# ON REPRESENTATIONS OF THE LIE SUPERALGEBRA $P(n)$

VERA SERGANOVA

ABSTRACT. We introduce a new way to study representation of the Lie superalgebra $p(n)$. Since the center of the universal enveloping algebra $U$ acts trivially on all irreducible representations, we suggest to study the quotient algebra $\bar{U}$ by the radical of $U$. We show that $\bar{U}$ has a large center which separates irreducible representations. We give a description of $\bar{U}$ factorized by a generic central character. Using this description we obtain character formulae of generic (infinite-dimensional) irreducible representations. We also show that $\bar{U}$ is a quantum deformation of the algebra of functions on some Poisson subvariety in the coadjoint representation.

## 1. Introduction

In this paper we study representations of the classical Lie superalgebra $p(n)$ introduced in [3]. Let us recall the definitions. Denote by $\widetilde{p}(n)$ the Lie superalgebra of all endomorphisms of $(n|n)$-dimensional vector superspace which preserve a fixed odd non-degenerate symmetric form. The Lie superalgebra $p(n)$ is the commutator of $\widetilde{p}(n)$. It is simple when $n \geq 2$.

The main difficulty in representation theory of $\widetilde{p}(n)$ and $p(n)$ is that the center of the universal enveloping superalgebra of $\widetilde{p}(n)$ is trivial. The center of the universal enveloping of $p(n)$ is non-trivial see [1], but it acts by the same central character on all irreducible representations. Therefore the powerful technique of central characters seems not applicable. We show that it still can work after some reduction.

The universal enveloping algebra $U$ of $\widetilde{p}(n)$ has a non-trivial radical $I$ as it was shown in [4]. Let $\bar{U}$ be the quotient algebra $U/I$. We suggest to study representation theory of $\bar{U}$ instead of $U$. Using this approach we do not loose any information about irreducible representations but in fact gain some additional information. Our main result is that $\bar{U}$ has a non-trivial center $Z$, which separates irreducible representation. In Theorem 3.4 below we describe $Z$ precisely. Furthermore we show that for a generic central character $\chi : Z \to \mathbb{C}$ the algebra $\bar{U}/\bar{U}\operatorname{Ker}\chi$ is isomorphic to the matrix algebra over $U(\mathfrak{g}_0)/U(\mathfrak{g}_0)\operatorname{Ker}\chi_0$ for a certain central character $\chi_0$ of $U(\mathfrak{g}_0)$. This allows to write character formulae for a generic irreducible weight representation.

The same method is applicable to $p(n)$.

In the last section we suggest some geometric motivation of our construction. We realize the graded algebra $\bar{U}$ as the algebra of function on a Poisson subvariety $Q$ in the coadjoint representation $\mathfrak{g}^*$. This $Q$ is the union of coadjoint orbits of all even

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elements. It is a specific feature of supergeometry that $Q$ does not coincide with $\mathfrak{g}^*$ but is a closed subvariety with singularities. Thus $\bar{U}$ is a quantum deformation of the superalgebra of functions on the Poisson supervariety $Q$.

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## 2. Preliminaries

Let $\mathfrak{g}$ be the Lie superalgebra $\widetilde{p}(n)\,(n \geq 2)$ over $\mathbb{C}$, $U$ be its universal enveloping algebra. Recall that $\mathfrak{g}_0 \cong gl(n)$, and $\mathfrak{g}$ has a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1}$ is isomorphic to $\Lambda^2 E^*$ and $\mathfrak{g}_1$ is isomorphic to $S^2 E$ as $\mathfrak{g}_0$-module, here $E$ stands for the standard module over $\mathfrak{g}_0$. Let $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0] \cong sl(n)$. The grading of $\mathfrak{g}$ induces the grading of $U$ in the natural way. Thus

$$U = \oplus_{i=-n(n-1)/2}^{n(n+1)/2} U_i.$$

We fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, which coincides with Cartan subalgebra of $\mathfrak{g}_0$ and identify $\mathfrak{h}$ with $\mathfrak{h}^*$ by means of the $\mathfrak{g}_0$-invariant form $(A, B) = \operatorname{tr} AB$ on $\mathfrak{g}_0$. If $\varepsilon_1, \dots, \varepsilon_n$ is the standard orthogonal basis in $\mathfrak{h}$, then the roots of $\mathfrak{g}$ are of the following form:

Roots of $\mathfrak{g}_0$: $\varepsilon_i - \varepsilon_j$, $i \neq j$, $1 \leq i, j \leq n$

Roots of $\mathfrak{g}_{-1}$: $-\varepsilon_i - \varepsilon_j$, $1 \leq i < j \leq n$

Roots of $\mathfrak{g}_1$: $\varepsilon_i + \varepsilon_j$, $1 \leq i \leq j \leq n$

Each root has multiplicity one. For each root $\alpha$ we fix an element $X_\alpha$ from the root space such that $[X_\alpha, X_{-\alpha}] = H_\alpha$ with $(H_\alpha, H_\alpha) = 2$. If $-\alpha$ is not a root we choose $X_\alpha$ being an arbitrary non-zero element of the root space. For $\mu \in \mathfrak{h}^*$ we denote by $\mu_i$ its coordinate in the standard basis, i.e. $\mu = \sum_{i=1}^n \mu_i \varepsilon_i$.

We choose a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ such that the positive roots are $\varepsilon_i - \varepsilon_j$ for $i < j$ and all roots of $\mathfrak{g}_1$. Let $\mathfrak{n} = [\mathfrak{b}.\mathfrak{b}]$. As it was shown in [3], all irreducible finite dimensional $\mathfrak{g}$-modules are modules of highest weight. An irreducible module with highest weight $\mu$ is finite-dimensional if and only if $\mu_i - \mu_j$ is a non-negative integer for all $i < j$.

Finally let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$. One can check that $\mathfrak{g}' = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0' \oplus \mathfrak{g}_1$. Let $U'$ denote the universal enveloping of $\mathfrak{g}'$.

## 3. Radical of $U$

In this section we will show that $U$ has a non-zero Jacobson radical and give some description of this radical. This radical was constructed in [4]. We put $d = n\,(n-1)\,/2$. Note that $d = \dim \mathfrak{g}_{-1}$.

Let $M_\mu^0$ denote Verma module over $\mathfrak{g}_0$ with highest weight $\mu$ and $M_\mu$ be Verma module over $\mathfrak{g}$. One can verify easily that

$$M_\mu = \operatorname{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} M_\mu^0, \tag{3.1}$$

assuming $\mathfrak{g}_1 M_\mu^0 = 0$.

Let $v$ be a highest vector of $M_\mu$ and $X = \Pi_{i<j} X_{\varepsilon_i+\varepsilon_j}$, $Y = \Pi_{i<j} X_{-\varepsilon_i-\varepsilon_j}$. Finally let $\Delta(\mu) = \Pi_{i<j}(\mu_i - \mu_j + j - i - 1)$.

**Lemma 3.1.** $XYv = \Delta(\mu)\, v$.

*Proof.* We order the roots $\varepsilon_i + \varepsilon_j$ in the following way

$$\varepsilon_i + \varepsilon_j < \varepsilon_p + \varepsilon_q \text{ iff } i < p \text{ or } i = p \text{ and } j < q.$$

According to this order we index our roots $\alpha_1, \dots, \alpha_d$.

Then whenever $\alpha < \beta < \gamma$ we have

$$[X_\alpha, X_{-\beta}] \in \mathfrak{b}_0,$$

$$[[X_\alpha, X_{-\beta}], X_{-\gamma}] \text{ is either zero or } X_{-\delta}, \text{ with } \gamma < \delta.$$

Now let $w_k = X_{-\alpha_k} \dots X_{-\alpha_d} v$, $w_{d+1} = v$. Using the above relations one verifies that

$$X_{\alpha_k} w_k = (\mu - \alpha_{k+1} - \dots - \alpha_d, H_{\alpha_k})\, w_{k+1}.$$

If $\alpha_k = \varepsilon_i + \varepsilon_j$, then $(\mu - \alpha_{k+1} - \dots - \alpha_d, H_{\alpha_k}) = \mu_i - \mu_j + j - i - 1$. Repeating this argument one obtains $Xw_1 = \Delta(\mu)\, w_{d+1}$. □

Let $\mathfrak{h}_0^* = \{\mu \in \mathfrak{h}^* \mid \mu_i - \mu_j \notin \mathbb{Z}, i \neq j\}$. Clearly $\mathfrak{h}_0^*$ is a Zariski dense set in $\mathfrak{h}^*$.

**Lemma 3.2.** *If $\mu \in \mathfrak{h}_0^*$ then $M_\mu$ is irreducible.*

*Proof.* If we put $\deg v = 0$, then $M_\mu$ inherits $\mathbb{Z}$-grading from $U$, such that $M_\mu = (M_\mu)_0 + \dots + (M_\mu)_{-d}$. Note that $(M_\mu)_{-d} = YM_\mu^0 = M_{\mu+\sigma}^0$, where $\sigma$ is the sum of all roots of $\mathfrak{g}_{-1}$.The highest vector of $M_{\mu+\sigma}^0$ is $Yv$. By (3.1) $M_\mu$ is free over $U(\mathfrak{g}_{-1})$. Therefore every non-zero submodule $N$ of $M_\mu$ has a nontrivial intersection with $M_{\mu+\sigma}^0$. The conditions on $\mu$ ensure that $M_{\mu+\sigma}^0$ is irreducible. Thus, $N$ must contain $Yv$, and therefore $XYv = \Delta(\mu)\, v$. As $\mu \in \mathfrak{h}_0^*$, $\Delta(\mu) \neq 0$, we have $v \in N$ and hence $N = M_\mu$. □

**Lemma 3.3.** *Let $P_\mu = \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M_\mu^0$.*

*(a) If $\mu \in \mathfrak{h}_0^*$, then $P_\mu$ has finite length with irreducible subquotients $M_{\mu+\gamma}$, where $\gamma$ runs over sums of non-repeating roots of $\mathfrak{g}_1$. Note that the length $s$ of $P_\mu$ is equal to $2^{n(n+1)/2}$.*

*(b) $\oplus_{\mu \in \mathfrak{h}_0^*} P_\mu$ is a faithful $U$-module.*

*Proof.* To prove (a) use

$$P_\mu = \operatorname{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} \left( \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}_0 \oplus \mathfrak{g}_1} M_\mu^0 \right).$$

A filtration of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}_0 \oplus \mathfrak{g}_1} M_\mu^0$ with $\mathfrak{g}_0$-irreducible quotients $M_{\mu+\gamma}^0$ induces the filtration on $P_\mu$ with irreducible quotients $M_{\mu+\gamma}$.

To show (b) recall that $\oplus_{\mu \in \mathfrak{h}_0^*} M_\mu^0$ is a faithful $U(\mathfrak{g}_0)$-module. Therefore $\oplus_{\mu \in \mathfrak{h}_0^*} P_\mu = \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \left( \oplus_{\mu \in \mathfrak{h}_0^*} M_\mu^0 \right)$ is a faithful $U$-module. □

The following theorem is proved in [4]. If $M$ is a module over an associative algebra $A$, then $\operatorname{Ann}_{\mathcal{A}} M$ denotes the annihilator of $M$ in $A$.

**Theorem 3.4.** *The ideal* $I = \cap_{\mu\in\mathfrak{h}_{\mu\in\mathfrak{h}}} \operatorname{Ann}_U M_\mu = \cap_{\mu\in\mathfrak{h}_0^*} \operatorname{Ann}_U M_\mu$ *coincides with the radical of* $U$.

*Proof.* The assertion $\cap_{\mu\in\mathfrak{h}^*} \operatorname{Ann}_U M_\mu = \cap_{\mu\in\mathfrak{h}_0^*} \operatorname{Ann}_U M_\mu$ follows from Zariski density of $\mathfrak{h}_0^*$. By Lemma 3.2 $M_\mu$ is irreducible if $\mu\in\mathfrak{h}_0^*$, and therefore $\operatorname{rad} U \subset I$. To show that $I \subset \operatorname{rad} U$ it suffices to prove that $I$ is nilpotent. Note that $I^s(P_\mu) = 0$ for all $\mu\in\mathfrak{h}_0^*$. Since $\oplus_{\mu\in\mathfrak{h}_0^*} P_\mu$ is faithful we obtain $I^s = 0$. $\square$

**Corollary 3.5.** *Let* $\bar U = U/I$. *Then the* $\mathbb{Z}$*-grading on* $U$ *induces the* $\mathbb{Z}$*-grading on* $\bar U$ *such that* $\bar U = \bar U_{-d} \oplus \cdots \oplus \bar U_d$. *The natural maps* $U(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) \to \bar U$ *and* $\mathfrak{g} \to \bar U$ *are injective. Finally* $\cap_{\mu\in\mathfrak{h}_0^*} \operatorname{Ann}_{\,\overline{U}} M_\mu = \{0\}$.

The following theorem can be proven in the same manner as Theorem 3.4.

**Theorem 3.6.** *The ideal* $I' = U' \cap \left(\cap_{\mu\in(\mathfrak{h}')^*} \operatorname{Ann}_{U'} M_\mu\right)$ *coincides with the radical of* $U'$.

## 4. The center of $\bar U$

In this section we will describe the center $Z$ of $\bar U$. As follows from Corollary 3.5 $U(\mathfrak{h})$ is a subalgebra of $\bar U$. Therefore the standard Harish-Chandra projection $h : U \to U(\mathfrak{h})$ with the kernel $\mathfrak{n}^- U + U\mathfrak{n}^+$ induces the map $\bar U \to S(\mathfrak{h})$. Thus, one can define the Harish-Chandra homomorphism $h : Z \to U(\mathfrak{h}) = S(\mathfrak{h})$. We identify $S(\mathfrak{h})$ with polynomial algebra on $\mathfrak{h}^*$. As in the usual case for any $z \in Z$

$$z|_{M_\mu} = h(z)(\mu)\operatorname{Id}. \tag{4.1}$$

Therefore the last statement of Corollary 3.5 implies that $h$ is injective. We will describe $h(Z)$.

Let $W$ be the Weyl group of $\mathfrak{g}_0$, note that $W$ is isomorphic to $S_n$. Define a $W$-action on $\mathfrak{h}^*$ by $\mu^w = w(\mu+\rho_0) - \rho_0$, where $\rho_0$ is the half sum of even positive roots. Denote by $S(\mathfrak{h})^W$ the subring of $W$-invariant polynomials on $\mathfrak{h}^*$. Let $\Theta(\mu) = \Pi_{i\neq j}((\mu+\rho_0, \varepsilon_i - \varepsilon_j) - 1)$. Note that $\Theta \in S(\mathfrak{h})^W$.

**Lemma 4.1.** $h(Z) \subset S(\mathfrak{h})^W$.

*Proof.* Let $\mathfrak{d} = \{\mu\in\mathfrak{h}^* \mid \mu_i - \mu_j \in \mathbb{Z}_{\geq 0}, i<j\}$. Then $\mathfrak{d}$ is a Zariski dense set in $\mathfrak{h}^*$. Furthermore for any $w\in W$

$$\operatorname{Hom}_{\mathfrak{g}}(M_{\mu^w}, M_\mu) = \operatorname{Hom}_{\mathfrak{g}_0}\left(M^0_{\mu^w}, M^0_\mu\right) = \mathbb{C}.$$

Therefore every element $z\in Z$ acts by the same scalar on $M_{\mu^w}$ and on $M_\mu$. By (4.1) every $p\in h(Z)$ satisfies $p(\mu^w) = p(\mu)$ for all $\mu\in\mathfrak{d}$. Since $\mathfrak{d}$ is Zariski dense, this implies that $p(\mu^w) = p(\mu)$ for all $\mu\in\mathfrak{h}^*$. $\square$

**Lemma 4.2.** *Let $\mu \in \mathfrak{h}^*$ be such that $\mu_{n-1} = \mu_n$ and $p \in h(Z)$. Then $p(\mu + t(\varepsilon_{n-1} + \varepsilon_n)) = p(\mu)$ for any $t \in \mathbb{C}$.*

*Proof.* Let $v$ be a highest vector of Verma module $M_\mu$. An easy calculation shows that $w = X_{-\varepsilon_n - \varepsilon_{n-1}} v$ is $\mathfrak{b}$-invariant, and therefore $\operatorname{Hom}_{\mathfrak{g}}\left(M_{\mu-\varepsilon_{n-1}-\varepsilon_n}, M_\mu\right) \neq 0$. By (4.1) this implies $p(\mu) = p(\mu - \varepsilon_{n-1} - \varepsilon_n)$. Applying this relation several times we obtain $p(\mu) = p(\mu + t(\varepsilon_{n-1} + \varepsilon_n))$ for all $t \in \mathbb{Z}$, which by Zariski density of $\mathbb{Z}$ in $\mathbb{C}$ implies the same relation for all $t \in \mathbb{C}$. $\square$

**Corollary 4.3.** *Let $(\mu + \rho_0, \varepsilon_i - \varepsilon_j) = 1$ and $p \in h(Z)$. Then $p(\mu) = p(\mu + t(\varepsilon_i + \varepsilon_j))$ for all $t \in \mathbb{C}$.*

**Lemma 4.4.** $h(Z) \subset \Theta S(\mathfrak{h})^W + \mathbb{C}$.

*Proof.* Let

$$l_{i,j} = \left\{\mu \in \mathfrak{h}^* \,|\, (\mu + \rho_0, \varepsilon_i - \varepsilon_j) = 1\right\}.$$

We want to show that if $p \in h(Z)$, then $p$ is constant on $\cup_{i\neq j} l_{i,j}$. Since $p$ is $W$-invariant, it suffices to show that $p$ is constant on one hyperplane $l_{i,j}$, for example on $l_{1,2}$. Let $\mu \in l_{1,2}$, i.e. $\mu_1 = \mu_2$. Put $\lambda = \mu + (\mu_3 - \mu_1)(\varepsilon_1 + \varepsilon_2)$. By Corollary 4.3 $p(\mu) = p(\lambda)$. Note that $\lambda \in l_{2,3}$. Put $\nu = \lambda + (\mu_1 - \mu_3 + 1)(\varepsilon_2 + \varepsilon_3)$. Then $p(\mu) = p(\lambda) = p(\nu)$. Let $w = (132) \in W$. Then $\nu^w = \mu + 2\varepsilon_3$. Thus we obtain the relation $p(\mu) = p(\mu + 2\varepsilon_3)$. In the same way as in the proof of lemma 5 we can get that $p(\mu) = p(\mu + t\varepsilon_3)$ for all $t \in \mathbb{C}$. Using $W$-invariance of $p$ we obtain that $p(\mu) = p(\mu + t\varepsilon_i)$ for all $t \in \mathbb{C}$ and $i = 3, \dots, n$. This implies that $p$ is constant on $l_{1,2}$ and therefore $p$ is constant on $\cup_{i\neq j} l_{i,j}$. $\square$

For any $x \in U$ denote by $x_0$ the image of $x$ under the projection onto $U(\mathfrak{g}_0)$ with the kernel $\mathfrak{g}_1 U + U\mathfrak{g}_{-1}$. Recall that $U = U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_0) \otimes U(\mathfrak{g}_{-1})$ and $U(\mathfrak{g}_1) = S(\mathfrak{g}_1) \subset U$.

**Lemma 4.5.** $\left(Y S^d(\mathfrak{g}_1)\right)_0 \neq 0$.

*Proof.* We will show that $(YX)_0 \neq 0$. Consider the Verma module $M_\mu$ with a highest vector $v$. Then $(YX)_0 Yv = YXYv = \Delta(\mu) Yv$ by Lemma 3.1. Therefore $(YX)_0 \neq 0$. $\square$

Let $U^k$ be the $k$-th term of the natural filtration of $U$. Lemma 4.5 enables us to define a non-trivial $\operatorname{ad}\mathfrak{g}_0'$-invariant map

$$\varphi : S^d(\mathfrak{g}_1) \to U^{2d}(\mathfrak{g}_0)$$

by

$$\varphi(x) = (Yx)_0.$$

Let $\{X_i\}$ be a basis in some $\operatorname{ad}\mathfrak{g}_0$-invariant subspace complementary to $\operatorname{Ker}\varphi$. Then $\{(YX_i)_0\}$ is a linearly independent set in $U^{2d}(\mathfrak{g}_0)$ and could be extended to some basis of $U^{2d}(\mathfrak{g}_0)$. Recall that $U^{2d}(\mathfrak{g}_0) \cong S^{\leq 2d}(\mathfrak{g}_0)$ as $\operatorname{ad}\mathfrak{g}_0$ module and it admits a non-degenerate $\operatorname{ad}\mathfrak{g}_0$-invariant bilinear symmetric form $B$. Use the notation $(YX_i)_0'$ for the vectors of the dual basis with respect to $B$. Let

$$T = \Sigma X_i (YX_i)_0'.$$

Then $T$ is $\operatorname{ad}\mathfrak{g}_0'$ invariant and

$$(YT)_0 = \Sigma \left(YX_i (YX_i)_0'\right)_0 = \Sigma (YX_i)_0 (YX_i)_0' \neq 0. \tag{4.2}$$

**Lemma 4.6.** *(a) Let $Y^{\mathrm{ad}} = \Pi_{i<j} \operatorname{ad} X_{-\varepsilon_i-\varepsilon_j}$ and $S = Y^{\mathrm{ad}}(T)$, and $\bar{S}$ be its image in $\bar{U}$. Then $\bar{S} \in Z$ and $h\left(\bar{S}\right)$ is proportional to $\Theta$.*

*(b) Let $z \in Z(U(\mathfrak{g}_0))$. Put $S_z = Y^{\mathrm{ad}}(zT)$, and let $\bar{S}_z$ be its image in $\bar{U}$. Then $\bar{S}_z \in Z$ and $h\left(\bar{S}_z\right) = h\left(\bar{S}\right) h(z)$.*

*Proof.* (a) First of all $\bar{S} \neq 0$. Indeed, if $v$ is a highest vector of $M(\mu)$ then

$$\bar{S}Yv = SYv = Y^{\mathrm{ad}}(T)Yv = YTYv = (YT)_0 Yv.$$

Since $(YT)_0 \neq 0$, $(YT)_0 Yv \neq 0$ for some $\mu$, and therefore $\bar{S} \neq 0$.

Note that $\bar{T} \in \bar{U}_d$, hence $\mathfrak{g}_1\bar{T} = \bar{T}\mathfrak{g}_1 = 0$, and $\operatorname{ad}\mathfrak{g}_0'\bar{T} = 0$. Thus, $\bar{T}$ is $\operatorname{ad}\mathfrak{b}$ invariant vector of weight $-\sigma$. Let $V$ be $\operatorname{ad}\mathfrak{g}$-submodule of $\bar{U}$ generated by $\bar{T}$. Since $\bar{S} \neq 0$, $V \cong \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1} \mathbb{C}\bar{T}$ and $\bar{S}$ generates the trivial $\operatorname{ad}\mathfrak{g}$-submodule of $V$. Thus, $\bar{S} \in Z$.

To find $h\left(\bar{S}\right)$ note that $\bar{S}$ acts by zero on the trivial $\mathfrak{g}$-module, hence $h\left(\bar{S}\right)(0) = 0$. On the other hand the degree of the polynomial $h\left(\bar{S}\right)$ is not greater than $2d$ as $\bar{S} \in \bar{U}^{2d}$. Therefore, by Lemma 4.4, $h\left(\bar{S}\right)$ must be proportional to $\Theta$.

(b) The arguments similar to (a) show that $\bar{S}_z \in Z$. Finally, if $v$ is a highest vector of $M_\mu$, then

$$h\left(\bar{S}_z\right)(\mu)v = \bar{S}_z v = Y^{\mathrm{ad}}(zT)v = zTYv = zY^{\mathrm{ad}}(T)v = zSv. \tag{4.3}$$

Note that $zSv = (h(z))(\mu)Sv$. Therefore $h\left(\bar{S}_z\right) = h\left(\bar{S}\right)h(z)$. □

**Corollary 4.7.** $\mathbb{C} + \Theta S(\mathfrak{h})^W \subset h(Z)$.

Lemma 4.4 and Corollary 4.7 immediately imply

**Theorem 4.8.** $h(Z) = \mathbb{C} + \Theta S(\mathfrak{h})^W$.

Let $\bar{U}' = U'/I'$, $Z'$ be the center of $\bar{U}'$ and $h\colon Z' \to S(\mathfrak{h}')^*$ be the Harish-Chandra homomorphism. One can prove the following theorem by repeating the arguments of this section.

**Theorem 4.9.** $h(Z') = \mathbb{C} + \Theta S(\mathfrak{h}')^W$.

## 5. The category of $\bar{U}$-modules

Let $\mu \in \mathfrak{h}^*$. Define $\chi_\mu : Z \to \mathbb{C}$ by the formula $\chi_\mu(z) = h(z)(\mu)$ for all $z \in Z$. We call $\mu$ *typical* if $\Delta(\mu) \neq 0$. Otherwise we call $\mu$ *atypical*. The following statement follows immediately from Theorem 4.8.

**Lemma 5.1.** *Let $\mu, \nu \in \mathfrak{h}^*$. Then $\chi_\mu = \chi_\nu$ iff either both $\mu$ and $\nu$ are atypical or $\mu$ and $\nu$ are both typical and $\mu = \nu^w$ for some $w \in W$*

Define $\bar{U}_\mu = \bar{U}/\bar{U} \operatorname{Ker} \chi_\mu$. Let $U_\mu^0 = U(\mathfrak{g}_0)/U(\mathfrak{g}_0) \operatorname{Ker} \chi_\mu^0$, where $\chi_\mu^0 : Z(U(\mathfrak{g}_0)) \to \mathbb{C}$ is defined using the Harish-Chandra projection restriction to $Z(U(\mathfrak{g}_0))$. By $\bar{U}_\mu -$ mod (respectively $U_\mu^0 -$ mod ) we denote the category of all left $\bar{U}_\mu$-modules (resp. $U_\mu^0$-modules). In the similar way one defines $\bar{U}'_\mu$ for $\mu \in (\mathfrak{h}')^*$.

We will prove the following

**Theorem 5.2.** *If $\mu \in \mathfrak{h}^*$ is typical then the categories $\bar{U}_\mu -$ mod and $U_\mu^0 -$ mod are equivalent.*

Let us introduce the following notations. Put $z_0 = \operatorname{diag}(1_n, -1_n) \in Z(\mathfrak{g}_0)$. We assume now that $\mu$ is typical and fixed. If $M$ is a $\mathfrak{g}$-module we put

$$M^{\mathfrak{g}_1} = \{m \in M \mid \mathfrak{g}_1 m = 0\}$$

$$M_0 = \{m \in M \mid z_0 m = \mu(z_0) m\}.$$

**Lemma 5.3.** *If $M$ is an $\bar{U}_\mu$-module then*

*(a) $M^{\mathfrak{g}_1}$ is an $U_\mu^0$-module;*

*(b) $M^{\mathfrak{g}_1} = M_0$;*

*(c) The functor* Inv : $\bar{U}_\mu -$ mod $\to U_\mu^0 -$ mod , *defined by* $\operatorname{Inv} M = M_0 = M^{\mathfrak{g}_1}$ *is faithful and exact.*

*Proof.* The first statement follows from the formula $\bar{S}_z m = z\bar{S}m$ for any $m \in M^{\mathfrak{g}_1}$, $z \in Z(U(\mathfrak{g}_0))$. Since $\bar{S}m = \chi_\mu(\bar{S}) m \neq 0$, we get

$$zm = \frac{\bar{S}_z m}{\chi_\mu(\bar{S})} = \chi_\mu^0(z) m.$$

By (a) $z_0 m = \mu(z_0) m$ for any $m \in M^{\mathfrak{g}_1}$, and therefore $M^{\mathfrak{g}_1} \subseteq M_0$. Let $m \in M_0$. Then $M' = U(\mathfrak{g}_1) m$ is a $z_0$-invariant finite-dimensional subspace of $M$ with $z_0$ weight decomposition $M' = \oplus_{i \geq 0} M'_{\mu(z_0)+i}$. In particular $M'_{\mu(z_0)} = \mathbb{C}m$. Since $(M')^{\mathfrak{g}_1} \neq 0$, and $(M')^{\mathfrak{g}_1} \subseteq M_0$, we obtain $m \in (M')^{\mathfrak{g}_1}$ and $M' = \mathbb{C}m$. Therefore $M^{\mathfrak{g}_1} = M_0$.

Faithfulness of Inv follows from the fact that $U(\mathfrak{g}_1)$ is isomorphic to finite-dimensional Grassman algebra. Exactness is a direct corollary of the identity $\operatorname{Inv} M = M_0$. □

Let $\bar{U}^+$ denote the image of $U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ in $\bar{U}$ under the natural projection. Since $U(\mathfrak{g}_0)$ is a subalgebra of $\bar{U}$ it is possible to define a $\bar{U}^+$-module structure on every $\mathfrak{g}_0$-module by putting $\mathfrak{g}_1 N = 0$.

Let $\operatorname{Ind} N = \bar{U} \otimes_{\bar{U}^+} N$.

**Lemma 5.4.** *Let $N$ be a $U_\mu^0$-module. Then*

*(a)* Ind $N$ *is $\bar{U}_\mu$-module.*

*(b)* $\operatorname{Ind} N \cong U(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} N$ *as $\mathfrak{g}_0$-module.*

*(c) The functor* Ind: $U_\mu^0 -$ mod $\to \bar{U}_\mu -$ mod *is exact and faithful.*

*(d)* $\operatorname{Inv}(\operatorname{Ind} N) \cong N$.

*Proof.* Since $\operatorname{Ind} N = \bar{U}N$, it suffices to check that $zv = \chi_\mu(z)\,v$ for every $z \in Z$ and $v \in N$. Define the projection $h_0 : \bar{U}_0 \to U(\mathfrak{g}_0)$ with the kernel $\bar{U}_0 \cap \bar{U}\mathfrak{g}_1$. One can easily verify that $h_0(Z) \subset Z(U(\mathfrak{g}_0))$, $zv = h_0(z)\,v$ and $\chi_\mu(z) = \chi^0_\mu(h_0(z))$. Now (a) follows immediately.

To prove $(b)$ recall that $U(\mathfrak{g}_{-1})$ is a subalgebra of $\bar{U}$ and $\operatorname{Ind} N$ is free over $U(\mathfrak{g}_{-1})$.

$(c)$ follows $(b)$.

To prove $(d)$ use that $\operatorname{Ind} N = N \oplus \mathfrak{g}_{-1} \operatorname{Ind} N$. Let

$$N' := \operatorname{Inv}(\operatorname{Ind} N) \cap (\mathfrak{g}_{-1} \operatorname{Ind} N).$$

It suffices to show that $N' = \{0\}$. Indeed, let $M' \subset \operatorname{Ind} N$ be a submodule generated by $N'$. Then $M' \subseteq \mathfrak{g}_{-1} \operatorname{Ind} N$, and therefore $TM' = 0$. But then $\bar{S}M' = 0$. Since $\mu$ is typical, $\bar{S}$ acts on $M'$ by a non-zero constant. Hence $M' = \{0\}$ and $N' = \{0\}$. ☐

**Lemma 5.5.** *The functors* $\operatorname{Inv} \colon U^0_\mu - \text{ mod } \to \bar{U}_\mu - \text{ mod }$ *and* $\operatorname{Ind} \colon \bar{U}_\mu - \text{ mod } \to U^0_\mu - \text{ mod }$ *establish equivalence of categories.*

*Proof.* By Lemma 5.4 $(d)$ $\operatorname{Inv}(\operatorname{Ind} N) \cong N$. Therefore it suffices to show that $\operatorname{Ind}(\operatorname{Inv} M) \cong M$. Since $\operatorname{Inv}(\operatorname{Ind}(\operatorname{Inv} M)) \cong \operatorname{Inv} M$ and Inv is faithful and exact, we obtain $\operatorname{Ind}(\operatorname{Inv} M) \cong M$. In fact we use only the left exactness of Inv, which is always true. ☐

Note that Lemma 5.5 implies Theorem 5.2.

*Remark* 5.6. Note that in the same way one can prove that the category of right $\bar{U}_\mu$-modules and the category of right $U^0_{\mu+\sigma}$-modules are equivalent. The shift by $\sigma$ comes from rewriting formula (4.3) for right $\mathfrak{g}$-modules.

**Theorem 5.7.** *For a typical* $\mu \in \mathfrak{h}^*$ *the algebra* $\bar{U}_\mu$ *is isomorphic to the matrix algebra* $U^0_\mu \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(U(\mathfrak{g}_{-1}))$.

*Proof.* We are using the following classical result (see [2]). Let A and $B$ be $\mathfrak{g}_0$-modules. Denote by $F(A,B)$ the subspace of $\operatorname{Hom}_{\mathbb{C}}(A,B)$ on which the adjoint action of $\mathfrak{g}_0$ is locally finite. The natural homomorphism $\rho_0 : U^0_\mu \to F\left(M^0_\mu, M^0_\mu\right)$ is an isomorphism. We will show that the natural homomorphism $\rho : \bar{U}_\mu \to F(M_\mu, M_\mu)$ is also an isomorphism. First of all since $M_\mu = \operatorname{Ind} M^0_\mu$ and $M^0_\mu$ is faithful, Morita equivalence of $U^0_\mu$ and $\bar{U}_\mu$ implies that $M_\mu$ must be faithful. Thus $\rho$ is injective. To show the surjectivity of $\rho$ consider $F(M_\mu, M_\mu)$ as a left $\bar{U}_\mu$-module. Then by Theorem 5.2

$$F(M_\mu, M_\mu) = \operatorname{Ind}(\operatorname{Inv} F(M_\mu, M_\mu)) = \operatorname{Ind} F\left(M_\mu, M^0_\mu\right).$$

Note that $F\left(M_\mu, M^0_\mu\right)$ is a right $\bar{U}_\mu$-module and therefore by Remark 5.6

$$F\left(M_\mu, M^0_\mu\right) = \operatorname{Ind}\left(\operatorname{Inv} F\left(M_\mu, M^0_\mu\right)\right) = \operatorname{Ind} F\left(M^0_{\mu+\sigma}, M^0_\mu\right).$$

Thus, it suffices to check that $F\left(M^0_{\mu+\sigma}, M^0_\mu\right) \subseteq \operatorname{Im}\rho$. Indeed, it follows from

$$F\left(M^0_{\mu+\sigma}, M^0_\mu\right) = F\left(M^0_\mu, M^0_\mu\right)\rho(T) = \rho(U(\mathfrak{g}_0)\,T).$$

Finally, the isomorphism of $\mathfrak{g}_0$-modules $M_\mu \cong M_\mu^0 \otimes_{\mathbb{C}} U(\mathfrak{g}_{-1})$ implies

$$F(M_\mu, M_\mu) \cong F\left(M_\mu^0, M_\mu^0\right) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(U(\mathfrak{g}_{-1})),$$

and that completes the proof of the theorem. □

**Corollary 5.8.** *Let $M_0$ be an irreducible $U_\mu^0$-module for some typical $\mu \in \mathfrak{h}^*$. Then $M = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} M_0$ is an irreducible $\mathfrak{g}$-module. Moreover, if $M_0$ is a semi-simple $\mathfrak{h}$-module with finite weight multiplicities, then $M$ is also $\mathfrak{h}$-semi-simple with finite weight multiplicities and $\mathrm{ch}\, M = \mathrm{ch}\, M_0 \Pi_{i \neq j}\left(1 + \varepsilon e^{-\varepsilon_i - \varepsilon_j}\right)$.*

**Theorem 5.9.** *Let $\nu \in (\mathfrak{h}')^*$ and $\mu \in \mathfrak{h}^*$ be such that $\mu_{|\mathfrak{h}'} = \nu$. Then the algebra $\bar{U}'_\nu$ is isomorphic to $\bar{U}_\mu$.*

*Proof.* Let $(U_\nu^0)' = U(\mathfrak{g}'_0) / U(\mathfrak{g}'_0) \mathrm{Ker}\, \chi_\nu^0$. One can prove repeating the above arguments that $\bar{U}'_\nu$ is isomorphic to $(U_\nu^0)' \otimes \mathrm{End}_{\mathbb{C}}(U(\mathfrak{g}_{-1}))$. Now theorem follows from the obvious isomorphism $(U_\nu^0)' \cong U_\mu^0$. □

## 6. Geometric realization of $\mathrm{Gr}\, \bar{U}$

As in classical case $\mathrm{Gr}\, U \cong S(\mathfrak{g})$ is a supercommutative algebra with Poisson bracket. It can be realized as the algebra of functions on $\mathfrak{g}^*$. Since $\mathrm{Gr}\, I$ is a Poisson ideal, the graded algebra $\mathrm{Gr}\, \bar{U}$ is also an algebra with Poisson bracket. We realize it as an algebra of functions on some Poisson supervariety. The reader can find all necessary details on supergeometry in [5] and [6]. For an affine supervariety $V$ we define by $\mathcal{O}(V)$ the ring of regular functions on $V$.

Consider $\mathfrak{g}^*$ as a supervariety with the natural action of the supergroup $G$. Its underlying variety is $\mathfrak{g}_0^* \subset \mathfrak{g}^*$. We define $Q$ as $G\mathfrak{g}_0^*$. We will show that $Q$ is a closed supervariety with singularities.

Recall that one can identify $\mathfrak{g}_0$ with $\mathfrak{g}_0^*$, and we denote by $R_0$ the open subset of all regular elements of $\mathfrak{g}_0^*$. By $R$ we denote the virtual supervariety $GR_0$.

**Lemma 6.1.** *Let $x \in \mathfrak{g}_0^*$ and $\mathfrak{g}_x$ be the Lie superalgebra of the stabilizer of $x$. Then $\dim(\mathfrak{g}_x \cap \mathfrak{g}_1) \geq n$. If $x \in R_0$ then $\dim(\mathfrak{g}_x \cap \mathfrak{g}_1) = n$ and $\mathfrak{g}_x \cap \mathfrak{g}_1 = \{0\}$. If $x \notin R_0$, then $\dim(\mathfrak{g}_x \cap \mathfrak{g}_1) > n$ and $\dim(\mathfrak{g}_x \cap \mathfrak{g}_{-1}) > 0$.*

*Proof.* We realize $x \in \mathfrak{g}_0^*$ by an $n \times n$ matrix, $a \in \mathfrak{g}_{-1}$ by a skewsymmetric matrix and $b \in \mathfrak{g}_1$ by a symmetric matrix. Then

$$\mathfrak{g}_x \cap \mathfrak{g}_{-1} = \left\{a \in \mathfrak{g}_{-1} \mid ax - x^t a = 0\right\}$$

and in the same way

$$\mathfrak{g}_x \cap \mathfrak{g}_1 = \left\{b \in \mathfrak{g}_1 \mid xb - bx^t = 0\right\}.$$

Since $b$ is symmetric and $xb - bx^t$ is skewsymmetric,

$$\dim \mathfrak{g}_x \cap \mathfrak{g}_1 \geq \dim \mathfrak{g}_1 - \dim \mathfrak{g}_{-1} = n.$$

That proves the first statement.

To check two other statements note that $\dim \mathfrak{g}_x \cap \mathfrak{g}_{-1}$ and $\dim \mathfrak{g}_x \cap \mathfrak{g}_1$ are constant on the conjugacy class of $x$. Therefore it is sufficient to prove the statements for a diagonal $x$ and for a nilpotent $x$ which can be easily done by direct calculations. $\square$

By Lemma 6.1 one should expect that $\dim Q = (n^2, n^2 - n)$ and that the singularities of $Q$ are at all non-regular elements. We will construct a resolution of $Q$.

Let $\mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_1$, $P$ be the subgroup with the Lie superalgebra $\mathfrak{p}$, and $K = G/P$. Then $K$ is an algebraic supervariety of dimension $(0|d)$. Therefore $K$ is isomorphic to $\mathfrak{g}_{-1}$ and $\mathcal{O}(K) \cong S(\mathfrak{g}^*_{-1})$. Let $V$ be the linear bundle on $K$ defined as $V = G \times_P (\mathfrak{g}/[\mathfrak{p},\mathfrak{p}])^*$. Then the cotangent bundle $\mathcal{T}^* \cong G \times_P (\mathfrak{g}/\mathfrak{p})^*$ is a subbundle of $V$. The underlying variety $V_0$ is isomorphic to $\mathfrak{g}_0^*$, and since it is affine, the supervariety $V$ splits. The embeddings $\mathcal{T}^* \to V$ and $\mathfrak{g}_0^* \cong V_0 \to V$ induce the isomorphism

$$V \cong \mathcal{T}^* \times \mathfrak{g}_0^*. \tag{6.1}$$

Now we define the map $r : V \to \mathfrak{g}^*$ in the obvious way. A point of $V$ is a pair $(\mathfrak{p}_y, \phi)$ where $\mathfrak{p}_y$ is a parabolic subalgebra conjugate to $\mathfrak{p}$ and $\phi$ is a linear functional on $\mathfrak{g}$ annihilating $[\mathfrak{p}_y, \mathfrak{p}_y]$. We put $r(\mathfrak{p}_y, \phi) = \phi$.

**Lemma 6.2.** *(a) $r$ is a $G$-equivariant map;*
*(b) The image of $r$ coincides with $Q$;*
*(c) $r$ is a rational isomorphism between $r^{-1}(R)$ and $R$.*

*Proof.* (a) is obvious from construction of $r$. It is also obvious that $r$ induces the identity map $\mathfrak{g}_0^* \to \mathfrak{g}_0^*$ of the underlying subvarieties. Therefore $(b)$ follows from (a). To check $(c)$ one calculates easily that $d_x r$ has a maximal rank for any regular point $x \in R_0$. $\square$

**Corollary 6.3.** *$V$ is a resolution of $Q$.*

The last corollary allows one to describe the structure sheaf $\mathcal{O}_Q$ as a subsheaf of $\mathcal{O}_V$. Indeed by (6.1)

$$\mathcal{O}_V = S(\mathfrak{g}^*_{-1}) \otimes S(\mathfrak{g}_{-1}) \otimes \mathcal{O}_{\mathfrak{g}_0^*}.$$

Identify $K$ with $\mathfrak{g}_{-1}$, $\mathcal{T}^*$ with $\mathfrak{g}_{-1} \times \mathfrak{g}^*_{-1}$, and $V$ with $\mathfrak{g}_{-1} \times \mathfrak{g}^*_{-1} \times \mathfrak{g}_0^*$. Then for any $x \in \mathfrak{g}_0^*$

$$r^{-1}(x) = (\mathfrak{g}_{-1} \cap \mathfrak{g}_x) \times \{0\} \times \{x\}.$$

For any $x \in \mathfrak{g}_0$ define

$$\zeta_x = \left\{\phi \in \mathfrak{g}^*_{-1} \mid \phi(\mathfrak{g}_{-1} \cap \mathfrak{g}_x) = 0\right\}.$$

By Lemma 6.1 $\zeta_x = \mathfrak{g}^*_{-1}$ if and only if $x \in R_0$. All functions lifted from $Q$ must be constant on the fibers of $r^{-1}(x)$. Therefore

$$\mathcal{O}_Q = \left\{f \in \mathcal{O}_V \mid f(x) \in S(\mathfrak{g}_{-1}) \oplus S(\mathfrak{g}^*_{-1})\,\zeta_x \otimes S(\mathfrak{g}_{-1}) \text{ for all } x \in \mathfrak{g}_0\right\}.$$

In particular, $\mathcal{O}_V(R_0) \cong \mathcal{O}_Q(R_0)$.

Now we are ready to prove the following theorem.

**Theorem 6.4.** *The Poisson superalgebras* $\operatorname{Gr}\bar{U}$ *and* $\mathcal{O}(Q)$ *are isomorphic.*

*Proof.* The restriction homomorphism $\xi\colon \operatorname{Gr} U \to \mathcal{O}(Q)$ is surjective. To calculate $\operatorname{Ker}\xi$ we are going to construct a homomorphism $\gamma : \operatorname{Gr} U \to \mathcal{O}(V)$ such that $\gamma = r^* \circ \xi$.

Let $\mathcal{D}$ be the algebra of differential operators on $K$ and $\bar{\mathcal{D}} = \mathcal{D} \otimes U(\mathfrak{g}_0)$. Since $\mathcal{O}(K) = S(\mathfrak{g}^*_{-1})$, $\mathcal{D} \cong \operatorname{End}(S(\mathfrak{g}^*_{-1}))$ and $\bar{\mathcal{D}} \cong \operatorname{End}(S(\mathfrak{g}^*_{-1})) \otimes U(\mathfrak{g}_0)$. Define the following $U$-module

$$\mathcal{F} = \operatorname{Hom}_{U(\mathfrak{g}_0+\mathfrak{g}_1)}(U, U(\mathfrak{g}_0)),$$

where $U(\mathfrak{g}_0)$ acts on $U(\mathfrak{g}_0)$ by left multiplications and $\mathfrak{g}_1$ acts on $U(\mathfrak{g}_0)$ trivially. Note that $\mathcal{F} \cong U(\mathfrak{g}^*_{-1}) \otimes U(\mathfrak{g}_0) \cong S(\mathfrak{g}^*_{-1}) \otimes U(\mathfrak{g}_0)$. Therefore one can define the natural injective homomorphism $i\colon \bar{\mathcal{D}} \to \operatorname{End}(\mathcal{F})$. Let $j\colon U \to \operatorname{End}(\mathcal{F})$ be the homomorphism induced by the representation of $U$. Then $j(U) \subseteq i(\bar{\mathcal{D}})$, and therefore the homomorphism $\bar{\gamma} : U \to \bar{\mathcal{D}}$ given by $\bar{\gamma} = i^{-1} \circ j$ is well defined.

**Lemma 6.5.** $\operatorname{Ker}\bar{\gamma} = I$.

*Proof.* Let $\mathcal{F}' = \operatorname{Hom}_{\bar{U}^+}(\bar{U}, U(\mathfrak{g}_0))$. Generally speaking $\mathcal{F}'$ is a submodule of $\mathcal{F}$. But since $U(\mathfrak{g}_0)$ and $U(\mathfrak{g}_{-1})$ are both subalgebras of $\bar{U}$, we obtain

$$\mathcal{F}' \cong U(\mathfrak{g}^*_{-1}) \otimes U(\mathfrak{g}_0) \cong S(\mathfrak{g}^*_{-1}) \otimes U(\mathfrak{g}_0) \cong \mathcal{F}.$$

Therefore $\mathcal{F}$ is well defined $\bar{U}$-module which implies that $I \subseteq \operatorname{Ann}_U \mathcal{F}$.

On the other hand, $\mathcal{F} \cong \operatorname{Ind} U(\mathfrak{g}_0) \otimes C_{-\sigma}$, where $C_{-\sigma}$ is the one-dimensional $\mathfrak{g}$-module of weight $-\sigma$. Hence $\mathcal{F}$ is a faithful $\bar{U}$-module, which implies $\operatorname{Ann}_U \mathcal{F} = I$. Since $\operatorname{Ker}\bar{\gamma} = \operatorname{Ann}_U \mathcal{F}$, we obtain the lemma. $\square$

The algebra $\bar{\mathcal{D}}$ has the filtration induced by the natural filtration of $U(\mathfrak{g}_0)$ and the natural filtration of $\mathcal{D}$. By (6.1) $\operatorname{Gr}\bar{\mathcal{D}} = \mathcal{O}(V)$. It is clear that $\bar{\gamma}$ preserves the filtration. Therefore $\bar{\gamma}$ induces the homomorphism $\gamma : \operatorname{Gr} U \to \mathcal{O}(V)$ of graded algebras.

**Lemma 6.6.** $\gamma = r^* \circ \xi$.

*Proof.* The homomorphism $\gamma$ induces the morphism $\theta : V \to \mathfrak{g}^*$ of affine supervarieties such that $\theta^* = \gamma$. We have to check that $\theta = r$. Let $V_0 \subset V$ be the underlying subvariety canonically embedded in $V$. Recall that $V_0$ is isomorphic to $\mathfrak{g}_0^*$. One can see from the definitions of $r$ and $\theta$ that $r(V_0) = \theta(V_0) = \mathfrak{g}_0^* \subset \mathfrak{g}^*$. Furthermore, $r$ and $\theta$ coincide on $V_0$. Note that $\theta$ is $G$-equivariant as well as $r$. Since $V = GV_0$, we obtain that $\theta = r$. $\square$

Since $r^*$ is injective, $\operatorname{Ker}\xi = \operatorname{Ker}\gamma = \operatorname{Gr} I$. Therefore $\operatorname{Gr}\bar{U}$ is isomorphic to $\mathcal{O}(Q)$. $\square$

## References

1. M. Gorelik, *The center of a simple p-type lie superalgebra*, Journal of Algebra.
2. A. Joseph, *Kostant's problem, Goldie rank and the Gelfand-Kirillov conjecture*, Inventiones Mathematicae **56** (1980), no. 3, 191–213.
3. V. G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.
4. E. Kirkman and J. Kuzmanovich, *Minimal prime ideals in enveloping algebras of lie superalgebras*, Proceedings of American Mathematical Society **124** (1996), 1693–1702.
5. Yu. Manin, *Gauge field theory and complex geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 289, Springer-Verlag, Berlin, 1988, Translated from the Russian by N. Koblitz and J. R. King.
6. I. Penkov, *Borel-Weil-Bott theory for classical lie supergroups*, Contemporary Problems in Mathematics, vol. 32, VINITI, Moscow, 1988.

Dept. of Mathematics, University of California at Berkeley, Berkeley, CA 94720
*E-mail address*: `serganov@math.berkeley.edu`