Plurisubharmonic functions and the Kempf-Ness theorem

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Let G be a complex Lie group and Ω a complex homogeneous space of G. A general problem in complex analysis is to give a description of plurisubharmonic functions invariant under a real subgroup K of G and of holomorphy hulls of $K-$ invariant domains in Ω : see, e.g. [18,10,12]. The present paper is a contribution to this problem which is inspired by the Kempf-Ness theorem [8]. They show that if G is a complex reductive group operating linearly on a vector space V, K a maximal compact subgroup of G and N the square of the norm function obtained from a $K-$ invariant Hermitian metric on V, then a $G-$ orbit Ω is closed if and only if the restriction of N to Ω has a critical point. Equivalently, the restriction of N to Ω is an exhaustion function for Ω if and only if it has a critical point. Now the function N is strictly plurisubharmonic and remains so on restriction to any complex submanifold of V. The following result can therefore be considered as an "intrinsic" generalization of the Kempf-Ness closedness criterion.

Theorem 1. *Let G be a complex reductive group, K a maximal compact subgroup of G and H a closed complex subgroup of G. If φ is a K-invariant strictly plurisubharmonic function on G/H with a critical point then H is reductive and φ is an exhaustion function for G/H.*

Our next result is related to results of D. Luna [13] and has similar applications to orbits of reductive groups operating on Stein manifolds.

Theorem 2. *Let L be a closed subgroup (not necessarily connected) of a compact connected group K and f an L-invariant function on $K^{\mathbf{C}}/L^{\mathbf{C}}$. The function f has $x_0 = eL^{\mathbf{C}}$ as a critical point if and only if its restriction to $N(L^{\mathbf{C}})/L^{\mathbf{C}}$ has x_0 as a critical point. In particular if $N(L^{\mathbf{C}})/L^{\mathbf{C}}$ is finite, then any L-invariant function on $K^{\mathbf{C}}/L^{\mathbf{C}}$ has a critical point*

In [2] it was shown that if L is a closed subgroup of a compact connected group K containing a maximal torus of K then the holomorphy hull of any $K-$ invariant domain in $K^{\mathbf{C}}/L^{\mathbf{C}}$ contains K/L. The main group theoretic ingredient was the characterization of K/L as the unique totally real $K-$ orbit in $K^{\mathbf{C}}/L^{\mathbf{C}}$. Here we give in § 3 a description of all totally real $K-$ orbits in $K^{\mathbf{C}}/L^{\mathbf{C}}$, L being any closed subgroup of the compact group K, and show that the number of such orbits is finite if and only if $N\left(\left(L^{\mathbf{C}}\right)^0\right)/\left(L^{\mathbf{C}}\right)^0$ is finite, and in this case there is only one such orbit. This implies as in [2] that the holomorphy hull of any $K-$ invariant domain in $K^{\mathbf{C}}/L^{\mathbf{C}}$ contains K/L whenever $N\left(\left(L^{\mathbf{C}}\right)^0\right)/\left(L^{\mathbf{C}}\right)^0$ is finite.

We give several applications in § 4 of our main results, mostly to orbits of reductive groups operating on Stein manifolds. In [16] R. Richardson shows the existence of a closed orbit in the closure of any orbit of a reductive group acting on a Stein manifold. In (4.2) we give a short proof of this result of Richardson. Corollary 4.1, in the algebraic category is due to Kempf and Ness [8]; corollaries (4.3) and (4.4) in the algebraic category are due to D. Luna [13]. A special case of theorem 1, namely when G is semisimple and H is the identity group, is stated without proof in Guillimin and Sternberg [3]. Before concluding this introduction we want to say a few words about the proof of theorem 1. Its proof depends on a convexity lemma, namely lemma 1.2, and a theorem of G.D. Mostow [14]. A version of this convexity lemma occurs first in M. Lasalle [10], later, in a more general sense in [12] and finally in [2] it occurs in more or less the same form as it is stated here. We have tried to give here a (hopefully) clearer version. A generalization of the Kempf-Ness theory in a different direction has been obtained by Richardson and Slodowy [17]. We take this opportunity to point out that part of the main result of [2] can also be obtained by using a suitable moment map and applying Kirwan [9, lemma 7.2]: see remarks at the end of section 1. We owe this observation essentially to P. Slodowy.

The reader is referred to [7, 11] for basic facts on plurisubharmonic functions. The notation is standard. In particular, if H is a subgroup of a group G then $N_G(H)$ and $Z_G(H)$ denote the normalizer and centralizer of H in G. Also H^0 denotes the connected component of H and x_y the conjugate xyx^{-1}.

1. Preliminary lemmas

We remind the reader that the Levi-form of a differentiable function φ defined on a complex manifold M is the Hermitian form associated to the 2-form $i\partial\overline{\partial}\varphi$. Denoting the Levi-form of φ by $L\varphi$ we have: $(L\varphi)_p(u,v) = (\partial\overline{\partial}\varphi)(p)(u,\bar{v})$, $p \in M$, $u,v \in T_p^{1,0}(M)$.

If the Levi-form of φ is positive definite, we say that φ is strictly plurisubharmonic.

Lemma 1.1. *If $f : \mathbf{R}^n \to \mathbf{R}$ is a differentible function whose restriction to each line through the origin is convex and has the origin as its only critical point, then* $\lim\limits_{\|x\|\to\infty} f(x) = +\infty$.

Proof: Fix $v \in \mathbf{R}^n$, $v \neq 0$. Let $h(t) = f(tv)(t \in \mathbf{R})$. By assumptions $h''(t) \geq 0$ and $h'(t) = 0$ only at $t = 0$. Therefore $h'(t) > 0$ for $t > 0$. Now $h(t) = \left(\int_1^t h'(x)dx\right) + h(1)$ so $h(t) \geq h'(1)(t-1) + h(1)$ if $t > 1$. From this inequality we see that (*) $f(tv) \geq (\nabla f)(v).v(t-1) + f(v)$, for all $v \neq 0$ and $t > 1$. Now $0 < h'(1) = (\nabla f)(v).v$, so if m is the minimum value of $(\nabla f)(v).v$ on the unit sphere in $\mathbf{R}^n$ then $m > 0$. Let k be the minimum value of f on the unit sphere in $\mathbf{R}^n$. From the inequality (*) we therefore have, for $\xi \in \mathbf{R}^n$ with $\| \xi \| > 1$

$$f(\xi) = f(\| \xi \| \xi / \| \xi \|) \geq m(\| \xi \| - 1) + k\,.$$

Therefore $\lim_{\|\xi\|\to\infty} f(\xi) = +\infty$.

Lemma 1.2 *Let G, K, H and ϕ be as in the statement of theorem 1. Assume that ϕ has a critical point at $\xi_0 = eH$. Fix $v \in Lie(K)$. If the function $\phi((\exp\ itv).\xi_0)$ $(t \in \mathbf{R})$ has a critical point at $t_0 \neq 0$ then the 1–parameter subgroup $\{\exp\ itv\}_{t\in\mathbf{R}}$ is contained in H .*

Proof: Consider the function $f(z) = \phi((\exp\ izv).\xi_0)(z \in \mathbf{C})$. The function f is subharmonic and by $K-$ invariance of ϕ it depends only on the real part $Re(z)$ of z . Since $\nabla^2 f \geq 0$ we see that if $g(t) = \phi((\exp\ itv).\xi_0)$ $(t \in \mathbf{R})$ then $g''(t) \geq 0$. By assumption $g'(0) = 0$ so g achieves its absolute minimum at zero. Assume that for some $t_0 \neq 0$ we have $g'(t_0) = 0$. By convexity of g the function g is constant on the segment joining 0 and t_0 . Assuming $t_0 > 0$, the complex curve $\gamma(z) = \exp\ izv.\xi_0, \quad 0 \leq Re(z) \leq t_0$, therefore lies on a level set of the function ϕ . Denoting the Levi form of ϕ at a point p by $L_p(\phi)$ we have, for $0 < Rez < t_0$, $L_{\gamma(z)}(\gamma'(z), \gamma'(z)) = 0$. Since ϕ is strictly plurisubharmonic this forces $\gamma'(z) = 0$ for $0 < Re(z) < t_0$, hence by continuity $\gamma(z)$ is constant on $0 \leq Re(z) \leq t_0$. In particular, for sufficiently small $t \in \mathbf{R}$, $e^{itv}.\xi_0 = \xi_0$, so the 1–parameter subgroup $\{e^{itv}\}_{t\in\mathbf{R}} \subset H$.

Remark As the proof shows, this lemma is valid for any complex Lie group G and a real form K thereof relative to which G factorizes as $G = KP$, where $P = \{\exp\ iX : X \in Lie(K)\}$.

Lemma 1.3 *If φ is a real valued differentiable function defined on a complex manifold M and N is a real submanifold of M contained in the critical set of φ , then N is totally isotropic relative to the form $i\partial\overline{\partial}\varphi$.*

Proof: Let $\omega = i\partial\overline{\partial}\varphi$ and $j : N \to M$ the inclusion map. We have to show that $j^*\omega = 0$. Now $\omega = dd^{\mathbf{C}}\varphi$, where $d = \partial + \overline{\partial}$ and $d^{\mathbf{C}} = {}^{\partial - \overline{\partial}}/_{2i}$. Moreover, for $p \in M$ and $v \in T_p(M)$ we have $(d^{\mathbf{C}}\varphi)_p(v) = (d\varphi)_p(Jv)$, where J is the complex structure tensor of M. Hence $j^*(d^{\mathbf{C}}\varphi) = 0$ as N is a critical submanifold for φ. Therefore $j^*\omega = d(j^*d^{\mathbf{C}}\varphi) = 0$, which is what had to be proved.

Corollary 1.4 *If Ω is a complex homogeneous space of a Lie group G and φ is a function on Ω invariant under a subgroup K of G whose Levi-form $i\partial\overline{\partial}\varphi$ is non-degenerate, then the K-orbits of critical points of φ are of dimension $\leq {}^{\dim(\Omega)}/_2$. In particular, if $\Omega = G/H$, with both G and H complex, K is a real form of G and $\xi_0 = eH$ is a critical point of φ then $\dim(K.\xi_0) = (\dim\ \Omega)/2$ and $K \cap H$ is a real form of H .*

Proof: The first statement follows from (1.3) taking into account the non-degeneracy of $i\partial\overline{\partial}\varphi$.

If G and H are complex and K is a real form of G then clearly $2\dim(K/K\cap H) \geq \dim(G/H)$, which combined with the first statement implies the remaining statements.

Remark. Our initial proof of theorem 1 used a lemma of Harvey and Wells [4], which we have replaced by lemma 1.3, and the main result of [2]. However, part of this result is implicit in Kirwan [9] and can be obtained as follows. The moment map for an exact form $w = d\eta$, η being invariant under a group K, is the contraction of η with the Killing vector fields induced by K [1, Th. 4.2.10]. If M is a complex manifold on which a complex reductive group G operates, K is a maximal compact subgroup of G and φ is a $K-$ invariant strictly plurisubharmonic function on M, then $dd^{\mathbf{C}}\varphi$ is a $K-$ invariant Kählerian form. Since the critical set of φ is contained in $\mu^{-1}(0)$, where μ is the moment map for $\omega = d\eta$, $\eta = d^{\mathbf{C}}\varphi$, the result follows from [9, lemma 7.2].

2. Proofs of main results

The ingredients of proof of Theorem 1 are the lemmas of § 1 and the following theorem of G.D. Mostow [14].

Theorem (Mostow) *Let L be a closed subgroup of a compact connected group K. There exists an L-invariant subspace m of $Lie(K)$ such that the mapping of $K\times_L im$ into $K^{\mathbf{C}}/L^{\mathbf{C}}$ defined by $(k\times_L v)\mapsto k.\exp(v).L^{\mathbf{C}}$ is an isomorphism of topological spaces.*

Proof of Theorem 1 Step (i). H is a reductive subgroup of G : Let $\xi_0 = eH$ and $a\xi_0 (a\in G)$ be a critical point of φ. The point ξ_0 is then a critical point of $\varphi\circ L_a$, where L_a is left translation by a. The function $\varphi\circ L_a$ is strictly plurisubharmonic and it is invariant under $a^{-1}Ka$. Hence without loss of generality we may assume that ξ_0 is a critical point of φ. By (1.4) we know that $K\cap H$ is a real form of H and therefore the connected component H^0 of H is reductive. As the natural map $\pi : G/H \to G/H^0$ is a local isomorphism, the function $\varphi\circ\pi$ is also a strictly plurisubharmonic. It is also $K-$ invariant and H/H^0 is in its critical set. By [1] or [9, lemma 7.2] the critical set is a single $K-$ orbit, so $K\cap H$ operates transitively on H/H^0. Therefore H/H^0 has representatives in K and so H/H^0 is finite. This means that H is reductive, as it is the complexification of the compact group $K\cap H$.

Step (ii) φ is an exhaustion function: By Step (i) we are in a position to apply Mostow's theorem. So let $L = K\cap H$ and $\mathfrak{m}\subset Lie(K)$ be as in the statement of Mostow's theorem. Fix $v\in\mathfrak{m}$, $v\neq 0$. As in (1.2) the function $g_v(t) = \phi(\exp itv.\xi_0)$ $(t\in\mathbf{R})$ is convex and has $t=0$ as a critical point. If g_v had another critical point $t_0\neq 0$ then by (1.2) $\exp itv.\xi_0$ would equal ξ_0 for all $t=\mathbf{R}$, in contradiction to the fact that the function $k\times_L v\to k(\exp iv).L^{\mathbf{C}}$ $(k\in K, v\in\mathfrak{m})$ is bijective. Therefore the function g_v has only $t=0$ as its critical point. Consider the function $f(v) = \phi(\exp iv.\xi_0)$ $(v\in\mathfrak{m})$. By what has just been shown the function f satisfies all the conditions of lemma (1.1). Hence $\lim_{||v||\to\infty} f(v) = +\infty$. We have to show that the sublevel sets $\phi \leq c\,(c \in \mathbf{R})$ are compact.

Let $\{k_n \exp iv_n.\xi_0\}$ be a sequence in $G/H = K^{\mathbf{C}}/L^{\mathbf{C}}$ with $k_n \in K$, $v_n \in \mathfrak{m}$ and $\phi(k_n \exp iv_n.\xi_0) \leq c$. Since ϕ is a $K-$ invariant we have $f(v_n) = \phi(\exp iv_n.\xi_0) \leq c$. Since f is unbounded at infinity, the sequence $\{v_n\} \subset \mathfrak{m}$ must be bounded. Extracting convergent subsequences of $\{k_n\}$ and $\{v_n\}$ we see that the sequence $\{k_n \exp iv_n.\xi_0\}$ contains a convergent subsequence. Therefore the sublevel sets $\phi \leq c$ are compact and ϕ is an exhaustion function.

__Proof of Theorem 2__ Let $G = K^{\mathbf{C}}$, $H = L^{\mathbf{C}}$ and f an $L-$ invariant function on $X = G/H$. Denoting the differential of f at a point p by $f_*(p)$ and using $L-$ invariance of f we have $\forall\, l \in L,\ \ v \in T_{x_0}(X) : f_*(lx_0)(l.v) = f_*(x_0)(v)$.

Since $lx_0 = x_0 \ \ (l \in L)$ this gives:

(a) $$f_*(x_0)(l.v) = f_*(x_0)(v),\ l \in L,\ v \in T_{x_0}(X)\,.$$

Let $< , >$ be a scalar product on $T_{x_0}(X)$ which is $L-$ invariant. Since $f_*(x_0)$ is a linear function on $T_{x_0}(X)$ we see that there exists a unique $h \in T_{x_0}(X)$ such that

(b) $$< h, v >= f_*(x_0)(v),\ \ \forall\, v \in T_{x_0}(X)\,.$$

Equation (a) then implies

(c) $$l.h = h \quad \forall\, l \in L\,.$$

Let $\pi : G \to G/H$ be the natural map. The differential π_* of π at e maps the Lie algebra $\dot{G}$ of G onto $T_{x_0}(X)$ and the kernel of π_* is the Lie algebra $\dot{H}$ of H. Let $y \in \dot{G}$ be such that $\pi_*(y) = h$.

Equality (b) is then equivalent to

(d) $$Ad(l)(y) = y\left(mod\ \dot{H}\right) \quad \forall\, l \in L\,.$$

By analytic continuation (d) holds for all $l \in L^{\mathbf{C}} = H$. Therefore if $z \in \dot{H}$ the curve $Ad(e^{tz})(y) - y$ lies in $\dot{H}$, hence by differentiation we obtain $[z, y] \in \dot{H}$. Therefore $y \in N\left(\dot{H}\right)$. Let $\bar{y}$ be the class of y in the Lie algebra $N\left(\dot{H}\right)/\dot{H}$. From (d) we get

(e) $$l\ \exp\, t\bar{y}\, l^{-1} \exp(-t\bar{y}) = 1\ \ in\ \ N(H^0)/H^0\ \ for\, all\ \ l \in H\,.$$

Hence $l\ \exp\ ty\, l^{-1} \exp(-ty) \in H^0\ \forall\, l \in H$, so $\exp\ ty \in N(H)$. In particular y is in the Lie algebra $N(H)^{\cdot}$ of $N(H)$. Denoting the image of an element $z \in \dot{G}$ in $\dot{G}/\dot{H}$ by $\bar{z}$ the equality (f) becomes

(b) $$< \bar{y}, \bar{z} >= f_*(x_0)(\bar{z}) \quad \left(z \in \dot{G}\right)$$

Now we have the decomposition

$$\dot{G}/\dot{H} = N(H)^{\cdot}/\dot{H} \oplus \left(N(H)^{\cdot}/\dot{H}\right)^{\perp},$$

the orthogonal complement being with respect to the $L-$ invariant inner product $< , >$. Therefore if the decomposition of $\bar{z}\left(z \in \dot{G}\right)$ is $\bar{z} = \bar{z}_1 + \bar{z}_2$, with $\bar{z}_1 \in N(H)^{\cdot}/\dot{H}$ and $\bar{z}_2 \in \left(N(H)^{\cdot}/\dot{H}\right)^{\perp}$ then from (f) we have

(g) $$f_*(x_0)(\bar{z}) = f_*(x_0)(\bar{z}_1)\,.$$

But this means that $x_0 = eH$ is a critical point of f if and only if it is a critical point of the restriction of f to $N(H)/H$.

In particular if $N(H)/H$ is finite then f has x_0 as a critical point. This completes the proof of theorem 2.

3. Totally real orbits in $K^{\mathbf{C}}/L^{\mathbf{C}}$

Let L be a closed subgroup of a compact connected group K . Let $G = K^{\mathbf{C}}$ and $H = L^{\mathbf{C}}$. The principal aim of this section is to prove the following result.

Proposition *K has finitely many totally real orbits in G/H if and only if $N(H^0)/H^0$ is finite, and in this case there is only one totally real $K-$orbit.*

Before giving a proof of this proposition we note that totally real $K-$orbits in G/H are precisely those of half the dimension of G/H . Moreover, if $\pi : G/H^0 \to G/H$ is the natural map and Ω is a totally real $K-$orbit in G/H then $\pi^{-1}(\Omega)$ is a union of totally real K-orbits which are permuted by the right action of the finite group H/H^0 on G/H^0 . Therefore to prove the proposition, we may assume that H is connected. The proof depends on the following lemmas.

Lemma 3.1 *If X is a Hermitian matrix and $e^{nX} (n \in \mathbf{Z}, n > 0)$ commutes with a matrix Y then e^X also commutes with Y .*

Proof This follows by elementary arguments taking into account that e^X has positive eigenvalues.

Let $G = KP$ be the Cartan decomposition of G .

Lemma 3.2 *If k, $k_1 \in K$ and $p \in P$ with $pkp^{-1} = k_1$ then $k = k_1$.*

Proof We have $pk = k_1 p$ so $pk = (k_1 p k_1^{-1}) k_1$, with $k_1 p k_1^{-1} \in P$. By unicity of the Cartan decomposition [6] we see that $k = k_1$.

Lemma 3.3 *If $p = e^X \in P$ centralizes $Y \in Lie(G)$ then the 1−parameter subgroup $\{e^{rX} : r \in \mathbf{R}\}$ also centralizes Y.*

Proof There is a faithful representation of G in $GL(n, \mathbf{C})$ in which K is represented by unitary matrices and P by Hermitian matrices [6]. By (3.1) $e^{qX} (q \in \mathbf{Q})$ centralizes Y and therefore so does $e^{rX} (r \in \mathbf{R})$.

Lemma 3.4 *If L is connected and $n \in N_G(H)$ then n factorizes as $n = kpx$, where $k \in K \cap N(H)$, $p \in P \cap Z_G(H)$ and $x \in H$ (recall that $H = L^{\mathbf{C}}$ and $G = K^{\mathbf{C}}$).*

Proof Let $n \in N(H)$. Now L is a maximal compact subgroup of H so by conjugacy of maximal compact subgroups we have ${}^{xn}L = L$ for some $x \in H$. Let $xn = kp$ be the Cartan decomposition xn with $k \in K$ and $p \in P$. The equation ${}^{p}L = {}^{k-1}L$ shows by (3.2) that p centralizes L and therefore H and k normalizes H. Hence $n = x^{-1}kp = k(k^{-1}x^{-1}k)p = kp(k^{-1}x^{-1}k) = kpx'$, where $k \in K \cap H$, $p \in P \cap Z_G(H)$ and $x' = k^{-1}x^{-1}k \in H$.

Lemma 3.5 *If L is connected and $N = N(H)/H$ is finite then N has representatives in K.*

Proof Let $n \in N(H)$ and let $n = kpx$ be the factorization of n given by (3.4). Let $p = e^X$. The 1−parameter subgroup $Z = \{e^{rX} : r \in \mathbf{R}\}$ is, by (3.3), in $Z_G(H)$. Since ZH/H is in the finite group $N(H)/H$, we must have $Z \subset H$. Therefore $N(H)/H$ has representatives in K.

Lemma 3.6 *For connected L, the orbits of $N_K(H).H/H$ on $N(H)/H$ parametrize the totally real K−orbits in G/H.*

Proof A K− orbit Ω in G/H is totally real if and only if $\dim(\Omega) = \dim(K/L)$. Let $\xi_0 = eH$ and let $Kx\xi_0$ be totally real in G/H. So $\dim(K \cap xHx^{-1}) = \dim(L)$ and therefore $\dim\left({}^{x^{-1}}K \cap H\right) = \dim(L)$. By conjugacy of maximal compact subgroups, the group $\left({}^{x^{-1}}K \cap H\right)^o$ is conjugate in $H = L^{\mathbf{C}}$ to L and therefore $(K \cap xHx^{-1})^o$ is conjugate in $G = K^{\mathbf{C}}$ to L, say by an element kp, where $k \in K$ and $p \in P$. By Lemma (3.2), p centralizes L and therefore $(K \cap xHx^{-1})^0 = k^{-1}Lk$. Hence $k^{-1}Lk \subset xHx^{-1}$ and $kx \in N(H)$. Therefore $x = k^{-1}n$ for some $n \in N(H)$. Conversely if $x = kn$

with $n \in N(H)$ then $K \cap xHx^{-1} = K \cap kHk^{-1} \cong K \cap H = L$. Therefore totally real $K-$orbits in G/H have representatives in $N(H)/H$. Finally, if $n_1, n_2 \in N(H)$ and $kn_1H = n_2H$ with $k \in K$, then clearly $k \in N_K(H)$. Therefore the orbits of the compact group $N_K(H).H/H \cong N_K(H)/L$ on $N(H)/H$ parametrize the totally real K-orbits in G/H.

Proof of the proposition: Suppose K has finitely many totally real orbits in G/H. Then K also has finitely many totally real orbits in G/H^0. Hence we may assume that H is connected. By Lemma 3.6 the compact group $N_K(H).H/H$ has finitely many orbits on the Stein manifold $N(H)/H$. Therefore $N(H)/H$ must be finite and by Lemma 3.5 it must have representatives in K. Hence there is a unique totally real $K-$orbit in G/H.

Corollary *If $N(H^0)/H^0$ is finite then the holomorphy hull of any $K-$invariant domain Ω in G/H contains the unique totally real orbit K/L.*

Proof This follows by repeating the argument given in Corollary 2 of [2] and using the proposition of this section.

4. Applications

In this section we give several applications of our results to orbits of a reductive group G operating on a Stein manifold, or more generally, on a manifold which has a strictly plurisubharmonic function. In this connection, an example of a non-algebraic Stein manifold with a non-linearizable $\mathbf{C}^*-$action is given in Heinzner [5]. Let K be a maximal compact subgroup of G.

4.1 *If G operates on a manifold M which has a strictly plurisubharmonic function (say φ), which by integrating over K we may assume to be $K-$invariant, then an orbit O in M is closed if the restriction of φ to O has a critical point. If φ is also an exhaustion function for M then an orbit O is closed if and only if the restriction of φ to O has a critical point.*

Proof Assume that the restriction ψ of φ to O has a critical point. By Theorem 1 ψ is an exhaustion function for O. Let $\{p_n\} \subset O$ be a sequence which converges to a point $p \in M$. Since $\psi(p_n)$ converges to $\varphi(p)$, the points p_n must lie in some sublevel set $\psi \leq c$. As ψ is an exhaustion function, the sequence $\{p_n\}$ must converge to a point in O. Hence O is closed. If φ is also an exhaustion function for M and an orbit O is closed then clearly the restriction ψ of φ to O achieves its minimum, hence ψ has a critical point.

4.2 Let M be a Stein manifold on which G operates holomorphically. Let O be a $G-$ orbit in M.

(a) The closure $\overline{O}$ of O in M contains a unqiue closed $G-$ orbit.
(b) If φ is any $K-$ invariant strictly plurisubharmonic exhaustion function for M, then the restriction of φ to $\overline{O}$ achieves its absolute minimum at a single $K-$ orbit, which is in the unique closed orbit in $\overline{O}$.

<u>Proof</u> The existence of at most one closed orbit in the closure of an orbit O in the Stein manifold M is classical. It is a consequence of Cartan's theorem B [7]. We reproduce the argument for the reader's convenience. Suppose O_1 and O_2 are two distinct closed orbits in $\overline{O}$. Let f be the function on $O_1 \cup O_2$ which is 0 on O_1 and 1 on O_2. By Theorem B, f extends to a holomorphic function $\hat{f}$ on M, which by integrating over K we may assume to be $K-$ invariant, and therefore $G-$ invariant. The function $\hat{f}$ restricted to $\overline{O}$ is constant as it is constant on O. By construction $\hat{f}$ is not constant on $O_1 \cup O_2 \subset \overline{O}$. Hence there can be at most one closed orbit in $\overline{O}$.

To show that there is at least one closed orbit in $\overline{O}$, take on the Stein manifold M a $K-$ invariant strictly plurisubharmonic exhaustion function ϕ. Take a sublevel set $\phi \leq c$ which intersects $\overline{O}$. Since $\{\phi \leq c\} \cap \overline{O}$ is compact, the restriction of ϕ to $\overline{O}$ achieves its aboslute minimum, say m, at some point ξ_0. By (4.1) the orbit $G.\xi_0$ is closed in M. By [2], the restriction of ϕ to $G.\xi_0$ achieves the value m at a single $K-$ orbit. Now if ξ_1 in $\overline{O}$ is such that $\phi(\xi_1) = m$ then the orbit $G.\xi_1 \subset \overline{O}$ is also closed. By uniqueness of a closed $G-$ orbit in $\overline{O}$ we see that $G.\xi_1 = G.\xi_0$. Hence the restriction of ϕ to $\overline{O}$ achieves its absolute minimum on a single $K-$ orbit which is contained in the unique closed $G-$ orbit in $\overline{O}$.

4.3 *If G operates on a manifold M which has a strictly plurisubharmonic function then the $G-$ orbit of a point x is closed if and only if the $N_G(H)$ orbit of x is closed, H being the stabilizer of x in G.*

<u>Proof</u> This follows from Theorem 2 and (4.1).

4.4 *Let H be a reductive subgroup of G with $N(H)/H$ is finite. If G operates on a manifold M which has a strictly plurisubharmonic function then all orbits of G with stabilizer isomorphic to H are closed in M.*

<u>Proof</u> This again follows from Theorem 2 and (4.1).

<u>Remarks</u> (a) The conditions of (4.4) hold if G is semisimple and H is a symmetric subgroup.

(b) If (K, L) is a symmetric pair with K compact, semisimple and $G = K^{\mathbf{C}}$, $H = L^{\mathbf{C}}$ then by theorems 1 and 2, any $K-$ invariant strictly plurisubharmonic function φ on G/H is an exhaustion function and its critical set by [2] or [9, lemma 7.2] is K/L. In other words, φ is a canonical exhaustion function in the sense of Patrizio-Wong [15]. This paper suggests many problems on the geometry of G/H.

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