On Anosov Energy Levels of Convex Hamiltonian Systems

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On Anosov Energy Levels of Convex Hamiltonian Systems*

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Abstract

Let $H : T^*M \to \mathbf{R}$ be an convex Hamiltonian. We show that Anosov energy levels of H can only arise at high energy levels, not containing conjugate points.

1 *Introduction*

Let M be a manifold without boundary, T^*M its cotangent bundle, $\pi : T^*M \to M$ the canonical projection and, if $\theta \in T^*M$, let $V(\theta) \subset T_\theta T^*M$ be the vertical fibre at θ, defined as usual as the kernel of $d\pi_\theta : T_\theta T^*M \to T_{\pi(\theta)}M$.

Let H be a Hamiltonian on T^*M, let $J\nabla H$ be its symplectic gradient and let ϕ_t denote its associated Hamiltonian flow. It is well known that ϕ_t preserves the canonical symplectic form of T^*M and leaves all the level sets $\Sigma_\sigma \stackrel{\text{def}}{=} H^{-1}(\sigma)$ invariant. We shall always assume that Σ_σ is connected and that $\phi_t|_{\Sigma_\sigma}$ is complete.

Recall that a Hamiltonian $H : T^*M \to \mathbf{R}$ is said to be *convex* if for each $q \in M$ the function $H(q, .)$ regarded as a function on the linear space T_q^*M has positive definite Hessian.

Our subject will be the *Anosov energy levels* of H, with H convex, i.e. regular values σ of H such that Σ_σ is compact and the flow $\phi_t|_{\Sigma_\sigma}$ is an Anosov flow. Our main result (in fact a corollary of a more general result to be stated below) is that **Anosov energy levels can only arise at high energy levels, not containing conjugate points.** More precisely:

Theorem 1.1 *If σ is an Anosov energy level, then:*
(a) $\pi(\Sigma_\sigma) = M$. In particular M is compact.
(b) Σ_σ does not contain conjugate points.

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Conjugate points, means, as usual, pair of points $\theta_1 \neq \theta_2 = \phi_t(\theta_1)$ such that $d\phi_t(V(\theta_1))$ intersects $V(\theta_2)$ non-trivially.

Suppose that for each $q \in M$, $\frac{\partial H}{\partial p}(q,0) = 0$, i.e. the Hamiltonian is "centered", then Property (a) implies $\sigma > \max_{q\in M} H(q,0)$ and it makes possible to see the dynamics of $\phi_t |_{\Sigma_\sigma}$ as the geodesic flow on the unit sphere bundle of an appropriatedly chosen Finsler structure on M. In particular if H is of the form kinetic energy plus a potential, then an Anosov energy level of H is equivalent to an Anosov geodesic flow of the associated (non-degenerate) Maupertuis metric corresponding to the fixed energy level.

We also note that for surfaces of genus > 1, given any Riemannian metric -since any metric is conformally equivalent to a metric of constant curvature -1- it is possible to add a potential to it so that we obtain Anosov energy levels.

For Hamiltonians arising from Riemannian metrics, Theorem 1.1 was proved by Klingenberg [6]. For a different proof and exposition of Klingenberg's result that also covers the non-compact case see [8].

Theorem 1.1 will be obtained as a corollary of a more general result (Theorem 1.2 below), where the Anosov hypothesis is replaced by a much weaker one, namely the existence of a continuous invariant Lagrangian subbundle, i.e. a continuous subbundle E of $T(T^*M)|_{\Sigma_\sigma}$ such that for all $\theta \in \Sigma_\sigma$ the fibre $E(\theta)$ is a Lagrangian subspace of $T_\theta T^*M$ and $E(\phi_t(\theta)) = d\phi_t(E(\theta))$ for all $t \in \mathbf{R}$. It is well known that under the Anosov hypothesis both the stable and unstable subbundles are Lagrangian. We will also see (cf. Remark 2.6) that Theorem 1.2 extends to the general symplectic setting, i.e. T^*M is replaced by an arbitrary symplectic manifold, (N^{2n}, ω) with ω^{n-1} exact and H is an optical Hamiltonian [2] respect to a fixed Lagrangian distribution.

In our last theorem (Theorem 1.3 below) we shall prove the same result dropping the compactness hypothesis but adding two assumptions: the symmetry of the Hamiltonian (i.e. the invariance of H under the involution $(q,p) \to (q,-p)$) and that every point in Σ_σ is non-wandering. We do not know whether the result still holds without the symmetry assumption. The recurrence hypothesis is necessary as we shall see through an example.

Theorem 1.2 *Suppose that σ is a regular value of H, that Σ_σ is compact and that $\phi_t |_{\Sigma_\sigma}$ admits a continuous invariant Lagrangian subbundle E. Then,*
(a) $E(\theta) \cap V(\theta) = \{0\}$ $\forall \theta \in \Sigma_\sigma$.
(b) $\pi(\Sigma_\sigma) = M$. In particular M is compact.
(c) Σ_σ contains no conjugate points.

Theorem 1.3 *Suppose that σ is a regular value of H, that every point in Σ_σ is non-wandering, that H is symmetric and that $\phi_t |_{\Sigma_\sigma}$ admits a continuous invariant Lagrangian subbundle. Then,*
(a) $E(\theta) \cap V(\theta) = \{0\}$ $\forall \theta \in \Sigma_\sigma$.
(b) $\pi(\Sigma_\sigma) = M$.
(c) Σ_σ contains no conjugate points.

As we mentioned before the recurrence hypothesis is necessary, even in the geodesic flow case. Consider the paraboloid of revolution $z = x^2+y^2$. The obvious circle action together with the field of the geodesic flow span a continuous invariant Lagrangian subbundle but there are conjugate points.

Let us remark that our results also include the results of A. Knauf in [7], who uses ideas similar to ours in the proof of Theorem 1.3.[1]

In [10] the second author showed that if the geodesic flow on a compact surface M is expansive, then there are no conjugate points. Recall that a flow $\varphi_t : X \to X$ on a compact metric space (X, d) is said to be *expansive* if given $\epsilon > 0$ there exists $\delta > 0$ such that if there is an homeomorphism $\tau : \mathbf{R} \to \mathbf{R}$, $\tau(0) = 0$, such that

$$d(\phi_{\tau(t)}(y), \phi_t(x)) < \delta,$$

for all $t \in \mathbf{R}$, then $y = \phi_{\bar{t}}(x)$ where $| \bar{t} | < \epsilon$. Anosov flows are expansive flows. Results analogous to Theorem 1.1 also hold for convex expansive Hamiltonians with two degrees of freedom [9].

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2 *Proof of Theorem 1.2*

The Theorem will be a consequence of Propositions 2.1, 2.3 and 2.4 below.

Let ϕ_t be a flow on a compact manifold W. We will say that $C \subset W$ is a *codimension one transversal cycle*, if C is a closed connected smooth stratified submanifold (in the sense of Whitney [5]), such that ϕ_t is transversal to C at every point, the top dimensional strata has codimension one in W, and all the other stratas have codimension ≥ 3 in W.

Proposition 2.1 *Let H be a Hamiltonian on the symplectic manifold (N^{2n}, ω) with ω^{n-1} exact. Then the Hamiltonian flow of H restricted to a compact regular energy level has no codimension one transversal cycle.*

Remark 2.2 We observe that Schwartzmann [11] showed that Hamiltonian flows on a symplectic manifold N have no smooth cross sections on N. However they may have them on energy levels (take a symplectic suspension); nevertheless if ω^{n-1} is exact this cannot happen either as the proposition shows. Our arguments are a variation of Schwartzmann's arguments and we include them for the sake of completeness. Note that in the case of (T^*M, ω_0) where ω_0 is the standard symplectic form, ω_0^{n-1} is always exact. Moreover if (T^*M, ω) is a *twisted cotangent bundle*, i.e. $\omega = \omega_0 + \pi^*\eta$, where $\eta \in H^2(M, \mathbf{R})$, then ω^{n-1} is exact, provided $n \geq 3$.

[1] We thank the referee for calling our atention to Knauf's paper.

Let Σ_σ be a compact regular energy level. Let Λ denote the Lagrangian-Grassmann bundle over Σ_σ, i.e. for each $\theta \in \Sigma_\sigma$, the fibre $\Lambda(\theta)$ consists of all Lagrangian subspaces of $T_\theta T^*M$. Let Λ_k denote the subbundle whose fibre at θ is the subset of $\Lambda(\theta)$ given by all Lagrangian subspaces that have intersection with $V(\theta)$ of dimension k. Now set $\Lambda_V = \cup_{k\geq 1}\Lambda_k$. Then Λ_V is a closed stratified submanifold where each strata Λ_k has codimension $\frac{k(k+1)}{2}$ [1, 2, 4].

Consider now the induced flow, $\phi_t^* : \Lambda \to \Lambda$, given by $\phi_t^*(\theta, E) = (\phi_t(\theta), d\phi_t(E))$. Let $\xi(\theta, E) \stackrel{\rm def}{=} \frac{d}{dt} |_{t=0} \phi_t^*(\theta, E)$. The following Proposition can be found in [2, Propositions 1.6, 1.8, 2.2] and [4].

Proposition 2.3 *Assume $H : T^*M \to \mathbf{R}$ is convex. The vector field ξ is transversal to Λ_V and moreover, there exists a number $r > 0$ so that for any $(\theta, E) \in \Lambda_V$ we have,*

$$dist(\xi(\theta, E), T_{(\theta,E)}\Lambda_V) > r,$$

where dist is any distance function compatible with the topology of $T\Lambda$.

Proposition 2.4 *Assume $H : T^*M \to \mathbf{R}$ is convex and let $E \subset T_\theta T^*M$ be a Lagrangian subspace. If $d\phi_t(E) \cap V(\phi_t(\theta)) = \{0\}$ for $t \in [0, a]$, then the segment $\{\phi_t(\theta),\ t \in [0, a]\}$ does not have conjugate points, i.e. the set of $t \in [0, a]$ such that $d\phi_t(x) \in V(\phi_t(\theta))$ consists at most of one point for all non-zero $x \in T_\theta T^*M$.*

Proof of Theorem 1.2:

We explain now how to derive Theorem 1.2 from Propositions 2.1, 2.3 and 2.4. First we observe:

Remark 2.5 Suppose that there exists a continuous invariant Lagrangian subbundle E of $T(T^*M)|_{\Sigma_\sigma}$. Then, there exists another one $\bar{E}$, satisfying $J\nabla H(\theta) \in \bar{E}(\theta)$, for all $\theta \in \Sigma_\sigma$. Just set $\bar{E}(\theta) = \{J\nabla H(\theta)\} + (E(\theta) \cap T_\theta\Sigma_\sigma)$. Thus from now on we shall assume without loss of generality that $J\nabla H(\theta) \in E(\theta)$.

We claim that to prove Theorem 1.2, it suffices to show Property (a), that is, for all $\theta \in \Sigma_\sigma$, $E(\theta) \cap V(\theta) = \{0\}$.

Proof of the claim: To show that $\pi(\Sigma_\sigma) = M$ observe that if for all $\theta \in \Sigma_\sigma$, $E(\theta) \cap V(\theta) = \{0\}$, then $\pi|_{\Sigma_\sigma}$ is a submersion. Indeed, when $d\pi_\theta|_{T_\theta\Sigma_\sigma}$ is not surjective, $J\nabla H(\theta) \in V(\theta)$ and then $E(\theta) \cap V(\theta) \supset \{J\nabla H(\theta)\} \neq \{0\}$. Then since $\pi(\Sigma_\sigma)$ is compact, it is open and closed and thus $\pi(\Sigma_\sigma) = M$. Also if for all $\theta \in \Sigma_\sigma$, $E(\theta) \cap V(\theta) = \{0\}$, then by Proposition 2.4, Σ_σ does not contain conjugate points concluding the proof of the claim.

Now note that a continuous Lagrangian subbundle E is a continuous section, $E : \Sigma_\sigma \to \Lambda$, of the bundle, $\Lambda \to \Sigma_\sigma$. Suppose now that E is invariant. Thus we can uniformly approximate E by a C^∞-section $\hat{E}$, so that the maps

$$\theta \to \frac{d}{dt}|_{t=0} \hat{E}(\phi_t\theta),$$

$$\theta \to \xi(\theta, \hat{E}(\theta)),$$

are also uniformly close (note that $\frac{d}{dt}\mid_{t=0} E(\phi_t\theta) = \xi(\theta, E(\theta))$). By the Transversality Theorem ([5, p. 38]) we can choose $\hat{E}$, so that $\hat{E}$ is transversal to Λ_V. On the other hand, if $\hat{E}$ is sufficiently close to E in the way described above, it follows from Proposition 2.3 that there exist a number $r' > 0$ so that, if $(\theta, \hat{E}(\theta)) \in \Lambda_V$, then

$$dist(\frac{d}{dt}\mid_{t=0} \hat{E}(\phi_t\theta), T_{(\theta,\hat{E}(\theta))}\Lambda_V) > r'. \tag{1}$$

Set $K := \{\theta \in \Sigma_\sigma : E(\theta) \cap V(\theta) \neq \{0\}\} = E^{-1}(\Lambda_V)$. Suppose that K is not empty, then if $\hat{E}$ is sufficiently close to E, the set $\hat{E}^{-1}(\Lambda_V)$ is also not empty. Indeed, if $\hat{E}$ is sufficiently close to E, it follows that E and $\hat{E}$ are homotopic. We will see in Proposition 3.3, that if K is not empty, there exists a closed curve, $\alpha : S^1 \to \Sigma_\sigma$, so that $E \circ \alpha : S^1 \to \Lambda$ has positive intersection number with Λ_V (i.e. positive Maslov index). It follows then, that $\hat{E}(\Sigma_\sigma)$ must intersect Λ_V otherwise the intersection number would be zero.

Next note that if $C = \hat{E}^{-1}(\Lambda_V)$ is not empty, it is a closed stratified submanifold of codimension one with low dimensional stratas of codimension ≥ 3. Moreover on account of equation (1), ϕ_t is transversal to C. Then if the set K is not empty, there exists a codimension one transversal cycle contradicting Proposition 2.1.

◇

Remark 2.6 Let (N^{2n}, ω) be a symplectic manifold with ω^{n-1} exact. We endow N with a Lagrangian distribution α, i.e. a section of the Lagrangian-Grassmann bundle over N. Suppose now H is an *optical* Hamiltonian on N respect to α [2] (optical Hamiltonians naturally generalize convex Hamiltonians on cotangent bundles). Then if on some compact regular energy level of H, the Hamiltonian flow of H, admits a continuous invariant Lagrangian subbundle E, then E and α must be transversal. The proof is exactly as the proof of Theorem 1.2 above, since Proposition 2.3 also hold in this general setting. We also note that if H is optical, proper and bounded from below, then necessarily ω^{n-1} is exact for $n \geq 3$. Indeed, since optical Hamiltonians are open, and the index of a non-degenerate critical point of an optical Hamiltonian does not exceed n [3], it follows from Morse theory that N has the homotopy type of a CW-complex with dimension $\leq n$. Thus $H^{2n-2}(M, \mathbf{R}) = 0$ for $n \geq 3$ and therefore any closed $2n-2$-form is exact.

Next, we prove Propositions 2.1 and 2.4.

Proof of Proposition 2.1: We first show:

Lemma 2.7 *If X is a C^1 vector field on a compact manifold W, and Ω is a volume form such that X is exact with respect to Ω (i.e. $i_X\Omega$ is exact), then X has no codimension one transversal cycles.*

Proof: If X had a codimension one transversal cycle C, then

$$\int_C i_X\Omega \neq 0,$$

but $i_X\Omega = d\eta$, hence

$$0 \neq \int_C i_X\Omega = \int_{\partial C} \eta = 0,$$

since C is a cycle. This contradiction proves the lemma.

◇

We complete now the proof of Proposition 2.1. The exactness hypothesis implies $\omega^{n-1} = d\lambda$, where λ is a $2n-3$-form. Take a vector field Y so that $\omega_x(Y(x), J\nabla H(x)) = 1$ for all $x \in H^{-1}(\sigma)$. Then denoting $X = J\nabla H$, we have

$$i_X i_Y(\underbrace{\omega \wedge \ldots \wedge \omega}_{n}) \mid_{H^{-1}(\sigma)} = \underbrace{\omega \wedge \ldots \wedge \omega}_{n-1}.$$

Hence defining the volume form $\Omega := i_Y(\underbrace{\omega \wedge \ldots \wedge \omega}_{n})$, it follows that

$$i_X\Omega = \underbrace{\omega \wedge \ldots \wedge \omega}_{n-1} = d\lambda.$$

Hence X is an exact vector field on $H^{-1}(\sigma)$ with respect to the volume form Ω. By the previous lemma $J\nabla H \mid_{H^{-1}(\sigma)}$ has no codimension one transversal cycle.

◇

Before proving Proposition 2.4 we recall briefly the linear equation and the Riccati equation associated with H.

Fix a Riemannian metric on M, then $T_\theta T^*M$ splits as a direct sum of two Lagrangian subspaces: the vertical subspace $V(\theta)$ and the horizontal subspace $H(\theta)$ given by the kernel of the connection map. Both subspaces can be canonically identified with $T_{\pi(\theta)}M$. Let $E \subset T_\theta T^*M$ be a subspace of dimension n and with the property $E \cap V(\theta) = \{0\}$. Then E is the graph of some linear map $S : H(\theta) \to V(\theta)$. It can be easily checked that E is Lagrangian if and only if S is symmetric with respect to the inner product given by the Riemmanian metric.

Take $\theta \in T^*M$ and $x = (h, v) \in T_\theta T^*M = H(\theta) \oplus V(\theta) = T_{\pi(\theta)}M \oplus T_{\pi(\theta)}M$. Now consider a variation

$$\alpha_s(t) = (q_s(t), p_s(t))$$

such that for each $s \in (-\epsilon, \epsilon)$, α_s is a solution of the Hamiltonian H such that $\alpha_0(0) = \theta$ and $\frac{d}{ds}\mid_{s=0} \alpha_s(0) = x$. Then if we write $(h(t), v(t)) = d\phi_t(v)$ we have that h and v verify the following linear equation

$$\dot{h} = H_{qp}h + H_{pp}v,$$

$$\dot{v} = -H_{qq}h - H_{pq}v, \tag{2}$$

where covariant derivatives are evaluated along $\pi(\alpha_0(t))$, and H_{pp}, H_{pq}, H_{qq} and H_{qp} are linear operators on $T_{\pi(\theta)}M$, that in local coordinates coincide with the matrices of partial derivatives $(\frac{\partial^2 H}{\partial p_i \partial p_j})$, $(\frac{\partial^2 H}{\partial p_i \partial q_j})$, $(\frac{\partial^2 H}{\partial q_i \partial q_j})$ and $(\frac{\partial^2 H}{\partial q_i \partial p_j})$. Moreover H_{pp} is positive definite if H is convex.

Next we will derive the Riccati equation. Let E be a Lagrangian subspace of $T_\theta T^*M$. Suppose for t in some interval $(-\epsilon, \epsilon)$, $d\phi_t(E) \cap V(\phi_t(\theta)) = \{0\}$. Then we can write $d\phi_t(E) = graph\ S(t)$, where $S(t) : H(\phi_t(\theta)) \to V(\phi_t(\theta))$ is a symmetric map. That is, if $x \in E$ then

$$d\phi_t(x) = (h(t), S(t)h(t)).$$

By means of the equation (2) we obtain:

$$\dot{S}h + S(H_{qp}h + H_{pp}Sh) = -H_{qq}h - H_{pq}Sh.$$

Since this works for every $x \in E$ we obtain the Riccati equation:

$$\dot{S} + SH_{pp}S + SH_{qp} + H_{pq}S + H_{qq} = 0. \tag{3}$$

Proof of Proposition 2.4:

Take the symmetric map $S(t)$ that gives $d\phi_t(E)$ as a graph and let $(h(t), v(t))$ represent $d\phi_t(x)$ $(x \in T_\theta T^*M)$. Suppose $h(c) = 0$ for some $c \in [0, a]$. Suffices to show that $h(t) \neq 0$ for all $t \in [0, a]$ different from c. Consider $Y(t)$ a family of linear isomorphisms satisfying

$$\dot{Y} = (H_{qp} + H_{pp}S)Y,$$

$$Y(0) = id.$$

If we take ω such that $(\omega, S(0)\omega) \in E$ and define

$$h_1(t) = Y(t)\omega,$$

$$v_1(t) = S(t)Y(t)\omega,$$

we get -using equation (3)- that (h_1, v_1) is a solution of the equation (2). Since $d\phi_t$ preserves the symplectic form we get

$$< h(t), v_1(t) > - < v(t), h_1(t) >= - < v(c), Y(c)\omega >$$

and hence

$$< Y^*(t)S(t)h(t), \omega > - < Y^*(t)v(t), \omega >= - < Y(c)^*v(c), \omega > .$$

Therefore

$$v(t) = S(t)h(t) + (Y^*)^{-1}(t)Y(c)^*v(c).$$

Since

$$\dot{h} = H_{qp}h + H_{pp}v,$$

we get

$$\dot{h} = (H_{qp} + H_{pp}S)h + H_{pp}(Y^*)^{-1}Y(c)^*v(c)$$

and hence

$$h(t) = Y(t)Y^{-1}(c)h(c) + Y(t)\int_c^t Y^{-1}(u)H_{pp}(Y^*)^{-1}(u)Y(c)^*v(c)du.$$

Since $h(c) = 0$ we obtain

$$< Y^{-1}(t)h(t), Y(c)^*v(c) >= \int_c^t < H_{pp}(Y^*)^{-1}(u)Y(c)^*v(c), (Y^*)^{-1}(u)Y(c)^*v(c) > du.$$

Then the convexity of H implies that $h(t) \neq 0$ for all $t \in [0, a]$ different from c and hence there are no conjugate points along the segment.

◇

3 *Proof of Theorem 1.3*

Let $H : T^*M \to \mathbf{R}$ be a convex Hamiltonian and let ϕ_t denote the flow associated to H. Let $X \subset T^*M$ be an arbitrary connected ϕ_t-invariant submanifold. Suppose a continuous invariant Langrangian subbundle E is given on X, i.e. for each $\theta \in X$, $E(\theta)$ is Lagrangian, $\theta \to E(\theta)$ is continuous and $E(\phi_t(\theta)) = d\phi_t(E(\theta))$.

For each continuous closed curve $\alpha : [0, T] \to X$ we can define the Maslov index of α, $\mu(\alpha)$, as the Maslov index of the curve $t \to (\alpha(t), E(\alpha(t)))$ in the Lagrangian-Grassmann bundle ([1, 2, 4, 8]). This index defines in turn an element $\mu \in H^1(X, \mathbf{Z})$ called the Maslov class of the pair (X, E).

We will say that a continuous closed curve $\alpha : [0, T] \to X$ is a *pseudo-orbit* of the Hamiltonian flow ϕ_t if for all $t_0 \in [0, T]$ where

$$E(\alpha(t_0)) \cap V(\alpha(t_0)) \neq \{0\},$$

there exists $\epsilon > 0$ such that

$$\alpha(t + t_0) = \phi_t(\alpha(t_0)),$$

for $t \in (-\epsilon, \epsilon)$ (for $t_0 = 0$ or $t_0 = T$ we take the continuous periodic extension of α to the real line).

The following lemma is an important consequence of the convexity of H and can be found in [2, 4, 8].

Lemma 3.1 *If $\alpha : [0,T] \to X$ is a closed pseudo-orbit, then $\mu(\alpha) \geq 0$. Moreover if there exists some $t_0 \in [0,T]$ for which $E(\alpha(t_0)) \cap V(\alpha(t_0)) \neq \{0\}$, then $\mu(\alpha) > 0$.*

Definition 3.2 Suppose X is invariant under the involution $(q,p) \to (q,-p)$. For a curve $\alpha : [0,T] \to X$, $\alpha(t) = (q(t), p(t))$, let $\bar{\alpha} : [0,T] \to X$ be the curve $\bar{\alpha}(t) = (q(T-t), -p(T-t))$.

Let $\Omega \subseteq X$ denote the set of non-wandering points of ϕ_t in X.

Proposition 3.3 *Suppose there exists $\theta \in \Omega$ so that $E(\theta) \cap V(\theta) \neq \{0\}$. Then there exists a closed pseudo-orbit $\alpha : [0,T] \to X$ so that $\alpha(0) = \theta$ and thus with $\mu(\alpha) > 0$ by Lemma 3.1. If in addition H is symmetric and X is invariant under the involution $(q,p) \to (q,-p)$, then α can be chosen so that $\bar{\alpha}$ is also a pseudo-orbit.*

Proof: (cf. also [8]) By Proposition 2.3 there exists $\epsilon > 0$ so that for all $t \in (-2\epsilon, 2\epsilon)$ and $t \neq 0$ we have $d\phi_t(E(\theta)) \cap V(\phi_t(\theta)) = \{0\}$ and the arc of trajectory $\phi_t(\theta)$, $t \in (-2\epsilon, 2\epsilon)$ is in a flow box. Let U_1 and U_2 be neighborhoods in X of $\phi_\epsilon(\theta)$ and $\phi_{-\epsilon}(\theta)$ respectively so that E does not touch the vertical non-trivially for any point in them. Since $\theta \in \Omega$ there exists $s > 0$ and $\eta \in U_1$ so that $\phi_s(\eta) \in U_2$. Connect now $\phi_\epsilon(\theta)$ with η by a path γ_1 contained in U_1, and $\phi_s(\eta)$ with $\phi_{-\epsilon}(\theta)$ by a path γ_2 contained in U_2.

Consider the closed curve α obtained by joining the following sequence of paths: $\{\phi_t(\theta) : t \in [-\epsilon, \epsilon]\}$, γ_1, $\{\phi_t(\eta) : t \in [0, s]\}$, γ_2. Clearly α is a pseudo-orbit through θ as desired.

If H is symmetric and X is invariant under the involution $(q,p) \to (q,-p)$, then if β is an orbit of ϕ_t, $\bar{\beta}$ is also an orbit. Then a small modification of the arguments above shows that the curve α can be chosen so that $\bar{\alpha}$ is also a pseudo-orbit (cf. [8]). $\diamond$

Proof of Theorem 1.3:

First note that if H is convex and symmetric, then $\pi(\Sigma_\sigma)$ is closed even if Σ_σ is not compact; indeed $\pi(\Sigma_\sigma) = U^{-1}(-\infty, \sigma]$, where $U : M \to \mathbf{R}$ is given by $U(q) = H(q, 0)$. Hence as in the proof of Theorem 1.2, we only have to show that $E(\theta) \cap V(\theta) = \{0\}$ for all $\theta \in \Sigma_\sigma$. Suppose that for some $\theta \in \Sigma_\sigma$, $E(\theta) \cap V(\theta) \neq \{0\}$ and we will derive a contradiction. By Proposition 3.3 there exists a closed pseudo-orbit α through θ so that $\mu(\alpha) > 0$. Let $\alpha^{-1}(t) = \alpha(T-t)$ and observe that α^{-1} and $\bar{\alpha}$ are homotopic,[2] where $\bar{\alpha}$ is the curve defined in Definition 3.2. Thus $0 < \mu(\alpha) = -\mu(\alpha^{-1}) = -\mu(\bar{\alpha})$. But since $\bar{\alpha}$ is also a pseudo-orbit (cf. again Proposition 3.3) it has non-negative Maslov index (Lemma 3.1) which is a contradiction.

$\diamond$

Remark 3.4 The argument above also shows that if the non-wandering set $\Omega \subseteq \Sigma_\sigma$ touches the zero section of T^*M then $\phi_t \mid_{\Sigma_\sigma}$ **does not** admit a continuous invariant Lagrangian subbundle.

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