The work of Kolyvagin on the arithmetic of elliptic curves

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Introduction

This paper gives a complete proof of a recent theorem of Kolyvagin [3, 4] on Mordell-Weil groups and Tate-Shafarevich groups of elliptic curves. Let E be an elliptic curve defined over **Q**, and assume that E is modular: for some integer N there is a nonconstant map defined over **Q**

$$\pi : X_0(N) \to E$$

which we may assume sends the cusp ∞ to 0. Here $X_0(N)$ is the usual modular curve over **Q** (see for example [8]) which over **C** is obtained by compactifying the quotient $\mathfrak{H}/\Gamma_0(N)$ of the complex upper half-plane $\mathfrak{H}$ by the group

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbf{Z}) \ : \ c \equiv 0 \ (\text{modulo } N) \right\}.$$

The points of $X_0(N)$ correspond to pairs (A, C) where A is a (generalized) elliptic curve and C is a cyclic subgroup of A of order N. Fix an imaginary quadratic field K in which all primes dividing N split, and an ideal $\mathfrak{n}$ of K such that $\mathcal{O}_K/\mathfrak{n} \cong \mathbf{Z}/N\mathbf{Z}$. Write H for the Hilbert class field of K and x_H for the point in $X_0(N)(\mathbf{C})$ corresponding to the pair

$$(\mathbf{C}/\mathcal{O}_K, \ \mathfrak{n}^{-1}/\mathcal{O}_K).$$

Fix an embedding of $\overline{\mathbf{Q}}$ into **C**; then the theory of complex multiplication shows that $x_H \in X_0(N)(H)$. Define $y_H = \pi(x_H) \in E(H)$, $y_K = \mathrm{Tr}_{H/K}(y_H) \in E(K)$, and $y = y_K - y_K^{\tau} \in E(K)$, where τ denotes complex conjugation on K.

Let $Ш_{E/\mathbf{Q}}$ denote the Tate-Shafarevich group of E over **Q**.

Theorem. (Kolyvagin [3, 4]) *Suppose* E *and* y *are as above. If* y *has infinite order in* E(K) *then* E(**Q**) *and* $Ш_{E/\mathbf{Q}}$ *are finite.*

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Remarks. 1. The proof of this theorem given below is organized differently from Kolyvagin's proof, and somewhat simplified, but the important ideas are all due to Kolyvagin and contained in [3, 4].

2. It is not difficult to show, using the Hecke operator w_N, that y has infinite order if and only if both y_K has infinite order and the sign in the functional equation of the L-function L(E, s) is +1.

3. The proof will give an annihilator of $Ш_{E/Q}$ which, via the theorem of Gross and Zagier [2], gives evidence for the Birch and Swinnerton-Dyer conjecture.

4. Observe that Kolyvagin's theorem makes no mention of the L-function of E. To relate his result to the Birch and Swinnerton-Dyer conjecture one needs the following:

Theorem. (Gross and Zagier [2]) *With* E *and* y *as above,* y *has infinite order in* E(K) *if and only if* $L(E, 1) \neq 0$ *and* $L'(E, \chi_K, 1) \neq 0$, *where* χ_K *is the quadratic character attached to* K.

Analytic Conjecture. *If* E *is a modular elliptic curve and the sign in the functional equation of* L(E, s) *is* +1, *then there exists at least one imaginary quadratic field* K, *in which all primes dividing* N *split, such that* $L'(E, \chi_K, 1) \neq 0$.

This analytic conjecture, as yet unproved, together with the theorems of Kolyvagin and Gross and Zagier, would imply:

(*) *For any modular elliptic curve* E, *if* $L(E, 1) \neq 0$ *then* E(Q) *and* $Ш_{E/Q}$ *are finite.*

Assertion (*) is known for elliptic curves with complex multiplication, by theorems of Coates and Wiles [1] (for E(Q)) and Rubin [6] (for $Ш_{E/Q}$).

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Notation. For any abelian group A, A_n will denote the n-torsion in A and $A_{n^\infty} = \bigcup_i A_{n^i}$. If A is a module for the appropriate Galois group, we will write $H^i(L/F, A)$ for $H^i(\mathrm{Gal}(L/F), A)$, $H^i(F, A)$ for $H^i(\overline{F}/F, A)$, and $H^i(F, E)$ for $H^i(F, E(\overline{F}))$.

Tools of the proof

Fix a prime number ℓ and a positive integer n. For any completion $\mathbf{Q}_v$ of $\mathbf{Q}$ we have the diagram

$$(1)\qquad \begin{array}{ccccccccc} 0 & \to & E(Q)/\ell^n E(Q) & \to & H^1(Q, E_{\ell^n}) & \to & H^1(Q, E)_{\ell^n} & \to & 0 \\ & & \downarrow & & \downarrow \mathrm{res}_v & & \downarrow \mathrm{res}_v & & \\ 0 & \to & E(Q_v)/\ell^n E(Q_v) & \to & H^1(Q_v, E_{\ell^n}) & \to & H^1(Q_v, E)_{\ell^n} & \to & 0 \end{array}$$

and we define the Selmer group $S^{(\ell^n)}$ and the ℓ^n-torsion in the Tate-Shafarevich group, $Ш_{\ell^n}$, by

$$S^{(\ell^n)} = \bigcap_v \mathrm{res}_v^{-1}(\text{image } E(Q_v)),$$

$$0 \to E(Q)/\ell^n E(Q) \to S^{(\ell^n)} \to Ш_{\ell^n} \to 0.$$

To prove Kolyvagin's theorem it will suffice to show that $S^{(\ell)} = 0$ for almost all ℓ, and that for other ℓ the order of $S^{(\ell^n)}$ is annihilated by a power of ℓ which is independent of n.

For $s \in S^{(\ell^n)}$ write s_v for the inverse image of $\mathrm{res}_v(s)$ in $E(Q_v)/\ell^n E(Q_v)$. Our main tool for bounding $S^{(\ell^n)}$ is the following, which is proved using the local Tate pairings.

Proposition 1. *Suppose* p *is a prime such that* $E(Q_p)_{\ell^n} \cong \mathbf{Z}/\ell^n\mathbf{Z}$, *and suppose that for some integer* k *there exists a cohomology class* $c_p \in H^1(Q, E)_{\ell^n}$ *satisfying*

(i) *for all* $v \neq p$, $\mathrm{res}_v(c_p) = 0$,

(ii) $\mathrm{res}_p(c_p)$ *has order* ℓ^{n-k}.

Then for every $s \in S^{(\ell^n)}$, $\ell^k s_p = 0$.

Proof. For any place v of $\mathbf{Q}$ let $\langle\ ,\ \rangle_v$ denote the local Tate pairing

$$\langle\ ,\ \rangle_v : E(\mathbf{Q}_v)/\ell^n E(\mathbf{Q}_v) \times H^1(\mathbf{Q}_v, E)_{\ell^n} \to \mathbf{Z}/\ell^n\mathbf{Z}.$$

For any $s \in S^{(\ell^n)}$ and $c \in H^1(\mathbf{Q}, E)_{\ell^n}$, let c' be any lift of c to $H^1(\mathbf{Q}, E_{\ell^n})$ in (1) and define an element $b(s, c)$ in the Brauer group of $\mathbf{Q}$ by the cup product

$$b(s, c) = s \cup c' \in H^2(\mathbf{Q}, E_{\ell^n} \otimes E_{\ell^n}) \cong H^2(\mathbf{Q}, \boldsymbol{\mu}_{\ell^n}) = \mathrm{Br}(\mathbf{Q})_{\ell^n}.$$

Here the isomorphism $E_{\ell^n} \otimes E_{\ell^n} \cong \boldsymbol{\mu}_{\ell^n}$ is given by the Weil pairing. By the definition of the Tate pairing ([5] §I.3, especially remark 3.5) we have

$$\langle s_v, \mathrm{res}_v(c)\rangle_v = \mathrm{inv}_v(b(s, c)).$$

Thus

$$\sum_v \langle s_v, \mathrm{res}_v(c)\rangle_v = \sum_v \mathrm{inv}_v(b(s, c)) = 0.$$

Applying this reciprocity law with a class c_p as in the statement of the proposition we conclude that $\langle s_p, \mathrm{res}_p(c_p)\rangle_p = 0$. But

$$E(\mathbf{Q}_p)/\ell^n E(\mathbf{Q}_p) \cong E(\mathbf{Q}_p)_{\ell^\infty}/\ell^n E(\mathbf{Q}_p)_{\ell^\infty} \cong \mathbf{Z}/\ell^n\mathbf{Z},$$

so if $\mathrm{res}_p(c_p)$ has order ℓ^{n-k} the nondegeneracy of the Tate pairing shows that $\ell^k s_p = 0$. //

It remains now to construct such a cohomology class c_p for sufficiently many p, with k bounded and usually 0. Kolyvagin constructs such a c_p using Heegner points. Write τ for the complex conjugation on $\overline{\mathbf{Q}}$ induced by our embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}$, and $[\tau]$ for its conjugacy class in $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. If A is any 2-divisible $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-module, the action of τ gives a decomposition $A = A^+ \oplus A^-$. From now on, for simplicity we will assume that $\ell \neq 2$, and if $K = \mathbf{Q}(\sqrt{-3})$ we also assume $\ell \neq 3$. Write D_K for the discriminant of K.

Lemma 2. *Suppose* p *is a prime not dividing* $\ell D_K N$, $r > 0$, *and* $\mathrm{Frob}_p(K(E_{\ell^r})/\mathbf{Q}) = [\tau]$. *Then if* $\tilde{E}$ *denotes the reduction of* E *modulo* p *and* $a_p = p + 1 - \#[\tilde{E}(\mathbf{F}_p)]$,

(i) $\ell^r \mid a_p$ *and* $\ell^r \mid p+1$,

(ii) p *remains prime in* K,

(iii) $E(\mathbf{Q}_p)_{\ell^r} \cong \tilde{E}(\mathbf{F}_p)_{\ell^r} \cong \mathbf{Z}/\ell^r\mathbf{Z}$, $(E(K_p)_{\ell^r})^- \cong (\tilde{E}(\mathbf{F}_{p^2})_{\ell^r})^- \cong \mathbf{Z}/\ell^r\mathbf{Z}$.

Proof. The characteristic polynomial of Frobenius acting on E_{ℓ^r} is $T^2 - a_pT + p$, and the characteristic polynomial of τ acting on $E_{\ell^r} = E(\mathbf{C})_{\ell^r}$ is $T^2 - 1$. Comparing these polynomials modulo ℓ^r proves (i). The second assertion holds because $\mathrm{Frob}_p(K/\mathbf{Q}) \neq 1$, and the third because $E(\mathbf{Q}_p)_{\ell^r} \cong (E_{\ell^r})^+ \cong E(\mathbf{R})_{\ell^r}$ and $E(K_p)_{\ell^r} \cong (E_{\ell^r})^+ \oplus (E_{\ell^r})^-$. //

Suppose p is a rational prime which remains prime in K and $p \nmid N$. Let $\mathcal{O}_p$ be the order of conductor p in $\mathcal{O}_K$, and x_p the point in $X_0(N)(\mathbf{C})$ corresponding to the pair

$$(\mathbf{C}/\mathcal{O}_p,\ (\mathfrak{n} \cap \mathcal{O}_p)^{-1}/\mathcal{O}_p).$$

The theory of complex multiplication shows that $x_p \in X_0(N)(K[p])$ where $K[p]$ denotes the ring class field of K modulo p. The field $K[p]$ is the abelian extension of K corresponding to the subgroup $K^\times \mathbf{C}^\times \prod_q (\mathcal{O}_p \otimes \mathbf{Z}_q)^\times$ of the ideles of K. It follows easily that $K[p]$ is a cyclic extension of H of degree $(p+1)/u_K$ where $u_K = \#(\mathcal{O}_K^\times)/2$, $K[p]/H$ is totally ramified at p and unramified everywhere else, and τ acts on $\mathrm{Gal}(K[p]/K)$ by -1. Define $y_p = \pi(x_p) \in E(K[p])$. The only facts about Heegner points which we will need (other than their natural fields of definition) are contained in the following proposition.

Proposition 3. i) $u_K \mathrm{Tr}_{K[p]/H}(y_p) = a_p y_H$.
ii) *For any prime* $\mathfrak{p}$ *of* $K[p]$ *above* p, $\tilde{y}_p = \tilde{y}_H^{\mathrm{Frob}} \in \tilde{E}(\mathbf{F}_{p^2})$, *where* ~ *denotes reduction modulo* $\mathfrak{p}$.

Proof. Fix an elliptic curve A defined over H, with complex multiplication by $\mathcal{O}_K$, so that $(A, A_{\mathfrak{n}})$ represents x_H. Without loss of generality we may assume that A has good reduction at all primes above $\mathfrak{p}$. The point x_p can be represented by $(A', A'_{\mathfrak{n}})$ where $A' = A/C_p$ is the quotient of A by a subgroup of order p. Let $\mathcal{C}$ denote the collection of the $p+1$ subgroups of A of order p. The Galois group $\mathrm{Gal}(K[p]/H)$ acts transitively on $\mathcal{C}/\mathrm{Aut}(E)$, which has order $(p+1)/u_K = [K[p]:H]$. Thus, writing T_p for the Hecke correspondence on $X_0(N)$,

$$T_p(x_H) = \sum_{C\in\mathcal{C}} (A/C, (A/C)_{\mathfrak{n}}) = u_K \sum_{\sigma\in \mathrm{Gal}(K[p]/H)} x_p^{\sigma}.$$

Projecting to E via π proves the first assertion, since $\pi \cdot T_p = a_p\pi$. For the second, consider the isogeny

$$\varphi : (A, A_{\mathfrak{n}}) \to (A', A'_{\mathfrak{n}})$$

of degree p. Since p remains prime in K, both A and A′ have supersingular reduction at $\mathfrak{p}$, so the reduced isogeny

$$\tilde{\varphi} : (\tilde{A}, \tilde{A}_{\mathfrak{n}}) \to (\tilde{A}', \tilde{A}'_{\mathfrak{n}})$$

must be, up to an automorphism, Frobenius ([9] II.2.12). This proves that $\tilde{x}_p = \tilde{x}_H^{\mathrm{Frob}}$ in $\tilde{X}_0(N)(\mathbf{F}_{p^2})$. By the universal property of the Neron model, π reduces to a morphism $\tilde{\pi}$ from $\tilde{X}_0(N)$ to $\tilde{E}$, and applying $\tilde{\pi}$ completes the proof. //

Remark. One can avoid using the universal property of the Neron model by requiring instead that p not belong to a certain finite set of primes. This restriction does not interfere with the proof of Kolyvagin's theorem.

Suppose p is a prime not dividing $\ell D_K N$, $r > 0$, and $\mathrm{Frob}_p(K(E_{\ell^r})/\mathbf{Q}) = [\tau]$. By Lemma 2, $\ell^r \mid a_p$ and $\ell^r \mid u_K[K[p]:H]$, so there is a (unique) extension H′ of H of degree ℓ^r in K[p]. Define

$$z_1 = u_K \mathrm{Tr}_{K[p]/H'}(y_p - y_p^{\tau}) - (a_p/\ell^r)(y_H - y_H^{\tau}) \in E(H').$$

Corollary 4. *Suppose* $p \nmid \ell D_K N$ *and* $\mathrm{Frob}_p(K(E_{\ell^r})/\mathbf{Q}) = [\tau]$, *and let* z_1 *be as above.*

(i) $\mathrm{Tr}_{H'/H}(z_1) = 0$.

(ii) *For any* $\sigma \in \mathrm{Gal}(H/K)$, *let* $\bar{\sigma}$ *denote any lift of* σ *to* $\mathrm{Gal}(H'/K)$. *Then*

$$\sum_{\sigma\in \mathrm{Gal}(H/K)} \widetilde{z_1^{\bar{\sigma}}} = -((p+1+a_p)/\ell^r)\tilde{y}.$$

Proof. This follows without difficulty from Proposition 3. //

For each place v of $\mathbf{Q}$ let $m_v = \#[H^1(Q_v^{unr}/Q_v, E(Q_v^{unr}))]$. By [5] Proposition I.3.8, each m_v is finite and all but finitely many are zero, so $m(\ell) = \sup\{\mathrm{ord}_\ell(m_v) : \text{all } v \text{ of } \mathbf{Q}\}$ is a well-defined integer, equal to zero for almost all ℓ.

Proposition 5. *Suppose* $p \nmid \ell D_K N$ *and* $\mathrm{Frob}_p(K(E_{\ell^r})/\mathbf{Q}) = [\tau]$, *where* $r = n + m(\ell)$. *Then there is an element* $c_p \in H^1(\mathbf{Q}, E)_{\ell^n}$ *such that*

i) $\mathrm{res}_v(c_p) = 0$ *for all* $v \neq p$,

ii) *the order of* $\mathrm{res}_p(c_p)$ *in* $H^1(\mathbf{Q}_p, E)_{\ell^n}$ *is equal to the order of* y *in* $E(K_p)/\ell^n E(K_p)$.

Proof. First suppose $\ell \nmid [H:K]$. Then there is a (unique) extension K' of K of degree ℓ^r in K[p], totally ramified at p and unramified at all other primes, and $H' = HK'$. Define

$$z = \mathrm{Tr}_{H'/K'}(z_1) \in E(K').$$

By Corollary 4, $\mathrm{Tr}_{K'/K}(z) = 0$. Fixing a generator σ of $\mathrm{Gal}(K'/K)$ gives rise to a group isomorphism (which is *not* τ-equivariant, see below)

$$\{\alpha \in E(K') : \mathrm{Tr}_{K'/K}(\alpha) = 0\}/(\sigma-1)E(K') \cong H^1(K'/K, E(K')).$$

Define

$$c_p' \in H^1(K'/K, E(K')) \subset H^1(K, E)_{\ell^r}$$

to be the image of z under this isomorphism.

Since τ commutes with $\mathrm{Tr}_{K[p]/K'}$, $z^\tau = -z$. Since τ also acts by -1 on $\mathrm{Gal}(K'/K)$, we conclude that $c_p'^{\tau} = c_p'$. Thus $c_p' \in (H^1(K, E)_{\ell^r})^+$. But for $\ell > 2$ the restriction map gives an isomorphism $H^1(\mathbf{Q}, E)_{\ell^r} \cong (H^1(K, E)_{\ell^r})^+$, so $c_p' \in H^1(\mathbf{Q}, E)_{\ell^r}$. Finally, define $c_p = \ell^{m(\ell)} c_p' \in H^1(\mathbf{Q}, E)_{\ell^n}$.

For $v \neq p$, since K'/K is unramified at v,

$$\mathrm{res}_v(c_p) = \ell^{m(\ell)} \mathrm{res}_v(c_p') \in \ell^{m(\ell)} H^1(Q_v^{unr}/Q_v, E(Q_v^{unr}))_{\ell^r} = 0$$

by definition of $m(\ell)$.

To complete the proof of the proposition we must determine the order of $\mathrm{res}_p(c_p)$ in $H^1(\mathbf{Q}_p, E)_{\ell^n}$. Write I_p for the inertia subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, and consider the sequence

$$H^1(\mathbf{Q}_p, E)_{\ell^n} \to H^1(I_p, E(\overline{\mathbf{Q}}_p))_{\ell^n} \to H^1(I_p, \tilde{E}(\overline{\mathbf{F}}_p))_{\ell^n} \to \mathrm{Hom}(\mathrm{Gal}(K'/K), \tilde{E}_{\ell^n}).$$

The first map is injective because its kernel, $H^1(Q_p^{unr}/Q_p, E(Q_p^{unr}))_{\ell^n}$, is 0 since E has good reduction at p. The second map is an isomorphism because the kernel of reduction modulo p is a pro-p group. The third map is an isomorphism because I_p acts trivially on $\tilde{E}(\overline{F}_p)$ and $K'Q_p^{unr}$ is the unique abelian extension of Q_p^{unr} of exponent ℓ^r. It is easy to see that the image of c_p under this sequence of injections is the homomorphism which sends the chosen generator σ of Gal(K′/K) to $\ell^{m(\ell)}\tilde{z}$. Thus the order of $res_p(c_p)$ in $H^1(Q_p, E)_{\ell^n}$ is the same as the order of $\ell^{m(\ell)}\tilde{z}$ in $\tilde{E}(F_{p^2})$.

Corollary 4 shows that

$$\ell^{m(\ell)}\tilde{z} = -((p+1+a_p)/\ell^n)\tilde{y}.$$

Up to a factor of 2, $\#[\tilde{E}(F_{p^2})^-] = \#[\tilde{E}(F_{p^2})]/\#[\tilde{E}(F_p)] = p+1+a_p$. By Lemma 2, $(\tilde{E}(F_{p^2})_{\ell^\infty})^-$ is cyclic, so we conclude that $(p+1+a_p)/\ell^n$ maps $\tilde{E}(F_{p^2})^-/\ell^n\tilde{E}(F_{p^2})^-$ isomorphically to $(\tilde{E}(F_{p^2})_{\ell^n})^-$. Therefore the order of $\ell^{m(\ell)}\tilde{z}$ in $\tilde{E}(F_{p^2})$ is the same as the order of y in $E(K_p)/\ell^nE(K_p) \cong \tilde{E}(F_{p^2})/\ell^n\tilde{E}(F_{p^2})$. This completes the proof when $\ell \nmid [H:K]$.

If $\ell \mid [H:K]$, there may not exist a field K′ as above. In that case, use the point z_1 to define $c'_{1,p} \in H^1(H, E)_{\ell^r}$. Then define c'_p to be the corestriction of $c'_{1,p}$ to $H^1(K, E)$ and proceed as above. //

Corollary 6. *Suppose* $p \nmid \ell D_K N$, *and* $Frob_p(K(E_{\ell^{n+m(\ell)}})/Q) = [\tau]$. *If* $k \geq 0$ *and* $y \notin \ell^{k+1}E(K_p)$, *then for all* $s \in S^{(\ell^n)}$, $\ell^k s_p = 0$.

Proof. This follows immediately from Propositions 1 and 4. //

For any $t \in H^1(K, E_{\ell^n})$, write $\hat{t}$ for the image of t under the restriction map

(2) $$H^1(K, E_{\ell^n}) \to Hom(Gal(\overline{K}/K(E_{\ell^{n+m(\ell)}})), E_{\ell^n})^{Gal(K(E_{\ell^{n+m(\ell)}})/K)}.$$

Lemma 7. *Suppose* $t \in H^1(K, E_{\ell^n})^{\pm}$ *and the image of* $\hat{t}$ *is cyclic. Then the order of* t *is at most* ℓ^{a+b}, *where* ℓ^a *is the order of the largest* Q*-rational cyclic subgroup of* E_{ℓ^∞} *and* ℓ^b *is the exponent of* $H^1(K(E_{\ell^{n+m(\ell)}})/K, E_{\ell^n})$.

Proof. Since $\hat{t}$ is $\mathrm{Gal}(K(E_{\ell^{n+m(\ell)}})/K)$-equivariant, its image is $\mathrm{Gal}(\overline{K}/K)$-invariant. Since τ acts on $\hat{t}$ by ± 1, the image is in fact rational over $\mathbf{Q}$. Thus if the image is cyclic, the order of $\hat{t}$ is at most ℓ^a. The kernel of the restriction map (2) is $H^1(K(E_{\ell^{n+m(\ell)}})/K, E_{\ell^n})$, so t has order at most ℓ^{a+b}. //

Proof of Kolyvagin's theorem

As above, we fix a prime ℓ not dividing $\#[\mathcal{O}_K^\times]$. Suppose y has infinite order in $E(K)$, and let $k = k(\ell)$ be the largest integer such that $y \in \ell^k E(K) + E(K)_{tors}$. Fix any integer $n \geq k + 1$. First assume that

(3) $\quad$ E has no ℓ-isogeny defined over $\mathbf{Q}$,

(4) $\quad$ $H^1(K(E_{\ell^{n+m(\ell)}})/K, E_{\ell^n}) = 0,$

both of which hold for all but a finite number of ℓ by Serre's theorem [7] or the theory of complex multiplication. Under these assumptions we will show that $\ell^k S^{(\ell^n)} = 0$.

Write $r = n + m(\ell)$. Fix $s \in S^{(\ell^n)}$, and as in Lemma 7 write $\hat{s}$ for the restriction of s to $\mathrm{Gal}(\overline{\mathbf{Q}}/K(E_{\ell^r}))$ and write $\hat{y}$ for the restriction of the image of y under the injection

$$E(K)^-/\ell^n E(K)^- \rightarrow H^1(K, E_{\ell^n})^-.$$

Fix a finite extension F of $K(E_{\ell^r})$ so that both $\hat{s}$ and $\hat{y}$ factor through $G = \mathrm{Gal}(F/K(E_{\ell^r}))$.

Choose any $\gamma \in G$, and choose any prime p, not dividing $\ell D_K N$, such that $\mathrm{Frob}_p(F/\mathbf{Q}) = [\gamma\tau]$. Then $\mathrm{Frob}_p(K(E_{\ell^r})/\mathbf{Q}) = [\tau]$, and $\mathrm{Frob}_p(F/K(E_{\ell^r})) \in [(\gamma\tau)^2]$ so

$$\ell^k s_p = 0 \Leftrightarrow \ell^k \hat{s}((\gamma\tau)^2) = 0, \quad \text{and} \quad y \in \ell^{k+1} E(K_p) \Leftrightarrow \ell^{n-k-1} \hat{y}((\gamma\tau)^2) = 0.$$

Since $\hat{s}^\tau = \hat{s}$, and $\hat{y}^\tau = -\hat{y}$,

$$\hat{s}((\gamma\tau)^2) = \hat{s}(\gamma) + \hat{s}(\tau\gamma\tau) = (1+\tau)\hat{s}(\gamma)$$

$$\hat{y}((\gamma\tau)^2) = \hat{y}(\gamma) + \hat{y}(\tau\gamma\tau) = (1-\tau)\hat{y}(\gamma)$$

By Corollary 6, we conclude that for every $\gamma \in G$, either $\ell^k \hat{s}(\gamma) \in (E_{\ell^n})^-$ or $\ell^{n-k-1} \hat{y}(\gamma) \in (E_{\ell^n})^+$. Therefore $G = (\ell^k \hat{s})^{-1}((E_{\ell^n})^-) \cup (\ell^{n-k-1} \hat{y})^{-1}((E_{\ell^n})^+)$. But a group cannot be the union of two proper subgroups, so either $\ell^k \hat{s}(G) \subset (E_{\ell^n})^-$ or $\ell^{n-k-1} \hat{y}(G) \subset (E_{\ell^n})^+$. By Lemma 7 (using assumptions (3) and (4)) we conclude that either

$\ell^k s = 0$ in $S^{(\ell^n)}$ or $\ell^{n-k-1} y = 0$ in $E(K)/\ell^n E(K)$. Since the latter is impossible by our definition of k, we have shown that $\ell^k S^{(\ell^n)} = 0$.

Since $k = 0$ for almost all ℓ, this proves Kolyvagin's theorem except for the finite number of ℓ-parts which we have ruled out above. Without assumptions (3) and (4), using Lemma 7 the proof above gives a somewhat weaker annihilator of $S^{(\ell^n)}$, but still one which is independent of n (again using [7] or the theory of complex multiplication to show that the exponent of $H^1(K(E_{\ell^{n+m(\ell)}})/K, E_{\ell^n})$ is bounded independent of n). Also, with a little more care, one obtains a suitable annihilator when $\ell \mid \#[\mathcal{O}_K^\times]$. This completes the proof. //

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