Kac-Peterson, Perron-Frobenius, and the Classification of Conformal Field Theories

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Dedicated to the memory of Rudelle Hall, teacher and friend

1. Introduction: The classification of conformal field theories. Conformal field theories (CFTs) and related structures have been of considerable value to mathematics, as for instance the work of Witten has shown. This paper is concerned with their classification. Fortunately, the problem has a simple expression in terms of the characters of Kac-Moody algebras (see (1.2) below), and requires no prior knowledge of CFT. Nevertheless, for reasons of motivation, in the following paragraphs we will sketch the definition of CFT.

Before discussing this background material, let us quickly state the actual mathematical problem addressed in this paper. The characters of an affine algebra at fixed level k define in a natural way a unitary representation of $\mathrm{SL}_2(\mathbb{Z})$ (see equations (3.3) below). The ultimate classification problem here is to find all matrices M which commute with the matrices of this representation, and which in addition obey relations (1.2b) and (1.2c) – such M are called *physical invariants*. In this paper we address the subproblem of finding all physical invariants which in addition satisfy (1.3b), where $\mathcal{S}$ is the group of all symmetries of the (extended) Coxeter-Dynkin diagram – these M we call $\mathcal{ADE}_7$*-type invariants*. Almost every physical invariant is expected to be a $\mathcal{ADE}_7$-type invariant. In this paper we develop a program to find all of these for any affine algebra, and apply it to explicitly find them for the algebra $A_r^{(1)}$.

The remainder of this introductory section is intended to explain the motivation for this problem. In the language of CFT (which will be touched on shortly), the classification of these physical invariants is equivalent to the classification of all possible *Wess-Zumino-Witten partition functions*. There is, we shall see, a fairly natural cut of this classification into two subproblems. One is to find all possible *chiral algebras* (these are essentially vertex operator algebras), and the other is to find all possible automorphisms of the corresponding *fusion rings* (these encode the tensor product structure of the algebra). In previous work [11,12] we accomplished the second subproblem for the case where the chiral algebra corresponds to an affine algebra; in this paper we generalize those arguments to the case where the chiral algebra is an extension of those by *simple currents* (see e.g. [6]). It is generally

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believed (for reasons given below) that 'almost every' chiral algebra finitely extending the chiral algebra of an affine algebra, will be of this form, and so this paper solves (for $A_r^{(1)}$) the second subproblem for what may be termed its *generic* chiral extensions.

According to Segal [25] (see also the presentation in [15]), a (two-dimensional) conformal field theory is a *representation* of the category $\mathcal{C}$ whose objects are disjoint unions of parametrized circles and whose morphisms are cobordisms – i.e. it is a functor $\mathcal{T}$ from $\mathcal{C}$ into the category of complex Hilbert spaces and trace-class operators. There exists a Hilbert space H such that $\mathcal{T}$ takes n circles to $H \otimes \cdots \otimes H$ (n times). Sewing together surfaces in $\mathcal{C}$ along boundary circles corresponds by $\mathcal{T}$ to composing operators. The detailed definitions and axioms are not important here, and would take us too far afield.

The data of a CFT decomposes into two *chiral halves*, related to the fact that the conformal maps in $\mathbb{C}$ consist of analytic functions and their complex conjugates. Of greatest interest are the rational conformal field theories (RCFTs), defined by Segal using the notion of a *modular functor*. The modular functor makes precise the constraints imposed on each chiral half: the key property of an RCFT is that the chiral data is labelled by a *finite* set (the primary fields of the theory).

$\mathcal{T}$ will map the closed torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\,\tau)$ to a complex number $\mathcal{Z}(\tau)$; $\mathcal{Z}$ is called the *partition function* for the theory. But different τ can correspond to the same torus; these τ are related by the modular group of the torus. Thus the partition function $\mathcal{Z}$ should be modular invariant, i.e. invariant under the natural action of $\mathrm{PSL}_2(\mathbb{Z})$ on the upper half complex plane.

Important examples of RCFTs are where the 'chiral labels' are given by representations of a Kac-Moody algebra $X_r^{(1)}$ at some fixed level $k \in \{1,2,3,\ldots\}$. These are called Wess-Zumino-Witten (WZW) models[1]. The partition function of a WZW model will be of the form

$$\mathcal{Z}(v) = \sum_{\mu,\nu} M_{\mu,\nu}\,\chi_\mu(v)\,\chi_\nu(v)^* \ , \tag{1.1}$$

where the parameter v can be taken to lie in a Cartan subalgebra of $X_r^{(1)}$. The sum in (1.1) is over all highest weights $P_+(X_r^{(1)},k)$; one of these weights, denoted $k\Lambda_0$, is distinguished. This differs the partition function $\mathcal{Z}(\tau)$ discussed earlier, only by depending on more variables. There is a natural action $v \mapsto Av$ of $\mathrm{SL}_2(\mathbb{Z})$ on the Cartan subalgebra [17]. The function in (1.1) obeys the following conditions:

$$\mathcal{Z}(Av) = \mathcal{Z}(v) \quad \text{for all } A \in \mathrm{SL}_2(\mathbb{Z}) \ ; \tag{1.2a}$$

$$M_{\mu,\nu} \in \{0,1,2,\ldots\} \ ; \tag{1.2b}$$

$$M_{k\Lambda_0,k\Lambda_0} = 1 \ . \tag{1.2c}$$

Any such $\mathcal{Z}$ or M is called a *physical invariant.*

WZW models have been extensively studied because they are simple enough to analyze, but complicated enough that the answers should be interesting and hopefully characteristic of more general RCFTs. They are generally regarded as building blocks, via the Goddard-Kent-Olive coset construction, for perhaps all other RCFTs.

[1] WZW is often used in the narrower sense of strings propagating on a group manifold, so the term *conformal current model* was proposed in [12] for the more general case of interest here.

One of the few remaining fundamental questions of WZW models is their classification. Because of Segal's sewing axiom, the higher genus behaviour of a CFT can be determined in principle from that of lower genus. In particular, a RCFT is uniquely determined by its two chiral algebras (which in most cases are taken to be isomorphic), the operator product structure coefficients (obtained in principle from $\mathcal{T}$ by selecting a disk with two punctures), and the partition function.

In this paper we address the classification of possible WZW partition functions (i.e. physical invariants). The first such result was for $A_1^{(1)}$, for all k [4]. It was found that the set of solutions to (1.2) for $A_1^{(1)}$ fall into the mysterious A-D-E pattern (see also [26]). An explanation has recently been announced by Ocneanu [21], using subfactor theory and path algebras on graphs. All physical invariants are also known for $A_2^{(1)}$ [10]. For it, no connection with A-D-E is known, but several unexplained coincidences have appeared (see e.g. [22]) between the $A_2^{(1)}$ classification and the Jacobians of Fermat curves. Zuber [28] and collaborators have explored using generalized Coxeter graphs to reinterpret and extend some of these observations. Classifying RCFTs is interesting in its own right, but what makes it more intriguing is the desire to understand and if possible generalize these apparent patterns.

Unfortunately physical invariants have resisted extensive attempts at their classification; only for $A_1^{(1)}$ [4], $A_2^{(1)}$ [10], and $(A_1 \oplus A_1)^{(1)}$ [9][2] has the classification been attained at all levels k. However there has been recent progress [11,12] toward the solution of this problem, and this paper takes us one step closer to this goal.

Let $\mathcal{S}$ denote the group of all symmetries of the (extended) Dynkin diagram of $X_r^{(1)}$. Any $A \in \mathcal{S}$ will induce a permutation $\lambda \mapsto A\lambda$ of the level k weights of $X_r^{(1)}$, by the action of A on the Dynkin labels. Write $\mathcal{S}\lambda$ for the orbit of λ by $\mathcal{S}$. The $A \in \mathcal{S}$ which fix the extended node are called *conjugations*; some of the remainder (defined in section 3 below) are called *simple currents*. It is easy to verify (see (3.5) below) that the modular behaviour of $\chi_{A\lambda}$ is closely related to that of χ_λ, for any symmetry $A \in \mathcal{S}$. So it is not surprising that these can be used to obtain new physical invariants from old ones [2]. Indeed it seems that most physical invariants can be obtained in this way from the identity matrix physical invariant $M = I$ – such physical invariants are called *simple current invariants* (and their conjugations). See (1.4) below.

As can be seen from (1.2c), as well as (3.4c) below, the weight $k\Lambda_0$ has special significance. A reasonable division of this classification problem into two subproblems is, on the one hand, to consider all possible values $M_{k\Lambda_0,\mu}$, $M_{\mu,k\Lambda_0}$ – these are severely constrained [10] – and on the other hand to find all physical invariants M which realize each of these possible choices for $M_{k\Lambda_0,\mu}$, $M_{\mu,k\Lambda_0}$. This is a restatement of the two subproblems mentioned in the third paragraph. In [11,12] we find all possible physical invariants satisfying the additional constraint

$$M_{\mu,k\Lambda_0} \neq 0 \text{ or } M_{k\Lambda_0,\mu} \neq 0 \quad \Longrightarrow \quad \mu = k\Lambda_0 \ . \tag{1.3a}$$

[2] However for this latter algebra an additional constraint beyond (1.2), involving the Knizhnik-Zamolodchikov equation, was assumed.

These are called *automorphism invariants*. In this paper we generalize those arguments to find all physical invariants obeying instead the more general constraint

$$M_{\mu,k\Lambda_0} \neq 0 \text{ or } M_{k\Lambda_0,\mu} \neq 0 \quad \Longrightarrow \quad \mu \in \mathcal{S}(k\Lambda_0) \ . \tag{1.3b}$$

We call these *$\mathcal{ADE}_7$-type invariants*. For example, in the $A_1^{(1)}$ classification [4], these consist of the physical invariants called $\mathcal{A}_\ell$ and $\mathcal{D}_\ell$, along with the exceptional $\mathcal{E}_7$. Based on the known classifications (e.g. [4,10,9,5]), together with various computer checks in the literature, it is reasonable to conjecture that almost all physical invariants are $\mathcal{ADE}_7$-type invariants. For example, for all but a small number of $A_r^{(1)}$, we expect all physical invariants for each $k \neq r-1, r+1, r+3$ to obey (1.3b). The $\mathcal{ADE}_7$-type invariants are interesting also because they include exceptional physical invariants (like $\mathcal{E}_7$ for $A_1^{(1)}$) which are notoriously difficult to obtain by standard constructions.

This paper is concerned with the classification of all $\mathcal{ADE}_7$-type invariants. We reduce the problem to the mechanical albeit tedious task of computing q-dimensions and tensor product coefficients. We complete the classification for the case of greatest interest: $A_r^{(1)}$. Up to conjugations, we find only 8 exceptional $\mathcal{ADE}_7$-type invariants. This is a significant step towards the classification of all WZW partition functions for the unitary algebras. The final step in that classification, namely solving the various constraints for $M_{\mu,k\Lambda_0}, M_{k\Lambda_0,\mu}$, will not be addressed here.

Some of the arguments in this paper are based on those for the automorphism invariant classification [11,12], as well as older classifications [9,10], but several new complications arise here. The main tools we use are the Kac-Peterson formula (3.4d) – which permits us to exploit the well-understood representation theory of finite-dimensional Lie algebras – and the Perron-Frobenius spectral theory for non-negative matrices.

A somewhat related problem is [19] to classify all physical invariants which for all weights μ, ν obey the constraint

$$M_{\mu,\nu} \neq 0 \Rightarrow \nu \in \mathcal{S}_{sc}\mu \ , \tag{1.4}$$

where $\mathcal{S}_{sc}$ is the subgroup of $\mathcal{S}$ consisting of all simple currents. These are called *simple current invariants*; they are a special case of the $\mathcal{ADE}_7$-type invariants considered here. Their classification has been accomplished for all RCFT, subject to a certain constraint on the modular S matrix (3.3c) [19] – it is found that there are no exceptional invariants of this form. Though this is clearly a major result, (1.4) is sufficiently stronger than (1.3b) that the arguments in [19] are not useful in our context.

In section 2 below we list all $\mathcal{ADE}_7$-type invariants for $A_r^{(1)}$. Section 3 establishes the basic results we need, and section 4 specializes to $A_r^{(1)}$ and outlines the argument for classifying all $A_r^{(1)}$ $\mathcal{ADE}_7$-type invariants. The problem reduces to some q-dimension calculations and computing some tensor product coefficients, which we do in sections 5 and 6 respectively. This completes the classification for almost all levels k of $A_r^{(1)}$; the finitely many trouble-making pairs (r, k) are explicitly handled in section 7.

2. The $\mathcal{ADE}_7$-type invariants of $A_r^{(1)}$.

In this section we explicitly list all of the $\mathcal{ADE}_7$-type invariants of $g = A_r^{(1)}$. The proof that this list is complete will be accomplished in the later sections. In the following section we will motivate and generalize many of the definitions made here; our purpose here is merely to state Theorem 2.1.

Fix the rank r and level k, and define $\bar{r} = r + 1$ and $\bar{k} = k + \bar{r}$. The level k highest weights of $A_r^{(1)}$ constitute the set P_+ of $\bar{r}$-tuples $\lambda = (\lambda_0, \ldots, \lambda_r)$ of non-negative integers λ_i obeying $\sum_{i=0}^r \lambda_i = k$. The extended Coxeter-Dynkin diagram of $A_r^{(1)}$ is a circle with $\bar{r}$ nodes, which we label counterclockwise 0 to r – 0 is called the *extended* node. Its $2\bar{r}$ symmetries (only $\bar{r}$, if $r = 1$) form the dihedral group $\mathcal{S}$; it is generated by an order 2 symmetry fixing 0 (the *conjugation* C), and an order $\bar{r}$ rotation taking i to $i + 1$ (the *simple current* J). This group acts on P_+ by permuting the indices of the weight:

$$C\lambda =(\lambda_0, \lambda_r, \lambda_{r-1}, \ldots, \lambda_1) \ , \tag{2.1a}$$

$$J\lambda =(\lambda_r, \lambda_0, \lambda_1, \ldots, \lambda_{r-1}) \ . \tag{2.1b}$$

A convenient quantity we will often use is the $\bar{r}$-ality t defined by

$$t(\lambda) \stackrel{\text{def}}{=} \sum_{j=1}^{r} j\lambda_j \ . \tag{2.2}$$

Together with Lemmas 3.1, 3.2 and 3.3, the following theorem is the main result of this paper. The $\mathcal{ADE}_7$-type invariants named in Theorem 2.1 are defined in equations (2.3a), (2.4), (2.5), and (2.7) below.

THEOREM 2.1. *The complete list of $\mathcal{ADE}_7$-type invariants for $A_r^{(1)}$ at level k is:*

- *for all $r, k \geq 1$, d dividing $\bar{r}$ and satisfying (2.3b), and $c = 0, 1$:* $C^c \cdot I[\mathcal{J}_d]$;
- *for $(r,k) \in \{(1,16), (3,8), (4,5), (7,4)\}$:* $\mathcal{E}^{(r,k)}$;
- *for $(r,k) \in \{(2,9), (8,3)\}$:* $\mathcal{E}^{(r,k)}$ *and* $C \cdot \mathcal{E}^{(r,k)}$;
- *for $(r,k) = (15,2)$:* $\mathcal{E}^{(15,2)}$, $\frac{1}{2}\, I[\mathcal{J}_4] \cdot \mathcal{E}^{(15,2)}$ *and* $C \cdot \mathcal{E}^{(15,2)}$.

Next, we explicitly define these $\mathcal{ADE}_7$-type invariants.

Denote by $\mathcal{J}_d$ the subgroup of $\mathcal{S}$ generated by J^d, when d divides $\bar{r}$. Each such subgroup can be used to construct a $\mathcal{ADE}_7$-type invariant. In particular, put $k' = \bar{k}$ if both k and $\bar{r}$ are odd, otherwise put $k' = k$. Define [23]

$$I[\mathcal{J}_d]_{\lambda,\mu} = \sum_{j=1}^{\bar{r}/d} \delta^{\bar{r}/d}(t(\lambda) + djk'/2)\ \delta_{\mu, J^{dj}\lambda} \ , \tag{2.3a}$$

where $\delta^y(x) = 1$ or 0 depending, respectively, on whether or not $x/y \in \mathbb{Z}$. Then $I[\mathcal{J}_d]$ will be a physical invariant iff [23]

$$k'd \equiv 0 \qquad (\text{mod } 2) \ . \tag{2.3b}$$

This can be readily proven using (4.1b) and (4.1c) below.

These $I[\mathcal{J}_d]$ were first explicitly given in [7], though some appeared earlier in [2]. Equation (2.3a) extends naturally to any $X_r^{(1)}$ (see [23]). Note that $d = \bar{r}$ always satisfies

(2.3b); it gives $I[\mathcal{J}_{\bar{r}}] = I$, the identity matrix. Incidently, for each divisor d of $\bar{r}$, there is a Lie group G_d whose simply-connected covering group is $G_1 \stackrel{\text{def}}{=} \mathrm{SU}_{\bar{r}}$, and which obeys $\|G_1/G_d\| = d$. For example, for $\bar{r} = 2$ $G_2 = \mathrm{SO}_3$. The existence of $I[\mathcal{J}_d]$ is intimately connected to that of $G_{\bar{r}/d}$ [7].

The conjugation C defines another $\mathcal{ADE}_7$-type invariant (see (4.1d), (4.1e)), which we will also denote by C:

$$C_{\lambda,\mu} = \delta_{\mu,C\lambda} \ , \tag{2.4}$$

where δ denotes the Kronecker delta. Moreover, the matrix product $C \cdot M$ of C with any other $\mathcal{ADE}_7$-type invariant M will also be a $\mathcal{ADE}_7$-type invariant, and $C^2 = I$.

In addition, there are a number of other $\mathcal{ADE}_7$-type invariants, called *$\mathcal{E}_7$-type exceptionals.* It is slightly more convenient to express these in terms of characters rather than their coefficient matrices M. It suffices to give the relevant subgroup $\mathcal{J}_d$, as well as the characters with the exceptional behaviour (the remaining characters combine exactly as in $I[\mathcal{J}_d]$). To help explain our notation, we will write out in full the two simplest such exceptionals:

$$\begin{aligned}\mathcal{E}^{(1,16)} =& |\chi_{16,0} + \chi_{0,16}|^2 + |\chi_{12,4} + \chi_{4,12}|^2 + |\chi_{10,6} + \chi_{6,10}|^2 \\ &+ (\chi_{14,2} + \chi_{2,14})\,\chi_{8,8}^* + \chi_{8,8}\,(\chi_{14,2} + \chi_{2,14})^* + |\chi_{8,8}|^2 \ ; \qquad (2.5a)\\ \mathcal{E}^{(2,9)} =& |\chi_{900} + \chi_{090} + \chi_{009}|^2 + |\chi_{522} + \chi_{252} + \chi_{225}|^2 + |\chi_{603} + \chi_{360} + \chi_{036}|^2 \\ &+ |\chi_{630} + \chi_{063} + \chi_{306}|^2 + |\chi_{144} + \chi_{414} + \chi_{441}|^2 + 2|\chi_{333}|^2 \\ &+ (\chi_{711} + \chi_{171} + \chi_{117})\,\chi_{333}^* + \chi_{333}\,(\chi_{711} + \chi_{171} + \chi_{117})^* \ . \qquad (2.5b)\end{aligned}$$

Here and elsewhere, we label a weight by its Dynkin labels.

For convenience write

$$\langle \chi_\lambda \rangle_d \stackrel{\text{def}}{=} \sum_{\mu \in \mathcal{J}_d \lambda} \chi_\mu \ . \tag{2.6a}$$

Also, write "$a * b$" as short-hand for "$ab^* + ba^*$". Note that we may capture all the information in (2.5a) by stating $d = 1$, and giving the 'exceptional' terms:

$$\langle \chi_{14,2} \rangle_1 * \chi_{8,8} + |\chi_{8,8}|^2 \ . \tag{2.6b}$$

The remaining terms in (2.5a) are exactly as in $I[\mathcal{J}_1]$. Similarly, (2.5b) can be summarized by stating $d = 1$, and giving the exceptional terms

$$2|\chi_{333}|^2 + \langle \chi_{711} \rangle_1 * \chi_{333} \ . \tag{2.6c}$$

The remaining $\mathcal{E}_7$-type exceptionals are expressed in this way as:

$$\begin{aligned}\mathcal{E}^{(3,8)}: \quad d = 1; \quad & 2|\chi_{2222}|^2 + (\langle \chi_{5012} \rangle_1 + \langle \chi_{5210} \rangle_1) * \chi_{2222} \\ & + \langle \chi_{6101} \rangle_1 * \langle \chi_{4040} \rangle_1 + |\langle \chi_{4040} \rangle_1|^2 \ ; \qquad (2.7a)\\ \mathcal{E}^{(4,5)}: \quad d = 1; \quad & \langle \chi_{31001} \rangle_1 * \chi_{11111} + 4|\chi_{11111}|^2 \ ; \qquad (2.7b)\\ \mathcal{E}^{(7,4)}: \quad d = 2; \quad & |\langle \chi_{20002000} \rangle_2|^2 + |\langle \chi_{02000200} \rangle_2|^2 + \langle \chi_{21000001} \rangle_2 * \langle \chi_{20002000} \rangle_2 \qquad (2.7c)\end{aligned}$$

$$+ \langle \chi_{12100000} \rangle_2 * \langle \chi_{02000200} \rangle_2 + (\langle \chi_{12000010} \rangle_2 + \langle \chi_{10100002} \rangle_2) * \chi_{01010101}$$
$$(\langle \chi_{21010000} \rangle_2 + \langle \chi_{20000101} \rangle_2) * \chi_{10101010} + 2|\chi_{01010101}|^2 + 2|\chi_{10101010}|^2 \};$$

$$\mathcal{E}^{(8,3)}: \quad d = 3; \quad \sum_{j=0}^{2} \left(2|\chi_{J^j(\Lambda_0+\Lambda_3+\Lambda_6)}|^2 + \langle \chi_{J^j(\Lambda_2+\Lambda_3+\Lambda_4)} \rangle_3 * \chi_{J^j(\Lambda_0+\Lambda_3+\Lambda_6)} \right) ; \qquad (2.7d)$$

$$\mathcal{E}^{(15,2)}: \quad d = 8; \quad \sum_{j=0}^{7} \left(|\chi_{J^j(\Lambda_0+\Lambda_8)}|^2 + \langle \chi_{J^j(\Lambda_3+\Lambda_5)} \rangle_8 * \chi_{J^j(\Lambda_0+\Lambda_8)} \right) . \qquad (2.7e)$$

$\mathcal{E}^{(1,16)}$ was first given in [4]; $\mathcal{E}^{(2,9)}$ in [20]; $\mathcal{E}^{(4,5)}$ in [24]; $\mathcal{E}^{(3,8)}$, $\mathcal{E}^{(7,4)}$, $\mathcal{E}^{(8,3)}$ in [8]. $\mathcal{E}^{(15,2)}$ is new but [8] obtained its projection: the matrix product $\frac{1}{2} I[\mathcal{J}_4] \cdot \mathcal{E}^{(15,2)}$, which has $d = 4$ and the exceptional terms

$$\sum_{j=0}^{3} \left(|\chi_{J^{2j}(\Lambda_0+\Lambda_8)}|^2 + \langle \chi_{J^{2j}(\Lambda_3+\Lambda_5)} \rangle_4 * \chi_{J^{2j}(\Lambda_0+\Lambda_8)} \right) . \qquad (2.7f)$$

Note the symmetry $(r,k) \leftrightarrow (k-1, r+1)$ in the list of ranks and levels of these exceptionals. This is not surprising, considering the *rank-level duality* (see (4.2), (4.3) below) exhibited by the Kac-Peterson S and T matrices.

Remarks 2.1. Note that the matrices M of all $\mathcal{ADE}_7$-type invariants here are symmetric: $M = M^T$. This is not always true for other g [13]. There are some redundancies in the list in Theorem 2.1. When $r = 1$ or $k \leq 2$, take $c = 0$ only – this is because there C will equal one of the $I[\mathcal{J}_d]$. Likewise, $C \cdot I[\mathcal{J}_d] = I[\mathcal{J}_d]$ for $(r,k,d) \in \{(2,3,1), (2,6,1), (4,5,1), (5,3,2)\}$. The final redundancy is $I[\mathcal{J}_1] = I[\mathcal{J}_2]$ for $\bar{r} = k = 2$.

3. Cyclotomy, Kac-Peterson and Perron-Frobenius. In this section we establish the fundamental lemmas which define our program to classify all $\mathcal{ADE}_7$-type invariants. We will state and prove them for any $g = X_r^{(1)}$ – indeed they continue to hold for any RCFT. The notation used here is standard; see e.g. [16] for more details. We will quickly review the basic facts, before heading into the statement and proof of Lemmas 3.1, 3.2, 3.3.

Let g be the non-twisted affine algebra $X_r^{(1)}$ derived from the finite-dimensional algebra $\bar{g} = X_r$. Let $L(\lambda)$ denote any irreducible integrable highest weight g-module, and let χ_λ be its normalized character with respect to a Cartan subalgebra $h = \bar{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ ($\bar{h}$ is a Cartan subalgebra of $\bar{g}$, and c the canonical central element, d a derivation, of g).

Let $\Lambda_0, \ldots, \Lambda_r \in h^*$ denote the fundamental weights of g. Then the highest weight $\lambda \in P_+(g,k)$ of $L(\lambda)$ can be taken to lie in

$$P_+ \stackrel{\text{def}}{=} \{ \sum_{j=0}^{r} \lambda_j \Lambda_j \,|\, \lambda_j \in \mathbb{Z},\ \lambda_j \geq 0,\ \sum_{j=0}^{r} a_j^\vee \lambda_j = k \} , \qquad (3.1)$$

where k is a positive integer called the level, and the positive integers $a_j^\vee$ are the co-labels of g. Write $h^\vee = \sum_i a_i^\vee$, and $\bar{k} = k + h^\vee$. The Weyl vector is $\rho = \sum_{i=0}^{r} \Lambda_i$. For convenience we will write Λ^0 for $k\Lambda_0$. Note that the projection

$$\lambda \mapsto \overline{\lambda} \stackrel{\text{def}}{=} \sum_{i=1}^{r} \lambda_i \bar{\Lambda}_i, \qquad \bar{\Lambda}_i \stackrel{\text{def}}{=} \Lambda_i - a_i^\vee \Lambda_0$$

produces a highest weight for the underlying algebra $\overline{g} = X_r$. This projection is orthogonal with respect to the invariant bilinear form $(-|-)$ (which we take to be normalized so that long roots have norm 2); in fact $(\lambda|\lambda) = (\bar{\lambda}|\bar{\lambda})$.

Let $Q^\vee$ denote the coroot lattice, and $\overline{W}$ the Weyl group of $\bar{g}$. The affine Weyl group W is isomorphic to the semi-direct product $T \cdot \overline{W}$, where T consists of the translations t_α, $\alpha \in Q^\vee$, defined on h^* (mod $\mathbb{C}\delta$) by

$$t_\alpha \Lambda_i \equiv \Lambda_i + a_i^\vee \alpha \qquad (\text{mod } \mathbb{C}\delta) \tag{3.2}$$

(δ here is an imaginary root of g). This is a central observation in the representation theory of affine Kac-Moody algebras. It permits an expression – the Weyl-Kac character formula – for the character χ_λ in terms of theta functions. This implies [17] that χ_λ will be a modular function: in particular, we may regard χ_λ as a function from h to $\mathbb{C}$; coordinatizing h in the usual way (i.e. $2\pi i(z - \tau\, d + u\, c) \in h$, where $z \in \bar{h}$, $\tau, u \in \mathbb{C}$), we obtain

$$\chi_\lambda(\tau+1, z, u) = \sum_{\mu \in P_+} T_{\lambda,\mu}\, \chi_\mu(\tau, z, u)\ , \tag{3.3a}$$

$$T_{\lambda,\mu} \stackrel{\text{def}}{=} \exp[\pi i \{ \frac{(\lambda+\rho|\lambda+\rho)}{\bar{k}} - \frac{(\rho|\rho)}{h^\vee} \}]\, \delta_{\lambda,\mu}\ ; \tag{3.3b}$$

$$\chi_\lambda(\frac{-1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}) = \sum_{\mu \in P_+} S_{\lambda,\mu}\, \chi_\mu(\tau, z, u)\ , \tag{3.3c}$$

$$S_{\lambda,\mu} \stackrel{\text{def}}{=} s \sum_{w \in \overline{W}} \det(w) \exp[-2\pi i \frac{(w(\lambda+\rho)|\mu+\rho)}{\bar{k}}]\ ; \tag{3.3d}$$

where in (3.3d) the normalization s is

$$s = i^{\|\overline{\Delta}_+\|}\, \bar{k}^{-r/2}\, \|Q^{\vee\,*}/Q^\vee\|^{-\frac{1}{2}}\ .$$

Here, $\|\overline{\Delta}_+\|$ denotes the number of positive roots of $\overline{g}$, and the weight lattice $Q^{\vee\,*}$ is the dual lattice of $Q^\vee$. Together, (3.3a),(3.3c) define the transformation properties of χ_λ with respect to $\mathrm{SL}_2(\mathbb{Z})$.

These *Kac-Peterson* matrices S and T have some special properties. They are unitary and symmetric. From the Weyl denominator formula we get, for any $\ell = 0, \ldots, k/(h^\vee - 1)$,

$$S_{\ell\bar{\rho},\lambda} = |s| \prod_{\bar{\alpha}>0} 2\sin(\pi \frac{(\bar{\lambda}+\bar{\rho}|\bar{\alpha})}{\bar{k}/(\ell+1)})\ , \tag{3.4a}$$

where by $\ell\bar{\rho}$ in (3.4a) we mean the weight $\ell\bar{\rho}+(k-\ell(h^\vee-1))\Lambda_0$, and where the product is over the positive roots $\bar{\alpha}\in\overline{\Delta_+}$ of $\bar{g}$. Usually we will take $\ell=0$ in (3.4a). This implies the following expression for the *q-dimensions:*

$$\mathcal{D}(\lambda)\stackrel{\text{def}}{=}\frac{S_{\lambda,\Lambda^0}}{S_{\Lambda^0,\Lambda^0}}=\prod_{\bar{\alpha}>0}\frac{\sin(\pi\,(\bar{\lambda}+\bar{\rho}|\bar{\alpha})/\bar{k})}{\sin(\pi\,(\bar{\rho}|\bar{\alpha})/\bar{k})}\ . \tag{3.4b}$$

From (3.4a) one can show that

$$S_{\lambda,\Lambda^0}\geq S_{\Lambda^0,\Lambda^0}>0,\quad \forall\lambda\in P_+\ . \tag{3.4c}$$

S also satisfies the important equation [17]

$$\frac{S_{\lambda,\mu}}{S_{\Lambda^0,\mu}}=\overline{ch}_{\overline{\lambda}}(-2\pi i\frac{\overline{\mu}+\overline{\rho}}{\bar{k}})\ , \tag{3.4d}$$

where $\overline{ch}_{\overline{\lambda}}$ is the Weyl character of the $\overline{g}$-module $\overline{L}(\overline{\lambda})$. Equation (3.4d) has many consequences, one of which has to do with the *fusion coefficients* of g. These can be taken to be defined by Verlinde's formula:

$$N_{\lambda,\mu}^{\nu}\stackrel{\text{def}}{=}\sum_{\gamma\in P_+}S_{\lambda,\gamma}\,\frac{S_{\mu,\gamma}}{S_{\Lambda^0,\gamma}}\,S^*_{\nu,\gamma}\ . \tag{3.4e}$$

Fusion coefficients have an algebraic interpretation in terms of the tensor products of representations of e.g. Hecke algebras and quantum groups at roots of unity, as well as a geometric interpretation involving moduli spaces of principle bundles over projective curves. In the language of RCFTs, they give the dimensions of the spaces of conformal blocks. The only relevant point here is that, because of (3.4d), they can be computed in terms of the tensor product multiplicities $\text{mult}_{\overline{\lambda}\otimes\overline{\mu}}(\overline{\nu})$ in $\bar{g}$ [27,16], and hence its weight multiplicities $m_{\overline{\lambda}}(\overline{\mu})\stackrel{\text{def}}{=}\dim\overline{L}(\overline{\lambda})_{\overline{\mu}}$:

$$N_{\lambda,\mu}^{\nu}=\sum_{w\in W}\det(w)\,m_{\overline{\mu}}(\overline{w.\nu}-\overline{\lambda})\ , \tag{3.4f}$$

where $w.\gamma\stackrel{\text{def}}{=}w(\gamma+\rho)-\rho$ (compare the Racah-Speiser algorithm for computing $\text{mult}_{\overline{\lambda}\otimes\overline{\mu}}(\overline{\nu})$).

The symmetries of the (extended) Coxeter-Dynkin diagram of g define the group $\mathcal{S}$. These play a major role in this paper. Those fixing the extended node are called *conjugations*. Another subgroup is $\mathcal{S}_{sc}=W_0^+$, defined as follows. Let T_0 denote the set of all translations t_α in (3.2) with $\alpha\in P^\vee$, where $P^\vee$ is the co-weight lattice. Define [18]

$$\mathcal{S}_{sc}=\{J\in T_0\cdot\overline{W}\,|\,J(\Delta_+)=\Delta_+\}\ ,$$

where Δ_+ are the positive roots of g. $\mathcal{S}_{sc}$ stabilizes $\Pi^\vee$, and defines a normal subgroup of $\mathcal{S}$ isomorphic to $Q^\vee/P^\vee$. Its elements are called *simple currents*. Both conjugations and simple currents act on P_+ by permuting the Dynkin labels, and together they generate $\mathcal{S}$.

All conjugations commute with S and T and fix Λ^0. The most important conjugation is called *charge conjugation*: it takes each weight λ to $C\lambda = {}'\lambda$, the weight contragredient to λ. It obeys the important relation

$$C = S^2 \ . \tag{3.5a}$$

The primary reason for the importance of $\mathcal{S}_{sc}$ is: let $J = t_\alpha w \in \mathcal{S}_{sc}$, then [18]

$$S_{J^a\lambda,J^b\mu} = \exp[2\pi i\,(a\,Q_J(\mu) + b\,Q_J(\lambda) + ab\,Q_J(J\Lambda^0))]\,S_{\lambda,\mu} \ , \tag{3.5b}$$

where by J^a we mean the a-fold composition $J \circ \cdots \circ J$, and where $Q_J(\mu) = -(\bar{\mu}|\alpha)$. The matrix T also behaves similarly under $\mathcal{S}_{sc}$:

$$\exp[2\pi i\,Q_J(\lambda)] = T_{\lambda,\lambda}T^*_{J\lambda,J\lambda}T_{J\Lambda^0,J\Lambda^0}T^*_{\Lambda^0,\Lambda^0} \ . \tag{3.5c}$$

Definition 3.1. By a *positive invariant* for a given algebra $X_r^{(1)}$ and level k we mean a matrix M commuting with the corresponding Kac-Peterson matrices S and T, with the additional property that each $M_{\lambda,\mu} \geq 0$. By a *physical invariant* we mean a positive invariant with each $M_{\lambda,\mu} \in \mathbb{Z}$, and obeying (1.2c). By a *$\mathcal{ADE}_7$-type invariant* we mean a physical invariant M satisfying (1.3b).

For example, any conjugation defines a $\mathcal{ADE}_7$-type invariant. Simple currents can also be used to construct them (see e.g. [2,23]) – an example is (2.3a). Any physical invariant not constructable in these standard ways out of simple currents and conjugations is called an *exceptional invariant*, and if it is in addition a $\mathcal{ADE}_7$-type invariant, we shall call it an $\mathcal{E}_7$-type exceptional (by analogy with the A-D-E classification in [4]).

The condition $TM = MT$ is equivalent to the 'selection rule'

$$M_{\lambda,\mu} \neq 0 \Rightarrow (\lambda+\rho\,|\,\lambda+\rho) \equiv (\mu+\rho\,|\,\mu+\rho) \pmod{2\bar{k}} \ . \tag{3.6a}$$

The other commutation condition, namely

$$S\,M = M\,S \ , \tag{3.6b}$$

or equivalently (since S is unitary)

$$S\,M\,S^\dagger = M \ , \tag{3.6c}$$

is much more subtle and interesting, and we will begin to explore its consequences in this section. Equations (3.6a) and (3.6b) are equivalent to the modular invariance condition (1.2a).

For a positive invariant M, define

$$\mathcal{J}_L(M) = \{J \in \mathcal{S}_{sc}\,|\,M_{J\Lambda^0,\Lambda^0} \neq 0\} \ ; \tag{3.7a}$$

$$\mathcal{P}_L(M) = \{\lambda \in P_+\,|\,\exists\mu \in P_+ \text{ such that } M_{\lambda,\mu} \neq 0\} \ ; \tag{3.7b}$$

and define $\mathcal{J}_R(M)$ and $\mathcal{P}_R(M)$ similarly (using the other subscript of M). Call $\lambda \in P_+$ a *fixed point* of $\mathcal{J} \subset \mathcal{S}_{sc}$ if $\|\mathcal{J}\lambda\| < \|\mathcal{J}\|$. Let $\mathcal{F}(\mathcal{J})$ denote the set of all fixed points of $\mathcal{J}$. For any $\mathcal{J} \subset \mathcal{S}_{sc}$, define

$$\mathcal{P}(\mathcal{J}) \stackrel{\text{def}}{=} \{\lambda \in P_+ \,|\, Q_J(\lambda) \equiv 0 \pmod 1 \quad \forall J \in \mathcal{J}\} . \tag{3.7c}$$

The remainder of this section is devoted to the statement and proof of the basic lemmas we will need. Exactly how to use these will be addressed in the following section.

Our first lemma is an easy consequence of (3.4c) and (3.5b). It tells us how $\mathcal{J}_L(M)$ and $\mathcal{J}_R(M)$ influence all other values of M.

LEMMA 3.1. (a) *Let M be any physical invariant, and $J, J' \in \mathcal{S}_{sc}$. Then the following statements are equivalent:*

(i) $M_{J\Lambda^0, J'\Lambda^0} \neq 0$;
(ii) $M_{J\Lambda^0, J'\Lambda^0} = 1$;
(iii) *for any $\lambda, \mu \in P_+$, if $M_{\lambda,\mu} \neq 0$ then $Q_J(\lambda) \equiv Q_{J'}(\mu)$ (mod 1)*;
(iv) $M_{J\lambda, J'\mu} = M_{\lambda,\mu}$ *for all $\lambda, \mu \in P_+$.*

(b) *Let M be any positive invariant satisfying*

$$M_{\Lambda^0,\mu} = \sum_{J \in \mathcal{J}_R} \delta_{\mu, J\Lambda^0}; \qquad M_{\lambda,\Lambda^0} = \sum_{J \in \mathcal{J}_L} \delta_{\lambda, J\Lambda^0} , \tag{3.7d}$$

for some $\mathcal{J}_L, \mathcal{J}_R$. Then

(i) $M_{J\lambda, J'\mu} = M_{\lambda,\mu}$ *for all $\lambda, \mu \in P_+$ and all $J \in \mathcal{J}_L$, $J' \in \mathcal{J}_R$.*
(ii) *$\mathcal{J}_L$ and $\mathcal{J}_R$ are groups and $\|\mathcal{J}_L\| = \|\mathcal{J}_R\|$.*
(iii) $\mathcal{P}_L(M) = \mathcal{P}(\mathcal{J}_L)$ *and* $\mathcal{P}_R(M) = \mathcal{P}(\mathcal{J}_R)$.

Proof. (a) Note that

$$M_{J\lambda, J'\mu} = \sum_{\beta,\gamma} S_{J\lambda,\beta}\, M_{\beta,\gamma}\, S^*_{\gamma, J'\mu} = \sum_{\beta,\gamma} \exp[2\pi i (Q_J(\beta) - Q_{J'}(\gamma))]\, S_{\lambda,\beta}\, M_{\beta,\gamma}\, S^*_{\gamma,\mu} . \tag{3.8a}$$

Applying this to $\lambda = \mu = \Lambda^0$, and using (3.4c), we get that $|M_{J\Lambda^0, J'\Lambda^0}| \leq |M_{\Lambda^0,\Lambda^0}|$ with equality iff the condition (iii) holds. Thus for any physical invariant M, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Statement (iii) implies (iv) by (3.8a), and (iv) $\Rightarrow$ (i) is trivial.

(b) The argument from (3.8a) with $\lambda = \mu = \Lambda^0$ and $J' = id.$ tells us that $J \in \mathcal{J}_L$ if $Q_J(\beta) \in \mathbb{Z}$ for all $\beta \in \mathcal{P}_L(M)$ – this implies, again from (3.8a), that $M_{J\lambda,\mu} = M_{\lambda,\mu}$ for all $J \in \mathcal{J}_L$, for all $\lambda, \mu \in P_+$. Similarly for $\mathcal{J}_R$. Hence $\mathcal{J}_L, \mathcal{J}_R$ are groups. The relation $\|\mathcal{J}_L\| = \|\mathcal{J}_R\|$ comes from the calculation

$$S_{\Lambda^0,\Lambda^0}\, \|\mathcal{J}_L\| = \sum_\gamma S_{\Lambda^0,\gamma}\, M_{\gamma,\Lambda^0} = \sum_\gamma M_{\Lambda^0,\gamma}\, S_{\gamma,\Lambda^0} = S_{\Lambda^0,\Lambda^0}\, \|\mathcal{J}_R\| . \tag{3.8b}$$

In addition, $(SM)_{\lambda,\Lambda^0} = (MS)_{\lambda,\Lambda^0}$ gives us

$$S_{\lambda,\Lambda^0} \sum_{J \in \mathcal{J}_L} \exp[2\pi i\, Q_J(\lambda)] = \sum_\gamma M_{\lambda,\gamma}\, S_{\gamma,\Lambda^0} . \tag{3.8c}$$

The r.h.s. is > 0 iff $\lambda \in \mathcal{P}_L(M)$. The l.h.s. is > 0 iff $Q_J(\lambda) \in \mathbb{Z}$ for all $J \in \mathcal{J}_L$, since $\mathcal{J}_L$ is a group. ■

Of course by Lemma 3.1(a), any $\mathcal{ADE}_7$-type invariant M will obey (3.7d) with $\mathcal{J}_L = \mathcal{J}_L(M)$ and $\mathcal{J}_R = \mathcal{J}_R(M)$. The converse however is false. We will state and prove the remaining results in this section for positive invariants obeying (3.7d), even though our primary interest is in $\mathcal{ADE}_7$-type invariants.

Definition 3.2. For a given positive invariant M obeying (3.7d), call the pair $(\lambda, \mu) \in \mathcal{P}_L(M) \times \mathcal{P}_R(M)$ *M-monogomous* if for all $\nu \in P_+$, both

$$M_{\lambda,\nu} \neq 0 \Rightarrow \nu \in \mathcal{J}_R\mu \ , \qquad \text{and} \qquad M_{\nu,\mu} \neq 0 \Rightarrow \nu \in \mathcal{J}_L\lambda \ .$$

In this case we also say λ (resp. μ) is *right-*(resp. *left-*)*M-monogomous.*

Note that (3.7d) says that (Λ^0, Λ^0) is M-monogomous. For another example, every $\lambda \in \mathcal{P}_L(M)$ for $M = I(\mathcal{J}_d)$ (see (2.3a)) is right-M-monogomous. We will find that M-monogomous pairs are the basic building blocks of $\mathcal{ADE}_7$-type invariants. When (λ, μ) is M-monogomous, the value of $M_{\lambda,\mu}$ is given by (3.9) below. Also, Lemma 3.2(b) below implies that whenever λ and μ are not fixed points of $\mathcal{J}_L$ and $\mathcal{J}_R$ respectively, then $M_{\lambda,\mu} \neq 0$, for some $\mathcal{ADE}_7$-type invariant M, means (λ, μ) must be M-monogomous.

Another tool we will use is the Perron-Frobenius theory of non-negative matrices – see e.g. [14]. By a non-negative matrix B is meant one whose entries are all non-negative real numbers. Such a matrix has an eigenvalue $r(B) \geq 0$ with the property that $r(B)$ is at least as large as the modulus of any other eigenvalue of B. An eigenvector corresponding to $r(B)$ is also non-negative. For example, if B is the $n \times n$ matrix satisfying $B_{ij} = m$ for all i, j, then $r(B) = mn$. There are other properties of non-negative matrices which we will need below; we will state them as we use them. The next lemma uses Perron-Frobenius to severely constrain the form M can take.

For Lemma 3.2 and elsewhere, it is convenient to introduce the direct sum decomposition

$$M = \oplus_j M_j = \begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_m \end{pmatrix} ,$$

where each M_j is indecomposable (i.e. cannot be written as $M_j' \oplus M_j''$). Let $\mathcal{I}(M_j)$ be the index set of M_j. We will always take M_1 to be the unique one with $\Lambda^0 \in \mathcal{I}(M_1)$.

By an *irreducible matrix* in Lemma 3.2(a) below we mean a matrix which cannot, under any simultaneous permutation of row and column indices, be written in the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

for submatrices A, B, D. Irreducible non-negative matrices have special properties [14], as we shall see in the proof of Lemma 3.3 given later.

LEMMA 3.2. (a) *Let M be any positive invariant satisfying (3.7d) for $\mathcal{J} = \mathcal{J}_L = \mathcal{J}_R$. Then for each i, either $M_i = (0)$ or $r(M_i) = \sqrt{r(M_i M_i^T)} = \|\mathcal{J}\|$. Moreover, each nonzero M_i is irreducible.*

(b) *Let M be any positive invariant satisfying (3.7d). If (λ,μ) are M-monogomous, then*

$$M_{J\lambda,J'\mu}=\frac{\|\mathcal{J}_L\|}{\sqrt{\|\mathcal{J}_L\lambda\|\,\|\mathcal{J}_R\,\mu\|}}\ ,\qquad \forall J\in\mathcal{J}_L\ ,\ J'\in\mathcal{J}_R\ . \tag{3.9}$$

Suppose $\beta,\gamma\in P_+$ satisfy $M_{\beta,\gamma}\geq\|\mathcal{J}_L\|/\sqrt{\|\mathcal{J}_L\beta\|\,\|\mathcal{J}_R\gamma\|}$. Then (β,γ) is M-monogomous.

Proof. (a) Consider $M(\ell)\stackrel{\text{def}}{=}(M/\|\mathcal{J}\|)^\ell$. The matrix $\|\mathcal{J}\|\,M(\ell)$ will also satisfy (3.7d). $M(\ell)$ will be a direct sum of $(M_i/\|\mathcal{J}\|)^\ell$. As $\ell\to\infty$, $(M_i/\|\mathcal{J}\|)^\ell$ will tend identically to 0 if $r(M_i)<\|\mathcal{J}\|$, and will be unbounded if $r(M_i)>\|\mathcal{J}\|$ – both these follow for example from the Jordan canonical form of M_i. That each $r(M_i)\leq\|\mathcal{J}\|$ then follows from the (crude) bound

$$\max\nolimits_{\lambda,\mu}M'_{\lambda,\mu}\leq\sum_{\lambda,\mu}M'_{\lambda,\mu}\leq\frac{1}{(S_{\Lambda^0,\Lambda^0})^2}\sum_{\lambda,\mu}S_{\Lambda^0,\lambda}\,M'_{\lambda,\mu}\,S_{\mu,\Lambda^0}=\frac{M'_{\Lambda^0,\Lambda^0}}{(S_{\Lambda^0,\Lambda^0})^2}\ , \tag{3.10}$$

valid for any positive invariant M' (see (3.4c)).

The bound (3.10) also means that the sequence $\{M(\ell)\}_{\ell=1}^\infty$ will have a limit point $\widetilde{M}$, by Bolzano-Weierstrass. $\|\mathcal{J}\|\,\widetilde{M}$ will be a positive invariant, and will also satisfy (3.7d) for $\mathcal{J}=\mathcal{J}_L=\mathcal{J}_R$. By Lemma 3.1(b), this means $\mathcal{P}_L(\widetilde{M})=\mathcal{P}_L(M)$, which forces $r(M_i)=\|\mathcal{J}\|$ whenever $M_i\neq 0$. That $r(M_iM_i^T)=\|\mathcal{J}\|^2$ follows by applying this result to $M\,M^T/\|\mathcal{J}\|$.

Finally, to see that each $M_i\neq 0$ is irreducible, it suffices to show [14] that both M_i and M_i^T have a strictly positive eigenvector corresponding to eigenvalue $\|\mathcal{J}\|$. Let v denote the vector with component $v_\mu=S_{\Lambda^0,\mu}$ for each $\mu\in\mathcal{I}(M_i)$. Then for each $\lambda\in\mathcal{I}(M_i)$,

$$\sum_{\mu\in\mathcal{I}(M_i)}(M_i)_{\lambda,\mu}\,v_\mu=\sum_{\mu\in P_+}M_{\lambda,\mu}\,S_{\mu,\Lambda^0}=\sum_{\mu\in P_+}S_{\lambda,\mu}\,M_{\mu,\Lambda^0}=\|\mathcal{J}\|\,S_{\lambda,\Lambda^0}=\|\mathcal{J}\|\,v_\lambda\ ,$$

so v is a positive eigenvector for M_i. The identical calculation and conclusion holds for M_i^T.

(b) Put $m=\|\mathcal{J}_L\|=\|\mathcal{J}_R\|$. Let $\oplus_iB_i$ be the direct sum decomposition of MM^T/m, where each B_i is indecomposable and $\Lambda^0\in\mathcal{I}(B_1)$. MM^T/m satisfies (3.7d) with $\mathcal{J}=\mathcal{J}_L$, so $r(B_i)=m$ for all nonzero B_i.

First let us prove (3.9). Suppose $\lambda\in\mathcal{I}(B_i)$. Note that by Lemma 3.1(b),

$$(B_i)_{\lambda,\lambda}=\frac{\|\mathcal{J}_R\,\mu\|}{m}\,M^2_{\lambda,\mu}\ ,$$

so by Lemma 3.1(b) and the hypothesis that (λ,μ) is M-monogomous, we get

$$m=r(B_i)=\frac{\|\mathcal{J}_R\,\mu\|}{m}\,M^2_{\lambda,\mu}\,\|\mathcal{J}_L\lambda\|\ .$$

Next, suppose $\beta\in\mathcal{I}(B_j)$. Note that, by Lemma 3.1(b), $B_j\geq B'_j$ (component-wise), where

$$(B'_j)_{\nu\nu'}=\begin{cases}m/\|\mathcal{J}_L\,\beta\| & \text{if }\nu,\nu'\in\mathcal{J}_L\beta\\ 0 & \text{otherwise}\end{cases}\ .$$

Because B_j is irreducible, we have [14] $r(B_j) \geq r(B'_j)$, with equality iff $B_j = B'_j$. However, $r(B'_j) = m$. Hence $B_j = B'_j$. A similar argument applied to $M^T M/m$ now concludes the proof that (β, γ) is M-monogomous. ∎

Definition 3.3. Given $\Gamma \subseteq \mathcal{P}(\mathcal{J})$, define

$$\Gamma^{(1)} = \{\lambda \in \mathcal{P}(\mathcal{J}) \,|\, \forall \mu \in \mathcal{P}(\mathcal{J}),\ \mu \notin \mathcal{J}\lambda,\ \exists \gamma \in \Gamma \text{ satisfying } \frac{S_{\gamma,\lambda}}{S_{\Lambda^0,\lambda}} \neq \frac{S_{\gamma,\mu}}{S_{\Lambda^0,\mu}} \}\ , \tag{3.11}$$

and $\Gamma^{(n)} = (\Gamma^{(n-1)})^{(1)}$. Γ is called a *fusion-generator* for $\mathcal{J}$ if $\Gamma^{(n)} = \mathcal{P}(\mathcal{J})$ for some n.

The name comes from the fact that the numbers $\frac{S_{\gamma,\lambda}}{S_{\Lambda^0,\lambda}}$ are the eigenvalues of the fusion matrix N_γ, whose entries are fusion coefficients (3.4e).

Lemma 3.3 below tells us that it suffices for most purposes to look at the Γ-rows and -columns of M. To find fusion-generators, by (3.4d) it is natural to look at the representation ring of X_r – see Proposition 4.1 below. For example, if $\mathcal{J} = \{id.\}$, a fusion-generator is formed from the lifts $(k - a_i^\vee)\Lambda_0 + \Lambda_i$ into P_+ of the horizontal fundamental weights $\bar{\Lambda}_i$, for all $i = 1, \ldots, r$ (provided $k \geq \max_i\{a_i^\vee\}$).

LEMMA 3.3. (a) *Let Γ be a fusion-generator for $\mathcal{J}$. Let M be a positive invariant obeying (3.7d) with $\mathcal{J}_L = \mathcal{J}_R = \mathcal{J}$, such that (γ, γ) is M-monogomous for all $\gamma \in \Gamma$. Then for all $\lambda \in \mathcal{P}(\mathcal{J})$, (λ, λ) is M-monogomous.*

(b) *Let M be any positive invariant obeying (3.7d). Let Γ_L and Γ_R be fusion-generators for $\mathcal{J}_L$ and $\mathcal{J}_R$, respectively. Suppose that each $\gamma \in \Gamma_L$ is right-M-monogomous, and each $\gamma \in \Gamma_R$ is left-M-monogomous. Then there exists a map $\sigma : \mathcal{P}(\mathcal{J}_L) \to \mathcal{P}(\mathcal{J}_R)$ such that $(\lambda, \sigma\lambda)$ is M-monogomous, and the induced map $\mathcal{J}_L\lambda \mapsto \mathcal{J}_R\sigma\lambda$ is a bijection between the $\mathcal{J}_L$-orbits in $\mathcal{P}(\mathcal{J}_L)$ and the $\mathcal{J}_R$-orbits in $\mathcal{P}(\mathcal{J}_R)$. In addition,*

$$S_{\lambda,\mu} = \sqrt{\frac{\|\mathcal{J}_R\sigma\lambda\|\,\|\mathcal{J}_R\sigma\mu\|}{\|\mathcal{J}_L\lambda\|\,\|\mathcal{J}_L\mu\|}}\, S_{\sigma\lambda,\sigma\mu}, \qquad \forall \lambda, \mu \in \mathcal{P}(\mathcal{J}_L)\ ; \tag{3.12a}$$

$$\sum_{J \in \mathcal{J}_L} N_{\lambda,\mu}^{J\nu} = \sqrt{\frac{\|\mathcal{J}_R\sigma\lambda\|\,\|\mathcal{J}_R\sigma\mu\|\,\|\mathcal{J}_R\sigma\nu\|}{\|\mathcal{J}_L\lambda\|\,\|\mathcal{J}_L\mu\|\,\|\mathcal{J}_L\nu\|}} \sum_{J' \in \mathcal{J}_R} N_{\sigma\lambda,\sigma\mu}^{J'\sigma\nu}, \quad \forall \lambda, \mu, \nu \in \mathcal{P}(\mathcal{J}_L). \tag{3.12b}$$

Finally, let M' be any other positive invariant obeying (3.7d) for the same $\mathcal{J}_L, \mathcal{J}_R$. If for all $\gamma \in \Gamma_L$, each $(\gamma, \sigma\gamma)$ is also M'-monogomous, then $M = M'$.

Proof. (a) Write $M = \oplus_i M_i$, where each M_i is irreducible (Lemma 3.2(a)). We get the equations

$$\|\mathcal{J}\|\, S_{\gamma,\lambda} = (MS)_{\gamma,\lambda} = (SM)_{\gamma,\lambda} = \sum_\mu S_{\gamma,\mu} M_{\mu,\lambda}, \quad \forall \lambda \in P_+\ , \tag{3.13}$$

and all $\gamma \in \{\Lambda^0\} \cup \Gamma$. In other words, writing v_i^γ for the vector with components $(v_i^\gamma)_\lambda = S_{\gamma,\lambda}$ for all $\lambda \in \mathcal{I}(M_i)$, (3.13) tells us that v_i^γ is a left-eigenvector of M_i with the Perron-Frobenius eigenvalue $\|\mathcal{J}\| = r(M_i)$. Since M_i is irreducible, this means [14] each v_i^γ must be a scalar multiple of $v_i^{\Lambda^0}$, i.e.

$$\frac{S_{\gamma,\lambda}}{S_{\Lambda^0,\lambda}} = \frac{S_{\gamma,\mu}}{S_{\Lambda^0,\mu}} \qquad \text{for all } \lambda, \mu \in \mathcal{I}(M_i)\ ,$$

for each $\gamma \in \Gamma$ and each submatrix M_i. Hence by (3.11), (λ, λ) will be M-monogomous for each $\lambda \in \Gamma^{(1)}$. Continuing recursively, we get the desired result.

(b) Let $\widetilde{M} = M^T M/\|\mathcal{J}_L\|$. Then $\widetilde{M}$ satisfies the hypotheses of Lemma 3.3(a) with $\mathcal{J} = \mathcal{J}_R$ and $\Gamma = \Gamma_R$, so for each $\mu \in \mathcal{P}(\mathcal{J}_R)$, (μ, μ) is $\widetilde{M}$-monogomous. Thus if $\mu' \notin \mathcal{J}_R\lambda$, then for any $\nu \in P_+$, $M_{\nu,\mu}M_{\nu,\mu'} = 0$. Similarly, for any $\lambda \in \mathcal{P}(\mathcal{J}_L)$ and $\nu \in P_+$, $M_{\lambda,\nu}M_{\lambda',\nu} = 0$ whenever $\lambda' \notin \mathcal{J}_L\lambda$. These statements force the existence of the map σ. Equations (3.6b), (3.9) now directly give (3.12a), and then (3.12b) follows from (3.4e).

Finally, let $\overline{M} = M\, M'^T/\|\mathcal{J}_R\|$; from Lemma 3.3(a) we find (λ, λ) is $\overline{M}$-monogomous for all $\lambda \in \mathcal{P}(\mathcal{J}_L)$. Thus, using (3.9), $M^T = M^T\overline{M}/\|\mathcal{J}_L\| = \widetilde{M}\, M'^T/\|\mathcal{J}_R\| = M'^T$. ■

Remark 3.1. For $A_r^{(1)}$, the case we are interested in in this paper, Lemma 3.1(b)(ii) forces $\mathcal{J}_L = \mathcal{J}_R$, and we can drop the hypothesis in Lemma 3.3(b) which says that (γ, γ) must be M-monogomous for all $\gamma \in \Gamma_R$ – this will follow from the Γ_L hypothesis. More generally, the same thing happens *whenever the number of $\mathcal{J}_L$-orbits in $\mathcal{P}(\mathcal{J}_L)$ equals the number of $\mathcal{J}_R$-orbits in $\mathcal{P}(\mathcal{J}_R)$*. That we can ignore Γ_R in this case follows from the proof of Lemma 3.3(b) given above.

4. The proof of the Theorem. In this section we specialize to the affine algebra $A_r^{(1)}$, and outline the proof of Theorem 2.1. We begin by collecting together results particular to $A_r^{(1)}$.

Fix the algebra $A_r^{(1)}$ and the level k. Let $\bar{r} = r + 1$, $\bar{k} = k + \bar{r}$. The set P_+ of weights is given by (3.1) with all $a_j^\vee = 1$. Recall the definitions of $J \in \mathcal{S}_{sc}$, C, and t, given in Section 2. $\mathcal{S}_{sc}$ is generated by J, and $\mathcal{S}$ by J and C. The Kac-Peterson matrices S and T are defined in (3.3b),(3.3d). We have:

$$t(J^a\lambda) \equiv ka + t(\lambda) \qquad (\mathrm{mod}\ \bar{r}) \tag{4.1a}$$

$$T_{J^a\lambda, J^a\mu} = \exp[\pi i\,(-2a\,t(\lambda) + ka\,(\bar{r} - a))/\bar{r}]\; T_{\lambda,\mu} \tag{4.1b}$$

$$S_{J^a\lambda, J^b\mu} = \exp[2\pi i\,(b\,t(\lambda) + a\,t(\mu) + kab)/\bar{r}]\; S_{\lambda,\mu} \tag{4.1c}$$

$$S_{C\lambda,\mu} = S^*_{\lambda,\mu} \tag{4.1d}$$

$$T_{C\lambda,C\mu} = T_{\lambda,\mu} \tag{4.1e}$$

(compare (4.1b),(4.1c) with (3.5b),(3.5c) – note that $Q_{J^d}(\lambda) = d\,t(\lambda)/\bar{r}$). The subgroups of $\mathcal{S}_{sc}$ are $\mathcal{J}_d$, which is generated by J^d where d divides $\bar{r}$. We will write $[\lambda]$ for the orbit of λ over $\mathcal{S}$, i.e. generated by J and C, and $[\lambda]_d$ for the orbit over $\mathcal{J}_d$, i.e. generated by J^d. Write $\mathcal{P}_d = \mathcal{P}(\mathcal{J}_d)$. Let $\mathcal{F}_d$ denote the fixed points of $\mathcal{J}_d$. They look, schematically, like $(\mu, \ldots, \mu)$ for some d'-tuple μ and some multiple $d' < \bar{r}$ of d.

The matrix S obeys a surprising relation called *rank-level duality*, related to the existence of the embedding $su(\bar{r}) \oplus su(k) \subset su(\bar{r}k)$. Given any $\lambda \in P_+$, define a weight $T(\lambda)$ of $A_{k-1}^{(1)}$ level $r + 1$, as follows. First construct the Young diagram of λ: for $1 \leq i \leq r$, its ith row consists of $\sum_{j=i}^{r} \lambda_j$ boxes. Take the transpose of this diagram; deleting all columns (if any) of length k, this will be the Young diagram of some level $r + 1$ $A_{k-1}^{(1)}$ weight which we will denote by $T(\lambda)$. For example, $T((k-1)\Lambda_0 + \Lambda_\ell) = (\bar{r} - \ell)\tilde{\Lambda}_0 + \ell\tilde{\Lambda}_1$ for all ℓ – to avoid confusion *we will always put tilde's over the quantities of $A^{(1)}_{k-1}$ level* $r+1$. It was shown in [1], by applying Laplace's determinant formula [14] to a determinant expression for the S matrix entries, that

$$S_{\lambda,\mu}=\sqrt{\frac{k}{\bar{r}}}\,\exp[\frac{2\pi i}{\bar{r}k}t(\lambda)\,t(\mu)]\;\tilde{S}^*_{T(\lambda),T(\mu)}\;. \tag{4.2a}$$

Note that this map T defines a bijection between $\mathcal{J}_1$-orbits in P_+ and $\tilde{\mathcal{J}}_1$-orbits in $\tilde{P}_+$.

(4.2a) takes a simpler form for $\lambda,\mu\in\mathcal{P}_1$: for $\lambda\in\mathcal{P}_1$, let $T'(\lambda)=\tilde{J}^{-t(\lambda)/\bar{r}}T(\lambda)$; then the obvious calculation from (4.2a) gives

$$S_{\lambda,\mu}=\sqrt{\frac{k}{\bar{r}}}\,\tilde{S}^*_{T'(\lambda),T(\mu)}\qquad\forall\mu\in P_+ \tag{4.2b}$$

$$=\sqrt{\frac{k}{\bar{r}}}\,\tilde{S}^*_{T'(\lambda),T'(\mu)}\qquad\forall\mu\in\mathcal{P}_1\;. \tag{4.2c}$$

Since, in addition, $\lambda\in\mathcal{P}_1$ implies

$$\tilde{t}(T'(\lambda))\equiv\bar{r}\,(-t(\lambda)/\bar{r})+\tilde{t}(T(\lambda))\equiv 0\pmod{k}\;,$$

we see that T' takes $\mathcal{P}_1$ to $\tilde{\mathcal{P}}_1$. In fact, (4.2b) implies that T' is a bijection.

Incidently, similar formulae hold for the matrix T. In particular,

$$T_{\lambda,\lambda}\,T^*_{\Lambda^0,\Lambda^0}=\exp[\frac{\pi i}{\bar{r}k}t(\lambda)(\bar{r}k-t(\lambda))]\;\tilde{T}^*_{T(\lambda),T(\lambda)}\,\tilde{T}_{\tilde{\Lambda}^0,\tilde{\Lambda}^0}\qquad\forall\lambda\in P_+ \tag{4.3a}$$

$$=\tilde{T}^*_{T'(\lambda),T'(\lambda)}\,\tilde{T}_{\tilde{\Lambda}^0,\tilde{\Lambda}^0}\qquad\forall\lambda\in\mathcal{P}_1\;. \tag{4.3b}$$

Choose any $\mathcal{ADE}_7$-type invariant M. Then by Lemma 3.1(b), $\mathcal{J}_L(M)$ is a subgroup of $\mathcal{S}_{sc}$, so equals $\mathcal{J}_d$ for some d. Lemma 3.1(b) also says $\mathcal{J}_d=\mathcal{J}_R(M)$ – write $\mathcal{J}(M)=\mathcal{J}_d$. Then by (3.6a),

$$k'd^2\equiv 0\qquad\pmod{2\bar{r}}\;, \tag{4.4}$$

where k' is defined in Section 2. (4.4) and (4.1b) together imply

$$(J^d\lambda+\rho\,|\,J^d\lambda+\rho)\equiv(\lambda+\rho\,|\,\lambda+\rho)\pmod{2\bar{r}}\qquad\forall\lambda\in\mathcal{P}_d\;.$$

The first important step in the proof is to find fusion-generators (Definition 3.3) for $\mathcal{J}_d$. Let

$$\lambda^i\stackrel{\text{def}}{=}(k-2)\Lambda_0+\Lambda_i+\Lambda_{\bar{r}-i}\qquad\text{for }1\le i\le\bar{r}/2\;, \tag{4.5a}$$

$$\mu^j\stackrel{\text{def}}{=}(k-3)\Lambda_0+\Lambda_1+\Lambda_j+\Lambda_{r-j}\qquad\text{for }1\le j\le r/2\;, \tag{4.5b}$$

$$\Lambda^i\stackrel{\text{def}}{=}(k-1)\Lambda_0+\Lambda_i\qquad\text{for }0\le i<\bar{r}\;,\qquad\Lambda^{\bar{r}}\stackrel{\text{def}}{=}\Lambda^0\;. \tag{4.5c}$$

PROPOSITION 4.1. (a) *For any divisor d of $\bar{r}$, let $\mathbb{F}_1$ denote the field of all rational functions over $\mathbb{Q}$ in the Weyl characters $\overline{ch}_{\bar{\nu}}$, for all horizontal weights $\bar{\nu}\in\cup_{k=1}^{\infty}\overline{\mathcal{P}_d}$. Then $\mathbb{F}_d$ is generated by the characters of the $\bar{r}$ weights $\bar{\lambda}^i$, $\bar{\mu}^j$, and $\bar{\Lambda}^{\bar{r}/d}$.*

(b) *The $\bar{r}$ weights in $\Gamma_d = \{\lambda^i, \mu^j, \Lambda^{\bar{r}/d}\}$ form a fusion-generator for $\mathcal{J}_d$. Choosing $m, \ell \in \mathbb{Z}$ such that $0 \leq \bar{r}/d - mk = \ell < k$, the k weights in $\Gamma'_d = \{\widetilde{T}'(\tilde{\lambda}^i), \widetilde{T}'(\tilde{\mu}^j), J^m\widetilde{T}(\tilde{\Lambda}^\ell)\}$ also form a fusion-generator for $\mathcal{J}_d$.*

Proof. (a) $\mathbb{F}_d$ is a subfield of the field of fractions of the representation ring of A_r, the latter being isomorphic by a theorem of Chevelley (see e.g. [3]) to the ring of polynomials $\mathbb{Q}[x_1, \ldots, x_r]$, the isomorphism sending

$$\bar{ch}_{\bar{\Lambda}_i} \mapsto x_i \ .$$

Thus $\mathbb{F}_d$ can be thought of as the field generated over $\mathbb{Q}$ by the monomials $x_1^{a_1} \cdots x_r^{a_r}$, where $t(a) \stackrel{\text{def}}{=} \sum j a_j \equiv 0 \pmod{\bar{r}/d}$. Consider the case $d = 1$; the result for general d follows immediately from this.

Note that the r monomials $x_1 x_r, x_2 x_r^2, \ldots, x_r^{\bar{r}}$ clearly generate $\mathbb{F}_1$:

$$x_1^{a_1} \cdots x_r^{a_r} = (x_r^{\bar{r}})^{-t(a)/\bar{r}} \prod_{j=1}^{r} (x_j x_r^j)^{a_j} \ .$$

Thus so do $x_i x_{\bar{r}-i}$ and $x_1 x_j x_{r-j}$: recursively,

$$x_\ell x_r^\ell = (x_{\ell-1} x_r^{\ell-1}) \cdot \frac{(x_1 x_r)\,(x_\ell x_{\bar{r}-\ell})}{(x_1 x_{\ell-1} x_{r-\ell+1})} \ .$$

Let $\overline{\Omega}(\bar{\mu})$ denote the dominant weights of $\bar{L}(\bar{\mu})$. Now, $\bar{\nu} \in \overline{\Omega}(\bar{\mu})$ iff $\bar{\mu} - \bar{\nu}$ is a sum of positive roots. From this we find

$$\overline{\Omega}(\bar{\lambda}^i) = \{0, \bar{\lambda}^1, \ldots, \bar{\lambda}^i\} \ ,$$
$$\overline{\Omega}(\bar{\mu}^j) = \{0, \bar{\lambda}^1, \ldots, \bar{\lambda}^{j+1}, \bar{\mu}^1, \ldots, \bar{\mu}^j\} \ .$$

Thus we can express $x_i x_{\bar{r}-i}$ and $x_1 x_j x_{r-j}$ in terms of the polynomials corresponding to $\overline{ch}_{\bar{\mu}}$, for $\bar{\mu} \in \{\bar{\lambda}^i, \bar{\mu}^j\}$.

(b) Define $X_d = \{\lambda \in \mathcal{P}_d \,|\, S_{\Lambda^i,\lambda} \neq 0 \ \forall i\}$. We begin by showing that $X_d \subseteq \Gamma_d^{(1)}$. Indeed, choose any $\lambda \in X_d$, $\mu \in \mathcal{P}_d$ with

$$\frac{S_{\gamma,\lambda}}{S_{\Lambda^0,\lambda}} = \frac{S_{\gamma,\mu}}{S_{\Lambda^0,\mu}} \qquad \forall \gamma \in \Gamma_d \ . \tag{4.6a}$$

Then by choosing $\gamma = \lambda^i$ for various i we find that also $\mu \in X_d$. Now, for any $\alpha, \beta \in \mathcal{P}_d$, (3.4d) and Prop. 4.1(a) tell us that $S_{\alpha,\beta}/S_{\Lambda^0,\beta}$ can be expressed as a rational function in the numbers $S_{\gamma,\beta}/S_{\Lambda^0,\beta}$, $\gamma \in \Gamma_d$; we find from the proof of Prop. 4.1(a) that this expression will be well-defined (i.e. not of the form 0/0) if $\beta \in X_d$. What this means is that (4.6a) implies

$$\frac{S_{\alpha,\lambda}}{S_{\Lambda^0,\lambda}} = \frac{S_{\alpha,\mu}}{S_{\Lambda^0,\mu}} \qquad \forall \alpha \in \mathcal{P}_d \ . \tag{4.6b}$$

Multiplying (4.6b) by $\sum_{J\in\mathcal{J}_d} S^*_{\alpha,J\lambda}$ (which vanishes for $\alpha\notin\mathcal{P}_d$) and summing over all $\alpha\in P_+$, gives us

$$\frac{\|\mathcal{J}_d\|}{\|\mathcal{J}_d\lambda\|}\,\frac{1}{S_{\Lambda^0,\lambda}}=\sum_{J\in\mathcal{J}_d}\frac{\delta_{J\lambda,\mu}}{S_{\Lambda^0,\mu}}\ ,$$

i.e. $\mu\in[\lambda]_d$. Therefore, we indeed have $X_d\subseteq\Gamma_d^{(1)}$.

Now we will show $\Gamma_d^{(2)}=\mathcal{P}_d$. Choose any $\lambda,\mu\in\mathcal{P}_d$ such that (4.6a) holds. Define the sums

$$s(\alpha,\beta)=\frac{S_{\Lambda^0,\mu}\,S_{\Lambda^0,\lambda}}{S_{\Lambda^0,\alpha}\,S_{\Lambda^0,\beta}}\sum_{\nu\in\mathcal{P}_d}S_{\nu,\alpha}\,S^*_{\nu,\beta}\prod_{i=1}^{r}|S_{\Lambda^i,\nu}|\ .$$

If $s(\lambda,\lambda)\neq s(\lambda,\mu)$, then $\exists\nu\in\mathcal{P}_d$ such that $S_{\Lambda^i,\nu}\neq0\ \forall i$, and $S^*_{\nu,\lambda}S_{\Lambda^0,\mu}/S_{\Lambda^0,\lambda}\neq S^*_{\nu,\mu}$ – by the previous result this would mean $\lambda\in\Gamma_d^{(2)}$. Similarly if $s(\nu,\nu)\neq s(\lambda,\nu)$. Thus we may assume $s(\lambda,\lambda)=s(\lambda,\nu)=s(\nu,\nu)$. But then the triangle inequality tells us

$$0=s(\lambda,\lambda)+s(\nu,\nu)-2\,s(\lambda,\mu)\geq\sum_{\nu\in\mathcal{P}_d}\{(\sqrt{\frac{S_{\Lambda^0,\mu}}{S_{\Lambda^0,\lambda}}}|S_{\nu,\lambda}|-\sqrt{\frac{S_{\Lambda^0,\lambda}}{S_{\Lambda^0,\mu}}}|S_{\nu,\mu}|)^2\}\prod_{i=1}^{r}|S_{\Lambda^i,\nu}|\geq0.$$

Therefore we must have $S_{\nu,\lambda}\,S^*_{\nu,\mu}\geq0\ \forall\nu\in\mathcal{P}_d$ (it is strictly larger for e.g. $\nu=\Lambda^0$). Now from (3.4e) applied to $N^{\mu'}_{\Lambda^0,\lambda}=\delta_{\mu',\lambda}$, we find

$$\sum_{\mu'\in[\mu]_d}\delta_{\mu',\lambda}=\sum_{\mu'\in[\mu]_d}\sum_{\nu\in\mathcal{P}_d}S_{\lambda,\nu}\,S^*_{\mu',\nu}=\|[\mu]_d\|\sum_{\nu\in\mathcal{P}_d}S_{\lambda,\nu}S^*_{\mu,\nu}>0\ .$$

This calculation forces $\mu\in[\lambda]_d$, which concludes the proof that Γ_d is a fusion-generator of $\mathcal{J}_d$.

We find that $\{\nu\in\mathcal{P}_d\,|\,S_{(k-i)\Lambda_0+i\Lambda_1,\nu}\neq0\ \forall i\}\subseteq\Gamma_d'^{(1)}$, by using (4.2b) and the Γ_1 argument. That $\Gamma_d'^{(2)}=\mathcal{P}_d$ now follows by the argument used for Γ_d, with Λ^i there replaced here with $(k-i)\Lambda_0+i\Lambda_1$. ■

Strictly speaking, Γ_d requires $k\geq3$: if $k=2$, simply drop the μ^j. We will discuss the trivial case $k=1$ at the end of this section. Similarly, Γ'_d requires $r\geq2$: if $r=1$ drop the $\widetilde{T}'(\tilde{\mu}^j)$.

When $\bar{r}\leq k$, we will choose the fusion-generator $\Gamma=\Gamma_d$; when $\bar{r}>k>1$ we will usually choose the smaller set $\Gamma=\Gamma'_d$. Note that $\widetilde{T}'(\tilde{\lambda}^i)=(k-2i)\Lambda_0+i\Lambda_1+i\Lambda_r$, and $\widetilde{T}'(\tilde{\mu}^j)=(k-2j-1)\Lambda_0+(j-1)\Lambda_1+\Lambda_2+(j+1)\Lambda_r$.

Suppose $M_{\lambda,\mu}\neq0$. Then using Lemma 3.1(a),

$$\frac{\bar{r}}{d}\,S_{\Lambda^0,\mu}=\sum_{\nu}M_{\Lambda^0,\nu}\,S_{\nu,\mu}=\sum_{\nu}S_{\Lambda^0,\nu}\,M_{\nu,\mu}\geq\|\mathcal{J}_d\lambda\|\,S_{\Lambda^0,\lambda}\,M_{\lambda,\mu}\ . \tag{4.7a}$$

Thus if we can show

$$\max_{\gamma\in\Gamma}\{\mathcal{D}^{r,k}(\gamma)\}<\frac{\|\mathcal{J}_d\varphi\|}{\bar{r}/d}\,\mathcal{D}^{r,k}(\varphi)\qquad\forall\varphi\in\mathcal{F}_d\ , \tag{4.7b}$$

where $\mathcal{D}^{r,k}(\lambda)$ is the q-dimension defined in (3.4b), then that would mean $M_{\gamma,\varphi} = M_{\varphi,\gamma} = 0$ for all fixed points φ and each $\gamma \in \Gamma$. In the following section we use this idea together with Lemmas 3.2(b) and 3.3(b) to prove:

PROPOSITION 5.1′. *For all but finitely many pairs (r,k), each $\lambda \in \mathcal{P}_d$ is right-M-monogomous, and neither λ^1 nor $\sigma\lambda^1$ will be fixed points of $\mathcal{J}_d$.*

From the previous discussion, this is not too surprising a result considering that for fixed r, the minimum of the r.h.s. of (4.7b) tends to ∞ as $k \to \infty$, while the l.h.s. tends to the dimension of some $\overline{L}(\overline{\gamma})$. The remaining (r,k) are treated in section 7.

Let σ denote the map in Lemma 3.3(b). Provided $\|[\lambda^1]_d\| = \|[\sigma\lambda^1]_d\|$ (this will hold e.g. if neither λ^1 nor $\sigma\lambda^1$ are fixed points), we see from (3.12a) that

$$\mathcal{D}^{r,k}(\lambda^1) = \mathcal{D}^{r,k}(\sigma\lambda^1) \ . \tag{4.8}$$

Using (4.8) and (3.6a), we prove in sections 5 and 7 that:

PROPOSITION 5.2′. *Suppose all $\lambda \in \mathcal{P}_d$ are right-M-monogomous, and (4.8) holds. Then $\sigma\lambda^1 \in [\lambda^1]$.*

We also know from (3.12b) that γ and $\sigma\gamma$ have similar fusions (3.4f), for $\gamma \in \Gamma_d$. From this we prove in section 6:

PROPOSITION 6.1′. *Suppose all $\lambda \in \mathcal{P}_d$ are right-M-monogomous, and (4.8) holds. Then for some $c = 0,1$, we have for all i,j that $(\lambda^i, C^c\lambda^i)$ and $(\mu^j, C^c\mu^j)$ are M-monogomous, and $\sigma\Lambda^{\bar{r}/d} \in [C^c\Lambda^{\bar{r}/d}]_1$.*

Putting these all together, we are now prepared to prove:

PROPOSITION 4.2. *Let M be an $\mathcal{ADE}_7$-type invariant with $\mathcal{J}_d = \mathcal{J}(M)$, and let Γ be any fusion-generator of $\mathcal{J}_d$. Suppose each $\gamma \in \Gamma$ is right-M-monogomous, and (4.8) holds. Then for some divisor d' of $\bar{r}$ for which $k'd'$ is even, and some $c = 0,1$, we have $M = C^c \cdot I(\mathcal{J}_{d'})$ (see (2.3)).*

Proof. By Remark 3.1, all $\lambda \in \mathcal{P}_d$ will be right-M-monogomous. Let $c = 0,1$ be as in Proposition 6.1′. For convenience replace M with $C^c \cdot M$. Then each (λ^i, λ^i) and (μ^j, μ^j) is M-monogomous, and $(\Lambda^{\bar{r}/d}, J^m\Lambda^{\bar{r}/d})$ is M-monogomous for some m. By Lemma 3.3(a) and (4.4) we are done if $d = 1$ (take $d' = 1$), so consider $d > 1$.

Equation (3.6a), and the fact that d divides $\bar{r}$, tells us

$$\frac{m}{d} + \frac{k'm^2}{2\bar{r}} \equiv 0 \qquad (\mathrm{mod}\ 1) \ . \tag{4.9a}$$

Put $d' = gcd\{m,d\}$. We may assume, by adding a multiple of d to m if necessary (see Lemma 3.1(a)), that $gcd\{m/d', 2\bar{r}\} = 1$. Then (4.9a) becomes

$$\frac{-2\bar{r}}{d} \equiv k'm \qquad (\mathrm{mod}\ 2\bar{r}/d') \ . \tag{4.9b}$$

Consider $M' = I[\mathcal{J}_{d'}]$. M' is a physical invariant because (2.3b) follows from (4.9b), and the facts that d divides $\bar{r}$ and m/d' is odd. Then $\mathcal{J}(M') = \mathcal{J}_d$ iff

$$\frac{2\bar{r}}{d} = gcd\{\frac{2\bar{r}}{d'}, d'k'\} . \tag{4.9c}$$

But (4.9c) follows from (4.9b) and the fact that d' divides d. Also, (4.9b) tells us that $M'_{\Lambda^{\bar{r}/d}, J^m\Lambda^{\bar{r}/d}} \neq 0$. Thus by Lemma 3.3(b), $M = M'$. ■

In particular, Propositions 4.2 and 5.1′ suffice to prove Theorem 2.1 for most pairs (r, k). The finitely many remaining pairs are handled in section 7.

Incidently, Proposition 4.2 permits an immediate proof of Theorem 2.1 for $k = 1$. It suffices to note that for $k = 1$: (i) $\mathcal{P}_d$ is generated by $\Lambda_{\bar{r}/d}$; (ii) $\mathcal{P}_d = [\Lambda_{\bar{r}/d}]_{\bar{r}/d}$; (iii) $\mathcal{P}_d$ has no fixed points. $k = 1$ was first proved in [5], though in a very different way. For convenience we will henceforth restrict attention to $k \geq 2$.

5. Q-dimension calculations. The point of this section is to prove, using q-dimensions (3.4b), that for most (r, k), (4.7b) will be satisfied, and for most (r, k), (4.8) implies $\sigma\lambda^1 \in [\lambda^1]$. We begin by listing some of the properties q-dimensions obey.

This section is the most technically complicated of the paper. Even so, q-dimensions are extremely well-behaved and amenable to analysis. Recall their definition in (3.4b) – the name comes from interpreting them as "q-deformed Weyl dimensions".

Note from (4.1c),(4.1d) that

$$[\lambda] = [\mu] \quad \Longrightarrow \quad \mathcal{D}^{r,k}(\lambda) = \mathcal{D}^{r,k}(\mu) . \tag{5.1a}$$

Also, an immediate consequence of rank-level duality (4.2a) is that

$$\mathcal{D}^{r,k}(\lambda) = \mathcal{D}^{k-1,r+1}(T(\lambda)) . \tag{5.1b}$$

By $C^{r,k}$ we mean the fundamental chamber

$$C^{r,k} \stackrel{\text{def}}{=} \{\sum_{i=0}^{r} x_i\Lambda_i \,|\, x_i \in \mathbb{R},\ x_i \geq 0,\ \sum_{i=0}^{r} x_i = k\} . \tag{5.2a}$$

Extend the domain of $\mathcal{D}^{r,k}$ from P_+ to $C^{r,k}$, using (3.4b). Choose any real $\bar{r}$-vectors a, b, $b \neq 0$, such that $a + bu \in C^{r,k}$ for all $u \in [u_0, u_1]$. Then for $u_0 \leq u' \leq u_1$, an easy calculation gives [11]

$$\frac{d}{du}\mathcal{D}^{r,k}(a + bu)|_{u=u'} = 0 \Rightarrow \frac{d^2}{du^2}\mathcal{D}^{r,k}(a + bu)|_{u=u'} < 0 . \tag{5.2b}$$

This implies the important fact:

$$\forall u_0 < u < u_1, \quad \mathcal{D}^{r,k}(a + bu) > \min\{\mathcal{D}^{r,k}(a + bu_0),\ \mathcal{D}^{r,k}(a + bu_1)\} . \tag{5.2c}$$

Elementary consequences of (5.2c) are, for all $\lambda \in P_+$,

$$\mathcal{D}^{r,k}(\lambda) \geq 1 \ , \tag{5.3a}$$
$$\mathcal{D}^{r,k}(\lambda) = 1 \quad \text{iff} \quad \lambda \in [\Lambda^0] \ . \tag{5.3b}$$

The following trigonometric identity, obtained from the factorisation of $y^m - 1$, will be used to simplify some expressions:

$$\prod_{\ell=0}^{m-1} \sin(x + \frac{\ell}{m}\pi) = \frac{\sin(mx)}{2^{m-1}} \ . \tag{5.4}$$

Ultimately, in comparing specific q-dimensions, we will have to estimate sizes of various products and quotients of sine's. Apart from the obvious trigonometric identities, a useful technique is to investigate the behaviour as k or r tends to ∞. A basic fact is that $x \cot x$ decreases: this means, for $a > 1$ and $0 < y < x \leq \pi/a$, that

$$a > \sin(ay)/\sin(y) > \sin(ax)/\sin(x) \ . \tag{5.5a}$$

For example consider a sequence $\beta^k \in C^{r,k}$ with constant projection $\overline{\beta} = \overline{\beta}^k$. Then (5.5a) and the Weyl dimension formula tells us

$$\mathcal{D}^{r,k}(\beta^k) < \prod_{\bar{\alpha}>0} \frac{(\overline{\beta} + \bar{\rho}|\bar{\alpha})}{(\bar{\rho}|\bar{\alpha})} = \dim \overline{L}(\overline{\beta}) \ , \tag{5.5b}$$

with the l.h.s. converging monotonically to the r.h.s. as $k \to \infty$. Another basic fact is the concavity of $\ln|\sin x|$:

$$\sin(a)\ \sin(b) < \sin(a-x)\ \sin(b+x) \ , \tag{5.5c}$$

provided $0 < b < a < \pi$ and $0 < x \leq (a-b)/2$.

Fix a divisor d of $\bar{r}$. As in section 4, for $\bar{r} \leq k$ choose the fusion-generator $\Gamma = \Gamma_d$ (cardinality $\bar{r}$), while for $\bar{r} > k$ choose $\Gamma = \Gamma'_d$ (cardinality k).

PROPOSITION 5.1. *All $r \geq 1$, $k \geq 1$ satisfy* (4.7b) *unless (r,k) equals*

(i) $(1,k)$ *for* $k \in \{2,4,6,\ldots,16\}$,
(ii) $(2,k)$ *for* $k \in \{3,6,9\}$,
(iii) $(3,k)$ *for* $k \in \{4,6,8,10\}$,
(iv) $(4,5)$,
(v) $(5,k)$ *for* $k \in \{6,8,10\}$,
(vi) (7,8),
(vii) $(r,6)$ *for* $r \in \{7,9\}$,
(viii) $(r,4)$ *for* $r \in \{5,7,9\}$,
(ix) $(r,3)$ *for* $r \in \{5,8\}$,
(x) $(r,2)$ *for* $r \in \{3,5,\ldots,15\}$.

Proof. Choose any fixed point $\varphi \in \mathcal{F}_d$. The period of φ (with respect to $\mathcal{J}_d$) is $p = d\,\|\mathcal{J}_d\varphi\|$: $\varphi_i = \varphi_j$ if $i \equiv j \pmod{p}$. Write

$$\varphi^p \stackrel{\text{def}}{=} \sum_{j=0}^{\bar{r}/p-1} \frac{kp}{\bar{r}}\,\Lambda_{pj}\ .$$

Putting $b = J^i\varphi^p - J^j\varphi^p$ in (5.2c) and using (5.1a), we find that $\mathcal{D}^{r,k}(\varphi) \geq \mathcal{D}^{r,k}(\varphi^p)$. Thus it suffices to consider only the fixed points of the form φ^p. By (5.4) and (3.4a):

$$S_{\Lambda^0,\varphi^p} = |s|\,2^{\bar{r}(p-1)/2}\,(\bar{r}/p)^{\bar{r}/2}\prod_{j=1}^{p-1}\sin(\frac{\pi\,j\bar{r}}{p\bar{k}})^{\bar{r}-j\bar{r}/p}\ . \tag{5.6}$$

Next, turn to the evaluation of the l.h.s. of (4.7b). Directly from (3.4b) we get

$$\mathcal{D}^{r,k}(\lambda^\ell) = \frac{\sin(\pi\,(\bar{r}-2\ell+1)/\bar{k})}{\sin(\pi\,(\bar{r}+1)/\bar{k})}\prod_{j=1}^{\ell}\frac{\sin^2(\pi\,(\bar{r}+2-j)/\bar{k})}{\sin^2(\pi\,j/\bar{k})} \tag{5.7a}$$

$$\mathcal{D}^{r,k}(\mu^\ell) = \frac{\sin(\pi\,(\bar{r}-\ell)/\bar{k})\,\sin(\pi\,(\bar{r}+2)/\bar{k})\,\sin(\pi\,(\bar{r}-2\ell)/\bar{k})\,\sin(\pi\,\ell/\bar{k})}{\sin(\pi/\bar{k})\,\sin^2(\pi\,(\ell+1)/\bar{k})\,\sin(\pi\,(\bar{r}+1)/\bar{k})}\times\prod_{j=1}^{\ell}\frac{\sin^2(\pi\,(\bar{r}+2-j)/\bar{k})}{\sin^2(\pi\,j/\bar{k})} \tag{5.7b}$$

$$\mathcal{D}^{r,k}(\Lambda^\ell) = \prod_{j=1}^{\ell}\frac{\sin(\pi\,(\bar{r}+1-j)/\bar{k})}{\sin(\pi\,j/\bar{k})}\ . \tag{5.7c}$$

Consider first the case $k \geq \bar{r}$. Stirling's formula tells us $\binom{m}{\ell} < 2^m/\sqrt{m}$ for all $m \geq \ell \geq 1$. Hence using (5.5b), we find

$$\mathcal{D}^{r,k}(\gamma) < 2^{2\bar{r}}\ , \tag{5.7d}$$

valid for all $\gamma \in \Gamma_d$, and all $\bar{r}$, k. By (5.5a), (5.6) and (3.4a) we see that

$$p\,2^{-2\bar{r}}\,\mathcal{D}^{r,k}(\varphi^p)\,/\,\bar{r} \tag{5.7e}$$

is an increasing function of k, for fixed $\bar{r}$ and p. Thus it suffices to show (5.7e) is greater than 1 at $k = \bar{r}$.

We begin by removing the dependence of (5.7e) (at $k = \bar{r}$) on p, by showing that the r.h.s. of (5.6) is a decreasing function of $p = 1, 2, \ldots$ at $k = \bar{r}$. Indeed, using (5.4), this is equivalent to showing

$$\frac{1}{\sqrt{2}}\frac{\prod_{j=1}^{p-1}\sin(\pi\,j/2p)^{j/p}}{\prod_{j=1}^{p}\sin(\pi\,j/(2p+2))^{j/(p+1)}} < 1\ . \tag{5.8a}$$

Now, elementary calculus tells us $f(x) = (\sin\frac{\pi}{2}x)^x$, for $0 < x < 1$, has a unique minimum at some point $x = x_m$ (≈ 0.25); f decreases for $x < x_m$, and increases for $x > x_m$. Moreover, $f(x_m) > 0.78$, so $\sqrt{2}\,f(x) > 1$ for all x. Now return to (5.8a). Choose the $\ell \in \{0,\ldots,p-1\}$ for which $\frac{\ell}{p} < x_m < \frac{\ell+1}{p}$. Because $\frac{j}{p+1} < \frac{j}{p} < \frac{j+1}{p+1}$, we may rewrite (5.8a) as

$$\Big(\prod_{j=1}^{\ell}\frac{f(j/p)}{f(j/(p+1))}\Big)\Big(\frac{1}{\sqrt{2}\,f((\ell+1)/(p+1))}\Big)\Big(\prod_{j=\ell+1}^{p-1}\frac{f(j/p)}{f((j+1)/(p+1))}\Big) < 1\,. \qquad (5.8b)$$

Thus we need only to consider $p = \bar{r}/2$ when $\bar{r}$ is even, and $p = r/2$ when $\bar{r}$ is odd. When $\bar{r}$ is even we get from (5.6) and (5.4) that

$$\begin{aligned}\mathcal{D}^{r,k}(\varphi^p) &= 2^{\bar{r}^2/4}\prod_{\substack{j=1\\ j\text{ odd}}}^{\bar{r}-1}\sin\Big(\frac{\pi j}{2\bar{r}}\Big)^j \geq 2^{\bar{r}^2/4}\cot\Big(\frac{\pi}{2\bar{r}}\Big)^{\bar{r}/2-1}\prod_{\substack{j=1\\ j\text{ odd}}}^{\bar{r}-1}\sin\Big(\frac{\pi j}{2\bar{r}}\Big)^{\bar{r}/2}\\ &> (.63\,\bar{r})^{\bar{r}/2-1}2^{\bar{r}/4} \end{aligned} \qquad (5.8c)$$

for $\bar{r} \geq 10$, using the fact that $\cot(\pi/20)/10 > 0.63$ ($x\,\cot x$ is a decreasing function). When $\bar{r}$ is odd the same argument works, except the (5.8b) calculation is needed to replace $f(\frac{j}{\bar{r}})$ with $f(\frac{j}{r})$ or $f(\frac{j-1}{r})$; $\bar{r}$ odd (and greater than 10) also obeys the lower bound of (5.8c) (in fact it obeys a somewhat stronger inequality).

Collecting everything, what we have shown thus far is that (for $\bar{r} \geq 10$)

$$\frac{\|\mathcal{J}_d\varphi\|}{\|\mathcal{J}_d\|}\,\frac{\mathcal{D}^{r,k}(\varphi)}{\mathcal{D}^{r,k}(\gamma)} > (.63\,\bar{r})^{\bar{r}/2-1}\,2^{-7\bar{r}/4}/\bar{r}\;, \qquad (5.8d)$$

valid for all $\varphi \in \mathcal{F}_d$, all $d < \bar{r}$ dividing $\bar{r}$, all $\gamma \in \Gamma_d$, and all $k \geq \bar{r}$. We find that the r.h.s. of (5.8d) is greater than 1 for $\bar{r} \geq 28$. To handle $\bar{r} < 28$, return to (5.7e), find the smallest $k \geq \bar{r}$, call it $k_{r,p}$, making (5.7e) at least 1 ($p < \bar{r}$ must divide $\bar{r}$, and $\bar{r}/p$ divide k). In most cases $k_{r,p} = \bar{r}$; when $k_{r,p} > \bar{r}$, tighten this estimate by going back to (5.7a), (5.7b), (5.7c), to compute each $\frac{p}{\bar{r}}\mathcal{D}^{r,k}(\varphi^p)/\mathcal{D}^{r,k}(\gamma)$. We find the results given in the statement of the proposition for $k \geq \bar{r}$.

By (5.1b), the proof for $1 < k < \bar{r}$ reduces to that of $k > \bar{r}$: when $\varphi^p \in P_+$, $T(\varphi^p) = \tilde{\varphi}^{kp/\bar{r}}$. ■

Proposition 5.1 and Lemmas 3.2(b), 3.3(b) give us Proposition 5.1′ as stated in section 4 – the exceptions are the pairs (r,k) listed in (i)-(x). Next we will see that in almost all cases, (4.8) forces $\sigma\lambda^1 \in [\lambda^1]$.

PROPOSITION 5.2. *Let $\mathcal{W}$ be the set of all $\nu \in P_+$, $\nu \notin [\lambda^1]$, with $\mathcal{D}^{r,k}(\nu) = \mathcal{D}^{r,k}(\lambda^1)$. For any $r \geq 1$, $k \geq 2$, the only nonempty $\mathcal{W}$ are:*

(a) $\mathcal{W} = [\Lambda^3]$ *for* $(r,k) = (8,3)$ *and* $(8,15)$;
(b) $\mathcal{W} = [(k-3)\Lambda_0 + 3\Lambda_1]$ *for* $(r,k) = (2,9)$ *and* (14,9);
(c) $\mathcal{W} = [(k-2)\Lambda_0 + 2\Lambda_2]$ *for* $(r,k) = (3,6)$ *and* (5,4);
(d) $\mathcal{W} = [\Lambda^4]$ *for* $(r,k) = (7,4)$ *and* (7,6);

(e) $\mathcal{W} = [(k-4)\Lambda_0 + 4\Lambda_1]$ *for* $(r,k) = (3,8)$ *and* $(5,8)$.

Proof. Begin by considering, for $\bar{r} \geq 3$ and $k \geq 3$, the quotient

$$\frac{\mathcal{D}^{r,k}((k-2)\Lambda_0 + \Lambda_1 + \Lambda_2)}{\mathcal{D}^{r,k}(\lambda^1)} = \frac{\sin(\pi\,\bar{r}/\bar{k})}{\sin(\pi\,3/\bar{k})} . \tag{5.9a}$$

This is always ≥ 1, with equality iff $\bar{r} = 3$ or $k = 3$ – in which case $(k-2)\Lambda_0+\Lambda_1+\Lambda_2 \in [\lambda^1]$.

Now suppose $\lambda \in P_+$ has at least 3 non-zero Dynkin labels λ_i (so necessarily $\bar{r}, k \geq 3$). By taking various $b = \Lambda_i - \Lambda_j$ in (5.2c), we find some weight μ of the form $\mu = (k-2)\Lambda_0 + \Lambda_m + \Lambda_n$, for $1 \leq m < n \leq r$, with $\mathcal{D}^{r,k}(\lambda) \geq \mathcal{D}^{r,k}(\mu)$, and equality iff $\lambda \in [\mu]$. Consider $T(\mu) = (\bar{r}-n)\tilde{\Lambda}_0 + (n-m)\tilde{\Lambda}_1 + m\tilde{\Lambda}_2$; taking $b = \tilde{\Lambda}_0 - \tilde{\Lambda}_1$ and $b = \tilde{\Lambda}_0 - \tilde{\Lambda}_2$ in (5.2c) and using (5.1b), we get

$$\mathcal{D}^{r,k}(\mu) \geq \min\{\mathcal{D}^{k-1,r+1}((\bar{r}-2)\tilde{\Lambda}_0 + \tilde{\Lambda}_1 + \tilde{\Lambda}_2),\, \mathcal{D}^{k-1,r+1}(\tilde{\lambda}^1)\} = \mathcal{D}^{r,k}(\lambda^1) ,$$

using (5.9a). Thus, $\mathcal{D}^{r,k}(\lambda) \geq \mathcal{D}^{r,k}(\lambda^1)$ for any such λ, with equality iff $\lambda \in [\lambda^1]$.

On the other hand, if λ has only 1 non-zero label, then $\mathcal{D}^{r,k}(\lambda) = \mathcal{D}^{r,k}(\lambda^1)$ would require $[\lambda^1] = [\Lambda^0] = [\lambda]$, by (5.3b). Therefore any $\nu \in \mathcal{W}$ must lie in the $\mathcal{S}$-orbit of some

$$\nu^{ab} \stackrel{\text{def}}{=} (k-a)\Lambda_0 + a\Lambda_b .$$

We can demand $1 \leq a \leq k/2$, and $1 \leq b \leq \bar{r}/2$. Note that $T(\nu^{ab}) = \tilde{\nu}^{ba}$.

Consider next

$$\frac{\mathcal{D}^{r,k}(\nu^{22})}{\mathcal{D}^{r,k}(\lambda^1)} = \frac{\sin(\pi/\bar{k})\,\sin^2(\pi\,\bar{r}/\bar{k})}{\sin^2(2\pi/\bar{k})\,\sin(3\pi/\bar{k})} . \tag{5.9b}$$

We may suppose $k \geq 4$, $\bar{r} \geq 4$. This can be analyzed using (5.5a), and we find that it is less than 1 iff $(\bar{r},k) = (4,4)$, (4,5), or (5,4) and equals 1 iff $(\bar{r},k) = (4,6)$ and (6,4). From this we deduce, in the now familiar way from (5.2c) and (5.1b), that the only $\nu^{ab} \in \mathcal{W}$ with both $a > 1$ and $b > 1$ is ν^{22}.

It thus suffices, using (5.1b), to consider $\nu^{1b} = \Lambda^b$ for $b \leq \bar{r}/2$. $\mathcal{D}^{r,k}(\Lambda^b)$ is computed in (5.7c). First note that

$$\mathcal{D}^{r,k}(\Lambda^1) < \mathcal{D}^{r,k}(\Lambda^2) < \cdots < \mathcal{D}^{r,k}(\Lambda^{\bar{r}/2}) . \tag{5.10a}$$

Write $\mathcal{Q}_b = \mathcal{D}^{r,k}(\Lambda^b)/\mathcal{D}^{r,k}(\lambda^1)$. When $\bar{r} \leq 3$ we need to consider only Λ^1: we find that $\mathcal{Q}_1 = 1$ iff $\Lambda^1 \in [\lambda^1]$. Assume now that $\bar{r} \geq 4$. From (5.5c), $\mathcal{Q}_1 < \mathcal{Q}_2 < 1$, except for $k = 2$, when $\Lambda^2 \in [\lambda^1]$. We find, using (5.5a), (5.5c), that $\mathcal{Q}_3 \geq 1$ except for $\bar{r} = 6$, 7 and 8, for all k, and $\bar{r} = 9$ for $4 \leq k \leq 14$. Thus from (5.10) the only other possible $\Lambda^b \in \mathcal{W}$ is Λ^4 for $\bar{r} = 8$ and any k, or for $\bar{r} = 9$ and $4 \leq k \leq 14$. These possibilities are handled in the usual way. ■

6. Fusion coefficient calculations. In this section we conclude the proof of Theorem 2.1 for most pairs (r,k). In particular, let M be a $\mathcal{ADE}_7$-type invariant with

$\mathcal{J}(M) = \mathcal{J}_d$. We will use the fusion-generator Γ_d for all r, k. Prop. 5.1′ says that for each $\gamma \in \Gamma_d$, there exist weights $\sigma\gamma \in \mathcal{P}_d$, $\sigma\gamma \notin \mathcal{F}_d$, such that $(\gamma, \sigma\gamma)$ is M-monogomous. Proposition 5.2 tells us $\sigma\lambda^1 \in [\lambda^1]$, for most (r, k). Our first task will be to show $\sigma\lambda^1 \in [\lambda^1]_d$. Using this, we will show that $\sigma\lambda^i \in [\lambda^i]_d$ and (replacing M if necessary by $C \cdot M$) $\sigma\mu^j \in [\mu^j]_d$. From this we can obtain $\sigma\Lambda^{\bar{r}/d} \in [\Lambda^{\bar{r}/d}]_1$. As indicated at the end of section 4, this will conclude the proof of Theorem 2.1 for those (r, k). Our main tool in this section will be *fusion coefficients* (3.4e).

$\overline{L}(\bar{\lambda}^1)$ has dominant weights $\bar{\lambda}^1$ (multiplicity 1) and 0 (multiplicity r, as seen by e.g. the Weyl dimension formula). Therefore (3.4f) implies

$$N^{\nu}_{\lambda^1,\mu} = \begin{cases} n(\nu) - 1 & \text{for} \quad \nu = \mu \\ 1 & \text{if } \nu = \Lambda_i - \Lambda_{i-1} - \Lambda_j + \Lambda_{j-1} + \mu \text{ for some } i \neq j \\ 0 & \text{otherwise} \end{cases}, \tag{6.1}$$

where $n(\nu)$ denotes the number of Dynkin labels $\nu_i \neq 0$, and where we put $\Lambda_{\bar{r}} = \Lambda_0$. From (3.12b) we see that, for all $\gamma, \gamma', \gamma'' \in \Gamma_d$, there is a $c(\gamma, \gamma', \gamma'') > 0$ such that

$$\sum_{J \in \mathcal{J}_d} N^{J\gamma''}_{\gamma,\gamma'} = c(\gamma\gamma'\gamma'') \sum_{J \in \mathcal{J}_d} N^{J\sigma\gamma''}_{\sigma\gamma,\sigma\gamma'} \tag{6.2}$$

(each $c(\gamma, \gamma', \gamma'') = 1$ when (r, k) avoids those pairs listed in Props. 5.1, 5.2, but we will be more general here).

PROPOSITION 6.1. *Suppose all $\gamma \in \Gamma_d$ are M-monogomous and $\sigma\lambda^1 \in [\lambda^1]$. Then there exists a $c \in \{0, 1\}$ such that, for all $\gamma \in \Gamma_d$, $\gamma \neq \Lambda^{\bar{r}/d}$, we have $\sigma\gamma \in [C^c\gamma]_d$.*

Proof. To begin we find from (6.1) that (defining $\lambda^0 = \Lambda^0$ and discarding μ^0)

$$N^{\nu}_{\lambda^1,\lambda^\ell} \neq 0 \quad \text{iff} \quad \nu \in \{\lambda^\ell, \lambda^{\ell-1}, \lambda^{\ell+1}, \mu^\ell, \mu^{\ell-1}, C\mu^\ell, C\mu^{\ell-1}, \lambda^1 + \lambda^\ell - \Lambda^0\}. \tag{6.3}$$

Therefore σ must permute the $\mathcal{J}_d$-orbits of these ν, if it fixes $[\lambda^1]_d$ and $[\lambda^\ell]_d$. Note also that

$$(\lambda^i + \rho \,|\, \lambda^i + \rho) = 2i\,(\bar{r} + 1 - i) + (\rho \,|\, \rho) \tag{6.4a}$$

$$(\mu^j + \rho \,|\, \mu^j + \rho) = 2\bar{r} + 2 + 2j\,(\bar{r} - j) + (\rho \,|\, \rho) \tag{6.4b}$$

$$(\lambda^1 + \lambda^h - \Lambda^0 + \rho \,|\, \lambda^1 + \lambda^h - \Lambda^0 + \rho) = 2\bar{r} + 4 + 2h\,(\bar{r} + 1 - h) + (\rho \,|\, \rho) \tag{6.4c}$$

for $0 \leq i \leq \bar{r}/2$, $0 < j \leq r/2$, and $0 < h \leq \bar{r}/2$. The point of (6.4) is (3.6a): σ must preserve norms (mod $2\bar{k}$).

Now let us show $\sigma\lambda^1 \in [\lambda^1]_d$. (3.4e) and (4.1c) tell us that

$$N^{J^{a+b}\nu}_{J^a\lambda, J^b\mu} = N^{\nu}_{\lambda,\mu}, \qquad \forall \lambda, \mu, \nu \in P_+. \tag{6.5}$$

We know from Proposition 5.2 that $\sigma\lambda^1 = J^a\lambda^1$, for some a. So, from (6.1), (6.2) and (6.5) we obtain

$$0 \neq \sum_{J \in \mathcal{J}_d} N^{J\lambda^1}_{\lambda^1,\lambda^1} = c(\lambda^1\lambda^1\lambda^1) \sum_{J \in \mathcal{J}_d} N^{JJ^a\lambda^1}_{J^a\lambda^1, J^a\lambda^1} = c(\lambda^1\lambda^1\lambda^1) \sum_{J \in \mathcal{J}_d} N^{JJ^{-a}\lambda^1}_{\lambda^1,\lambda^1}. \tag{6.6}$$

Using (6.3), (6.4) and (6.6) we get $\sigma\lambda^1 \in [\lambda^1]_d$.

By induction on ℓ, we find from (6.3) and (6.4) that σ must fix each $[\lambda^\ell]_d$, and $[\sigma\mu^\ell]_d \in \{[\mu^\ell]_d, [C\mu^\ell]_d\}$. Replacing M if necessary with $C \cdot M$, we may suppose that σ also fixes $[\mu^1]_d$.

Now note from (6.3) that

$$N_{\lambda^1,\mu^\ell}^{\nu} \neq 0 \quad \text{iff} \quad \overline{\nu} \in \{\overline{\mu}^\ell, \overline{\mu}^{\ell+1}, \overline{\mu}^{\ell-1}, \overline{\lambda}^\ell, \overline{\lambda}^{\ell+1}, \overline{\Lambda}_2 + \overline{\Lambda}_{\ell-1} + \overline{\Lambda}_{r-\ell},$$
$$\overline{\Lambda}_2 + \overline{\Lambda}_\ell + \overline{\Lambda}_{r-\ell-1}, 2\overline{\Lambda}_1 + \overline{\Lambda}_{\ell-1} + \overline{\Lambda}_{r-\ell}, 2\overline{\Lambda}_1 + \overline{\Lambda}_\ell + \overline{\Lambda}_{r-\ell-1},$$
$$\overline{\Lambda}_2 + \overline{\Lambda}_\ell + \overline{\Lambda}_{r-\ell} + \overline{\Lambda}_r, \overline{\lambda}^1 + \overline{\lambda}^\ell, \overline{\lambda}^1 + \overline{\lambda}^{\ell+1}, \overline{\lambda}^1 + \overline{\mu}^\ell\} \,. \tag{6.7}$$

Label these weights consecutively $\nu^1 = \mu^\ell, \ldots, \nu^{13} = \lambda^1 + \mu^\ell - \Lambda^0$. When we must make ℓ explicit, we will write these as $\nu^a(\ell)$. Assume inductively that $\sigma\mu^1 = \mu^1, \ldots, \sigma\mu^\ell = \mu^\ell$, and suppose for contradiction that $\sigma\mu^{\ell+1} = C\mu^{\ell+1}$. We are interested here in $k \geq 3$, $\bar{r} \geq 5$, $1 \leq \ell \leq \frac{\bar{r}}{2} - 1$. The question is, when can $C\mu^{\ell+1} \in [\nu^a(\ell)]_d$ for some $a \geq 6$?

Solving this is straightforward once one realizes that we can ignore $\nu^6(1)$, $\nu^6(2)$, $\nu^7(1)$, $\nu^8(1)$, $\nu^{10}(1)$ – in these cases $\nu^a(\ell)$ equals one of $\lambda^{\ell+1}, \mu^\ell, \mu^{\ell+1}$, or $\lambda^1 + \lambda^{\ell+1} - \Lambda^0$. This means that the $\overline{\Lambda}_i$ for each $\overline{\nu}$ in (6.7) are written in non-decreasing order of indices, making comparison with $J^m C\mu^{\ell+1}$ easy. Note also that we can ignore ν^{11} and ν^{12} for all ℓ, as $C\mu^{\ell+1}$ lies in their $\mathcal{J}_d$-orbit iff $\mu^{\ell+1}$ does. Also, for $k \geq 7$ we find that $(C\mu^{\ell+1})_0 = k-3 > \nu_i^a$ for $i > 0$. So the only possibilities are $3 \leq k \leq 6$.

One solution is $k = 4$, $\bar{r} = 12$, and $\ell = 3$, where we have $C\mu^{\ell+1} = J^{-3}\nu^{10}$. But $\sigma\mu^4 \neq C\mu^4$ because both $d = 1$ and $d = 3$ violate (4.4). Similarly, when $k = \bar{r} = 6$ and $\ell = 1$, we have $C\mu^2 = J^{-1}\nu^{13}$, but $d = 1$ violates (4.4).

The only other solution is $k = 3$, $\ell = \bar{r}/3$: we find $C\mu^{\ell+1} = J^{\bar{r}/3}\nu^6$. However, if $\sigma\mu^{\ell+1} = C\mu^{\ell+1}$ in this case, then by (6.7) and (3.12b) we would need to have $\mu^{\ell+1} \in [\nu^a(\ell-1)]_d$ for some a, d, and this does not happen here.

Thus in all cases $\sigma\mu^{\ell+1} \in [\mu^{\ell+1}]_d$, and our proposition is proved. ∎

From (3.12a), we find that Proposition 6.1 tells us

$$\frac{S_{\gamma,\lambda}}{S_{\Lambda^0,\lambda}} = \frac{S_{\gamma,\sigma\lambda}}{S_{\Lambda^0,\sigma\lambda}} \quad \forall \gamma \in \Gamma_d,\ \gamma \neq \Lambda^{\bar{r}/d},\ \forall \lambda \in \mathcal{P}_d \,. \tag{6.8a}$$

But these γ form a fusion-generator for $\mathcal{J}_1$, so (6.8a) implies $\sigma\lambda \in [\lambda]_1$ for all $\lambda \in \mathcal{P}_1$, by the recursive argument in the proof of Lemma 3.3(a). By the argument given in the proof of Prop. 4.1(b), we also see that (6.8a) implies

$$\frac{S_{\lambda,\mu}}{S_{\Lambda^0,\mu}} = \frac{S_{\lambda,\sigma\mu}}{S_{\Lambda^0,\sigma\mu}} \quad \forall \lambda \in \mathcal{P}_1 \,, \tag{6.8b}$$

and all $\mu \in X_d$. These two observations, together with the fact (see the proof of Prop. 4.1(b)) that $\lambda \in X_d^{(1)} = \mathcal{P}_d$, imply $\sigma\lambda \in [\lambda]_d$ for all $\lambda \in \mathcal{P}_1$. Hence by (3.12a), (6.8b) holds in addition for all $\mu \in \mathcal{P}_d$. Multiplying (6.8b) by $\sum_{J \in \mathcal{J}_1} S^*_{\lambda,J\mu}$ (which vanishes for $\lambda \notin \mathcal{P}_1$) and summing over all $\lambda \in P_+$, we find $\sigma\mu \in [\mu]_1$. In particular this holds for $\mu \in \Lambda^{\bar{r}/d}$, which gives us Proposition 6.1′ as stated in section 4.

7. Anomolous ranks and levels. In this section we conclude the proof of Theorem 2.1 by addressing the few pairs (r,k) which slipped through the previous arguments. In the process we will explicitly construct the $\mathcal{E}_7$-type exceptionals listed in section 2. Recall Proposition 4.2.

7.1. In this subsection we use norm arguments and tighten the q-dimension arguments to discard almost all remaining pairs (r,k).

Consider first Proposition 5.2. It is trivial to verify that (3.6a) is violated by the choice $\lambda=\lambda^1$, $\mu=\nu$ for any ν and (r,k) listed in (a)-(e). This gives us Proposition 5.2′ stated in section 4.

Next we turn to (i)-(x) in Proposition 5.1. Again (3.6a) is a severe constraint, as is the requirement (see (4.7a)) that

$$M_{\lambda,\mu}\neq 0 \Rightarrow \frac{\|\mathcal{J}_d\|}{\|\mathcal{J}_d\lambda\|}\geq\frac{\mathcal{D}^{r,k}(\lambda)}{\mathcal{D}^{r,k}(\mu)}\geq\frac{\|\mathcal{J}_d\mu\|}{\|\mathcal{J}_d\|}\ . \tag{7.1a}$$

Moreover, if in addition $\|\mathcal{J}_d\|/\|\mathcal{J}_d\mu\|<(\mathcal{D}^{r,k}(\mu)+1)/\mathcal{D}^{r,k}(\lambda)$ then

$$M_{\lambda,\nu}=\delta_{[\nu]_d,[\mu]_d}\qquad \forall\nu\in P_+\ . \tag{7.1b}$$

Once again we will choose the fusion-generator $\Gamma=\Gamma_d$ when $\bar{r}\leq k$, and $\Gamma=\Gamma_d'$ when $\bar{r}>k>1$. Using (3.6a), (7.1a), (4.4) and Proposition 5.1, we find that $M_{\gamma,\varphi}=0$ for all $\gamma\in\Gamma$ and $\varphi\in\mathcal{F}_d$, except possibly for:

(a)′ $(r,k,d)=(1,4,1)$, $\gamma=\varphi=\lambda^1$;
$(r,k,d)=(1,16,1)$, $\gamma=\lambda^1$ and $\varphi=(8,8)$;

(b)′ $(r,k,d)=(2,3,1)$, $\gamma=\varphi=\lambda^1$;
$(r,k,d)=(2,9,1)$, $\gamma=\lambda^1$ and $\varphi=(3,3,3)$;

(c)′ $(r,k,d)=(3,2,2)$, $\gamma=\varphi=\lambda^1$;
$(r,k,d)=(3,4,2)$, $\gamma=\varphi=\lambda^2$;
$(r,k,d)=(3,8,d)$ for $d=1$ or 2, $\gamma=\lambda^1$ and $\varphi\in[(4,0,4,0)]_1$, and (if $d=1$) $\gamma=\mu^1$ and $\varphi=(2,2,2,2)$;

(d)′ $(r,k,d)=(4,5,1)$, $\gamma=\lambda^1$ and $\varphi=(1,1,1,1,1)$;

(e)′ $(r,k,d)=(5,6,2)$, $\gamma=\mu^1$, $\varphi=(2,0,2,0,2,0)$;

(f)′ $(r,k,d)=(7,4,d)$ for $d=2$ or 4, $\gamma=\lambda^1$ and $\varphi\in[2\Lambda_0+2\Lambda_4]_2$, and (if $d=2$) $\gamma=\widetilde{T}'\tilde{\mu}^1$ and $\varphi=\Lambda_1+\Lambda_3+\Lambda_5+\Lambda_7$;

(g)′ $(r,k,d)=(8,3,3)$, $\gamma=\lambda^1$ and $\varphi=\Lambda_0+\Lambda_3+\Lambda_6$;

(h)′ $(r,k,d)=(15,2,4)$, $\gamma=\lambda^1$ and $\varphi=\Lambda_0+\Lambda_8$;
$(r,k,d)=(15,2,8)$, $\gamma=\lambda^1$ and $\varphi\in[\Lambda_0+\Lambda_8]_4$.

7.2. In this subsection we find all the $\mathcal{E}_7$-type exceptionals for $A_1^{(1)}$. This was first done in [4]. By the previous subsection it suffices to consider $(k,d)=(4,1)$ and (16,1).

First consider $k=4$. The problem here is that λ^1 is a fixed point. However, $\mathcal{P}_1=[\Lambda^0]_1\cup[\lambda^1]_1$, so (λ^1,λ^1) must be M-monogomous. Thus there is no exceptional here.

Next, consider $k=16$. Write $\varphi=(8,8)$. Suppose M is exceptional. Then by Proposition 4.2, both $M_{\lambda^1,\varphi}\neq 0$ and $M_{\varphi,\lambda^1}\neq 0$. By (7.1b) we find that in fact

$$M_{\lambda^1,\nu}=M_{\nu,\lambda^1}=\delta_{\nu,\varphi},\qquad\forall\nu\in P_+\ . \tag{7.2a}$$

The remaining entries of M are fixed by Lemma 3.1, (3.9) and (3.6a), except for $M_{\varphi,\varphi}$. $M_{\varphi,\varphi}=1$ is forced by Lemma 3.2(a): we must have the eigenvalue

$$r_m \stackrel{\rm def}{=} r(\begin{pmatrix} m & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}) \tag{7.2b}$$

equal to $\bar{r}/d=2$; by [14], r_m is a strictly increasing function of $m\geq 0$, and an easy calculation gives $r_1=2$.

7.3. In this subsection we find all the $\mathcal{E}_7$-type exceptionals for $A_2^{(1)}$. This was first done in [10]. By subsection 7.1 it suffices to consider $(k,d)=(3,1)$ and (9,1). $k=3$ is handled by the identical argument used on $k=4$ in subsection 7.2.

So consider $k=9$, and assume M is exceptional. Write $\varphi=(3,3,3)$. Exactly as before we must have (7.2a) satisfied. There are precisely 7 $\mathcal{J}_1$-orbits in $\mathcal{P}_1$; (3.6a) and Lemma 3.2(b) force (λ,λ) to be M-monogomous for each $\lambda\in\mathcal{P}_1$ except for $\lambda\in[\lambda^1]\cup[\varphi]\cup[(0,3,6)]$; either (λ,λ) or $(\lambda,C\lambda)$ will be M-monogomous for each $\lambda\in[(0,3,6)]$. The value of $M_{\varphi,\varphi}$ again is fixed by the (7.2b) argument. Note that $((0,3,6),\ (0,3,6))$ being M-monogomous implies $M=\mathcal{E}^{(2,9)}$, while $((0,3,6),\ C(0,3,6))$ being M-monogomous implies $M=C\cdot\mathcal{E}^{(2,9)}$.

7.4. In this subsection we find all the $\mathcal{E}_7$-type exceptionals for $A_3^{(1)}$. By the previous subsection it suffices to consider $(k,d)=(2,2)$, (4,2), (8,2) and (8,1). For both $k=2$ and $k=4$, the problem is that $\lambda^{k/2}$ is a fixed point. However in both cases (3.6a) forces $(\lambda^{k/2},\lambda^{k/2})$ to be M-monogomous, and so there are no exceptionals.

Consider next $k=8$ and $d=2$, and suppose M is exceptional. Then $M_{\lambda^1,\varphi}\neq 0$ for some $\varphi\in[(4,0,4,0)]_1$. By (7.1b) we again have the analogue of (7.2a) satisfied. We find, by (3.6a) and (7.1a), that $(2\rho,2\rho)$ must be M-monogomous. We can show (see e.g. (3.4a)) that

$$S_{2\rho,\lambda^1}=-S_{2\rho,\varphi}\neq 0\ . \tag{7.3}$$

However, this violates (3.6b) evaluated at $(\lambda^1,2\rho)$. Thus M cannot be exceptional.

Finally, consider $k=8$ and $d=1$, and let M be exceptional. $\mathcal{P}_1$ consists of exactly 12 $\mathcal{J}_1$-orbits; (3.6a) and (7.1a) tell us that the only way to have both $M_{\lambda,\mu}\neq 0$ and $\lambda\notin[\mu]_1$ is if either $\lambda,\mu\in[\mu^1]_1\cup[C\mu^1]_1\cup[2\rho]_1$ or $\lambda,\mu\in[\lambda^1]_1\cup[\varphi]_1$, where $\varphi=(4,0,4,0)$. Because M commutes with $C=S^2$, we see that $M_{\mu^1,2\rho}=M_{C\mu^1,2\rho}$. By (7.1b) we find that $M_{\mu^1,2\rho}\neq 0$ iff

$$M_{\mu,\lambda}=M_{\lambda,\mu}=\delta_{\lambda,2\rho},\qquad \forall\mu\in[\mu^1],\quad \forall\lambda\in P_+\ . \tag{7.4}$$

When $M_{\mu^1,2\rho}\neq 0$, we fix the value $M_{2\rho,2\rho}=2$ by the usual argument as in (7.2b). Similarly, if $M_{\lambda^1,\varphi}\neq 0$, then we know all $[\lambda^1]\cup[\varphi]$-rows and -columns of M.

The only remaining question is whether $M_{\mu^1,2\rho}\neq 0$ iff $M_{\lambda^1,\varphi}\neq 0$. That this is so follows immediately from (7.3), and the fact that (2.7a) will satisfy (3.6b). Hence $M=\mathcal{E}^{(3,8)}$.

7.5. In this subsection we find all the $\mathcal{E}_7$-type exceptionals for $A_4^{(1)}$ at $k=5$, $d=1$. There are precisely 6 $\mathcal{J}_1$-orbits in $\mathcal{P}_1$, and exactly one fixed point: $\varphi=\rho$. The argument is exactly as for $(r,k,d)=(1,16,1)$, and we get that $M=\mathcal{E}^{(4,5)}$.

7.6. In this subsection we show there are no $\mathcal{E}_7$-type exceptionals for $A_5^{(1)}$ at $k=6$, $d=2$. Put $\varphi=(2,0,2,0,2,0)$. By (e)′ and Prop. 4.2, any exceptional M would have both $M_{\mu^1,\varphi}$ and M_{φ,μ^1} nonzero. Put $\lambda=4\Lambda_0+\Lambda_1+\Lambda_2$; by the usual arguments we find that in fact $(\lambda,J^aC^b\lambda)$ is M-monogomous for some $a,b\in\{0,1\}$. Replacing M with $I(\mathcal{J}_3)^a\cdot C^b\cdot M$ ($I(\mathcal{J}_3)$ is an invertible matrix), we may assume (λ,λ) is M-monogomous. From (3.6b) we get $3\,S_{\lambda,\mu^1}=S_{\lambda,\varphi}$. However its l.h.s. is non-real, while the r.h.s. is real. So such an M cannot exist.

7.7. In this subsection we find all the $\mathcal{E}_7$-type exceptionals for $A_7^{(1)}$. By the previous subsection it suffices to consider $(k,d)=(4,4)$ and (4,2). The case $k=d=4$ is handled exactly as $(r,k,d)=(3,8,2)$ was (the analogue of (7.3) here is simply its rank-level dual – see (4.2c)).

When $(k,d)=(4,2)$, there are precisely 24 J-orbits, including four orbits of fixed points. Equations (7.1a) and (4.6a) require $(J\Lambda^0,J\Lambda^0)$ to be M-monogomous, so by Lemma 3.1(a) $M_{\lambda,\mu}=M_{J\lambda,J\mu}$ for all λ,μ. The remainder of the argument is exactly as for $(r,k,d)=(3,8,1)$.

7.8. In this subsection we find all the $\mathcal{E}_7$-type exceptionals for $A_8^{(1)}$, when $k=d=3$. There are 21 $\mathcal{J}$-orbits, and three fixed points. We find from (7.1a) and (4.6a) that, replacing M with $C\cdot M$ if necessary, $(J\Lambda^0,J\Lambda^0)$ must be M-monogomous, so by Lemma 3.1(a) we know that $M_{\lambda,\mu}=M_{J\lambda,J\mu}$ for all $\lambda,\mu\in P_+$, and

$$M_{\lambda,\mu}\neq 0\Rightarrow t(\lambda)\equiv t(\mu)\qquad(\mathrm{mod}\ 9)\ . \tag{7.5}$$

Applying (7.5), (7.1a) and (4.6a) to the remaining weights in $\mathcal{P}_3$, and using the familiar arguments, we can fix all values of M, except for determining whether (μ^1,μ^1) or $(\mu^1,C\mu^1)$ is M-monogomous. However the former must hold, because otherwise $M-\mathcal{E}^{(8,3)}+I(\mathcal{J}_3)$ would violate Proposition 4.2.

7.9. In this subsection we find all the $\mathcal{E}_7$-type exceptionals for $A_{15}^{(1)}$, when $k=2$ and $d=8$ or 4. Start with $d=8$. There are 40 $\mathcal{J}_4$-orbits, with 8 containing fixed points. But everything simplifies, as we once again find (conjugating if necessary) that $M_{\lambda,\mu}=M_{J\lambda,J\mu}$ and

$$M_{\lambda,\mu}\neq 0\Rightarrow t(\lambda)\equiv t(\mu)\qquad(\mathrm{mod}\ 16) \tag{7.6}$$

for all $\lambda,\mu\in P_+$ by the usual arguments. We can now quickly force $M=\mathcal{E}^{(15,2)}$.

The case $d=4$ is completely analogous to $(r,k,d)=(1,16,1)$.

8. Conclusion. The problem of classifying all *physical invariants* (see Definition 3.1) for a given nontwisted affine algebra $X_r^{(1)}$ and level k is a key step in the classification problem for RCFTs. Though this problem remains open, progress is being made, and with this paper the end could be in sight, at least for the algebras $A_r^{(1)}$. In particular, there is a natural division of the problem into two pieces, based on the structure of the physical invariant about the distinguished weight $k\Lambda_0$. One subproblem is to classify all physical invariants M – they are called *$\mathcal{ADE}_7$-type invariants* – whose $k\Lambda_0$-row and column reflect the symmetries of the Dynkin diagram of $X_r^{(1)}$ (see (1.3b) for a more precise statement). Almost all physical invariants are expected to be of this type, including what seem to be the most elusive *exceptional* physical invariants. In this paper we develop an approach for achieving the classification of $\mathcal{ADE}_7$-type invariants for any algebra $X_r^{(1)}$, and explicitly solve it for the algebra $A_r^{(1)}$ at all levels k. We find that most of these physical invariants are built up in a natural way from the symmetries of the Coxeter-Dynkin diagram, but some are not – the so-called *$\mathcal{E}_7$-type exceptionals*. These exceptional invariants are surprisingly rare.

The second subproblem, which remains completely open apart from some minor special cases (most notably $A_1^{(1)}$ [4] and $A_2^{(1)}$ [10]), is to find those $X_r^{(1)}$ and k for which a physical invariant can have 'irregular' values of $M_{k\Lambda_0,\lambda}, M_{\lambda,k\Lambda_0}$. In the language of conformal field theory, this is the problem of finding all possible exceptional *chiral extensions* for the given affine algebra $X_r^{(1)}$ at level k. These also seem to be quite rare: for $A_r^{(1)}$ at level k, these occur at $(r,k) = (r,r-1)$, $(r,r+1)$, and $(r,r+3)$, and probably only finitely many other pairs (r,k). The solution to these two subproblems would quickly imply the classification of all physical invariants.

A natural follow-up to this paper will be to extend the results here to the remaining affine algebras. $A_r^{(1)}$ is special because its Coxeter-Dynkin diagram is so symmetrical. In some ways this makes the classification of $\mathcal{ADE}_7$-type invariants more difficult (e.g. we are inundated with possibilities at most steps of the solution), but in other ways it makes things much simpler (e.g. q-dimensions are easier to handle). In any case, it can be expected that the classification for the remaining $g = X_r^{(1)}$ should follow the method used here [13].

Another follow-up is the classification of all physical invariants for $A_r^{(1)}$ at levels 2 and 3 (until now only level 1 is known). This follows from Theorem 2.1 in this paper, and the work in [10], and will be reported elsewhere.

Of course it would be preferable for a "uniform proof" of the classification of $\mathcal{ADE}_7$-type invariants for all $X_r^{(1)}$ and k. At present the closest we can come are the lemmas in section 3. In fact it is far from clear that even a "uniform" *list* of $\mathcal{ADE}_7$-type invariants is possible – the problem of course are the $\mathcal{E}_7$-type exceptionals.

Another disappointing feature of the proof given here is that it is computer-assisted (although in a minor way): some q-dimensions (3.4b) were computed for sections 5 and 7, and an S matrix element was needed in section 7.6. Round-off error is not a problem in these calculations (2 decimal places suffice). Nevertheless it would be nice if these computations could be replaced by more conceptual arguments. Again, the existence of the $\mathcal{E}_7$-type exceptionals makes this more difficult.

This paper hints that the classification of all physical invariants, at least for simple X_r, may not be far away. In the process of solving this problem we are being forced to investigate properties of the Kac-Peterson matrices in some detail, and it can be hoped that our analysis will also be of use to other problems involving the modular behaviour of the affine characters.

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