On a class of Dirichlet series associated to the ring of representations of a Weil group

by

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§ 1. Introduction

In early fifties Yu.V. Linnik circulated the following problem among his colleagues and students (cf. [2], [14]): let

$$L(s,\chi_j) = \sum_{n=1}^{\infty} a_n(\chi_j)n^{-s} , \quad 1 \le j \le r ,$$

be a Hecke L-function with a grossencharacter χ_j in the field k_j ; can one continue meromorphically the function

$$s \longmapsto \sum_{n=1}^{\infty} n^{-s} \prod_{j=1}^{r} a_n(\chi_j)$$

to the half-plane $\operatorname{Re} s < 1$? According to Yu.V. Linnik, this problem would have interesting applications to the multi-dimensional arithmetic in the sense of E. Hecke, [5] (cf. also [9], [13]). After some preliminary results summarised in the author's thesis (summer 1964) A.I. Vinogradov, [35], had obtained the meromorphic continuation to the half-plane $\operatorname{Re} s > \frac{1}{2}$ under an additional assumption that the fields $k_1,\ldots,k_r$ are linear disjoint (over $\mathbb{Q}$) . A few years later O.M. Fomenko, [2], had continued this function meromorphically to the whole complex plane in the case of two quadratic fields (that is, when $r \doteq [k_1 : \mathbb{Q}] = [k_2 : \mathbb{Q}] = 2$). Generalising slightly this construction, suppose that $k \subseteq k_j$ for each j and write

$$L(s,\chi_j) = \sum_{\mathfrak{n}} a_{\mathfrak{n}}(\chi_j) N_{k/\mathbb{Q}}\mathfrak{n}^{-s} , \quad 1 \le j \le r , \tag{1}$$

where $\mathfrak{n}$ ranges over integral ideals of k ; one defines the scalar product of Hecke L-functions (1) over k as a Dirichlet series

$$L(s,\vec{\chi}) = \sum_{\mathfrak{n}} N_{k/\mathbb{Q}}\mathfrak{n}^{-s} \prod_{j=1}^{r} a_{\mathfrak{n}}(\chi_j) \tag{2}$$

convergent absolutely for $\operatorname{Re} s > 1$. In his thesis, [1], P.K.J. Draxl had continued the scalar product $L(s,\vec{\chi})$ meromorphically to the half-plane $\operatorname{Re} s > 0$. On the other hand, it follows from the general theory developed in [6] that in the case of two quadratic extensions (that is, when $[k_1 : k] = [k_2 : k] = r = 2$) this function admits a meromorphic continuation to the whole complex plane. Independently and about the same time several authors, [3], [11], [15] (cf. also [16]), had noticed that the case of two quadratic extensions can be treated elementary and had expressed $L(s,\vec{\chi})$ in terms of the ordinary Hecke functions in this case. In 1977 N. Kurokawa, [10], (*) showed that Draxl's result was, in fact, the best possible. To state the results of Kurokawa's let us assume, as we may without loss of generality, that

$$k_j \neq k \quad \text{for} \quad 1 \leq j \leq r . \tag{3}$$

If (3) holds and either $r > 2$ or $r = 2$ but

(*) This preprint was kindly communicated to us by the late Professor P.K.J. Draxl in May 1979.

$$[k_1 : k] + [k_2 : k] > 4 ,$$

then the line $\mathrm{Re}\, s = 0$ is the natural boundary of $L(s,\vec{\chi})$ and this function admits no analytic continuation to the half-plane $\mathrm{Re}\, s < 0$. This statement has been proved in [10] under an additional assumption that each of the characters χ_j , $1 \leq j \leq r$, is of finite order. We have removed this assumption at first under the Grand Riemann Hypothesis, [17], then, [18, § II.2], [26], [23, Theorem 1 on p.110], unconditionally. Recently N. Kurokawa, [12], has also obtained this result.(*) It is the goal of this paper to give a shorter proof of the discussed theorem based on a new Primzahlsatz proved in [24]. To make this exposition self-contained we shall recall the main construction described in [13] and thereby give a new proof of Draxl's theorem. Such a proof has been announced in [11] and has been presumably given in the second part of [10], non-available to us (cf. also [17]).

To conclude this introduction let us recall the well known articles, [28], [27], on scalar product of Dirichlet series associated to modular forms which have been reconsidered from

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(*) The reader is advised to disregard the Remark on p.45 in [12] as making no sense at all. In particular, Lemma 20 in [17] is correct and the number of primes satisfying conditions (27) of this lemma tends to infinity as $\nu \to \infty$ and $\varepsilon = \nu^{-2}$ (contrary to the statement of this Remark).

representation-theoretical point of view in [6]. This new point of view advocated by R. Langlands and his school suggests that one should define "convolution" of L-functions associated to automorphic forms locally and then build the corresponding Euler product (cf., for example, [8]). When translated in terms of Dirichlet series, [7], this operation lacks the elegance of Rankin's convolution but, unfortunately, the scalar product of Dirichlet series defined by the assignment

$$\left(\sum_{n=1}^{\infty} a_n n^{-s} , \sum_{n=1}^{\infty} b_n n^{-s} \right) \longmapsto \sum_{n=1}^{\infty} a_n b_n n^{-s}$$

does not have the desirable analytic properties in the general case. It remains to refer to [22] for a review of some classic examples of Dirichlet series with natural boundary and to draw the reader's attention to the class of L-functions defined in [1] whose properties deserve further investigation. The arithmetical applications of the scalar product of Hecke L-functions "mit Größencharakteren" have been described in [35] and [19] - [21] (cf. also [23, Ch.III]). One should mention also an article by K. Chandrasekharan and R. Narasimhan in Math.Ann. 152 (1963), p.30-64, where some scalar products have been studied.

§ 2. Statement of the main results

Let k be an algebraic number field of finite degree over $\mathbb{Q}$, and let $W(k)$ be the (absolute) Weil group of k

(as usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}_+$, $\mathbb{R}$, $\mathbb{C}$ denote the set of natural numbers, the ring of rational integers, the multiplicative group of positive real numbers, the real number field and the complex number field, respectively; when it is necessary k is regarded as a subfield of a fixed algebraic closure of $\mathbb{Q}$, never explicitely mentioned), and let $W(K|k)$ denote the relative Weil group of the finite normal extension $K|k$ with Galois group $G(K|k)$. We embed $\mathbb{R}_+$ diagonally into the infinite component of the idèle-class group of C_K of the field K. Such an embedding leads to an isomorphism

$$C_K \cong C_K^1 \times \mathbb{R}_+ ,$$

where C_K^1 denotes the subgroup of idèle-classes having unit volume, so that

$$W(K|k) \cong W_1(K|k) \times \mathbb{R}_+ ,$$

where $W_1(K|k)$ is a compact group isomorphic to the extension of the Galois group $G(K|k)$ by C_K^1 which is determined by the canonical cohomology class of class field theory. The group $W(k)$ may be defined as the projective limit of the groups $W(K|k)$ when K varies over all the finite normal extensions of k, [33], [30]. Let

$$\rho : W(k) \to GL(V) \qquad (4)$$

be a continuous representation of $W(k)$ into the group of invertible linear operators of a finite dimensional complex vector space V. There is a finite Galois extension K of k such that ρ factors through $W(K|k)$; if $\mathbb{R}_+ \subseteq \operatorname{Ker} \rho$, we say that ρ is <u>normalised</u>. Let X_1 be the set of continuous normalised representations (4) and let Y be the ring of virtual characters generated by the set of characters

$$\{\chi | \chi = \operatorname{tr} \rho, \ \rho \in X_1\} .$$

Consider a polynomial

$$\Phi(t) = 1 + \sum_{j=1}^{\ell} t^j a_j, \ a_j \in Y \tag{5}$$

in $Y[t]$ and let

$$\Phi_g(t) = 1 + \sum_{j=1}^{\ell} t^j a_j(g) \tag{6}$$

for $g \in W(k)$. The polynomial (5) is said to be <u>unitary</u>(*), if $\Phi_g(\alpha) \neq 0$ as soon as $|\alpha| \neq 1$, $\alpha \in \mathbb{C}$, $g \in W(k)$. Any ρ in X_1 may be regarded as a representation of a compact group $W_1(K|k)$, therefore it is semi-simple. Hence one can write

$$a_j = \sum_{\chi} m_j(\chi)\chi \ , \ m_j(\chi) \in \mathbb{Z} ,$$

(*) This concept has been introduced in [10].

where χ varies over simple characters of $W(k)$. Moreover, the set

$$X_\circ(\Phi) = \{\rho | m_j(\mathrm{tr}\ \rho) \neq 0 \ \text{ for some } \ j\}$$

is finite. Given a prime divisor p in k , let σ_p and I_p denote the Frobenius class and the inertia subgroup in $W(k)$ at the place p . Let $\rho \in X_1$ and let, as in (4), V be the representation space of ρ . Consider the subspace

$$V^{I_p} = \{v | v \in V \ , \ \rho(g)v = v \ \text{ for } \ g \in I_p\}$$

of I_p-invariant vectors in V . Since the restriction

$$\rho(g)\Big|_{V^{I_p}} \ \text{ of the operator } \ \rho(g) \ \text{ to } \ V^{I_p}$$

does not depend on the choice of g in σ_p , we may set

$$\rho(\sigma_p) = \rho(g)\Big|_{V^{I_p}} \ , \ g \in \sigma_p \ , \tag{7}$$

and extend (7), by linearity, to Y . Furthermore, let

$$\Phi_p(t) = 1 + \sum_{j=1}^{\ell} t^j \, a_j(\sigma_p) \ . \tag{8}$$

By (6) - (8), if $V^{I_p} = V$ for each ρ in $X_\circ(\Phi)$, then

$$\Phi_p(t) = \Phi_g(t) \ \text{ for any } \ g \ \text{ in } \ \sigma_p \ . \tag{9}$$

In particular, relation (9) is satisfied for all but a finite number of primes p in k . Let F be a finite extension of $\mathbb{Q}$; we write

$$|\mathfrak{a}| := N_{F/\mathbb{Q}}\mathfrak{a} \tag{10}$$

for a fractional ideal $\mathfrak{a}$ in the ring of integers of F . In these notations, let

$$L(s,\Phi) = \prod_{p} \Phi_p(|p|^{-s})^{-1} \text{ , Res} > 1 \text{ , } s \in \mathbb{C} \text{ ,} \tag{11}$$

where the product in (11) is extended over all the prime divisors p in k .

<u>Theorem 1</u>. The function $s \longmapsto L(s,\Phi)$, defined for $\text{Re}s > 1$ by an absolutely convergent product (11), can be meromorphically continued to the half-plane $\mathbb{C}_+ = \{s \mid \text{Re}s > 0\}$. If Φ is unitary, this function can be meromorphically continued to the whole complex plane $\mathbb{C}$; if Φ is not unitary, then the function $L(s,\Phi)$ has a natural boundary $\mathbb{C}^\circ = \{s \mid \text{Re}s = 0\}$ and admits no analytic continuation to the left half-plane $\mathbb{C}_- = \{s \mid \text{Re}s < 0\}$.

Take, in particular, $\Phi(t) = \det(1-t\rho)$ for some ρ in X_1 , then equation (11) defines the Weil's L-function, [33],

$$L(s,\rho) = \prod_{p} \det(1-|p|^{-s}\rho(\sigma_p))^{-1} \text{ , Res} > 1 \text{ ,} \tag{12}$$

associated to ρ . We develop the product (12) in an absolutely convergent for $\mathrm{Re}\, s > 1$ Dirichlet series

$$L(s,\rho) = \sum_{\mathfrak{n}} c(\mathfrak{n},\chi)\,|\mathfrak{n}|^{-s} \ , \ \chi := \mathrm{tr}\ \rho \ ,$$

where $\mathfrak{n}$ ranges over all the integral divisors of k . Given r representations ρ_j , $1 \leq j \leq r$, in X_1 with characters $\chi_j = \mathrm{tr}\ \rho_j$, let

$$L(s,\vec{\chi}) = \sum_{\mathfrak{n}} \prod_{j=1}^{r} c(\mathfrak{n},\chi_j)\,|\mathfrak{n}|^{-s} \ , \ \mathrm{Re}\, s > 1 \ , \tag{13}$$

be the scalar product of the L-functions $L(s,\rho_j)$, $1 \leq j \leq r$. Let d_j denote the dimension of the representation ρ_j and assume, without a loss of generality, that

$$d_1 \geq \ldots \geq d_r \geq 2 \ , \ r \geq 2 \ . \tag{14}$$

Theorem 2. The function $s \longmapsto L(s,\vec{\chi})$ defined for $\mathrm{Re}\, s > 1$ by an absolutely convergent Dirichlet series (13) can be meromorphically continued to $\mathbb{C}_+$. If either $r > 2$ or $d_1 > 2$, then this function has a natural boundary $\mathbb{C}^\circ$ and admits no analytic continuation to $\mathbb{C}_-$.

Consider now r finite extensions k_j , $1 \leq j \leq r$, of k and let $d_j = [k_j : k]$. Given a grössencharacter ψ_j in k_j , one defines an L-function

$$L(s,\psi_j) = \sum_{\mathfrak{a}} \psi_j(\mathfrak{a})\,|\mathfrak{a}|^{-s} = \sum_{\mathfrak{n}} c(\mathfrak{n},\psi_j)\,|\mathfrak{n}|^{-s} \ , \ \mathrm{Re}\, s > 1 \ ,$$

where $\mathfrak{a}$ and $\mathfrak{n}$ range over the integral ideals of k_j and k, respectively. In particular,

$$c(\mathfrak{n},\psi_j) = \sum_{\mathfrak{a}} \psi_j(\mathfrak{a}) \ , \ N_{k_j/k}\mathfrak{a} = \mathfrak{n} \ ,$$

is a finite sum extended over the integral ideals $\mathfrak{a}$ in k_j subject to the condition $N_{k_j/k}\mathfrak{a} = \mathfrak{n}$. Let

$$L(s,\vec{\psi}) = \sum_{\mathfrak{n}} |\mathfrak{n}|^{-s} \prod_{j=1}^{r} c(\mathfrak{n},\psi_j) \ , \ \operatorname{Re} s > 1 \ .$$

The grossencharacter ψ_j can be regarded as an one-dimensional representation of $W(k_j)$; let ρ_j be the representation of $W(k)$ induced by ψ_j . Then

$$L(s,\psi_j) = L(s,\rho_j)$$

and therefore

$$L(s,\vec{\psi}) = L(s,\vec{\chi}) \ , \ \vec{\chi} = (\chi_1,\ldots,\chi_2) \ , \ \chi_j = \operatorname{tr} \rho_j \ .$$

The following statement is an immediate consequence of theorem 2.

<u>Theorem</u>. 1) The function $s \longmapsto L(s,\vec{\psi})$ can be meromorphically continued to $\mathbb{C}_+$.

2) If the degrees d_j of k_j over k satisfy (14) and either $r > 2$ or $d_1 > 2$, then $\mathbb{C}^0$ is the natural boundary of $L(s,\vec{\psi})$ and this function admits no analytic continuation to $\mathbb{C}_-$.

§ 3. On polynomials associated to representations of compact groups

Consider a compact group G and let X be set of all the irreducible representations of G. Let

$$Y = \{ \sum_{\chi} m(\chi)\chi \mid m(\chi) \in \mathbf{Z} , \chi = \operatorname{tr} \rho , \rho \in X \}$$

be the ring of virtual characters of G, so that m ranges over all the functions $m : \check{X} \longrightarrow \mathbf{Z}$ on the set

$$\check{X} = \{\chi | \chi = \operatorname{tr} \rho , \rho \in X\}$$

of irreducible characters of G for which the set

$$\{\chi | m(\chi) \neq 0\}$$

is finite. Given a polynomial $\Phi(t)$ of the form (5), we define $\Phi_g(t)$ by (6) and let

$$\Phi_g(t) = \prod_{j=1}^{\ell} (1-\alpha_j(g)t) , g \in G , \tag{15}$$

be the decomposition of $\Phi_g(t)$ in $\mathbb{C}[t]$. Let, moreover,

$$\gamma = \sup\{|\alpha_j(g)| \mid 1 \leq j \leq \ell, g \in G\} . \tag{16}$$

By lemma 14 in [17], we have

$$1 \leq \gamma < \infty \,. \tag{17}$$

A polynomial $\Phi(t)$ in $Y[t]$ is said to be unitary, if $\gamma = 1$. By (16) and (17), $\Phi(t)$ is unitary if and only if

$$\Phi_g(\alpha) \neq 0 \quad \text{whenever} \quad |\alpha| \neq 1 \quad \text{and} \quad g \in G \,,\ \alpha \in \mathbb{C} \,. \tag{18}$$

Write $a_j = \sum_\chi m_j(\chi)\chi$ with $\chi \in \check{X}$ and let

$$X_\circ(\Phi) = \{\varphi | \varphi \in X \,,\ m_j(\mathrm{tr}\varphi) \neq 0 \quad \text{for some} \quad j\}$$

for a polynomial Φ of the form (5).

<u>Proposition 1</u>. Let $\Phi(t) \in Y[t]$ and suppose that $\Phi(0) = 1$. There is a sequence of integer valued functions

$$b_n : X \longrightarrow \mathbb{Z} \,,\ 1 \leq n < \infty \,,$$

satisfying the following conditions: the set

$$X_n(\Phi) = \{\varphi | \varphi \in X \,,\ b_n(\varphi) \neq 0\} \quad \text{is finite} \quad ; \tag{19}$$

identity

$$\Phi(t) = \prod_{n=1}^{\infty} \prod_{\varphi \in X} \det(1 - t^n \varphi)^{b_n(\varphi)} \tag{20}$$

holds formally in the ring of formal power series $Y[[T]]$ with coefficients in Y; for each g in G the product

$$\Phi_g(t) = \prod_{n=1}^{\infty} \prod_{\varphi \in X} \det(1-t^n\varphi(g))^{b_n(\varphi)} \tag{21}$$

converges absolutely in the circle $|t| < \gamma^{-1}$, and the following estimates hold:

$$\left| \sum_{\varphi \in X} b_n(\varphi) \operatorname{tr}\varphi(g) \right| \leq \frac{\tau(n)}{n} \ell\gamma^n , \quad n \in \mathbb{N} , \quad g \in G , \tag{22}$$

and

$$\sum_{n \geq M} \sum_{\varphi \in X} \left| \log \det(1-t^n\varphi(g))^{b_n(\varphi)} \right| \leq \frac{\ell(|t|\gamma)^M}{(1-\gamma|t|)^2} \quad \text{when } |t|<\gamma^{-1}, \tag{23}$$

where $\tau(n)$ denotes the number of positive divisors of n and ℓ is the degree of $\Phi(t)$.

Proof. To deduce (20) one constructs inductively two sequences

$$\{b_n | b_n : X \longrightarrow \mathbb{Z} , \ 1 \leq n \leq \infty\}$$

and

$$\{F_n | F_n(t) \in Y[t] , \ 1 \leq n < \infty\}$$

satisfying the following relations:

$$F_n(t) \equiv \Phi(t) \pmod{t^{n+1}} \tag{24}$$

and

$$F_n(t) = \prod_{\nu=1}^{n} \prod_{\varphi \in X} \det(1-t^{\nu}\varphi)^{b_\nu(\varphi)} . \qquad (25)$$

Let $F_o(t) = 1$ and suppose that (24), (25) hold.

It follows from (24) that

$$F_n(t) \equiv (1 + bt^{n+1})\ \Phi\ (t)\ (\mathrm{mod}\ t^{n+2})\ ,\ b \in Y\ , \qquad (26)$$

since $\Phi(0) = 1$. In view of (26), one can define b_{n+1} by the relation :

$$b = \sum_{\varphi \in X} b_{n+1}(\varphi)\mathrm{tr}\varphi\ ;$$

let

$$F_{n+1}(t) = F_n(t) \prod_{\varphi \in X} \det(1-t^{n+1}\varphi)^{b_{n+1}(\varphi)} .$$

Then (19) holds by construction, while (20) follows from (25). Write $\Phi(t)$ in the form (5) and define ℓ functions

$$\alpha_j : G \longrightarrow \mathbb{C}\ ,\ 1 \leq j \leq \ell\ ,$$

by (15); then (20) may be rewritten as

$$\prod_{j=1}^{\ell} (1-t\alpha_j) = \prod_{n=1}^{\infty} \prod_{\varphi \in X} \det(1-t^n\varphi)^{b_n(\varphi)} . \tag{27}$$

We apply the operator

$$- t\frac{\partial}{\partial t} \log : Y[[t]] \to Y[[t]]$$

to the both sides of (27) and obtain an identity

$$\sum_{j=1}^{\ell} \frac{t\alpha_j}{1-t\alpha_j} = \sum_{n=1}^{\infty} \sum_{\varphi \in X} nb_n(\varphi) \mathrm{tr}(t^n\varphi(1-t^n\varphi)^{-1}) \tag{28}$$

in $Y[[t]]$. Let

$$\sigma(m,g) = \sum_{j=1}^{\infty} \alpha_j(g)^m , \quad h_n(g) = n \sum_{\varphi \in X} b_n(\varphi) \mathrm{tr}\varphi(g)$$

for $g \in G$.

It follows from (28) that, for any g in G ,

$$\sum_{m=1}^{\infty} t^m \sigma(m,g) = \sum_{m,n=1}^{\infty} t^{nm} h_n(g^m) \quad \text{in } \mathbb{C}[[t]] ,$$

or equivalently,

$$\sigma(n,g) = \sum_{mm'=n} h_m(g^{m'}) , \quad m \in \mathbb{N} , m' \in \mathbb{N} . \tag{29}$$

Introducing the Möbius function $\mu : \mathbb{N} \to \{0,\pm 1\}$ one obtains from (29) an equation

$$\sum_{r|n} \mu(r)\, \sigma\left(\frac{n}{r}, g^r\right) = h_n(g) \quad , \; n \in \mathbb{N} \; . \tag{30}$$

Since $|\sigma(m,g)| \leq \ell\gamma^m$, estimate (22) follows from (30). Estimate (23) is an easy consequence of (22) and the well known operator identity log · det = tr · log . The absolute convergence of (21) for $|t| < \gamma^{-1}$ follows from (23). This proves the proposition.

<u>Proposition 2</u>. If Φ is unitary, then there is n_0 such that

$$b_n(\varphi) = 0 \quad \text{for} \quad n > n_0 \quad , \tag{31}$$

and therefore

$$\Phi(t) = \prod_{n=1}^{n_0} \prod_{\varphi \in X_n(\Phi)} \det(1 - t^n \varphi)^{b_n(\varphi)} \tag{32}$$

<u>Proof</u>. By condition, $\gamma = 1$. Therefore it follows from (22) that one can find n_0 in $\mathbb{N}$ for which

$$\left| \sum_{\varphi \in X} b_n(\varphi) \operatorname{tr} \varphi(g) \right| < 1 \quad \text{whenever} \quad n > n_0 \; , \; g \in G \; . \tag{33}$$

In view of the orthogonality relations, (31) follows from (33) and (19). Identity (32) is a formal consequence of (20) and (31).

§ 4. Continuation of $L(s,\Phi)$ to $\mathbb{C}_+$

We return now to notations of § 2. In view of the remarks made in § 2, any polynomial Φ in $Y[t]$ may be regarded as a polynomial with coefficients in the ring of virtual characters of a compact group $G = W_1(K|k)$ for some finite Galois extension $K \supseteq k$. Given a representation (1) we denote by

$$S(\rho) = \{p \mid V^{I_p} \neq V\}$$

the set of all the primes p in k at which ρ is ramified. It follows from the definitions, [33], that $S(\rho)$ is a finite set. Therefore the set

$$S(\Phi) = \{p \mid p \in S(\rho) \text{ for some } \rho \text{ in } X_0(\Phi)\}$$

is also finite. Moreover, by (9),

$$\Phi_p(t) = \Phi_g(t) \quad \text{for } p \notin S(\Phi),\ g \in \sigma_p \ . \tag{34}$$

<u>Proposition 3</u>. If Φ is an unitary polynomial and $\Phi(0) = 1$, then $L(s,\Phi)$ can be meromorphically continued to the whole plane $\mathbb{C}$.

<u>Proof</u>. It follows from the relations (11), (12), (32) and (34) that

$$L(s,\Phi) = \prod_{n=1}^{n_0} \prod_{\rho\in X_n(\Phi)} (L^{\Phi}(ns,\rho))^{b_n(\rho)} \prod_{p\in S(\Phi)} \Phi_p(|p|^{-s})^{-1} , \qquad (35)$$

where

$$L^{\Phi}(s,\rho) := L(s,\rho) \prod_{p\in S(\Phi)} \det(1-\rho(\sigma_p)|p|^{-s}) .$$

Since $L(s,\rho)$ is a meromorphic function, [33], and the set $X_n(\Phi)$ is finite, the assertion follows from (35).

Choose two rational integers M and N subject to the condition:

$$M > 0,\ \gamma^M < N,\ N > |p| \text{ for each } p \text{ in } S(\Phi) \qquad (36)$$

with γ defined by (16) and let, in notations of (20) and (7),

$$f_{n,p}(t) = \prod_{\varphi\in X_1} \det(1-t^n\varphi(\sigma_p))^{b_n(\varphi)} . \qquad (37)$$

We define, generalising the construction of [10], two finite products

$$Z_N(s) = \prod_{|p|<N} \Phi_p(|p|^{-s})^{-1} , \qquad (38.1)$$

$$R_{N,M}(s) = \prod_{\substack{p\notin S(\Phi)\\ |p|<N}} \prod_{n<M} f_{n,p}(|p|^{-s}) , \qquad (38.2)$$

and two infinite products

$$U_M(s) = \prod_{n<M} \prod_{p \notin S(\Phi)} f_{n,p}(|p|^{-s})^{-1} , \qquad (38.3)$$

$$T_{N,M}(s) = \prod_{n \geq M} \prod_{|p| \geq N} f_{n,p}(|p|^{-s})^{-1} . \qquad (38.4)$$

It follows from (38) and (20) that

$$L(s,\Phi) = Z_N(s)\ R_{N,M}(s)\ U_M(s)\ T_{N,M}(s) \qquad (39)$$

as a formal Euler product. Moreover, it follows from (37) and (38.3) that

$$U_M(s) = \prod_{n<M} \prod_{\rho \in X_n(\Phi)} L(ns,\rho)^{b_n(\rho)} \prod_{p \in S(\Phi)} f_{n,p}(|p|^{-s}) , \qquad (40)$$

since, by (19), $b_n(\rho) = 0$ when $\rho \notin X_n(\Phi)$.

Lemma 1. The functions

$$s \mapsto R_{N,M}(s) \ , \ s \mapsto Z_N(s) \ , \ s \mapsto U_M(s)$$

are meromorphic in $\mathbb{C}$.

Proof. Since $L(s,\rho)$ is meromorphic in $\mathbb{C}$, [33], the assertion follows from (38.1), (38.2), (40) and (19).

Lemma 2. Suppose that M , N satisfy (36). Then the product $T_{N,M}(s)$ converges absolutely for $\operatorname{Re} s > \frac{1}{M}$.

Proof. By (36), we have

$$\gamma |p|^{-\operatorname{Re} s} < 1 \quad \text{for} \quad \operatorname{Re} s > \frac{1}{M} , \ |p| \geq N . \tag{41}$$

In view of (41), we deduce from (23) and (37) that

$$\sum_{n \geq M} |\log f_{n,p}(|p|^{-s})| \leq \frac{\ell(\gamma |p|^{-\operatorname{Re} s})^M}{(1-\gamma |p|^{-\operatorname{Re} s})^2} \quad \text{for} \quad \operatorname{Re} s > \frac{1}{M} .$$

Therefore, if $\operatorname{Re} s > \frac{1}{M}$ then

$$\sum_{n \geq M} \ \sum_{|p| \geq N} |\log f_{n,p}(|p|^{-s})| \leq \frac{\ell \gamma^M [k:\mathbb{Q}]}{(1-\gamma N^{-1/M})^2} \sum_{n=1}^{\infty} n^{-M \operatorname{Re} s} , \tag{42}$$

since there are no more than $[k : \mathbb{Q}]$ prime divisors p in k such that $|p| = n$, $n \in \mathbb{N}$. The assertion of lemma 2 follows from (42) and (38.4).

Proposition 4. Let $\Phi(t) \in Y[t]$, $\Phi(0) = 1$. The function $L(s,\Phi)$ defined by (11) for $\operatorname{Re} s > 1$ can be meromorphically continued to the right half-plane $\mathbb{C}_+$.

Proof. Choose M,N satisfying (36). By lemma 1 and lemma 2, equation (39) defines a meromorphic continuation of $L(s,\Phi)$ to the half-plane

$$\mathbb{C}_{1/M} = \{s | \operatorname{Re} s > \frac{1}{M}\} .$$

Therefore the assertion follows from an obvious relation:

$$\mathbb{C}_+ = \bigcup_{M=1}^{\infty} \mathbb{C}_{1/M} .$$

§ 5. A new Primzahlsatz

Let $\mathfrak{m}$ be a finite subset of X_1 and choose an element g in $W(k)$ and a real number ε in the interval $0 < \varepsilon < 1$. Let

$$\check{\mathfrak{m}} = \{\chi \mid \chi = \mathrm{tr}\rho \ , \ \rho \in \mathfrak{m}\} .$$

Consider the set $\Pi(g,\varepsilon)$ of all the prime divisors of k satisfying the condition

$$|\chi(\sigma_\rho) - \chi(g)| < \varepsilon \quad \text{for each } \chi \text{ in } \check{\mathfrak{m}} , \tag{43}$$

and let, for $x \in \mathbb{R}_+$,

$$\pi(g,\varepsilon;x) = \mathrm{card}\, \{p \mid p \in \Pi(g,\varepsilon),\ |p| < x\} .$$

<u>Primzahlsatz</u>. The following relation holds:

$$\pi(g,\varepsilon;x) = c_0(\mathfrak{m};g,\varepsilon) \int_2^x \frac{du}{\log u} + O(x \exp(-c_1 \sqrt{\log x})) , \tag{44}$$

where

$$c_0(\mathfrak{M};g,\varepsilon) \geq c_2\, \varepsilon^{c_3}, \quad c_j \in \mathbb{R}_+ \quad \text{for} \quad 1 \leq j \leq 3 . \tag{45}$$

Here the constants c_j and the implied by the O-symbol constant depend at most on $\mathfrak{M}$, but not on ε, g and x.

Proof. Let $H(\mathfrak{M}) = \bigcap_{\rho \in \mathfrak{M}} \operatorname{Ker} \rho$. It follows from the definition of Weil's group that the group

$$G := W(k)/_{H(\mathfrak{M})}$$

fits into an exact sequence

$$1 \longrightarrow T \longrightarrow G \longrightarrow G_1 \longrightarrow 1 ,$$

where T is a finite-dimensional real torus and G_1 is a finite group. Let

$$\varphi : W(k) \longrightarrow G$$

be the natural surjective homomorphism (with $\operatorname{Ker} \varphi = H(\mathfrak{M})$) and let χ_φ denote the character of G defined by the equation:

$$\chi_\varphi(\varphi(t)) = \chi(t) \quad \text{for} \quad \chi \in \check{\mathfrak{M}} , \; t \in W(k) .$$

Consider the finite set

$$N = \{\chi_\varphi | \chi \in \check{\mathfrak{M}}\}$$

and define a set

$$\mathcal{L} = \{h \mid h \in G \;,\; |\psi(h) - \psi(\varphi(g))| < \varepsilon \quad \text{for} \quad \psi \in \mathfrak{N}\} \;.$$

In [24] we have deduced from a theorem of Yomdin's, [36], on volumes of tubes the following estimate (here μ denotes the Haar measure on G normalised by the condition $\mu(G) = 1$) :

$$\pi(g,\varepsilon;x) = \mu(\mathcal{L}) \int_2^x \frac{du}{\log u} + O\,(x \exp(-c_1\sqrt{\log x}))\,,\; c_1 > 0 \;, \tag{46}$$

with c_1 and the implied O-constant depending at most on $\mathfrak{M}$. To estimate $\mu(\mathcal{L})$ write, for brevity, $\varphi(g) = \bar{g}$ and let

$$\psi|T = \sum_{i=1}^{n(\psi)} \lambda_i^{(\psi)}$$

be the decomposition of the restriction of ψ to T into the sum of simple (hence one-dimensional) characters of T . Then

$$\psi(h\bar{g}) = \sum_{i=1}^{n(\psi)} \lambda_i^{\psi}(h)\,a_i(\psi) \quad \text{for} \quad h \in T$$

with some $a_i(\psi)$ depending, of course, on $\bar{g}$. Moreover,

$$|a_i(\psi)| \leq 1 \quad \text{for} \quad 1 \leq i \leq n(\psi) \;,\; \psi \in \mathfrak{N} \;,$$

since G is a compact group and therefore ψ may be regarded as a character of an unitary representation. Thus

$$|\psi(h\bar{g})-\psi(\bar{g})| \leq \sum_{i=1}^{n(\psi)} |\lambda_i^{\psi}(h)-1| \quad \text{for} \quad h \in T .$$

Therefore the set (we let here $m = \max\{n(\psi) | \psi \in \mathfrak{N}\}$)

$$\mathcal{C}_1 = \{h\bar{g} | h \in T , |\lambda_i^{\psi}(h)-1| < \frac{\varepsilon}{m} \text{ for } 1 \leq i \leq n(\psi) , \psi \in \mathfrak{N}\}$$

is contained in $\mathcal{C}$. In particular,

$$\mu(\mathcal{C}_1) \leq \mu(\mathcal{C}) .$$

Let $\{\nu_1,\ldots,\nu_\ell\}$ be a system of generators of the group of characters $\hat{T}$ of the torus T (so that T is an ℓ-dimensional torus) and let

$$\lambda_i^{\psi} = \prod_{j=1}^{\ell} \nu_j^{b_j(i,\psi)} , b_j(i,\psi) \in \mathbb{Z} .$$

Then the set

$$\mathcal{C}_2 = \{h\bar{g} | h \in T , |\nu_j(h)-1| < \frac{\varepsilon}{C(\mathfrak{m})} \text{ for } 1 \leq j \leq \ell\}$$

is contained in $\mathcal{C}_1$ and, in particular,

$$\mu(\mathcal{C}_2) \leq \mu(\mathcal{C}) ,$$

as soon as $C(\mathfrak{m})$ is chosen to be large enough (compared to $b_j(i,\psi)$ and m). On the other hand, we have

$$\mu(\mathcal{C}_2) \geq c_4\left(\frac{\varepsilon}{C(\mathfrak{m})}\right)^{\ell} \mu(T) , c_4 > 0 , \tag{47}$$

with c_4 depending on ℓ only. Relations (44) and (45) follow from (46) and (47), respectively. This proves the Primzahlsatz.

Remark. The Primzahlsatz proved in this paragraph generalises both the Chebotarev density theorem and the Primzahlsatz for grossencharacters due to E. Hecke, [5], and seems to be of independent interest (cf. also, [29, Appendix to Chaper I]).

§ 6. Proof of theorem 1

Consider a rectangle

$$D_\nu(\delta,t_\circ) = \{s \mid s \in \mathbb{C} \;,\; \frac{1}{\nu+1} < \mathrm{Re}\, s < \frac{1}{\nu} \;,\; t_\circ < \mathrm{Im}\, s \leq t_\circ + \delta\}$$

in the complex plane (here Re s and Im s denote the real and imaginary parts of a complex number s, respectively; the real parameters ν, $t_\circ$, δ are subject to the conditions: $\delta > 0$, $\nu > 0$). Let $\Phi(t) \in Y[t]$ and $\Phi(0) = 1$. Suppose, as in § 4, that each representation in $X_\circ(\Phi)$ factors through $W_1(K|k)$ for a finite Galois extension $K \supseteq k$, so that Φ may be regarded as a polynomial with coefficients in the ring of virtual characters of $W_1(K|k)$.

Proposition 5. If Φ is not unitary, then there is ν_0 in $\mathbb{R}_+$ such that the function $s \mapsto L(s,\Phi)$ has at least one pole in $D_\nu(\delta,t_0)$ as soon as $\nu > \nu_0$.

We retain the notations of § 4. In particular, let $N,M \in \mathbb{N}$ and suppose that (36) is satisfied, so that equation (39) defines a meromorphic continuation of $L(s,\Phi)$ to $\mathbb{C}_{1/M}$. Let, moreover, $M = \nu+1$ so that $D_\nu(\delta,t_0) \subseteq \mathbb{C}_{1/M}$.
Let $a_1(\nu;\delta,t_0)$ and $a_2(\nu;\delta,t_0)$ denote the number of distinct zeros of U_M in $D_\nu(\delta,t_0)$ and the number of distinct poles of Z_N in $D_\nu(\delta,t_0)$, respectively. To simplify our notations let us assume that $t_0(t_0+\delta) \geq 0$. Let $gr(K)$ denote the set of all the normalised grossen-characters of K . Let, in notations of § 5 ,

$$\mathfrak{m} = X_0(\Phi) . \tag{48}$$

By construction, there is an element g in $W_1(K|k)$ such that

$$|\alpha(g)| = \gamma \tag{49}$$

and

$$\Phi_g(t) = (1-\alpha(g)t)^b \,\tilde{\Phi}_g(t) \ , \ b \geq 1 \ , \ \tilde{\Phi}_g(\alpha(g)^{-1}) \neq 0 \ , \tag{50}$$

so that $\alpha(g)^{-1}$ is a root of $\Phi_g(t)$ whose multiplicity is equal to b . Let

$$P(g,\varepsilon) = \Pi\,(g,\varepsilon) \setminus S(\Phi) .$$

<u>Lemma 3</u>. There is an ε_0 in $\mathbb{R}_+$ such that for every ε in the interval $0 < \varepsilon < \varepsilon_0$ and for each p in $P(g,\varepsilon^{b+2})$ the polynomial Φ_p has a root $\kappa(p)^{-1}$ satisfying the condition

$$|\log|\kappa(p)| - \log \gamma| < \varepsilon \ . \tag{51}$$

<u>Proof</u>. Choose ε_1 in the interval $0 < \varepsilon_1 < 1$ in such a way that $\tilde{\Phi}_g(t) \neq 0$ in the circle: $|t - \alpha(g)^{-1}| \leq \varepsilon_1$ and let w be the minimum of $|\tilde{\Phi}_g(t)|$ in this circle. Obviously, $w > 0$. Choose $w_1 > 0$ so that

$$|a_j(p) - a_j(g)| < w_1\varepsilon \quad \text{for} \quad p \in P(g,\varepsilon),\ 1 \leq j \leq \ell,\ \varepsilon > 0\ , \tag{52}$$

where

$$\Phi_g(t) = 1 + \sum_{j=1}^{\ell} t^j a_j(g),\ \Phi_p(t) = 1 + \sum_{j=1}^{\ell} t^j a_j(p)\ .$$

For each ε in the interval $0 < \varepsilon < \varepsilon_1$ we get an estimate

$$|\Phi_g(t)| \geq w\gamma^b\varepsilon^b \quad \text{on the circle: } |t - \alpha(g)^{-1}| = \varepsilon\ .$$

Write $\Phi_p(t) = \Phi_g(t) + h_p(t)$. By (52), for $p \in P(g,\varepsilon^b)$ we have

$$|h_p(t)| < w_1(1+\gamma)^{\ell}\ell\varepsilon^b \quad \text{on the circle: } |t - \alpha(g)^{-1}| = \varepsilon \ ,$$

as soon as $0 < \varepsilon < 1$. Therefore there is a positive ε_2 such that

$$|h_p(t)| < |\Phi_g(t)| \quad \text{when} \quad p \in P(g,\varepsilon^{b+1}) \quad \text{and} \quad |t-\alpha(g)^{-1}| = \varepsilon, \qquad (53)$$

as soon as $0 < \varepsilon < \varepsilon_2$. By a well known lemma (cf., e.g., [32], § 3.42), it follows from (53) that Φ_p has a root $\kappa(p)^{-1}$ satisfying the inequality $|\kappa(p)^{-1}-\alpha(g)^{-1}| \le \varepsilon$. This implies the assertion of lemma 3.

<u>Lemma 4</u>. Suppose that $\gamma > 1$. There are two positive numbers c_5 and c_6 such that

$$a_2(\nu;\delta,t_\circ) > \exp(c_5\sqrt{\nu}) \quad \text{when} \quad \nu > c_6 \ . \qquad (54)$$

<u>Proof</u>. Let $\varepsilon = e^{-\sqrt{\nu}}$ and let

$$Q(\nu) = \{p|p \in P(g,\varepsilon^{b+2}), \ (\gamma e^{\varepsilon})^{\nu} \le |p| < (\gamma e^{-\varepsilon})^{\nu+1}\} \ .$$

By Lemma 3, there is $\kappa(p)$ satisfying (51) and such that

$$\Phi_p(\kappa(p)^{-1}) = 0 \ , \qquad (55)$$

as soon as $\varepsilon < \varepsilon_\circ$ and $p \in Q(\nu)$. Let

$$\kappa(p) = |p|^{s(p)} \qquad (56)$$

and let

$$t_\circ < \operatorname{Im} s(p) \leq t_\circ + \delta \, . \tag{57}$$

Since condition (56) defines $\operatorname{Im} s(p)$ only modulo $\frac{2\pi}{\log|p|}\mathbb{Z}$, we can satisfy condition (57) if

$$\frac{2\pi}{\log|p|} < \delta \, .. \tag{58}$$

It follows from (56) and (51) that

$$\frac{1}{\nu+1} < \operatorname{Re} s(p) < \frac{1}{\nu} \quad \text{for} \quad p \in Q(\nu) \, . \tag{59}$$

Let

$$\nu > \max \left\{ \frac{2\pi}{\delta \log \gamma} \, , \, (\log \varepsilon_\circ)^2 \right\} , \tag{60}$$

then $\varepsilon < \varepsilon_\circ$ and (58) holds. Therefore in view of (36), (38.1) and our choice of $M(= \nu+1)$ it follows from (55) - (57) and (59) that

$$s(p) \quad \text{is a pole of} \quad Z_N(s) \quad \text{for} \quad p \in Q(\nu) \, . \tag{61}$$

Moreover, if ν satisfies (60) then it follows from (59), (56) and (51) that

$$\Big| \log|p| - \log|q| \Big| < 2\varepsilon(\nu+1) \quad \text{whenever} \quad s(p) = s(q) \tag{62}$$

for p and q in $Q(\nu)$. Let us divide $Q(\nu)$ into disjoint subsets

$$Q_j(\nu) = \{p \mid p \in P(g,\varepsilon^{b+2}) \ , \ 1 \leq \frac{|p|}{\gamma^{\nu}\exp(j\lambda+\nu\varepsilon)} < e^{\lambda}\} \ ,$$

where $\lambda := 2\varepsilon(\nu+1)$ and j ranges over the set

$$J = \{j \mid j \in \mathbb{Z} \ , \ 0 \leq j \leq \frac{\log \gamma}{\lambda} - 2\} \ .$$

We notice that

$$Q_j(\nu) \subseteq Q(\nu) \quad \text{for} \quad j \in J \ . \tag{63}$$

It follows from (61) and (63) that

$$s(p) \ \text{ is a pole of } \ Z_N(s) \ \text{ whenever } \ p \in Q_j(\nu),\ j \in J \ . \tag{64}$$

Moreover, by (62) ,

$$s(p) \neq s(q) \quad \text{if} \quad p \in Q_j(\nu),\ q \in Q_{j'}(\nu),\ |j-j'| \geq 2 \ . \tag{65}$$

The Primzahlsatz of § 5 shows that

$$Q_j(\nu) \neq \emptyset \quad \text{for} \quad j \in J \ , \ \nu > c_7 \tag{66}$$

when c_7 is chosen to be large enough. By definition, there is a constant c_8 such that

$$\text{card } J > 2 \exp(c_5\sqrt{\nu}) \ , \ c_5 > 0 \ , \text{ for } \ \nu > c_8 \ . \tag{67}$$

Relation (54) with $c_6 = \max \{ \frac{2\pi}{\delta \log\gamma}, (\log \varepsilon_\circ)^2 , c_7, c_8 \}$ is a consequence of (64) - (67). This proves lemma 4.

<u>Lemma 5</u>. There are c_9 and c_{10} such that

$$a_1(\nu;\delta,t_\circ) < c_9 \ \nu^{c_{10}} \ . \tag{68}$$

<u>Proof</u>. Making use of the Brauer's induction theorem we decompose each of the characters $\text{tr } \rho \ , \ \rho \in X_\circ(\Phi)$, in a linear combination of monomial characters and write, in notations of (5),

$$a_j = \sum_{\chi \in Y_1} \ell_j(\chi)\chi \ , \ \ell_j(\chi) \in \mathbb{Z} \ ,$$

where Y_1 denotes the set of the characters of monomial representations of $W_1(K|k)$. Let

$$Y_1(\Phi) = \{\chi | \chi \in Y_1 \ , \ \ell_j(\chi) \neq 0 \ \text{ for some } \ j \ \}$$

and let

$$Y_n(\Phi) = \{ \prod_{\chi \in Y_1(\Phi)} \chi^{e(\chi)} \ \Big| \ e(\chi) \geq 0 \ , \ \sum_{\chi \in Y_1(\Phi)} e(\chi) \leq n \ \} \ .$$

By a theorem of Mackey's, any character in $Y_n(\Phi)$ is equal to a sum of monomial characters, so that one can write:

$$\chi = \sum_{\psi} \ell'_\chi(\psi)\text{tr}(\text{Ind } \psi), \ \ell'_\chi(\psi) \geq 0 \ \text{ for } \ \chi \in Y_n(\Phi) \ ,$$

where ψ ranges over grossencharacters of the intermediate fields k_ψ and $\operatorname{Ind}\psi$ stands for the induced representation

$$\operatorname{Ind}\psi := \operatorname{Ind}_{W(K|k_r)}^{W(K|k)}(\psi), \psi \in gr(k_\psi),\ k \subseteq k_\psi \subseteq K\ . \qquad (69)$$

Finally, let

$$Y_n'(\Phi) = \{\psi | \psi \in gr(k_\psi),\ k \subseteq k_\psi \subseteq K,\ \ell_\chi'(\psi) \neq 0 \ \text{ for some } \chi \text{ in } Y_n(\Phi)\}.$$

By construction, the set $Y_n(\Phi)$ and the set $Y_n'(\Phi)$ are finite for each n. Given χ in Y_1, we write $\chi = tr(\operatorname{Ind}\psi)$ for some ψ in $gr(k_\psi)$ and denote by $F(\chi)$ the conductor of the grossencharacter $\psi \circ N_{K/k_\psi}$ in $gr(K)$. Choose an integral divisor A in K satisfying the following condition:

$$A \equiv 0 (\operatorname{mod} F(\chi)) \quad \text{for each } \chi \text{ in } Y_1(\Phi)\ .$$

Let

$$G(A) = \{\psi | \psi \in gr(K)\ ,\ F_\psi | A\}$$

be the subgroup of $gr(K)$ consisting of those grossencharacters ψ whose conductor F_ψ divides A. By a theorem of Hecke's, [5],

$$G(A) = G_1(A) \times H(A)\ ,$$

where $G_1(A)$ is a free abelian group of rank $m := [K:\mathbb{Q}]-1$ and

$H(A)$ is the finite subgroup of grossencharacters of finite order. Let $\psi \in Y_n'(\Phi)$. Then the character $\psi \circ N_{K/k_\psi}$ lies in $G(A)$ and, moreover,

$$\psi \circ N_{K/k_\psi} = \prod_{j=1}^{m} \psi_j^{m_j} \psi_\circ \quad \text{with} \quad \psi_\circ \in H(A),\ m_j = O(n) \ , \tag{70}$$

where

$$\{\psi_1, \ldots, \psi_m\}$$

is a fixed system of generators of $G_1(A)$. Therefore

$$\text{card } Y_n'(\Phi) = O(n^m) \tag{71}$$

with an O-constant depending at most on Φ and A (but not on n !). The power sums $\sigma(\ell,g)$ can be expressed as polynomials in the coefficients of Φ , by the well known formulae of Newton's. This procedure,applied to the identity (30), shows that

$$h_n = \sum_{\chi \in Y_n(\Phi)} c_n(\chi)\chi,\ c_n(\chi) \in \mathbb{Z};\ h_n := n \sum_{\varphi \in X_n(\Phi)} b_n(\varphi) \operatorname{tr} \varphi \ .$$

Thus

$$\prod_{\varphi \in X_n(\Phi)} L(ns,\varphi)^{nb_n(\varphi)} = \prod_{\psi \in Y_n'(\Phi)} L(ns,\psi)^{c_n'(\psi)}$$

with some $c_n'(\psi)$ in $\mathbb{Z}$ .Therefore it follows from (40) (with

$M = \nu+1$) that

$$U_M(s) = g(s) \prod_{n=1}^{\nu} \prod_{\psi \in Y'_n(\Phi)} L(ns,\psi)^{c'_n(\psi)} , \quad c'_n(\psi) \in \mathbf{Z} , \tag{72}$$

where

$$g(s) := \prod_{p \in S(\Phi)} f_{n,p}(|p|^{-s}) .$$

Let $N(\psi,T)$ denote the number of zeroes of the function $s \longmapsto L(s,\psi)$ in the rectangle

$$\{s | s \in \mathbb{C},\ 0 \le \mathrm{Re} s \le 1,\ 0 \le |\mathrm{Im} s| \le |T|,\ T(\mathrm{Im} s) \ge 0\} .$$

A classical argument, [31, § 9.2] (cf. also [23, equation (19) on p.55]), shows that

$$N(\psi,T+1) = N(\psi,T) + O(\log(\alpha(\psi)(1+T)^m)) \tag{73}$$

for $\psi \in G(A)$, where, in notations of (70),

$$\alpha(\psi) := \prod_{j=1}^{m} (3+|m_j|) . \tag{74}$$

Since $g(s) \neq 0$ for $\mathrm{Re} s > 0$, it follows from (72) that

$$a_1(\nu;\delta,t_\circ) \le \sum_{n=1}^{\nu} \sum_{\psi \in Y'_n(\Phi)} (N(\psi,n(t_\circ+\delta)) - N(\psi,nt_\circ)) . \tag{75}$$

In view of the estimates (70) and (71), relations (73) - (75) imply (68) as soon as one takes c_{10} to satisfy the inequality:

$$c_{10} > [K : \mathbb{Q}] . \tag{76}$$

Proof of Proposition 5. It follows from (38.2) and lemma 2 that, in notations of (38),

$$R_{N,M}(s)T_{N,M}(s) \neq 0 \quad \text{for} \quad s \in D_\nu(\delta,t_0) \,. \tag{77}$$

By (54) of lemma 4 and (68) of lemma 5, there is ν_0 for which

$$a_2(\nu;\delta,t_0) > a_1(\nu;\delta,t_0) \quad \text{when} \quad \nu > \nu_0 \,. \tag{78}$$

The assertion of Proposition 5 follows from (77), (78) and (39).

Corollary 1. If Φ is not unitary, then $\mathbb{C}^0 = \{s \mid \operatorname{Re} s = 0\}$ is the natural boundary of the function $s \mapsto L(s,\Phi)$ defined in $\mathbb{C}_+$ by the sequence of equations (39) when M varies over $\mathbb{N}$.

Proof. Let $s \in \mathbb{C}^0$. Each neighbourhood of s contains a set $D_\nu(\delta,t_0)$ for some δ in $\mathbb{R}_+$, some t_0 in $\mathbb{R}$, and some $\nu > \nu_0$; therefore, by Proposition 5, it contains a pole of $L(s,\Phi)$. Thus $\mathbb{C}^0$ is contained in the closure of the set of poles of $L(s,\Phi)$, and the assertion follows.

Theorem 1 is a direct consequence of Proposition 3, Proposition 4, and Corollary 1.

§ 7. Proof of theorem 2

We start with a few simple remarks concerning convolution of L-functions (cf. [17]; [18, Ch.II § 1,2]). Given r power series

$$f_j(t) = \sum_{n=0}^{\infty} a(n,j)t^n , \quad 1 \leq j \leq r ,$$

one defines their Hadamard convolution (cf. [4]) as follows:

$$(f_1*\dots*f_r)(t) = \sum_{n=0}^{\infty} \left(\prod_{j=1}^{r} a(n,j)\right)t^n .$$

<u>Proposition 6</u>. Suppose that f_j , $1 \leq j \leq r$, has the form

$$f_j(t) = \prod_{i=1}^{d_j} (1 - \alpha(i,j)t)^{-1} , \quad \alpha(i,j) \in \mathbb{C} ,$$

and let

$$d = \prod_{j=1}^{r} d_j,\ d_1 \geq \dots \geq d_r , \quad n = \left(\sum_{j=1}^{r} d_j\right) - r+1 .$$

The following identity holds formally in $\mathbb{C}[[t]]$:

$$(f_1*\dots*f_r)(t)=(f_1\circ\dots\circ f_r)(t)h(t),\quad h(t) \equiv 1 \pmod{t^2} , \tag{79}$$

where $h(t)$ is a polynomial of degree not higher than $d-1$ and

$$(f_1{}^\circ \ldots {}^\circ f_r)(t) := \prod_\nu \left(1 - t \prod_{j=1}^{r} \alpha(\nu(j),j)\right)^{-1}$$

with ν ranging over the set of sequences

$$\{(\nu(1),\ldots,\nu(r)) \mid 1 \leq \nu(j) \leq d_j \ , \ \nu(j) \in \mathbb{N}\} \ .$$

In particular, if $f_j(t) = (1-t)^{-d_j}$, $1 \leq j \leq r$, so that $\alpha(i,j) = 1$ for each pair (i,j) , then

$$(f_1*\ldots*f_r)(t) = (1-t)^{-n} h_r(t) \, , \ h_r(t) \equiv 1+(d-n)t \pmod{t^2} \ , \qquad (80)$$

where $h_r(t)$ is a polynomial of degree not higher than $n-d_1$.

Proof. It can be deduced by formal computations in $\mathbb{C}[[t]]$ (cf. [17, § 3]).

Corollary 2. If either $r > 2$ or $d_1 > 2$ and condition (14) is satisfied, then the polynomial $h_r(t)$ in (80) has a root β with $|\beta| < 1$.

Proof. By (80), we can write

$$h_r(t) = \prod_{j=1}^{n-d_1} (1+\beta_j t) \ , \quad \sum_{j=1}^{n-d_1} \beta_j = d - n \ ,$$

so that

$$\max_j |\beta_j| \geq \frac{d-n}{n-d_1} \ .$$

On the other hand, if (14) holds and either $r > 2$ or $d_1 > 2$, then

$$\frac{d-n}{n-d_1} > 1 ,$$

and the assertion follows.

To prove the theorem 2 let, for $\rho \in X_1$,

$$f_p(\rho,t) = \det(1-t\,\rho(\sigma_p))^{-1}$$

and let, in notations of (13),

$$f_p(\vec{\chi},t) = f_p(\rho_1,t)*\dots*f_p(\rho_r,t)$$

and

$$f_p^0(\vec{\chi},t) = f_p(\rho_1,t)\circ\dots\circ f_p(\rho_r,t) ,$$

where p ranges over the prime divisors of k. Let furthermore,

$$\rho = \rho_1 \otimes \dots \otimes \rho_r$$

and let

$$S(\vec{\chi}) = \bigcup_{j=1}^{r} S(\rho_j) .$$

It follows from the definitions that

$$f_p^0(\vec{\chi},t) = \det(1-t\rho_1(\sigma_p) \otimes \ldots \otimes \rho_r(\sigma_p))^{-1} \quad ;$$

therefore, recalling (7) and the definition of $S(\rho)$, we get

$$f_p^0(\vec{\chi},t) = f_p(\rho,t) \quad \text{for} \quad p \notin S(\vec{\chi}) \ . \tag{81}$$

By (79), there is $h_p(t)$ in $\mathbb{C}[t]$ for which

$$f_p(\vec{\chi},t) = f_p^0(\vec{\chi},t)h_p(t) \ . \tag{82}$$

Lemma 6. There is a polynomial Φ in $Y[t]$ such that $S(\Phi) \subseteq S(\vec{\chi})$ and

$$h_p(t) = \Phi_p(t) \quad \text{for} \quad p \notin S(\vec{\chi}) \ .$$

Moreover, if (14) holds and either $r > 2$ or $d_1 > 2$, then Φ is not unitary.

Proof. Let $T^m A$ and $\Lambda^m A$ denote the m-th symmetric and exterior powers of a linear operator A in a finite dimensional complex vector space. By well known identities of linear algebra,

$$\det(1+At) = \sum_{m=0}^{\infty} t^m \mathrm{tr}(\Lambda^m A) \ , \ \det(1-At)^{-1} = \sum_{m=0}^{\infty} t^m \mathrm{tr}(T^m A)$$

in $\mathbb{C}[[t]]$. Since, by Proposition 6, the degree of $h_p(t)$ does not exceed $d-1$, it follows from (82) and (81) that

$$h_p(t) = \Phi_p(t) \quad \text{for} \quad p \notin S(\vec{\chi}) ,$$

where $\Phi(t) = 1 + \sum_{\ell=1}^{d-1} b_\ell t^\ell$ with

$$b_\ell = \sum_{\ell_1=0}^{\ell} (-1)^{\ell_1} tr(\Lambda^{\ell_1}\rho) \prod_{j=1}^{r} tr(T^{\ell-\ell_1}\rho_j) .$$

In particular, taking g to be the unit element in $W_1(K|k)$ one obtains

$$\Phi_g(t) = ((1-t)^{-d_1} * \dots * (1-t)^{-d_r})(1-t)^d .$$

Therefore, by Corollary 2, Φ is not unitary when $r > 2$ or $d_1 > 2$ and (14) holds. This proves the lemma.

We notice now that, by definition,

$$L(s,\vec{\chi}) = \prod_p f_p(\vec{\chi}, |p|^{-s}) ,$$

where p varies over the prime divisors of k . Therefore (81) and (82) give:

$$L(s,\vec{\chi}) = L(s,\rho) \prod_{p \in S(\vec{\chi})} \ell_p(|p|^{-s}) \prod_p h_p(|p|^{-s}) , \qquad (83)$$

where

$$\ell_p(t) = f_p^0(\vec{\chi},t) \det (1-t\rho(\sigma_p)) \quad . \tag{84}$$

The assertion of Theorem 2 follows from (84), (83), lemma 6 and Theorem 1.

§ 8. Correction and acknowledgement

Theorem II.2.1 in [23, p.89] is incorrect as it stands; it should be replaced by Proposition 1 of this paper. Accordingly, lemma II.4.2 in [23, p.100] should be replaced by lemma 5 of this paper, and in lemma II.4.4, [23, p.103], one should obtain a sharper estimate

$$a_2(\nu;\delta,t_o) > \exp(A_H^{(1)}(\delta,t_o)\sqrt{\nu}) \quad ,$$

as in lemma 4 of this paper.

It is my pleasant duty to thank Professor N. Kurokawa for pointing out an error in the theorem II.2.1 of [23] and to acknowledge my intellectual debt to his unpublished work [10].

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