THE SMALL QUANTUM GROUP AND THE SPRINGER RESOLUTION

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Abstract. In [ABG] the derived category of the principal block in modules over the Lusztig quantum algebra at a root of unity is related to the derived category of equivariant coherent sheaves on the Springer resolution $\widetilde{\mathcal{N}}$. In the present paper we deduce a similar relation between the derived category of the principal block for the *small (reduced) quantum algebra* u and the derived category of (non-equivariant) coherent sheaves on $\widetilde{\mathcal{N}}$. As an application we get a geometric description of Hochschild cohomology (in particular, the center) of the regular block for u, and use it to give an explicit description of a certain subalgebra in the center (obtained previously by another method and under more restrictive assumptions in [La]). We also briefly explain the relation of our result to the geometric description [BK] of the derived category of modules over the De Concini – Kac quantum algebra.

To the memory of Iosef Donin.

Contents

1. Quantum algebras

1.1. **Basic notations and conventions.** Throughout the paper $\Bbbk$ is an algebraically closed field of zero characteristic.

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Let R be a finite reduced root system in a $\Bbbk$-vector space E and fix a basis of simple roots $S = \{\alpha_i, i \in I\}$. Let $\check{\alpha}$ denote the coroot corresponding to the root $\alpha \in R$. The Cartan matrix is given by $a_{ij} = \langle \alpha_i, \check{\alpha}_j \rangle$, where $\langle \cdot, \cdot \rangle$ is the canonical pairing $E^* \times E \to \Bbbk$. Let W be the Weyl group of R. There exists a unique W-invariant scalar product in E such that $(\alpha, \alpha) = 2$ for any short root $\alpha \in R$. Set $d_i = \frac{1}{2}(\alpha_i, \alpha_i) \in \{1, 2, 3\}$ for each $i \in I$. We denote by $\mathbb{Y} = \mathbb{Z}R$ the root lattice, and by $\mathbb{X} = \{\mu \in E : \langle \mu, \check{\alpha} \rangle \in \mathbb{Z} \ \ \forall \alpha \in R\}$ the weight lattice corresponding to R. The coweight lattice is $\check{\mathbb{Y}} = \mathrm{Hom}(\mathbb{Y}, \mathbb{Z}) \in E^*$. Let R^+ be the set of positive roots, define the dominant weights by $\mathbb{X}^+ = \{\mu \in \mathbb{X} : \langle \mu, \check{\alpha} \rangle \geq 0 \ \ \forall \alpha \in R^+\}$ and set $\mathbb{Y}^+ = \mathbb{Y} \cap \mathbb{X}^+$.

Let G be a connected semisimple group of adjoint type over $\Bbbk$ with the Lie algebra $\mathfrak{g}$ corresponding to the root system R. Let B be a Borel subgroup in G, and N its unipotent radical. Let $\mathfrak{b}$ and $\mathfrak{n}$ be their respective Lie algebras.

1.2. **Quantum algebras at a root of unity.** Let $\Bbbk(q)$ denote the field of rational functions in the variable q. We denote by $U_q(\mathfrak{g}) = U_q$ the Drinfeld-Jimbo quantized enveloping algebra of $\mathfrak{g}$. It is generated over $\Bbbk(q)$ by $E_i, F_i, i \in I$ and $K_\mu^{\pm 1}, \mu \in \check{\mathbb{Y}}$ subject to well-known relations, see e.g. [L]. We will write K_i for $K_{d_i \check{\alpha}_i}$. The algebra U_q is a Hopf algebra over $\Bbbk(q)$.

Fix an odd positive integer l which is greater than the Coxeter number of the root system, prime to the index of connection $|\mathbb{X}/\mathbb{Y}|$ and prime to 3 if R has a component of type G_2. Choose a primitive l-th root of unity $\xi \in \Bbbk$ and let $\mathcal{A} \subset \Bbbk(q)$ be the ring localized at ξ, and $\mathfrak{m}$ the maximal ideal of $\mathcal{A}$. For any $n \in \mathbb{N}$ set $[n]_d = \frac{q^{dn} - q^{-dn}}{q^d - q^{-d}}$ and $[n]_d! = \prod_{s=1}^n \frac{q^{ds} - q^{-ds}}{q^d - q^{-d}}$.

In U_q consider the divided powers of the generators $E_i^{(n)} = E_i^n/[n]_{d_i}!, F_i^{(n)} = F_i^n/[n]_{d_i}!, i \in I, n \geq 1$, and $\left[{K_\mu, m \atop n} \right]$ as defined in [L]. The Lusztig's integral form $\mathsf{U}_\mathcal{A}$ is defined as an $\mathcal{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by these elements. $\mathsf{U}_\mathcal{A}$ is a Hopf subalgebra of U_q. The Lusztig quantum algebra at a root of unity U is defined by specialization of $\mathsf{U}_\mathcal{A}$ at ξ: $\mathsf{U} = \mathsf{U}_\mathcal{A}/\mathfrak{m}\mathsf{U}_\mathcal{A}$. It has a Hopf algebra structure over $\Bbbk$.

Another version of the quantum algebra $\mathfrak{U}_\mathcal{A}$ was introduced in [DK]. This is the $\mathcal{A}$-subalgebra of U_q generated by $E_i, F_i, \frac{K_i - K_i}{q^{d_i} - q^{-d_i}}, \ i \in I$ and $K_\mu, \mu \in \check{\mathbb{Y}}$. The elements K_i^l are central in $\mathfrak{U}_\mathcal{A}$. The De Concini-Kac quantum algebra is defined as $\mathfrak{U} = \mathfrak{U}_\mathcal{A}/(\mathfrak{m}\mathfrak{U}_\mathcal{A} + \sum_{i \in I}(K_i^l - 1) \cdot \mathfrak{U}_\mathcal{A})$. It is a Hopf algebra over $\Bbbk$.

By definition $\mathfrak{U}_\mathcal{A} \subset \mathsf{U}_\mathcal{A}$. After the specialization at ξ, the imbedding of $\mathcal{A}$-forms induces a (not injective) Hopf algebra homomorphism $\mathfrak{U} \longrightarrow \mathsf{U}$. The image of this homomorphism is the *small quantum group* u. Equivalently, u is a subalgebra in U generated by the elements $E_i, F_i, \frac{K_i - K_i}{q^{d_i} - q^{-d_i}}, \ i \in I$ and $K_\mu, \mu \in \check{\mathbb{Y}}$. Since we have assumed l to be odd, u is a Hopf algebra over $\Bbbk$.

1.3. **Quantum Lusztig-Frobenius map.** Let $\{e_i, f_i, h_i\}_{i \in I}$ be the standard Chevalley generators of the Lie algebra $\mathfrak{g}$. Lusztig proved that the map $E_i^{(l)} \longrightarrow e_i$, $E_i \longrightarrow 0$, $F_i^{(l)} \longrightarrow f_i$, $F_i \longrightarrow 0$ for all $i \in I$ can be extended to a well-defined surjective algebra homomorphism $\phi : \mathsf{U} \longrightarrow \hat{U}(\mathfrak{g})$ which is called the *quantum Frobenius map*. Here $\hat{U}(\mathfrak{g})$ stands for a certain completion of the universal enveloping $U(\mathfrak{g})$ (see [L]) such that the representation category of finite dimensional $\hat{U}(\mathfrak{g})$-modules may be identified with that of the group G. The kernel of this map coincides with the two-sided ideal in U generated by the augmentation ideal of the small quantum group, u_ϵ. One has an exact sequence of algebras

$$0 \longrightarrow (\mathsf{u}_\epsilon) \longrightarrow \mathsf{U} \xrightarrow{\phi} \hat{U}(\mathfrak{g}). \tag{1}$$

We let $\mathrm{Rep}(G)$ denote the category of finite dimensional algebraic G-modules, and $\mathrm{Rep}(\mathsf{U})$ be the category of finite dimensional $\mathbb{Y}$-graded U-modules.

The pull-back of the Frobenius homomorphism gives rise to the functor between the tensor categories:

$$\phi^* : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(\mathsf{U}), \quad V \longrightarrow V^\phi. \tag{2}$$

2. Functor to a derived category of u-modules

In the next three subsections we recall the construction of the functor introduced in [ABG] from a derived category of coherent G-equivariant sheaves on the Springer resolution corresponding to G to a certain derived category of representations of U.

2.1. **The principal block.** The category $\mathrm{Rep}(\mathsf{U})$ is an abelian artinian category, and therefore is a direct sum of its indecomposable abelian subcategories, or blocks.

We write $L(\nu)$ for a simple finite dimensional U-module of highest weight $\nu \in \mathbb{Y}^+$. For any $\lambda \in \mathbb{Y}$ let L_λ be the finite dimensional simple U-module with highest weight $w(l\lambda+\rho)-\rho$, where $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$ and $w \in W$ is the unique element such that $w(l\lambda+\rho)-\rho$ is a dominant weight.

We write $\mathsf{block}(\mathsf{U})$ for the block of $\mathrm{Rep}(\mathsf{U})$ which contains the trivial representation (the principal block). Equivalently, $\mathsf{block}(\mathsf{U})$ is the full subcategory of the abelian category of left U-modules formed by the modules M such that all simple subquotients of M are of the form L_λ, $\lambda \in \mathbb{Y}$.

We let $D^b\mathsf{block}(\mathsf{U})$ denote the corresponding bounded derived category.

2.2. **Springer resolution and U-modules.** Using the adjoint action of B on the nilradical $\mathfrak{n}$ of $\mathfrak{b} = \mathrm{Lie}(B)$, one defines the *Springer resolution* $\widetilde{\mathcal{N}} = G \times_B \mathfrak{n}$ as the quotient of $G \times \mathfrak{n}$ by the action $h \cdot (g,x) = (gh^{-1}, Ad_h(x))$. Thus $\widetilde{\mathcal{N}}$ is an algebraic variety equipped with an algebraic action of G. The multiplicative group $\mathbb{C}^*$ acts on $\widetilde{\mathcal{N}}$ by $t : (g,x) \mapsto (g, t^2x)$ along the fibers.

Let $Coh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})$ (resp. $Coh^{\mathbb{C}^*}(\widetilde{\mathcal{N}})$) denote the abelian category of $G\times\mathbb{C}^*$-equivariant (respectively $\mathbb{C}^*$-equivariant) coherent sheaves on $\widetilde{\mathcal{N}}$. For an abelian category C we write D^bC for its bounded derived category.

The following result was obtained in [ABG], Corollary 1.4.4.

Theorem 3. *There exists a triangulated functor*

$$F : D^bCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}}) \longrightarrow D^b\mathsf{block}(\mathsf{U}),$$

such that:

1. $F(z^i \otimes \mathscr{F}) = F(\mathscr{F})[i]$ *for any* $\mathscr{F} \in D^bCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})$ *and any* $i \in \mathbb{Z}$.
2. *The functor F induces, for any* $\mathscr{F}, \mathscr{F}' \in D^bCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})$, *canonical isomorphisms*

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^\bullet_{D^bCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})}(\mathscr{F}, z^i\otimes\mathscr{F}') \xrightarrow{\sim} \mathrm{Hom}^\bullet_{D^b\mathsf{block}(\mathsf{U})}(F(\mathscr{F}), F(\mathscr{F}')).$$

3. *The image of F generates the target category as a triangulated category.*

(4) *The functor intertwines the natural action of the tensor category* $\mathrm{Rep}(G)$ *on* $D^bCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})$ *with the action on* $D^b\mathsf{block}(\mathsf{U})$ *coming from the Lusztig-Frobenius homomorphism* ϕ*; i.e., for every* $V \in \mathrm{Rep}(G)$ *we have an isomorphism* $F(V\otimes\mathcal{F}) \cong \phi^*(V)\otimes F(\mathcal{F})$*, satisfying the natural compatibilities.*

Here $z^i \otimes \mathscr{F}$ denotes the $\mathbb{C}^*$-equivariant sheaf $\mathscr{F}$ with the $\mathbb{C}^*$-equivariant structure twisted by the character $z \to z^i$. The notation $\mathscr{F}[k]$ stands for the homological shift of $\mathscr{F}$ by k in the derived category.

2.3. **Springer resolution and u-modules.** We would like to obtain a similar functor into a derived category of u-modules.

We let $\mathrm{Rep}(\mathsf{u})$ be the category of finite dimensional u-modules. The category $\mathrm{Rep}(\mathsf{u})$ is an abelian artinian category, and therefore is a direct sum of blocks. Let $\mathsf{block}(\mathsf{u})$ denote the block containing the trivial representation.

Theorem 4. *There exists a triangulated functor*

$$F_u : D^bCoh^{\mathbb{C}^*}(\widetilde{\mathcal{N}}) \longrightarrow D^b\mathsf{block}(\mathsf{u})$$

fitting into the commutative diagram

$$\begin{array}{ccc} D^bCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}}) & \xrightarrow{F} & D^b\mathsf{block}(\mathsf{U}) \\ \text{Forget}\downarrow & & \downarrow\text{Res} \\ D^bCoh^{\mathbb{C}^*}(\widetilde{\mathcal{N}}) & \xrightarrow{F_u} & D^b\mathsf{block}(\mathsf{u}) \end{array}$$

The functor F_u *satisfies properties (1–3) stated in Theorem 3.*

Proof. We need to recall a relation between the categories $\mathsf{block}(\mathsf{U})$ and $\mathsf{block}(\mathsf{u})$ established in [AG].[1]

Recall that we have the restriction functor $\mathrm{Res} : \mathrm{Rep}(\mathsf{U}) \to \mathrm{Rep}(\mathsf{u})$. The right adjoint functor to Res is defined if we pass to the categories of ind-objects. The category on ind-objects in $\mathrm{Rep}(\mathsf{U})$ can be identified with the category of locally finite $\mathbb{Y}$-graded modules over U; we denote it by $\mathrm{Rep}^{lf}(\mathsf{U})$. Thus we have the "locally finite induction" functor $\overline{\mathrm{Ind}} : \mathrm{Rep}(\mathsf{u}) \to \mathrm{Rep}^{lf}(\mathsf{U})$. An object in $\mathrm{Rep}^{lf}(\mathsf{U})$ $\overline{\mathrm{Ind}}(M)$, $M \in \mathsf{block}(\mathsf{u})$ carries an additional structure; namely, the object $\overline{\mathrm{Ind}}(M)$ of the tensor category $\mathrm{Rep}^{lf}(\mathsf{U})$ is naturally a comodule over the coalgebra $\mathcal{O}(G)$. Here $\mathcal{O}(G)$ is the algebra of regular functions on the algebraic group G viewed as a coalgebra in the tensor category of $\mathfrak{g}$-modules. To make sense of the notion of an $\mathcal{O}(G)$-comodule in $\mathrm{Rep}^{lf}(\mathsf{U})$ we need to fix an action of the tensor category $\mathrm{Rep}(G)$ on $\mathrm{Rep}^{lf}(\mathsf{U})$. Such an action is given by $V : M \mapsto V^\phi \otimes M$.

A comodule structure on a locally finite module M amounts to a collection h_V of isomorphisms $V^\phi \otimes M \to \underline{V} \otimes M$ fixed for every algebraic G-module V; here $\underline{V}$ denotes the vector space underlying the representation V. The collection of isomorphisms h_V has to satisfy a certain compatibility condition spelled out, e.g., in [AG].

[1] [AG] concentrate on the case of an *even* root of unity. However, the (simpler) case of an odd root of unity considered in the present paper is also covered by the general Theorem 2.8 of *loc. cit.* (it is easy to show that the assumptions of [AG], Theorem 2.8 are satisfied in our case).

For a finite dimensional module $M \in \mathrm{Rep}(\mathsf{u})$ the $\mathcal{O}(G)$ comodule $\overline{\mathrm{Ind}}(M)$ is finitely (co)generated. We let $\mathrm{coMod}_{\mathrm{Rep}^{lf}(\mathsf{U})}(\mathcal{O}(G))$ denote the category of finitely generated $\mathcal{O}(G)$ comodules in $\mathrm{Rep}^{lf}(\mathsf{U})$.

It follows from the results of [AG] (or from the much more general Barr-Beck theorem, see e.g. [M]) that the functor from $\mathrm{Rep}(\mathsf{u})$ to $\mathrm{coMod}_{\mathrm{Rep}^{lf}(\mathsf{U})}(\mathcal{O}(G))$ is an equivalence.

It is not hard to show that the functor Res sends $\mathsf{block}(\mathsf{U})$ to $\mathsf{block}(\mathsf{u})$, while $\overline{\mathrm{Ind}}$ sends of $\mathsf{block}(\mathsf{u})$ to $\mathsf{block}^{lf}(\mathsf{U})$, where $\mathsf{block}^{lf}(\mathsf{U})$ is the category of ind-objects in $\mathsf{block}(\mathsf{U})$ (identified with the category of locally finite graded U-modules, which are unions of modules in $\mathsf{block}(\mathsf{U})$). It follows that we have an equivalence

$$\mathsf{block}(\mathsf{u}) \cong \mathrm{coMod}_{\mathsf{block}^{lf}(\mathsf{U})}(\mathcal{O}(G)),$$

where $\mathrm{coMod}_{\mathsf{block}^{lf}(\mathsf{U})}(\mathcal{O}(G))$ is the category of finitely generated $\mathcal{O}(G)$-comodules in the category $\mathsf{block}^{lf}(\mathsf{U})$. (Notice that the action $\mathrm{Rep}(G)$ on $\mathrm{Rep}^{lf}(\mathsf{U})$ preserves $\mathsf{block}^{lf}(\mathsf{U})$, thus the notion of an $\mathcal{O}(G)$-comodule in this category is well-defined).

Furthermore, Res, $\overline{\mathrm{Ind}}$ are exact, hence $\overline{\mathrm{Ind}}$ sends injective objects to injective ones. It is not hard to deduce that the above equivalences are inherited by the derived categories, i.e. we have

$$D^b\mathrm{Rep}(\mathsf{u}) \cong \mathrm{coMod}_{D^b\mathrm{Rep}^{lf}(\mathsf{U})}(\mathcal{O}(G));$$

$$D^b\mathsf{block}(\mathsf{u}) \cong \mathrm{coMod}_{D^b\mathsf{block}^{lf}(\mathsf{U})}(\mathcal{O}(G)).$$

Here $\mathrm{coMod}_{D^b\mathrm{Rep}^{lf}(\mathsf{U})}(\mathcal{O}(G))$, $\mathrm{coMod}_{D^b\mathsf{block}^{lf}(\mathsf{U})}(\mathcal{O}(G))$ denote the categories of $\mathcal{O}(G)$-comodules in the corresponding bounded derived categories.

A similar relation exists between the (derived) categories of equivariant and non-equivariant (or equivariant under a smaller group) coherent sheaves. More precisely, we have a pair of adjoint functors $R : D^bCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}}) \to D^bCoh^{\mathbb{C}^*}(\widetilde{\mathcal{N}})$ and $Av : D^bCoh^{\mathbb{C}^*}(\widetilde{\mathcal{N}}) \to D^bQCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})$, where R stands for the restriction of equivariance functor, and the right adjoint Av is the "averaging" functor a_*pr^*, where $pr, a : G \times \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$ are, respectively, the projection and the action map. An object $Av(\mathcal{F})$, $\mathcal{F} \in Coh^{\mathbb{C}^*}(\widetilde{\mathcal{N}})$ carries an additional structure of a finitely generated $\mathcal{O}(G)$-comodule. As above, the structure of an $\mathcal{O}(G)$-comodule on a quasi-coherent equivariant sheaf amounts to the data of an isomorphism $h_V : V \otimes \mathcal{F} \xrightarrow{\sim} \underline{V} \otimes \mathcal{F}$ fixed for every algebraic G-module V and satisfying the compatibilities of [AG]. Here the action of $\mathrm{Rep}(G)$ on $QCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})$ is given by $V : \mathcal{F} \mapsto V \otimes \mathcal{F}$, where the G-equivariant structure on the sheaf $V \otimes \mathcal{F}$ is the tensor product from the equivariant structure on $\mathcal{F}$ and the action of G on V.

It is elementary to check that this way we obtain an equivalence

$$Coh^{\mathbb{C}^*}(\widetilde{\mathcal{N}}) \cong \mathrm{coMod}^{fl}_{QCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})}(\mathcal{O}(G)),$$

where $\mathrm{coMod}^{fl}_{QCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})}(\mathcal{O}(G))$ is the category of finitely generated $\mathcal{O}(G)$-comodules in $QCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})$.

Moreover, the functors Res, $\overline{\mathrm{Ind}}$ are exact, and the functor $\overline{\mathrm{Ind}}$ sends injective ind-objects to injective ind-objects. It follows that

$$D^bCoh^{\mathbb{C}^*}(\widetilde{\mathcal{N}}) \cong \mathrm{coMod}^{fl}_{D^bQCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})}(\mathcal{O}(G)),$$

where the category in the right hand side is the category of finitely generated comodules for $\mathcal{O}(G)$ in $D^bQCoh^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})$.

It remains to notice that in view of property (4) of Theorem 3 the functor F intertwines the actions of the tensor category $\mathrm{Rep}(G)$, thus it induces a functor between the categories of finitely generated comodules. It is immediate to see that properties (1–3) of the functor F from Theorem 3 yield similar properties of the induced functor between the comodule categories. □

2.4. **Connection to the Kac-De Concini algebra and a result of** [BK]**.** In this subsection we sketch an alternative way to prove and somewhat strengthen Theorem 4, which relies on the result of [BK] and elementary theory of differential-graded schemes. The material of this subsection is not used elsewhere in the paper; the details are omitted.

Recall that $\mathfrak{U}$ denotes the *De Concini – Kac* algebra. The center $Z(\mathfrak{U})$ contains two subalgebras, the *Harish-Chandra center* Z_{HC}, and the *l-center* Z_l. The Harish-Chandra center Z_{HC} is obtained from the center of the quantized enveloping algebra U_q by specialization. The center $Z(U_q)$ is isomorphic by the quantum Harish-Chandra map to $((U_q)_0)^{W\ltimes\Gamma}$, where $(U_q)_0$ is the subalgebra of U_q generated by $\{K_\mu\}_{\mu\in\check{\mathbb{Y}}}$, and Γ is the group of homomorphisms $\check{\mathbb{Y}}\to\{\pm1\}$. The l-center Z_l is the central subalgebra of $\mathfrak{U}_l$ generated by the l-th powers of the generators $\{E_i, F_i\}_{i\in I}$, $\{K_\mu\}_{\mu\in\check{\mathbb{Y}}}$. Then we have $Z(\mathfrak{U})\simeq Z_l\otimes_{Z_l\cap Z_{HC}} Z_{HC}$ (see e.g. [DKP]).

Let $\mathrm{Rep}(\mathfrak{U})_0$, respectively, $\mathrm{Rep}(\mathfrak{U})_{\widehat{0}}$ denote the full subcategory in $\mathrm{Rep}(\mathfrak{U})$ consisting of modules killed by the augmentation ideal in Z_l (respectively, by some power of this ideal). Similarly, let $\mathrm{Rep}(\mathfrak{U})^0$, $\mathrm{Rep}(\mathfrak{U})^{\widehat{0}}$ be the full subcategories in $\mathrm{Rep}(\mathfrak{U})$ consisting of modules killed by the augmentation ideal in Z_{HC} (respectively, by some power of this ideal). We also set $\mathrm{Rep}(\mathfrak{U})_0^0=\mathrm{Rep}(\mathfrak{U})_0\cap\mathrm{Rep}(\mathfrak{U})^0$ etc.

We have $\mathsf{u}=\mathfrak{U}\otimes_{Z_l}\Bbbk$, thus $\mathrm{Rep}(\mathfrak{U})_0=\mathrm{Rep}(\mathsf{u})$. It is not hard to show that $\mathrm{Rep}(\mathfrak{U})_0^{\widehat{0}}\cong\mathsf{block}(\mathsf{u})$.

The main theorem of [BK] yields an equivalence

$$D^b\,\mathrm{Rep}(\mathfrak{U})_{\widehat{0}}^{\widehat{0}}\cong D^bCoh_{G/B}(\widetilde{\mathfrak{g}}). \tag{5}$$

Here $\widetilde{\mathfrak{g}}=G\times^B\mathfrak{b}$, and $Coh_{G/B}(\widetilde{\mathfrak{g}})$ is the full subcategory in $Coh(\widetilde{\mathfrak{g}})$ consisting of sheaves set-theoretically supported on the zero section. The completion $\widehat{Z}_l$ of Z_l at the augmentation ideal is naturally identified with the completion $\widehat{\mathcal{O}}(\mathfrak{g})$ of the polynomial algebra $\mathrm{Sym}(\mathfrak{g})$ at the augmentation ideal (here we use an identification $\mathfrak{g}\cong\mathfrak{g}^*$ provided by an invariant quadratic form). The equivalence (5) intertwines the action of $\widehat{Z}_l$ on the derived $\mathfrak{U}$-module category with the action of $\widehat{\mathcal{O}}(\mathfrak{g})$ on the derived category of coherent sheaves coming from the Grothendieck-Springer map $\widetilde{\mathfrak{g}}\to\mathfrak{g}$.

One can show that the completion of $\mathfrak{U}$ at the augmentation ideal of $Z_l\cdot Z_{HC}$ is flat over $\widehat{Z}_l$.

Furthermore, one can deduce by $\widehat{Z}_l=\widehat{\mathcal{O}}(\mathfrak{g})$" a base change argument an equivalence

$$D^b\,\mathrm{Rep}(\mathfrak{U})_0^{\widehat{0}}\cong DGCoh(\widetilde{\mathfrak{g}}\overset{\mathrm{L}}{\times}_{\mathfrak{g}}\{0\}). \tag{6}$$

Here $\widetilde{\mathfrak{g}}\overset{\mathrm{L}}{\times}_{\mathfrak{g}}\{0\}$ is the *differential graded (DG) scheme*, which is the *derived fiber product* of the schemes $\widetilde{\mathfrak{g}}$ and $\{0\}=Spec(\Bbbk)$ over $\mathfrak{g}$, while $DGCoh$ denotes the derived category of sheaves of DG $\mathcal{O}$-modules over the DG scheme (see e.g. [Ka] for the definitions).

By the definition, the structure sheaf of the DG scheme $\widetilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \{0\}$ is a sheaf of DG-algebras on G/B, which is well-defined up to a quasiisomorphism. A possible construction of a representative of the quasi-isomorphism class is as follows: $\mathcal{O}_{\widetilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \{0\}} = \pi^* \mathbb{K}_{\mathfrak{g}}$, where $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is the Grothendieck-Springer map, and $\mathbb{K}_{\mathfrak{g}} = \Lambda^\bullet(\mathfrak{g}) \otimes \mathrm{Sym}^\bullet(\mathfrak{g})$ is the Koszul complex of $\mathfrak{g}$.

It is not hard to show that for a vector bundle (locally free sheaf) $\mathcal{E}$ on an algebraic variety together with an embedding of vector bundles $\mathcal{E} \subset V \otimes \mathcal{O}$ of $\mathcal{E}$ into the trivial vector bundle, the sheaf of DG-algebras $\mathcal{O}_{E \overset{\mathrm{L}}{\times}_V \{0\}}$ is canonically quasiisomorphic to the sheaf of DG-algebras with zero differential

$$\Lambda((V \otimes \mathcal{O}/\mathcal{E})^*[1]) = \underline{\mathrm{Tor}}_\bullet^{\mathcal{O}(V)}(\mathcal{O}_E, \Bbbk).$$

Here E denotes the total space of $\mathcal{E}$.

In particular we see that $\mathcal{O}_{\widetilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \{0\}}$ can be represented by the DG algebra with zero differential $\Lambda^\bullet(\Omega^1_{G/B}[1])$, where $\Omega^1_{G/B}$ is the locally free sheaf of 1-forms on G/B. Thus we get an equivalence

$$D^b \mathsf{block}(\mathsf{u}) \cong DGCoh(\Lambda^\bullet(\Omega^1_{G/B}[1])).$$

Finally, the standard Koszul (or $S - \Lambda$) duality, see e.g. [BGS] (cf. also [ABG], §3.3), gives a canonical equivalence

$$DGCoh(\Lambda^\bullet(\Omega^1_{G/B}[1])) \cong DGCoh(\mathrm{Sym}^\bullet(T_{G/B}[-2])),$$

where $T_{G/B}$ is the tangent sheaf of G/B. Thus we get

$$D^b \mathsf{block}(\mathsf{u}) \cong DGCoh(\mathrm{Sym}^\bullet(T_{G/B}[-2])).$$

Notice that the relative spectrum of the sheaf of commutative rings $\mathrm{Sym}^\bullet(T_{G/B})$ on G/B is nothing but $\widetilde{\mathcal{N}}$. Thus the last equivalence implies Theorem 4.

This method can also be used to provide a similar description for the derived category of the regular block for the restricted enveloping algebra of a semi-simple Lie algebra over a field of positive characteristic; the reference to [BK] should then be replaced by a reference to [BMR].

3. Hochschild cohomology of the principal block of u

The finite dimensional Hopf algebra u decomposes as a left u-module into a finite direct sum of finite dimensional submodules. Denote by u_0 the largest direct summand for which all its simple subquotients belong to the principal block of the category $\mathrm{Rep}(\mathsf{u})$. Then u_0 is a two-sided ideal in u, which will be called the principal block of u.

Let z denote the center of u. It decomposes into a direct sum of ideals according to the block decomposition of u. Set $\mathsf{z}_0 = \mathsf{z} \cap \mathsf{u}_0$.

The rest of this section contains a computation of the Hochschild cohomology of the principal block of u and a description of the center z_0.

3.1. **The result.** Recall that we have a G-equivariant isomorphism of vector bundles $G \otimes_B \mathfrak{n} \cong T^*(G/B) = \widetilde{\mathcal{N}}$ and that the multiplicative group acts on $\widetilde{\mathcal{N}}$ by dilations on the fibers: an element $t \in \mathbb{C}^*$ acts on n by multiplication by t^2. Consider the coherent sheaf of poly-vector fields $\Lambda^\bullet T(\widetilde{\mathcal{N}})$ on $\widetilde{\mathcal{N}}$. The direct image of this sheaf to G/B is in fact bi-graded. Here the first grading is the natural grading

$\Lambda^\bullet T(\widetilde{\mathcal{N}}) = \oplus_{j=0}^{2\dim(\widetilde{\mathcal{N}})} \Lambda^j T(\widetilde{\mathcal{N}})$. The second grading comes from the induced action of $\mathbb{C}^*$ on $\widetilde{\mathcal{N}}$. We will write $\Lambda^j T(\widetilde{\mathcal{N}})^k$ for the (j,k)-th component with respect to this bi-grading; this is a locally free G-equivariant coherent sheaf on G/B. Notice that $\Lambda^j T(\widetilde{\mathcal{N}})^k = 0$ for odd k.

Theorem 7. *There exists an isomorphism of algebras between the total Hochschild cohomology of the principal block* u_0 *and the total cohomology of* $\widetilde{\mathcal{N}}$ *with coefficients in* $\Lambda^\bullet T(\widetilde{\mathcal{N}})$*; here the algebra structure on the second space comes from multiplication in the exterior algebra* $\Lambda^\bullet T(\widetilde{\mathcal{N}})$*. The isomorphism is compatible with the grading as follows:*

$$\mathrm{HH}^s(\mathsf{u}_0) \cong \oplus_{i+j+k=s} \mathrm{H}^i(\Lambda^j T(\widetilde{\mathcal{N}}))^k.$$

Remark 8. "Morally" the Theorem is a direct consequence of Theorem 4. More precisely, suppose it were possible to define for a triangulated category $\mathcal{T}$ a triangulated category of endo-functors $End(\mathcal{T})$ with good properties. Some of the expected properties are as follows: if $\mathcal{T}$ is the derived category of modules over an algebra, then $End(\mathcal{T})$ is the derived category of bi-modules over the same algebra; while if $\mathcal{T}$ is the derived category of (equivariant) coherent sheaves on an algebraic variety X, then $End(\mathcal{T})$ is the derived category of (equivariant) coherent sheaves on X^2. The relation between the categories $D^b(Coh^{\mathbb{C}^*}(\widetilde{\mathcal{N}}))$ and $D^b(\mathsf{block}(\mathsf{u}))$ explained in Theorem 4 would then imply a similar relation between the endomorphism categories. Expanding property (2) of Theorem 3 for $\mathcal{F}$ and $\mathcal{F}'$ being the identity functor we would get Theorem 7.

It is well known that the naive category of endo-functors of a triangulated category does not, in fact, carry a natural triangulated structure and does not satisfy the properties indicated above. One can probably define an appropriate category of endomorphisms by working with differential graded categories (or in another rigid setting, such as that of A_∞ categories). We found it more effective to derive Theorem 7 by a more elementary ad hoc argument.

Corollary 9. *The principal block of the center of* u *is isomorphic as an algebra to*

$$\mathsf{z}_0 \cong \oplus_{i+j+k=0} \mathrm{H}^i(\Lambda^j T(\widetilde{\mathcal{N}}))^k.$$

The Corollary is immediate by setting $s = 0$ in Theorem 7.

The proof of Theorem 7 is based on the following standard statement, which is an algebraic version of the Hochschild-Kostant-Rosenberg Theorem, see, e.g. [Sw].

Lemma 10. *setting Let* X *be a smooth variety and* $\delta : X \to X \times X$ *the diagonal imbedding. Then there is an algebra isomorphism*

$$\mathrm{Ext}^q_{Coh(X\times X)}(\delta_*\mathcal{O}_X, \delta_*\mathcal{O}_X) \cong \bigoplus_{i+j=q} H^i(\Lambda^j T_X).$$

In what follows we identify $\mathsf{u} \cong \mathsf{u}^{op}$ by means of the antipode of the Hopf algebra u, and we identify the category of u-bimodules with that of $\mathsf{u}\otimes\mathsf{u}$-modules. We let $\mathsf{block}(\mathsf{u}^2)$ denote the block of the trivial representation in the category of u-bimodules.

In view of the Lemma, Theorem 7 follows immediately from the following

Proposition 11. *There exists a functor* $\Phi : D^b Coh^{\mathbb{C}^*}(\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}) \to D^b\mathsf{block}(\mathsf{u}^{\otimes 2})$*, which satisfies the properties (1–3) of Theorem 3, and such that* $\Phi(\delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}})) \cong R$*;*

here R is the maximal summand in the regular bimodule for u *belonging to the principal block, and* $\delta : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$ *is the diagonal embedding.*

3.2. **The proof.** The rest of the section is devoted to the proof of the Proposition 11. We start with some auxiliary statements.

Recall a monoidal structure on the derived category of bimodules and an action of this monoidal category on the derived category of modules. More precisely, let a be an associative ring, and $\mathsf{a}-\mathrm{mod}$, $\mathsf{a}-\mathrm{bimod}$ be the categories of a-modules and of a-bimodules respectively. We set $B \star M = B \overset{\mathrm{L}}{\otimes}_{\mathsf{a}} M$; here $B \in D^-(\mathsf{a}-\mathrm{bimod})$, and M is either an object of the same category, or $M \in D^-(\mathsf{a}-\mathrm{mod})$.

In the first case we get a monoidal structure on $D^-(\mathsf{a}-\mathrm{bimod})$, while the second one gives an action of this monoidal category on $D^-(\mathsf{a}-\mathrm{mod})$.

We also have the dual operation $B \overset{!}{\star} M = R\mathrm{Hom}_{\mathsf{a}}(B, M)$. This formula defines functors $D^+(\mathsf{a}-\mathrm{bimod}) \times D^+(\mathsf{a}-\mathrm{bimod}) \to D^+(\mathsf{a}-\mathrm{bimod})$, $D^+(\mathsf{a}-\mathrm{bimod}) \times D^+(\mathsf{a}-\mathrm{mod}) \to D^+(\mathsf{a}-\mathrm{mod})$.

Lemma 12. *Let* a *be a (left and right) Noetherian associative ring, and* $B \in D^b(\mathsf{a}-\mathrm{bimod})$ *be such that the image of* B *in* $D^b(\mathsf{a}-\mathrm{mod})$, $D^b(\mathsf{a}^{op}-\mathrm{mod})$ *under the functors of forgetting the right (respectively, left) action is a perfect complex (i.e. can be represented by a finite complex of finitely generated projective modules).*

a) The functor $M \mapsto B \star M$ *from* $D^b(\mathsf{a}-\mathrm{mod}) \to D^b(\mathsf{a}-\mathrm{mod})$ *has a right adjoint given by* $M \mapsto B \overset{!}{\star} M$.

b) We have a canonical isomorphism

$$(B \overset{!}{\star} C) \star D \cong B \overset{!}{\star} (C \star D)$$

Here $C \in D^b(\mathsf{a}-\mathrm{bimod})$, *and* D *either lies in* $D^-(\mathsf{a}-\mathrm{bimod})$, *or in* $D^-(\mathsf{a}-\mathrm{mod})$.

c) Assume moreover that the functor $D^b(\mathsf{a}-\mathrm{mod}) \to D^b(\mathsf{a}-\mathrm{mod})$, $M \mapsto B \star M$ *is an equivalence. Then the functor* $D^b(\mathsf{a}-\mathrm{bimod}) \to D^b(\mathsf{a}-\mathrm{bimod})$, $M \mapsto B \overset{!}{\star} M$, *is an equivalence sending* B *to the regular bimodule.*

Proof. a) and b) are standard. To check (c) observe that right adjoint to an equivalence is the inverse equivalence. Thus the composition of endo-functors $M \mapsto B \star M$ and $M \mapsto B \overset{!}{\star} M$ of $D^b(\mathsf{a}-\mathrm{mod})$ is isomorphic to identity. This composition is given by

$$M \mapsto B \overset{!}{\star} (B \star M) \cong (B \overset{!}{\star} B) \star M,$$

where the isomorphism is provided by part (b).

Thus setting $C = (B \overset{!}{\star} B)$ we see that the endo-functor of $D^b(\mathsf{a}-\mathrm{mod})$, $M \mapsto C \star M$ is isomorphic to identity. This is easily seen to imply that C is isomorphic to the regular bimodule. It remains to show that the endofunctor of $D^b(\mathsf{a}-\mathrm{bimod})$, $M \mapsto B \overset{!}{\star} M$ is an equivalence.

Its left adjoint functor $M \mapsto B \star M$ (where the adjunction is provided by part (a) of the Lemma) is also its left inverse: this is clear from (b) and the established isomorphism between $B \overset{!}{\star} B$ and the regular bimodule. This implies that the endofunctor of $D^b(\mathsf{a}-\mathrm{bimod})$, $M \mapsto B \star M$ is a full embedding, and the category $D^b(\mathsf{a}-\mathrm{bimod})$ admits a semi-orthogonal decomposition

$$D^b(\mathsf{a}-\mathrm{bimod}) = \mathrm{Im}(M \mapsto B \star M) * \mathrm{Im}(M \mapsto B \star M)^{\perp}.$$

Here $\mathrm{Im}(M \mapsto B\star M)^{\perp} \subset D^b(\mathsf{a}-\mathrm{bimod})$ is the full subcategory of objects $N \in D^b(\mathsf{a}-\mathrm{bimod})$ satisfying $\mathrm{Hom}(B\star M, N)=0$ for all M, and the displayed formula means that for every object $M\in D^b(\mathsf{a}-\mathrm{bimod})$ there exists a (canonical) distinguished triangle

$$M_1 \to M \to M_2, \qquad M_1 \in \mathrm{Im}(M\mapsto B\star M),\ M_2\in \mathrm{Im}(M\mapsto B\star M)^{\perp}.$$

Thus to show that $M\mapsto B\star M$, $M\mapsto B\overset{!}{\star}M$ are auto-equivalences of $D^b(\mathsf{a}-\mathrm{bimod})$ it is enough to show $\mathrm{Im}(M\mapsto B\star M)^{\perp}=\{0\}$. We have $\mathrm{Im}(M\mapsto B\star M)^{\perp}=\mathrm{Ker}(M\mapsto B\overset{!}{\star}M)$. Thus it suffices to show that $B\overset{!}{\star}M\neq 0$ provided $M\neq 0$. The endo-functors of $D^b(\mathsf{a}-\mathrm{bimod})$, $D^b(\mathsf{a}-\mathrm{mod})$ defined by $M\mapsto B\overset{!}{\star}M$ are obviously compatible via the functor of forgetting the right action. Since the endo-functor of $D^b(\mathsf{a}-\mathrm{mod})$, $M\mapsto B\overset{!}{\star}M$ is an auto-equivalence, it kills no non-zero objects; hence the same is true for the endo-functor of $D^b(\mathsf{a}-\mathrm{bimod})$. $\square$

Replacing $\mathfrak{g}$ by $\mathfrak{g}\oplus\mathfrak{g}$ in Theorem 4, we obtain the functor

$$F_{u^2}: D^bCoh^{\mathbb{C}^*}(\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}}) \to D^b\mathsf{block}(\mathsf{u})^{\otimes 2}$$

satisfying properties (1–3) of Theorem 3.

Lemma 13. *Set $B=F_{u^2}(\delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}}))$. Then we have a functorial isomorphism*

$$F_u(\mathcal{F})\cong B\star F_u(\mathcal{F}^{\check{}})^*$$

for $\mathcal{F}\in D^bCoh^{\mathbb{C}^*}(\widetilde{\mathcal{N}})$; here $\mathcal{F}^{\check{}}=R\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{O}_{\widetilde{\mathcal{N}}})$ is the dual sheaf, and $*$ denotes the usual duality for (complexes of) modules.

Proof. It will be convenient to use the abbreviation $\mathrm{Hom}^\bullet(M,N)=\bigoplus_i \mathrm{Hom}(M,N[i])$ for $M,N\in D^b\mathsf{block}(\mathsf{u})$ and $\mathrm{Hom}^\bullet(\mathcal{F},\mathcal{G})=\bigoplus_{i,j}\mathrm{Hom}(\mathcal{F},z^j\otimes\mathcal{G}[i])$ for $\mathcal{F},\mathcal{G}\in D^bCoh^{\mathbb{C}^*}(\widetilde{\mathcal{N}})$. Thus property (2) of Theorem 3 asserts that

$$\mathrm{Hom}^\bullet(\mathcal{F},\mathcal{G})\xrightarrow{\sim}\mathrm{Hom}^\bullet(F_u(\mathcal{F}),F_u(\mathcal{G}))$$

for $\mathcal{F},\mathcal{G}\in D^bCoh^{\mathbb{C}^*}(\widetilde{\mathcal{N}})$. Notice that this isomorphism preserves the grading, where elements of $\mathrm{Hom}(M,N[i])$ are assigned degree i, while elements of $\mathrm{Hom}(\mathcal{F},z^j\otimes\mathcal{G}[i])$ are assigned degree $i+j$.

It suffices to construct a functorial isomorphism

$$\mathrm{Hom}^\bullet(\mathcal{F},\mathcal{G})\cong\mathrm{Hom}^\bullet(B\star F_u(\mathcal{F}^{\check{}})^*,F_u(\mathcal{G})),$$

preserving the grading. Indeed, then plugging in $\mathcal{F}=\mathcal{G}$ we get a morphism $F_u(\mathcal{F}^{\check{}})^*\to F_u(\mathcal{F})$, which induces an isomorphism between the spaces of homomorphisms to $F_u(\mathcal{G})$ for any $\mathcal{G}$. Since the image of F_u generates $D^b\mathsf{block}(\mathsf{u})$, we see that this morphism is an isomorphism.

We have:

$$\mathrm{Hom}^\bullet((B\star F_u(\mathcal{F}^{\check{}})^*,F_u(\mathcal{G}))\cong\mathrm{Hom}^\bullet_{D^b\mathsf{block}(\mathsf{u}^2)}(B,F_u(\mathcal{F}^{\check{}})\boxtimes F_u(\mathcal{G}))\cong \mathrm{Hom}^\bullet(\delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}}),\mathcal{F}^{\check{}}\boxtimes\mathcal{G});$$

here we used the obvious compatibility of F_u with external products, and also the adjunction $\mathrm{Hom}^\bullet(B\star M,N)\cong\mathrm{Hom}^\bullet(B,M^*\boxtimes N)$ valid for any bimodule B and modules M, N (or complexes of such). Applying duality $\mathcal{F}\mapsto R\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{O}_{\widetilde{\mathcal{N}}^2})$ we get

$$\mathrm{Hom}^\bullet(\delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}}),\mathcal{F}^{\check{}}\boxtimes\mathcal{G})\cong\mathrm{Hom}^\bullet(\mathcal{F}\boxtimes\mathcal{G}^{\check{}},\delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}})^{\check{}}).$$

An easy calculation shows that $\delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}})^{\check{}} \cong z^{2d}\otimes\delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}})[-2d]$, where $d = \dim G/B$. Thus the latter Hom space can be rewritten as

$$\mathrm{Hom}^\bullet(\mathcal{F}\boxtimes\mathcal{G}^{\check{}}, z^{2d}\otimes\delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}})[-2d]) \cong \mathrm{Hom}^\bullet(\mathcal{F}, z^{2d}\otimes\mathcal{G}[-2d]) \cong \mathrm{Hom}^\bullet(\mathcal{F},\mathcal{G}).$$

It is clear that the last isomorphism preserves the grading (the twists of the $\mathbb{C}^*$ action and the homological shifts cancel). □

Proof of Proposition 11. We start with the functor F_{u^2} (introduced before Lemma 13), and set $B = F_{u^2}(\delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}}))$ as in Lemma 13.

Lemma 13 implies that B is of finite projective dimension as left and right u-module, and $M \mapsto B\star M$ is an auto-equivalence of $D^b\mathsf{block}(\mathsf{u})$. Indeed, objects of the form $F_u(\mathcal{F}^{\check{}})^*$ generate $D^b\mathsf{block}(\mathsf{u})$ as a triangulated category, since the image of F_u does. Since the functor $M\mapsto B\star M$ sends such objects into the bounded derived category (rather than into $D^-\mathsf{block}(\mathsf{u})$), it also sends the whole of $D^b\mathsf{block}(\mathsf{u})$ into itself. It follows that B has a finite projective dimension as a right u-module. The involution of switching the two factors in $\mathsf{u}\otimes\mathsf{u}$ obviously sends B into itself, thus B also has a finite projective dimension as a left u-module. Finally, the functor $M\mapsto B\star M$ sends the set $\{F_u(\mathcal{F}^{\check{}})^* \mid \mathcal{F}\in D^b(Coh^{\mathbb{C}^*}(\widetilde{\mathcal{N}}))\}$ generating $D^b\mathsf{block}(\mathsf{u})$ as a triangulated category into another generating set, and it induces an isomorphism on Hom's between objects in the generating set. Hence it is an equivalence.

Thus by Lemma 12 there exists an autoequivalence A of $D^b\mathsf{block}(\mathsf{u}^2)$, sending B to the regular bimodule. Then $\Phi = A\circ F_{u^2}$ is readily seen to satisfy the required properties. □

Remark 14. One can ask for a more explicit description of the object $B = F_{u^2}(\delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}}))$. This question is similar to the question of describing the bimodule over the classical enveloping algebra $U(\mathfrak{g})$ obtained as the global sections of the D-module $\Delta_*(\mathcal{O}_{G/B})$; here $\Delta: G/B\to(G/B)^2$ is the diagonal embedding, and Δ_* denotes the direct image in the category of D-modules. In both cases one can show that the endo-functor of the derived category of modules coming from this bimodule can be described in terms of the action of the braid group on the derived category of modules by *intertwining functors* [BB] (cf. also [ABG], §4.1). More precisely, it coincides with the action of the canonical lifting to the braid group of the longest element in the Weyl group. We neither prove nor use this fact in the present paper.

3.3. **An explicit subalgebra in the principal block of the center of the small quantum group.** We will describe an easily computable subalgebra in z_0. To state the answer, define a commutative algebra $\mathfrak{H}$ of dimension $2|W|-1$ as follows. Endow the space $H^\bullet(G/B,\Bbbk)\oplus H^\bullet(G/B,\Bbbk)$ with a commutative algebra structure given by $(h_1,h_2)\cdot(h_1',h_2') = (h_1h_1', \epsilon(h_1)h_2' + \epsilon(h_1')h_2$, where $\epsilon: H^\bullet(G/B)\to\Bbbk = H^0(G/B)$ is the augmentation. We let $\mathfrak{H}$ be the quotient of this algebra, $\mathfrak{H} = H^\bullet(G/B)\oplus H^\bullet(G/B)/\Bbbk(\omega,-1)$, where ω, 1 are the canonical generators of $H^{\mathrm{top}}(G/B)$, $H^0(G/B)$ respectively.

Proposition 15. *The principal block of the center z_0 contains a canonically defined subalgebra isomorphic to $\mathfrak{H}$.*

Proof. The fibration $pr: \widetilde{\mathcal{N}}\to G/B$ induces a short exact sequence of $G\times\mathbb{C}^*$ equivariant sheaves on $\widetilde{\mathcal{N}}$:

$$0\to T^{vert}_{\widetilde{\mathcal{N}}} = z^{-2}\otimes pr^*T^*_{\mathcal{G}/B}\to T_{\widetilde{\mathcal{N}}}\to T^{hor}_{\widetilde{\mathcal{N}}} = pr^*T_{G/B}\to 0,$$

where $T^{vert}_{\widetilde{\mathcal{N}}}$, $T^{hor}_{\widetilde{\mathcal{N}}}$ are, respectively, the horizontal and the vertical tangent spaces. In particular we get embeddings

$$z^{-2i} pr^* \Omega^i_{G/B} \hookrightarrow \Lambda^i T_{\widetilde{\mathcal{N}}},$$

$$\Lambda^d(T^{vert}_{\widetilde{\mathcal{N}}}) \otimes \Lambda^i(pr^*(T_{G/B})) \hookrightarrow \Lambda^{d+i}(T_{\widetilde{\mathcal{N}}}),$$

where $d = \dim(G/B)$. Notice that $\Lambda^d(T^{vert}_{\widetilde{\mathcal{N}}}) \cong z^{-2d} \otimes pr^*(\Omega^d_{G/B})$. Also a standard isomorphism $\Lambda^d(V^*) \otimes \Lambda^i(V) \cong \Lambda^{d-i}(V^*)$ for a d-dimensional vector space V yields $\Omega^d(G/B) \otimes \Lambda^i(T_{G/B}) \cong \Omega^{d-i}_{G/B}$. Thus the second embedding above can be rewritten as:

$$z^{-2d} \otimes pr^* \Omega^{d-i}_{G/B} \hookrightarrow \Lambda^{d+i} T_{\widetilde{\mathcal{N}}}.$$

In fact, the two embeddings are easily seen to give isomorphisms of sheaves on G/B:

$$\Lambda^i(T_{\widetilde{\mathcal{N}}})^{-2i} \cong \Omega^i_{G/B} \cong \Lambda^{d+i}(T_{\widetilde{\mathcal{N}}})^d.$$

It is clear that the subsheaf $\bigoplus\limits_i \Lambda^i(T_{\widetilde{\mathcal{N}}})^{-2i} \subset pr_*(\Lambda^\bullet(T_{\widetilde{\mathcal{N}}}))$ is a subsheaf of subalgebras isomorphic to $\Omega^\bullet_{G/B}$. Thus, using Corollary 9 we get a subalgebra $\bigoplus\limits_i H^i(\Lambda^i(T_{\widetilde{\mathcal{N}}})^{-2i}) \subset \mathsf{z}_0$, isomorphic to $H^\bullet(G/B) = \bigoplus H^i(\Omega^i_{G/B})$.

Similarly we get a subsheaf $\bigoplus\limits_i \Lambda^{d+i}(T_{\widetilde{\mathcal{N}}})^d \subset pr_*(\Lambda^\bullet(T_{\widetilde{\mathcal{N}}}))$ with zero multiplication; multiplication of this sheaf by the sheaf $\Lambda^i(T_{\widetilde{\mathcal{N}}})^{-2i}$ also vanishes for $i > 0$. Thus we get a subspace

$$H^\bullet(\bigoplus_i \Lambda^{d+i}(T_{\widetilde{\mathcal{N}}})^d) \cong \bigoplus H^i(\Omega^i_{G/B}) = H^\bullet(G/B) \subset \mathsf{z}_0.$$

Together with previously constructed subalgebra $H^\bullet(\bigoplus\limits_i \Lambda^i(T_{\widetilde{\mathcal{N}}})^{-2i})$ it clearly generates a subalgebra canonically isomorphic to $\mathfrak{H}$. $\square$

A $|W|$-dimensional subalgebra $\underline{\mathsf{z}_0} \subset \mathsf{z}_0$ isomorphic to $H^\bullet(G/B)$ was described in [BG], where it was obtained by ramification of the center of the De Concini-Kac quantum algebra $\mathfrak{U}$. The subalgebra $\underline{\mathsf{z}_0}$ can be defined as the intersection of $Z_{HC} \otimes_{Z_l \cap Z_{HC}} \Bbbk$ with the principal block u_0 and is usually referred to as the Harish-Chandra part of the center of u_0.

In case when the root system of $\mathfrak{g}$ is simple and simply laced, another $|W|$-dimensional subspace $\overline{\mathsf{z}_0} \subset \mathsf{z}_0$ was constructed in [La] using the quantum Fourier transform [LM]. The subspace $\overline{\mathsf{z}_0}$ is shown to be an ideal in z_0 with zero multiplication, such that $\mathrm{Nilrad}(\mathsf{z}_0) \cdot \overline{\mathsf{z}_0} = 0$. The the intersection $\underline{\mathsf{z}_0} \cap \overline{\mathsf{z}_0}$ is one-dimensional and the subalgebra $\underline{\mathsf{z}_0} + \overline{\mathsf{z}_0} \subset \mathsf{z}_0$ is isomorphic to $\mathfrak{H}$.

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