}{\partial R'_{ij}} \qquad (1 \leq i, j \leq n).$$

It is now easy to see that the operators $L_{i,j}^{(d_1)}$ and $L'^{(d_2)}_{i,j}$ describe the action of $\Delta_{i,j}(X)$ and $\Delta_{i,j}(X')$, respectively on associated polynomials. More precisely, let $Q \in \mathcal{Q}_{n,v}(2)$ be the associated polynomial of a polynomial $P \in \mathcal{P}_{n,v}(d_1, d_2)$, *i.e.* $Q(XX^t, X'X'^t) = P(X, X')$ with $X \in M_{n,d_1}$ and $X' \in M_{n,d_2}$, then one has

$$\begin{aligned} \Delta_{i,j}(X)P(X,X') &= (L_{i,j}^{(d_1)}Q)(XX^t, X'X'^t) \\ \Delta_{i,j}(X')P(X,X') &= (L'^{(d_2)}_{i,j}Q)(XX^t, X'X'^t). \end{aligned}$$

By the remarks in the last section a "homogeneous" polynomial is pluri-harmonic if and only if it is harmonic. Hence a necessary and sufficient condition for a polynomial $Q_{n,2v} \in \mathcal{Q}_{n,2v}(2)$ being in $\mathcal{H}_{n,2v}(d_1, d_2)$ is given by

$$\sum_{i=1}^{n} (L_{i,i}^{(d_1)} + L'^{(d_2)}_{i,i})Q_{n,2v} = 0.$$

In order to evaluate this equation we have to calculate the action of the Laplacians $L_{i,i}^{(d_1)}$ and $L'^{(d_2)}_{i,i}$ on products of P_α's. Let us first describe the action of these operators on a product of two polynomials Q and Q'

$$\begin{aligned} L_{i,i}^{(d_1)}(QQ') &= (L_{i,i}^{(d_1)}Q)Q' + Q(L_{i,i}^{(d_1)}Q') + 8(Q,Q')_{i,R}, \\ L'^{(d_2)}_{i,i}(QQ') &= (L'^{(d_2)}_{i,i}Q)Q' + Q(L'^{(d_2)}_{i,i}Q') + 8(Q,Q')_{i,R'} \end{aligned}$$

where

$$4(Q,Q')_{i,R} = R_{i,i}(D_{i,i}Q)(D_{i,i}Q') + \sum_{1\leq l\leq n} R_{l,i}(D_{l,i}Q)(D_{i,i}Q') +$$

$$
\begin{aligned}
& + \sum_{1\le m\le n} R_{m,i}(D_{i,i}Q)(D_{m,i}Q') + \sum_{1\le l,m\le n} R_{l,m}(D_{l,i}Q)(D_{m,i}Q')\,, \\
4(Q,Q')_{i,R'} \quad = \quad & R'_{i,i}(D'_{i,i}Q)(D'_{i,i}Q') + \sum_{1\le l\le n} R'_{l,i}(D'_{l,i}Q)(D'_{i,i}Q') + \\
& + \sum_{1\le m\le n} R'_{m,i}(D'_{i,i}Q)(D'_{m,i}Q') + \sum_{1\le l,m\le n} R'_{l,m}(D'_{l,i}Q)(D'_{m,i}Q').
\end{aligned}
$$

To state the corresponding formulas for $Q = P_\alpha$ and $Q' = P_\beta$ let us define polynomials $(P_{i_1,\ldots,i_g;j_1,\ldots,j_g})$ depending on two matrix variables $R, R' \in M_{n,n}$ and the variable λ as the determinant of the matrix $R+\lambda R'$ with the rows $i_1,\ldots,i_g$ and columns $j_1,\ldots,j_g$ removed. Furthermore, we denote by $(P_{i_1,\ldots,i_g;j_1,\ldots,j_g})_\alpha$ the coefficient of λ^α in $(P_{i_1,\ldots,i_g;j_1,\ldots,j_g})$.

Then one has the following result.

Lemma 3.2 *With the notations above one has*

$$
\begin{aligned}
L^{(d_1)}_{i,i} P_\alpha &= 2(d_1+1-n+\alpha)\ (P_{i;i})_\alpha \qquad (0\le\alpha\le n-1)\,, \\
L'^{(d_2)}_{i,i} P_\alpha &= 2(d_2+1-\alpha) \qquad (P_{i;i})_{\alpha-1} \qquad (1\le\alpha\le n)\,, \\
L^{(d_1)}_{i,i} P_n &= L'^{(d_2)}_{i,i} P_0 = 0\,, \\
(P_\alpha,P_\beta)_{i,R} &= P_\alpha (P_{i;i})_\beta - P_{\beta+1}(P_{i;i})_{\alpha-1} + (P_{\alpha-1},P_{\beta+1})_{i,R}\,, \\
(P_\alpha,P_\beta)_{i,R'} &= P_\beta (P_{i;i})_{\alpha-1} - P_{\alpha-1}(P_{i;i})_\beta + (P_{\alpha-1},P_{\beta+1})_{i,R'}
\end{aligned}
$$

where in the two last equations $\alpha\le\beta$ and we have set $P_\gamma = (P_{i;i})_\gamma = 0$ for $\gamma<0$ or $\gamma>n$.

Using these formulas one can now calculate $\sum_{i=1}^n (L^{(d_1)}_{i,i} + L'^{(d_2)}_{i,i})$ of any monomial in the P_α's. To be able to extract the equations which the coefficients $C(a)$ have to satisfy if $Q_{n,v}$ is pluri-harmonic we need to know that the polynomials $\sum_{i=1}^n (P_{i;i})_\beta$ are linearly independent over $\mathcal{Q}_n(2)$.

Lemma 3.3 *For any $n\ge 1$, the polynomials $\sum_{i=1}^n (P_{i;i})_\alpha$ $(0\le\alpha\le n-1)$ are linearly independent over $\mathcal{Q}_n(2)$.*

We can now use the last three lemmas to calculate the equations satisfied by the coefficients $C(a)$.

Theorem 3.4 *Let $d_1\ge n$ and $d_2\ge n$. With the notations as above the polynomial*

$$
Q_{n,2v} = \sum_{a\in I_{n,2v}} C(a) \prod_{\alpha=0}^{n} P_\alpha^{a_\alpha}
$$

is a pluri-harmonic of weight $2v$, i.e. $Q_{n,2v}$ is in $\mathcal{H}_{n,2v}(d_1,d_2)$, if the coefficients $C(a)$ satisfy

$$
\begin{aligned}
(d_1+1-n+i+2(a_i-1))a_i\ C(a) \qquad &= -(d_2-i)(a_{i+1}+1)\ C(a-e_i+e_{i+1}) \\
&\quad -2 \sum_{\substack{i<l\le l' \\ l'+l-i\le g}} \tilde{a}(i,l,l')_l\ (\tilde{a}(i,l,l')_{l'} - \delta_{l,l'})\ C(\tilde{a}(i,l,l')) \\
&\quad +2 \sum_{\substack{i<l\le l' \\ l'+l-i-1\le g}} \hat{a}(i,l,l')_l\ (\hat{a}(i,l,l')_{l'} - \delta_{l,l'})\ C(\hat{a}(i,l,l'))
\end{aligned}
$$

for all $a \in I_{n,2v}$ *such that* $i := \min\{j | a_j \neq 0\} < n$. *Here we have used* $\tilde{a}(i,l,l') = a - e_i + e_l + e_{l'} - e_{l+l'-i}$, $\hat{a}(i,l,l') = a - e_i + e_l + e_{l'} - e_{l+l'-i-1}$ *and* $e_j \in \{0,1\}^{n+1}$ *for the vector with components* $(e_j)_l = \delta_{j,l}$. *Note that in the above formula we have set* $C(a) = 0$ *if* $a_j < 0$ *for some* $0 \leq j \leq n$.

Firstly, note that the equations determine the coefficients $C(a)$ uniquely as multiples of $C((0,\ldots,0,v))$. To see this define an order for the elements of $I_{n,v}$ by saying that $a < b$ for some $a, b \in I_{n,v}$ if there exists some j with $0 \leq j \leq n$ so that $a_j < b_j$ and $a_i = b_i$ for all $0 \leq i < j$. Looking at the above recursion equations it is obvious that, for $d_1 \geq n$, they can be used to express $C(a)$ in terms of coefficients $C(b)$ with $b < a$. Hence we obtain inductively that once given $C((0,\ldots,0,v))$ all other coefficients are uniquely determined (assuming still $d_1 \geq n$). This implies that there is exactly one solution to the equations which is indeed easy to calculate for any given *numerical* values of n and v. It would of course be desirable to give closed explicit formulas for the coefficients $C(a)$ in general but we have not succeeded in doing so; several special cases where we obtain such formulas are discussed in §5.

As, by Proposition 2.4, the dimension of $\mathcal{H}_{n,2v}(d_1,d_2)$ is equal to one for $d_1, d_2 \geq n$ Theorem 3.4 gives a (more or less explicit) description of $\mathcal{H}_{n,2v}(d_1,d_2)$.

Finally, the differential operators obtained from non-constant invariant pluri-harmonic polynomials in $\mathcal{H}_{n,v}(d_1,d_2)$ map two modular forms to a cusp form.

Proposition 3.5 *Let* Γ *be a subgroup of* $\mathrm{Sp}(2n,\mathbb{Q})$ *which is commensurable with* $\mathrm{Sp}(2n,\mathbb{Z})$ *and let* F *and* G *be Siegel modular forms on* Γ *of weight* k *and* l*, respectively. Let* D *be a covariant differential operator obtained from a non-constant invariant pluri-harmonic polynomial in* $\mathcal{H}_{n,v}(2k,2l)$. *Then* $D(F(Z_1)G(Z_2))|_{Z_1=Z_2}$ *is a cusp form.*

In section 4 we will give the proofs of the Lemmas 3.1, 3.2 and 3.3 as well as the proof of Theorem 3.4 and Proposition 3.5.

4 Proofs

In this section we have collected those proofs which have not been given so far.

4.1 Proof of Proposition 2.4

Proof of Proposition 2.4. Roughly speaking, the irreducible representations of $O(d)$ are parametrised by "Young diagrams" $(f_1,\ldots,f_k)_+$ and $(f_1,\ldots,f_k)_-$ where $f_1 \geq f_2 \geq \cdots \geq f_k \geq 0$, $k = d/2$, and, where, $+$ and $-$ coincide if $f_k \neq 0$. The space of pluri-harmonic polynomials $P(X,X')$ ($(X,X') \in M_{n,d_1} \times M_{n,d_2}$) such that $P(A^tX, A^tX') = \det(A)^v P(X,X')$ gives an irreducible representation of $O(d)$ ($d = d_1 + d_2$) corresponding to the Young diagram $(v,\ldots,v)_+$ of depth n (*cf.* the notation of M. Kashiwara and M. Vergne in ref. [6]). If we take the restriction of this representation to $O(d_1) \times O(d_2)$ then $\mathcal{H}_{n,v}(d_1,d_2)$ is the subspace which corresponds to the trivial representation of $O(d_1) \times O(d_2)$. So, what we should do is to count the multiplicity of the trivial representation in the restriction of $(v,\ldots,v)_+$ to $O(d_1) \times O(d_2)$. The irreducible decomposition of a similar restriction has already been worked out by K. Koike and I. Terada in Theorem 2.5 and Corollary 2.6 on p. 115 of ref. [7] but here a subtle point is different. For the sake of simplicity let us denote by $R(O(d))$ the ring of those characters of $SO(d)$ which can be obtained as restrictions of characters of irreducible representations of $O(d)$. K. Koike and I. Terada take $\lambda_{SO(d)} \in R(O(d))$ and give the following formula for the restriction of $\lambda_{SO(d)}$ to $O(d_1) \times O(d_2)$.

$$\lambda_{SO(d)} = \sum_{\beta,\mu,\kappa,\nu} L^{\lambda}_{\beta,\mu} L^{\beta}_{2\kappa,\nu} \pi_{SO(d_1)}(\mu_{SO}) \times \pi_{SO(d_2)}(\nu_{SO})),$$

where $L^a_{b,c}$ are the so-called Littlewood-Richardson coefficients, which can be calculated explicitly in principle, where the parameters β, κ, μ, ν are any partitions or "universal characters" and, where $\pi_{SO(c)}$ is a "specialisation" homomorphism whose image is contained in $R(O(c))$ (*cf. loc. cit.*). Then what we should do is as follows. First we calculate the coefficients for those pairs μ and ν where $\pi_{SO(d_1)}(\mu)$ and $\pi_{SO(d_1)}(\nu)$ are the trivial characters.

If $d_1 > n$ and $d_2 > n$ it is not difficult to show that this occurs only when μ and ν are trivial and that the coefficient is one. This is seen as follows. There are exactly two irreducible representations of $O(c)$ whose restriction to $SO(c)$ is trivial: the trivial representation and the determinant representation. We must exclude the latter possibility. Since we are assuming $d_1 > n$ and $d_2 > n$ the fundamental theorem on invariants (*cf.* Theorem 2.9A on p. 53 of ref. [10]) implies that any $SO(d_i)$ invariant vector is also $O(d_i)$ invariant. This means in our case that if the restriction to $SO(d_i)$ is trivial then it comes from the trivial representation of $O(d_i)$. This proves our assertion for $d_1 > n$ and $d_2 > n$.

When $d_1 = n$ or $d_2 = n$ the proof is more involved. First of all, for $\lambda = (v, \ldots, v)_+$, we pick up pairs of μ and ν such that $L^{\lambda}_{\beta,\mu} L^{\beta}_{2\kappa,\nu} \neq 0$ and $\pi_{SO(d_1)}(\mu) \times \pi_{SO(d_2)}(\nu)$ is trivial. Under the assumption that $d_1 \geq n$ and $d_2 \geq n$ these pairs can be described as follows (for the sake of simplicity we set $\rho_a = (a, \ldots, a)$ with depth n).

(1) If v is even and μ and ν are trivial then the coefficient is one.

(2) If $d_1 = n$ and v is odd then the coefficient is one for $\mu = \rho_1$, ν trivial, $\beta = \rho_{v-1}$ and $\kappa = \rho_{(v-1)/2}$.

(3) If $d_2 = n$ and v is odd then the coefficient is one for μ trivial, $\nu = \rho_1$, $\beta = \lambda$ and $\kappa = \rho_{(v-1)/2}$.

(4) If $d_1 = d_2 = n$ and v is even with $v \geq 2$ then the coefficient is one for $\mu = \rho_1$, $\nu = \rho_1$, $\beta = \rho_{v-1}$ and $\kappa = \rho_{(v-2)/2}$.

These possibilities exhaust all cases giving the trivial representation of $SO(d_1) \times SO(d_2)$. We have already shown in §3 that v is even in our case so that the cases (2) and (3) do not occur.

If $d_1 > n$ or $d_2 > n$ then the pluri-harmonic polynomial in question is invariant by $O(d_1)$ or $O(d_2)$, respectively. Indeed, if this were not the case then, by the theory of invariants, it is of the form $\det(X')Q(R, R')$ ($d_2 = n$ and $X' \in M_n$) or $\det(X)Q(R, R')$ ($d_1 = n$, $X \in M_n$), respectively, where $Q \in \mathcal{Q}_{n,v-1}(2)$. But since $Q = 0$ unless $v - 1$ is even this cannot be the case.

Therefore, the only remaining case is $d_1 = d_2 = n$ and we must show that the trivial representation of $O(n) \times O(n)$ occurs exactly once. We prove this by showing that there exists exactly one $\det(g) \times \det(h)$ representation of $O(d_1) \times O(d_2)$ in our restriction. Since, for $d_1 = d_2 = n$ and v even, the multiplicity of the trivial representation of $SO(d_1) \times SO(d_2)$ is two this indeed implies that our restriction contains the trivial representation of $O(n) \times O(n)$

exactly once.

For even v assume that $P(X, X^{'})$ (X, $X^{'} \in M_n$) is a pluri-harmonic polynomial such that $P(A^tX, A^tX^{'}) = \det(A)^v P(X, X^{'})$. By the above considerations this polynomial is invariant both by $O(d_1)$ and $O(d_2)$ or odd invariant for both $O(d_1)$ and $O(d_2)$, *i.e.* $P(Xg, X^{'}h) = P(X, X^{'})$ or $P(Xg, X^{'}h) = \det(g)\det(h)P(X, X^{'})$, respectively. We show the latter case occurs just for one polynomial in question (up to multiplication by a constant). By the classical theorem of invariants (Weyl, *loc.cit.*) we find in the latter case $P(X, X^{'}) = \det(X)\det(X^{'})Q(R, R^{'})$ for some $Q \in \mathcal{Q}_{n,v-2}(2)$. By applying $\Delta_{11} = \Delta_{11}(X) + \Delta_{11}(X^{'})$ to this polynomial it is easy to see that

$$\Delta_{11}(\det(X)\det(X^{'})Q(R, R^{'})) = (\det(X)\det(X^{'}))(\Delta_{11}Q + 2(\frac{\partial Q}{\partial R_{11}} + \frac{\partial Q}{\partial R^{'}_{11}})).$$

Hence the polynomial is pluri-harmonic if and only if $\Delta_{11}Q + 2(\frac{\partial Q}{\partial R_{11}} + \frac{\partial Q}{\partial R^{'}_{11}}) = 0$. The latter action, however, is nothing but the action of $L^{(d_1+1)}_{1,1} + L^{'(d_2+1)}_{1,1}$. Since $d_1 + 1 > n$ and $d_2 + 1 > n$ we have already shown that the kernel of $L^{(d_1+1)}_{1,1} + L^{'(d_2+1)}_{1,1}$ is one dimensional. □

4.2 Proof of Lemmas 3.1-3.3, Theorem 3.4 and Proposition 3.5

This subsection contains the proofs of the three lemmas stated in section 3 which we then use to prove Theorem 3.4. We also give a proof of Proposition 3.5.

Proof of Lemma 3.1. For any polynomial $Q(R, R') \in \mathcal{Q}_{n,v}(2)$ the polynomial $Q(\mathbb{1}_n, T)$ is a polynomial of the coefficients of T where T is symmetric and $\mathbb{1}_n$ is the unit matrix of size n. Since $Q(\mathbb{1}_n, T) = Q(\mathbb{1}_n, O^{-1}TO)$ for any orthogonal matrix O the polynomial $Q(\mathbb{1}_n, T)$ is a polynomial in the functions $\mu_i(T)$ $(1 \leq i \leq n)$ defined by $\det(t\mathbb{1}_n + T) = \sum_{i=0}^n \mu_i(T)t^i$. Indeed, if T is a diagonal matrix then $Q(\mathbb{1}_n, T)$ is a symmetric function of the diagonal components and hence a function of the $\mu_i(T)$. Since $\mu_i(T)$ is invariant under conjugation of T we obtain that $Q(\mathbb{1}_n, T)$ is a polynomial in the $\mu_i(T)$. Furthermore, for $0 \leq i \leq n$, we find that $\det(R)\mu_i(T) = P_i(R, R')$ where we have set $R = A^tA$ and $T = (A^t)^{-1}R'A^{-1}$ (strictly speaking we are working here over an algebraic extension of our ring $\mathbb{C}[R_{i,j}]$ $(R = (R_{i,j}))$ which allows to find such a matrix A with $R = A^tA$). Hence, for any non-negative integers e_i $(1 \leq i \leq n)$ and $l_0 = \sum_{i=1}^n e_i$ we get

$$\det(R)^{l_0} \prod_{i=1}^{n} \mu_i(T)^{e_i} = \prod_{i=1}^{n} P_i^{e_i}(R, R').$$

Now, any linear combination of the above "monomials" for fixed l_0 and various e_i $(i \geq 1)$ is not divisible by $\det(R)$ since even if some column of R is a zero vector such a linear combination does not vanish. This is seen as follows. Choose $R' = \mathbb{1}_n$ and R as a diagonal matrix with $R_{1,1} = 0$. Then, each P_i $(1 \leq i \leq n-1)$ becomes an elementary symmetric function of the $R_{i,i}$ $(2 \leq i \leq n)$. This means that the above linear combination does not vanish identically. Furthermore, it also means that, for a fixed v, any linear combination of the "monomials" $\det(R)^l \prod_{i=1}^n P_i^{e_i}(R, R')$ with $l + \sum_{i=1}^n e_i = v$ where l is an integer and the e_i are non-negative integers is a polynomial if and only if l is positive. Since any element in

$Q_{n,v}(2)$ is a linear combination of "monomials" of the above type $Q_{n,v}(2)$ is generated by the P_α $(0 \leq \alpha \leq n)$. Furthermore, the P_i $(1 \leq i \leq n)$ are obviously algebraic independent with each other. This completes the proof of the lemma. □

Proof of Lemma 3.2. To prove the equations in Lemma 3.2 we choose without loss of generality $i = 1$.

Firstly, note that one has

$$D_{l,1} P_\alpha = (2 - \delta_{1,l})(-1)^{l+1} (P_{1;l})_\alpha \qquad (0 \leq \alpha \leq n)$$

(here we regard $(P_{1;j})_n = 0$) and

$$D_{l',1} (P_{1;l})_\alpha = (-1)^{l'} (P_{1,l';1,l})_\alpha \qquad (l' \neq 1).$$

This directly implies

$$L_{1,1}^{(d_1)} P_\alpha = 2 d_1 (P_{1;1})_\alpha - 2 \sum_{l,l' \neq 1} (-1)^{l+l'} R_{l,l'} \, (P_{1,l';1,l})_\alpha .$$

Secondly, we show that

$$\sum_{l,l' \neq 1} (-1)^{l+l'} R_{l,l'} \, (P_{1,l';1,l})_\alpha = (n - 1 - \alpha)(P_{1;1})_\alpha .$$

This obviously implies the formula stated in the lemma.

The last equality can be proven as follows. Multiplying both sides with λ^α and summing over α gives, as an equivalent equation,

$$\lambda \frac{d}{d\lambda} P_{1;1} = (n-1) \; P_{1;1} - \sum_{j,k \neq 1} (-1)^{j+k} R_{j,k} P_{1,j;1,k} .$$

Instead of proving this equation is suffices to prove the equation with $P_{1;1}$ replaced by P and $n-1$ replaced by n, *i.e.* to prove

$$\lambda \frac{d}{d\lambda} P = n \, P - \sum_{j,k=1}^{n} (-1)^{j+k} \; R_{j,k} P_{j;k} .$$

Expanding the determinants on the right hand side gives

$$\begin{aligned} & n \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} (R_{i,\sigma(i)} + \lambda R'_{i,\sigma(i)}) \\ & - \sum_{k=1}^{n} \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} (R_{i,\sigma(i)} + \lambda (1 - \delta_{i,k}) R'_{i,\sigma(i)}) \\ = \; & \sum_{k=1}^{n} \sum_{\sigma \in S_n} (-1)^\sigma \lambda R'_{k,\sigma(k)} \prod_{i \neq k} (R_{i,\sigma(i)} + \lambda R'_{i,\sigma(i)}) \end{aligned}$$

where S_n is the symmetric group of n elements. Note that the last expression on the r.h.s. is equal to $\lambda \frac{d}{d\lambda} P$ so that we have proven the desired equality.

The analogous equation for $L'^{(d_2)}_{i,i}$ follows directly from the symmetry $R \leftrightarrow R'$, $\alpha \leftrightarrow n-\alpha$.

We now prove the equation equation for $(P_\alpha, P_\beta)_{1,R}$. Using again the equality $D_{l,1}P_\alpha = (2-\delta_{1,l})(-1)^{l+1}(P_{1;l})_\alpha$ we immediately get

$$(P_\alpha, P_\beta)_{1,R} = \sum_{l,m=1}^{n} (-1)^{l+m} R_{l,m} (P_{1;l})_\alpha (P_{1;m})_\beta.$$

Furthermore, we find

$$\sum_{m=1}^{n} (-1)^{l+m} R_{l,m} (P_{1;m})_\alpha = \delta_{1,l} P_\alpha - \sum_{m=1}^{n} (-1)^{l+m} R'_{l,m} (P_{1;m})_{\alpha-1}$$

and

$$\sum_{l=1}^{n} (-1)^{l+m} R'_{l,m} (P_{1;l})_\beta = \delta_{m,1} P_{\beta+1} - \sum_{l=1}^{n} (-1)^{l+m} R_{l,m} (P_{1;l})_{\beta+1}.$$

Finally, collecting the above formulas gives the desired equation

$$\begin{aligned}
(P_\alpha, P_\beta)_{1,R} &= \sum_{l=1}^{n} \delta_{l,1} P_\alpha (P_{1;l})_\beta - \sum_{l,m=1}^{n} (-1)^{l+m} R'_{l,m} (P_{1;m})_{\alpha-1} (P_{1;l})_\beta \\
&= P_\alpha (P_{1;1})_\beta \\
&\qquad - \sum_{m=1}^{n} \left(\delta_{m,1} P_{\beta+1} (P_{1;m})_{\alpha-1} - \sum_{l=1}^{n} (-1)^{l+m} R_{l,m} (P_{1;m})_{\alpha-1} (P_{1;l})_{\beta+1} \right) \\
&= P_\alpha (P_{1;1})_\beta - P_{\beta+1} (P_{1;1})_{\alpha-1} + \sum_{l,m=1}^{n} (-1)^{l+m} R_{l,m} (P_{1;m})_{\alpha-1} (P_{1;l})_{\beta+1} \\
&= P_\alpha (P_{1;1})_\beta - P_{\beta+1} (P_{1;1})_{\alpha-1} + (P_{\alpha-1}, P_{\beta+1}).
\end{aligned}$$

The corresponding equation for $(P_\alpha, P_\beta)_{1,R'}$ follows, again, directly from the symmetry $R \leftrightarrow R'$, $\alpha \leftrightarrow n-\alpha$. □

Proof of Lemma 3.3. For the sake of simplicity we denote by $(P_{i;i})^n_\beta$ the differentiation of $(P_{i;i})_\beta$ by $\frac{\partial}{\partial R_{nn}}$. (In other words, $(P_{i;i})^n_\beta$ is obtained from $(P_{i;i})_\beta$ by substituting $R_{n,j} = 0$ $(n \neq j)$, $R_{n,n} = 1$ and $R'_{n,j} = 0$ for all j with $1 \leq j \leq n$.)

For β with $0 \leq \beta \leq n-1$, we take polynomials Q_β of $n+1$ variables and assume that they satisfy

$$\sum_{\beta=0}^{n-1} Q_\beta(P_0, ..., P_n)(\sum_{i=1}^{n} (P_{i;i})_\beta) = 0.$$

We show that $Q_\beta = 0$ for all β. We set the last row (and the last column) of R' to 0 and also set $R_{n,j} = R_{j,n} = 0$ for all $j \neq n$. Then P_n becomes 0 and $(P_{n;n})_\beta$ is unchanged for any β. Furthermore, P_α becomes equal to $R_{n,n}(P_{n;n})_\alpha$ for $\alpha \leq n-1$ and $(P_{i;i})_\alpha$ becomes equal to $R_{nn}(P_{i,i})^n_\alpha$ for $i \leq n-1$. Hence we have

$$\sum_{\beta=0}^{n-1} Q_\beta(R_{n,n}(P_{n;n})_0, ..., R_{n,n}(P_{n;n})_{n-1}, 0)(\sum_{i=1}^{n-1} R_{n,n}(P_{i;i})^n_\beta + (P_{n;n})_\beta) = 0.$$

Note that $(P_{n;n})_\beta$ does not contain $R_{n,n}$ so that by comparing the degree of $R_{n,n}$ on both sides we obtain

$$\sum_{\beta=0}^{n-1} Q_\beta(R_{n,n}(P_{n;n})_0, ..., R_{n,n}(P_{n;n})_{n-1}, 0)(\sum_{i=1}^{n-1} R_{n,n}(P_{i;i})_\beta^n) = 0.$$

Now we set $R_{n,n} = 1$ and use induction by n. Since for $n = 2$ the lemma is easily verified we can assume that the lemma holds for $n-1$. Then the above induction step gives nothing but the relation for $n-1$ and we obtain $Q_\beta(X_0, ..., X_{n-1}, 0) = 0$. Therefore, all Q_β are divisible by X_n. By repeating this process we find that $Q_\beta = 0$. □

Proof of Theorem 3.4. Using Lemma 3.2 we can calculate the Laplacian of $Q_{n,v}$ and obtain, since the $\sum_{i=1}^n (P_{i;i})_\alpha$ $(0 \le \alpha \le n-1)$ are linearly independent over $\mathcal{Q}_n(2)$ by Lemma 3.3, the following set of equations for the coefficients $C(a)$

$$\begin{aligned}(d_1+1-g+i)a_iC(a) \quad + \quad & (d_2-i)(a_{i+1}+1)C(a-e_i+e_{i+1}) = \\ & -2\sum_{l\le l'\le i} \tilde{a}(i,l,l')_l\ (\tilde{a}(i,l,l')_{l'} - \delta_{l,l'})\ C(\tilde{a}(i,l,l')) \\ & -2\sum_{i\le l\le l'} \hat{a}(i,l,l')_l\ (\hat{a}(i,l,l')_{l'} - \delta_{l,l'})\ C(\hat{a}(i,l,l')) \\ & +2\sum_{i< l\le l'} \tilde{a}(i,l,l')_l\ (\tilde{a}(i,l,l')_{l'} - \delta_{l,l'})\ C(\tilde{a}(i,l,l')) \\ & +2\sum_{l\le l'< i} \hat{a}(i,l,l')_l\ (\hat{a}(i,l,l')_{l'} - \delta_{l,l'})\ C(\hat{a}(i,l,l'))\end{aligned}$$

for all $0 \le i \le n-1$. Choosing $i = \min\{j|a_j \ne 0\}$ gives the equations in the formulation of the theorem. (These equations are equivalent to the vanishing of coefficient in front of $(\sum_{k=1}^n (P_{k;k})_i)\prod_{j=0,n} P_j^{a_j-\delta_{i,j}}$.)

Finally, note that by Lemma 3.1, Proposition 2.4 and the remarks after Theorem 3.4 we know that the other equations, *i.e.* those with $i \ne \min\{j|a_j \ne 0\}$, do not contain any further information about the coefficients $C(a)$. □

Proof of Proposition 3.5. Firstly, we show that if R and R' are positive semi-definite symmetric real matrices and if $\det(R+R') = 0$ then $\det(R+\lambda R') = 0$ and hence $P_\alpha(R, R') = 0$ for all α with $0 \le \alpha \le n$. In general, if A and B are positive semi-definite symmetric matrices so is $A+B$ and $\det(A+B) \ge \det(A) \ge 0$. Now, let λ be any real number between 0 and 1. Then we obtain $0 = \det(R+R') \ge \det(R+\lambda R') \ge 0$ so that $\det(R+\lambda R')$ vanishes identically as $\det(R+\lambda R')$ is a polynomial in λ.

Secondly, let F and G be Siegel modular forms of weight k and l, respectively. We consider $(F|_M^k)(Z_1)$ and $(G|_M^l)(Z_2)$ for $M \in \mathrm{Sp}(2n, \mathbb{Q})$. As these are modular forms on $M^{-1}\Gamma M$ there exists some natural number N such that they have a Fourier expansion of the form

$$\begin{aligned}(F|_M^k)(Z_1) &= \sum_{T_1} a_1(T_1)\exp(2\pi i\ \mathrm{tr}(T_1Z_1/N)), \\ (G|_M^l)(Z_2) &= \sum_{T_2} a_2(T_2)\exp(2\pi i\ \mathrm{tr}(T_2Z_2/N)),\end{aligned}$$

where T_1 and T_2 run over all positive semi-definite half-integral matrices. We set $H = (F|_M^k)(Z_1)(G|_M^l)(Z_2)$. Assume now that D is obtained from an associated polynomial $Q \in \mathcal{H}_{n,v}(2k, 2l)$. Then we easily obtain

$$(DH)|_{Z_1=Z_2} = \sum_{T_1,T_2} a_1(T_1)a_2(T_2)Q(T_1/N, T_2/N)\exp(2\pi i \ \mathrm{tr}((T_1+T_2)Z_2/N)),$$

where, again, T_1 and T_2 run over all positive semi-definite half-integral matrices, and $Q(T_1/N, T_2/N) = 0$ if $\det(T_1+T_2) = 0$ by the discussion above. By Theorem 2.3 we also know that

$$(D(FG)|_{Z_1=Z_2=Z})|_M^{k+l+v} = D(H)|_{Z_1=Z_2=Z}.$$

This means that $\Phi((D(FG)|_{Z_1=Z_2=Z})|_M^{k+l+v}) = 0$ for all $M \in \mathrm{Sp}(2n,\mathbb{Q})$ where Φ is the Siegel operator and hence $D(FG)|_{Z_1=Z_2=Z}$ is a cusp form. □

5 Some explicit examples

In this section we discuss some special cases of Theorem 3.4. In particular we recover the results of ref. [2] for $n=1$ (see also the examples in ref. [4]) and ref. [1] for $n=2$. Furthermore, we give closed explicit formulas for $Q_{n,v}$ for $v = 2, 4$ and general n.

5.1 The case $v = 2$

By the discussion in section 3 we can write $Q_{n,2}$ as

$$Q_{n,2} = \sum_{\alpha=0}^{n} C(\alpha)P_\alpha.$$

The recursion equations given in Theorem 3.4 simplify in this case to

$$(d_2-\alpha)C(\alpha) = -(d_1+n-\alpha+1)C(\alpha+1) \qquad (0 \le \alpha \le n-1).$$

Hence we find that, up to a constant multiple, $Q_{n,2}$ is given by

$$Q_{n,2} = \sum_{\alpha+\beta=n} (-1)^\alpha \, \alpha! \, \beta! \binom{d_2-\alpha}{\beta}\binom{d_1-\beta}{\alpha} P_\alpha.$$

5.2 The case $v = 4$

We consider the case $v=4$ for general n. Writing

$$Q_{n,4} = \sum_{i,j=0}^{n} C_{i,j}P_iP_j$$

with $C_{i,j} = C_{j,i}$ one finds from Theorem 3.4 the following recursion equations

$$(d_1+1-n+i+2\delta_{i,j})C_{i,j} + (d_2-i+2\delta_{i<j})C_{i+1,j} = 2\sum_{r=1}^{j-i-1}(C_{i+r,j-r} - C_{i+r,j+1-r})$$

where $0 \leq i \leq j \leq n$ and $i \leq n-1$.

One can give an explicit solution to these equations [1]. The coefficients $C_{r,s}$ $(0 \leq r,\, s \leq n)$ are given by the closed formula

$$C_{r,s} = (-1)^{r-s}\,\frac{(d_1-n+r)!(d_1-n+s)!(d_2-r)!(d_2-s)!}{(d_1-n)!(d_1-n+2)!(d_2-n)!(d_2-n+2)!}\,p_{r-s}(\kappa_1(\frac{r+s}{2}),\,\kappa_2(\frac{r+s}{2}))\,,$$

where $\kappa_1(r) = d_1+2-(n-r)$, $\kappa_2(r) = d_2+2-r$ and $p_e(x,y)$ is the polynomial of degree 4 in e^2, x and y given by

$$p_e(x,y) = x^2y^2 + (e^2-1)\,xy(x+y+\frac{e^2-6}{6}) + \frac{e^2(e^2-4)}{12}\,(x^2+y^2-\frac{1}{4})\,.$$

The first few values of this polynomial are given by

$$\begin{aligned}
p_0(x,y) &= x(x-1)y(y-1)\,,\\
p_1(x,y) &= (x+\tfrac{1}{2})(x-\tfrac{1}{2})(y+\tfrac{1}{2})(y-\tfrac{1}{2})\,,\\
p_2(x,y) &= xy(xy+3x+3y-1)\,,\\
p_3(x,y) &= (x+\tfrac{1}{2})(y+\tfrac{1}{2})(xy+\tfrac{15}{2}x+\tfrac{15}{2}y-\tfrac{15}{4})\,,\\
p_4(x,y) &= x^2y^2+15x^2y+15xy^2+16x^2+25xy+16y^2-4\,.
\end{aligned}$$

To check the correctness of the formula, it is convenient to replace the recursion relation above by the simpler 4-term recurrence

$$\kappa_1(r+3)\,C_{r,s} + \kappa_2(r+2)\,C_{r+1,s} = \kappa_1(r-2)\,C_{r-1,s+1} + \kappa_2(r-3)\,C_{r,s+1}\,.$$

It is easy to verify that the solution given above indeed satisfies this recursion relation.

Note that the denominator in the formula for $C_{r,s}$ is of course just conventional. One could also make the choice $(d_1-n)!^2(d_2-n)!^2$, which gives the simpler expression

$$C_{r,s} = (-1)^{r-s}(n-1-d_1)_r(n-1-d_1)_s(n-1-d_2)_{n-r}(n-1-d_2)_{n-s}p_{r-s}(\kappa_1(\frac{r+s}{2}),\kappa_2(\frac{r+s}{2}))$$

(here we have used $(x)_n$ for $\prod_{i=0}^{n-1}(x-i)$).

5.3 The genus one case

In this case $\mathcal{Q}_1(2)$ is generated by P_0 and P_1. Writing $C_{r,s}$ for $C(a)$ with $a=(r,s)$ the recursion equations in Theorem 3.4 for $C(a)$ become

$$2(d_1/2+r-1)r\;C_{r,s} = -d_2(s+1)\;C_{r-1,s+1}+2s(s+1)\;C_{r-1,s+1} = -2(d_2/2-s)(s+1)\;C_{r-1,s+1}$$

for $1 \leq r \leq v$. Hence we find that, up to a constant multiple, $Q_{1,2v}$ is given by

$$Q_{1,2v} = \sum_{r+s=v}(-1)^r\binom{v+d_2/2-1}{r}\binom{v+d_1/2-1}{s}\,P_0^r\,P_1^s.$$

The differential operators corresponding to $Q_{1,2v}$ are precisely the operators studied by H. Cohen in ref. [2]. Note that this case has already been discussed in the context of pluri-harmonic polynomials in ref. [4].

[1] This explicit form of the solution is due to D. Zagier

5.4 The genus two case

To make contact with the results of ref. [1] we use $P_0^* = P_0, P_2^* = P_2$ and $P_1^* = P_0 + P_1 + P_2$ instead of P_0, P_1 and P_2 to generate $\mathcal{Q}_2(2)$. Writing $Q_{2,2v}$ as

$$Q_{2,2v} = \sum_{r+s+p=v} C_{r,s,p} \, (P_0^*)^r \, (P_2^*)^s \, (P_1^*)^p$$

and applying Lemma 3.2 and Lemma 3.3 we find the following two recursion relations for the coefficients $C_{r,s,p}$ (*cf.* the equations on page 13 of *loc. cit.*)

$$\begin{aligned} 0 &= (r+1)((d_1-3)/2+r+1)C_{r+1,s,p} + (p+1)((d_1+d_2-3)/2+p+1)C_{r,s,p+1}\,, \\ 0 &= (s+1)((d_2-3)/2+s+1)C_{r,s+1,p} + (p+1)((d_1+d_2-3)/2+p+1)C_{r,s,p+1}. \end{aligned}$$

Hence we find that, up to a constant multiple, $Q_{2,2v}$ is given by

$$\begin{aligned} Q_{2,2v} \quad &= \sum_{r+s+p=v} \frac{1}{r!s!p!}(d_1/2-3/2+v)_{v-r} \; (d_2/2-3/2+v)_{v-s} \\ &\qquad (-((d_1+d_2)/2-3/2+v))_{v-p} \; (P_0^*)^r (P_2^*)^s (P_1^*)^p \end{aligned}$$

where we have, again, used $(x)_n$ for $\prod_{i=0}^{n-1}(x-i)$.

Since it is obvious that

$$\begin{aligned} &\mathbb{D}^p(\mathbb{D}^r F(Z) \; \mathbb{D}^s G(Z)) = \\ &((P_1^*(\partial_{Z_1}, \partial_{Z_2}))^p \; (P_0^*(\partial_{Z_1}, \partial_{Z_2}))^r \; (P_2^*(\partial_{Z_1}, \partial_{Z_2}))^s F(Z_1)G(Z_2)) \,|_{Z=Z_1=Z_2} \end{aligned}$$

where $\mathbb{D} = \det(\partial_Z)$ we obtain exactly the formula in Theorem 1.2 of *loc. cit.*.

6 A vector valued generalisation

In this section we describe vector valued differential operators D such that

$$D(F(Z_1)G(Z_2))_{|Z_1=Z_2}$$

is a vector valued modular form if F and G are modular forms.

Let us introduce some notation first. For any representation ρ of $\mathrm{GL}(n,\mathbb{C})$ we denote by $d(\rho)$ its dimension. Furthermore, for any even positive integers d_1 and d_2, we denote by $\mathcal{H}_{n,\rho}(d_1,d_2)$ the space of $d(\rho)$ dimensional vectors of $O(d_1)\times O(d_2)$-invariant pluri-harmonic polynomials $P(X,X') = (P_i(X,X'))_{1\le i\le d(\rho)}$ such that $P(AX,AX') = \rho(A)P(X,X')$.

The main result of ref. [4] not only includes the case of invariant pluri-harmonic polynomials in $\mathcal{H}_{n,v}(d_1,d_2)$ (*c.f.* Theorem 2.3) but also applies to the case of pluri-harmonic polynomials in $\mathcal{H}_{n,\rho}(d_1,d_2)$. It is shown in *loc. cit.* that the vector valued differential operators corresponding to the latter polynomials map two modular forms to a vector valued modular form. We will, therefore, concentrate on giving some examples of pluri-harmonic polynomials in $\mathcal{H}_{n,\rho}(d_1,d_2)$.

More precisely, we will consider only the representations of $\mathrm{GL}(n,\mathbb{C})$ which are of the form $\rho_{m,v} = \det^v \mathrm{Sym}^m$ where Sym^m is the symmetric tensor representation of degree m. When $n=2$ these representations exhaust all polynomial representations of $\mathrm{GL}(n,\mathbb{C})$.

For given m and v the Young diagram corresponding to $\rho_{m,v}$ is given by $(v+m, v, ..., v)$ where the depth is n if $v \neq 0$ and 1 if $v = 0$. For the sake of simplicity, we assume from now on that $d_1 \geq 2n$ and $d_2 \geq 2n$. Then one can easily see that $\dim(\mathcal{H}_{n,\rho_{m,v}}(d_1, d_2)) = 1$ if and only if m and v are even. We want to give explicit bases of (some of) the one dimensional spaces $\mathcal{H}_{n,\rho_{m,v}}(d_1, d_2)$.

Firstly, we consider the simplest case, *i.e.* the case of the symmetric representation $\rho_{m,0}$. In this case everything can be reduced to the genus one case which was already discussed in §5.3.

Let $u_1, ..., u_n$ be n independent variables and, for any multi-index $\nu = (\nu_1, ..., \nu_n) \in \mathbb{N}_0^n$, write u^ν for $\prod_{i=1}^n u_i^{\nu_i}$ and $|\nu|$ for $\nu_1 + \cdots + \nu_n$. Denote by $I(m)$ the set of all multi-indices with $|\nu| = m$ and set $S = \sum_{i,j=1}^n R_{ij} u_i u_j$ and $S' = \sum_{i,j=1}^n R'_{ij} u_i u_j$.

Proposition 6.1 *Let m be even and let $Q(r, r')$ be a basis of $\mathcal{H}_{1,m}(d_1, d_2)$. For each multi-index ν with $|\nu| = m$ we define $Q_\nu(R, R')$ by $Q(S, S') = \sum_{|\nu|=m} Q_\nu(R, R') u^\nu$. Then the vector $(Q_\nu(R, R'))_{\nu \in I(m)}$ gives a basis of the one dimensional space $\mathcal{H}_{n,\rho_{m,0}}(d_1, d_2)$.*

Proof. Since it is clear that

$$(Q_\nu(ARA^t, AR'A^t))_{\nu \in I(m)} = \rho_{m,0}(A)(Q_\nu(R, R'))_{\nu \in I(m)}$$

all we should do is to prove that the polynomials $Q_\nu(R, R')$ are harmonic. This is easily proved as follows. Using that $Q(r, r')$ is harmonic one obtains

$$\left(2d_1 \frac{\partial Q(r, r')}{\partial r} + 4r \frac{\partial^2 Q(r, r')}{\partial r^2}\right) + \left(2d_2 \frac{\partial Q(r, r')}{\partial r'} + 4r' \frac{\partial^2 Q(r, r')}{\partial r'^2}\right) = 0.$$

Furthermore, a simple calculation shows that

$$\begin{aligned} L_{ij}^{(d_1)}(Q(S, S')) &= u_i u_j (2d_1 \frac{\partial Q}{\partial r} + 4S \frac{\partial^2 Q}{\partial r^2}), \\ L_{ij}^{(d_2)}(Q(S, S')) &= u_i u_j (2d_2 \frac{\partial Q}{\partial r'} + 4S' \frac{\partial^2 Q}{\partial r'^2}) \end{aligned}$$

so that the proposition becomes obvious. □

Finally, we give some examples for the case of representations of mixed type when $n = 2$. For $(m+v, v) = (m+2, 2)$ with even m an invariant pluri-harmonic polynomial in $\mathcal{H}_{2,\rho_{m,2}}$ is given by $(Q_\nu(R, R'))_{\nu \in I(m)}$ where $Q_\nu(R, R')$ is the coefficients of u^ν of the following polynomial $Q(S, S')$.

$$\begin{aligned} Q(S, S') &= Q_{2,d_1,d_2}(R_1, R_2) F_{m,d_1+2,d_2+2}(S, S') \\ &\quad + \frac{1}{2}((d_2-1)P_0 S - (d_1-1)P_2 S')(\frac{\partial F_{m.d_1+2,d_2+2}}{\partial r} - \frac{\partial F_{m.d_1+2,d_2+2}}{\partial s})(S, S'). \end{aligned}$$

Here we have denoted by Q_{v,d_1,d_2} the non-zero invariant pluri-harmonic polynomial in $\mathcal{H}_{2,v}(d_1, d_2)$ normalised as in section 5.4 and by $F_{m,d_1,d_2}(r, s)$ a non-zero polynomial in $\mathcal{H}_{1,m}(d_1, d_2)$.

When $(m+2, 2) = (4, 2)$, for example, then Q this is given by

$$\begin{aligned} Q(S, S') = \; & (d_2-1)d_2(d_2+2)P_0 S - (d_1-1)(d_2-1)(d_2+2)P_1 S \\ & + (d_1-1)(d_1+2)(d_4+4)P_2 S - (d_1+4)(d_2-1)(d_2+2)P_0 S' \\ & + (d_1-1)(d_1+2)(d_2-1)P_1 S' - d_1(d_1-1)(d_1+2)P_2 S'. \end{aligned}$$

7 Conclusion

In this paper we have described certain spaces of invariant pluri-harmonic polynomials. These polynomials are in one-to-one correspondence with Rankin-Cohen type differential operators which, in the case of non-constant polynomials, map two Siegel modular forms to a cusp form. In particular, we have derived a set of recursion equations (Theorem 3.4) which uniquely (up to multiplication by non-zero elements in $\mathbb{C}^*$) determine an invariant pluri-harmonic polynomial $Q_{n,v}$ in $\mathcal{H}_{n,v}(d_1, d_2)$ (for $d_1 \geq n$ and $d_2 \geq n$). Although the recursion equations can easily be solved for any *numerical* values of n and v we have not been able to give the closed explicit formulas for the solutions for general n and v. However, in several examples we have obtained such formulas for the solutions. In addition, we have discussed certain vector valued bilinear differential operators.

Let us conclude with a few remarks and point out some interesting open questions in connection with our results.

Firstly, the polynomials $Q_{n,v} \in \mathcal{H}_{n,v}(d_1, d_2)$ are in one-to-one correspondence with the differential operators D defined in Theorem 2.3 only if $d_1 \geq n$ and $d_2 \geq n$. Throughout our analysis we have always assumed that this condition is satisfied. For fixed n and v the polynomial $Q_{n,v}$ depends only on d_1 and d_2 and, with a suitable normalisation, is well defined and non-vanishing even if $d_1 < n$ or $d_2 < n$. Therefore, one might speculate that the differential operators corresponding to $Q_{n,v}$ in the latter cases also map any two automorphic forms to an automorphic form (for $n = 1, 2$ this follows from the results in ref. [2, 1]). In contrast to the situation for $d_1 \geq n$ and $d_2 \geq n$ the dimension of $\mathcal{H}_{n,v}(d_1, d_2)$ can be larger than one if d_1 or d_2 are 'small'. For example if $d_1 = d_2 = n - 1$ and $v = 2$ then $\mathcal{H}_{n,2}(n-1, n-1)$ is spanned by P_0 and P_n (note that, in this case, the pluri-harmonic polynomials $P_0(XX^t, X'X'^t)$ and $P_n(XX^t, X'X'^t)$ are identically zero). In this case the differential operators corresponding to P_0 and P_n satisfy the commutation relation (*c.f.* §2). For odd n, however, all modular forms of weight $(n-1)/2$ on congruence subgroups are singular so that these differential operators act as zero (this special case is already contained in the results of Kapitel III, §6 of ref. [3] and ref. [9]). Furthermore, note that if d_1 and d_2 are such that $d_1 + d_2 < 2n$ then there is no pluri-harmonic polynomial in $\mathcal{P}_{n,v}(d_1, d_2)$ (*c.f.* the discussion in ref. [6]). The last remarks show that it would be interesting to compute the dimension of $\mathcal{H}_{n,v}(d_1, d_2)$ for $d_1 + d_2 \geq 2n$ and either d_1 or d_2 small and to understand the relation between covariant differential operators and invariant pluri-harmonic polynomials in this case.

Secondly, the differential operators D give rise to differential operators for Jacobi forms of higher degree. The relation between the Rankin-Cohen type operators for $n = 2$ and the corresponding differential operators for Jacobi forms has been discussed in detail in ref. [1]. One would expect that –like in the case $n = 2$– the dimension of the space of covariant differential operators for higher degree Jacobi forms is, for fixed v, generically greater than one. It would be interesting if one could compute this dimension and obtain, as in the case of $n = 2$, closed explicit formulas.

Thirdly, in ref. [11] certain algebraic structures –called Rankin-Cohen algebras– have been defined using only the Rankin-Cohen operators for $n = 1$. There is an obvious generalisation of this definition for the case of arbitrary n using the corresponding Rankin-Cohen type differential operators studied in this paper. Therefore, one is naturally led to the question whether one can describe the structure of these generalised Rankin-Cohen algebras in an independent way analogous to the case $n = 1$ (*cf.* the Theorem in *loc. cit.*). Furthermore, it would be interesting to know if other examples of these algebraic structures can be found in mathematical nature.

Fourthly, it seems quite natural to look for applications of the Rankin-Cohen type operators constructed in this paper which involve theta series and/or Eisenstein series.

Finally, let us mention that there are other very interesting types of pluri-harmonic polynomials. The polynomials considered in this paper correspond to the second case in ref. [4]. The first case considered in *loc. cit.* is concerned with differential operators D acting on automorphic functions F on $\mathbb{H}_n$ such that the restriction of $D(F)$ to $\mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_r}$ with $n = n_1 + \cdots + n_r$ is again an automorphic form and the corresponding pluri-harmonic polynomials [5].

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References

[1] Y. Choie, W. Eholzer *Rankin-Cohen Operators for Jacobi and Siegel Forms*, preprint DAMTP-96-106 (**alg-geom/9611032**).

[2] H. Cohen, *Sums Involving the Values at Negative Integers of L Functions of Quadratic Characters*, Math. Ann. 217 (1975), 271-285.

[3] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren der mathematischen Wissenschaften 254, Springer, Berlin–Heidelberg–New York, 1983.

[4] T. Ibukiyama, *On Differential Operators on Automorphic Forms and Invariant Pluri-Harmonic Polynomials*, preprint.

[5] T. Ibukiyama, D. Zagier, *Higher Spherical Polynomials*, in preparation.

[6] M. Kashiwara and M. Vergne. *On the Segal-Shale-Weil Representation and Harmonic Polynomials*, Invent. Math. 44 (1978), 1-47.

[7] K. Koike and I. Terada, *Young Diagrammatic Methods for the Restriction of Representations of Complex Classical Lie Groups to Reductive Subgroups of Maximal Rank*, Advances in Mathematics 79 (1990), 104-135.

[8] R. Rankin, *The Construction of Automorphic Forms from the Derivatives of a given Form*, J. Indian Math. Soc. 20 (1956), 103-116.

[9] H. L. Resnikoff, *On a Class of Linear Differential Equations for Automorphic Forms in Several Complex Variables*, Amer. J. Math. 95 (1973), 321–332.

[10] H. Weyl, *The Classical Groups*, Princeton University Press, 1946.

[11] D. Zagier, *Modular Forms and Differential Operators*, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 104, No. 1 (1994), 57-75.