Moduli spaces of convex projective structures on surfaces

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To the memory of Yan Kogan

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1 Introduction

Let S be an orientable smooth surface. A projective structure on S is defined by an atlas whose transition functions are given by (restrictions) of projective transformations. For any projective structure S one can associate a developing map

$$\varphi : \widetilde{S} \to \mathbb{RP}^2$$

where $\widetilde{S}$ is a universal cover of S. It defined up to the right action of $PSL_3(\mathbb{R})$.

A domain D in $\mathbb{RP}^2$ is *convex* if any line intersects D by a connected segment, possibly empty. A projective structure on S is *convex* if the developing map φ is an embedding and $\varphi(\widetilde{S})$ is a convex domain. Since φ is injective, $\varphi(\widetilde{S})$ is orientable. Conversely, let Δ be a discrete subgroup of $PSL_3(\mathbb{R})$ isomorphic to a surface group, which acts on $\mathbb{RP}^2$ preserving a convex domain D. Then the quotient D/Δ is a surface with a convex projective structure.

The space of convex projective structures on smooth compact surfaces without boundary was studied by W.Goldman, S.Choi [7], [1], F.Labourie [12], J.Loftin [14] and implicitly N.Hitchin [10].

On the other hand, given a split semi-simple algebraic group G with trivial center and a surface S with non-empty boundary, we defined in [3] the Higher Teichmüller space $\mathcal{X}^+_{G,S}$. It comes equipped with a distinguished collection of coordinate systems, parametrized by the set of the isotopy classes of trivalent graphs embedded to S and homotopically equivalent to S, plus some extra data. The mapping class group acts in this coordinates in an explicit way. We proved in Section 10 of loc. cit that for $G = PSL_2$ we recover the classical Teichmüller theory.

In this paper we investigate convex projective structures on hyperbolic surfaces with a non-empty geodesic boundary. We introduce a distinguished collection of global coordinate systems on the corresponding moduli space $\mathcal{T}_3^+(S)$. Each of them identifies it with $\mathbb{R}^{8\chi(S)}$. We show that the moduli space $\mathcal{T}_3^+(S)$ is naturally isomorphic to the space $\mathcal{X}_{G,S}^+$ for $G = PGL_3$. We prove that the corresponding monodromy representations are discrete and regular hyperbolic. We introduce and study the universal higher Teichmüller space $\mathcal{T}_3^+$ which contains the moduli spaces $\mathcal{T}_3^+(S)$ for all surfaces S. The Thompson group $\mathbb{T}$ acts by it automorphisms. We show that the universal higher Teichmüller space $\mathcal{T}_3^+$ has a cluster ensemble structure, defined in [4]. Precisely, $\mathcal{T}_3^+$ is the positive part of the $\mathcal{X}$-space of certain cluster ensemble. Moreover the Thompson group is a subgroup of the mapping class group $\mathbb{T}_3$ of this cluster ensemble. For the spaces $\mathcal{T}_3^+(S)$ the situation is similar but a bit more complicated: they have orbi-cluster structure, and the mapping class group of S is a subgroup of the corresponding group for the cluster ensemble. We quantise (= define a non-commutative q-deformation of) the above moduli spaces. The quantum universal Teichmüller space $\mathcal{T}_3^q$ can be viewed as a combinatorial version of the W-algebra for SL_3.

We made exposition self-contained and as elementary as possible. Therefore this paper can serve as an elementary introduction to [3], where the results of this paper were generalized to the case when PGL_3 is replaced by an arbitrary split reductive group G.

2 Convex projective structures with geodesic boundary

1. The moduli space of framed convex projective structures with geodesic boundary on S and its versions. A *convex curve* in $\mathbb{RP}^2$ is a curve such that any line intersects it either by a connected line segment, or in no more then two points. If every line intersects a curve in no more then two points, it is a *strictly convex curve* in $\mathbb{RP}^2$.

There is a natural bijective correspondence between convex curves and convex domains in $\mathbb{RP}^2$. Indeed, let D be a convex domain in $\mathbb{RP}^2$. Then its boundary ∂D is a convex curve. Conversely, let K be a convex curve. Then barring the two trivial cases when K is a line or an empty set, the complement to K is a union of two convex domains. One of them is not orientable, and contains lines. The other is orientable, and does not contain any line. The latter is called *interier* of the convex curve K, and denoted D_K. If K is a line or the empty set, the convex domain D_K, by definition, is $\mathbb{R}^2$, or $\mathbb{RP}^2$.

Given a convex domain $D \subset \mathbb{RP}^2$ we define the projectively dual domain $\widehat{D} \subset \widehat{\mathbb{RP}}^2$ as the set of all lines in $\mathbb{RP}^2$ which do not intersect D. It is a convex domain. The dual to $\widehat{D}$ is D.

A curve on a surface S with projective structure is called *geodesic* if it is a straight line segment in any projective coordinate system. Therefore a geodesic develops into an infinite collection of line segments. They are permuted by the monodromy group Δ.

Let us investigate the case of a boundary geodesic γ. The monodromy along γ provides a conjugacy class $[g_\gamma]$ in $PSL_3(\mathbb{R})$. We claim that it is the conjugacy class of a matrix with real eigenvalues. Indeed, otherwise an element g_γ from the conjugacy class has a unique invariant line L and a unique invariant point P in $\mathbb{RP}^2$. Then $\varphi(\widetilde{S})$ contains at the boundary either the line L or the point P. In the first case, since $\varphi(\widetilde{S})$ is convex, it must coincide with $\mathbb{RP}^2$. But then P is a stable point of the monodromy group $\Delta \subset PSL_3(\mathbb{R})$, so the quotient $\varphi(\widetilde{S})/\Delta$ is not a manifold, and thus can not be diffeomorphic to S. Now let us assume that the boundary of $\varphi(\widetilde{S})$ contains the point P. Since the group Δ is discrete, the element g_γ must be of finite order. But then, if it is non-trivial, it can not preserve the domain $\varphi(\widetilde{S})$. So we rulled out the possibility of g_γ having complex eigenvalues.

If the surface S has a boundary, the space of projective structures on S is infinite dimensional even with convexity requirement since any projective invariant of the image of any subsegment of the boundary under φ is an invariant of the projective structure. Therefore we impose the following requirement on the behavior of the projective structure at the vicinity of the boundary: either the projective structure is expendable to the boundary and the boundary is geodesic, or it is the limiting case of the previous one, and we say that we have a *cuspidal end* of S. The latter case is called degenerate. So in the non-degenerate case the image of a boundary component under φ form a line interval connecting two of the three possible stable points of the monodromy around this component. We will show that the degeneration amounts to shrinking of the interval to a point, and keeping an invariant flag at this point. Thus the limit of the monodromy is an element with a Jordan block. The boundary of this type is called *geodesic*. *Framing* of such a projective structure is an orientation of all non-degenerate boundary components.

Definition 2.1 *$\mathcal{T}_3^+(S)$ is the moduli space of framed convex real projective structures on an oriented surface S with geodesic boundary considered up to the action of the group* $\mathrm{Diff}^0(S)$ *of diffeomorphisms isotopic to the identity.*

(The index 3 in $\mathcal{T}_3^+(S)$ indicates the group $PSL(3,\mathbb{R})$, and h stays for oriented holes).

The space $\mathcal{T}_3^+(S)$ is a $2^s : 1$ cover of the space $\mathcal{T}_3(S)$ of non-framed convex real projective structures on S with geodesic boundary, where s is the number of the holes, ramified over the surfaces which have at least one cuspidal end. On the other hand the orientation of S provides orientations of the boundary components. Thus there is a canonical embedding $\mathcal{T}_3(S) \hookrightarrow \mathcal{T}_3^+(S)$ as a subspace with corners.

There are the following related maps structures:

1. There is an embedding $i : \mathcal{T}_2^+(S) \hookrightarrow \mathcal{T}_3^+(S)$, where $\mathcal{T}_2^+(S)$ is the classical Teichmüller space parametrising complex structures on S modulo the action of $\mathrm{Diff}^0(S)$, with chosen orientation of holes. Indeed, due to the Poincaré uniformisation theorem any complex surface can be represented as a quotient of the hyperbolic plane by a discrete group. Consider the Klein model of the hyperbolic plane in the interior of a conic in $\mathbb{RP}^2$. So the geodesics are straight lines in this model. The quotient inherits the real projective structure. The orientations of holes are inherited trivially.

2. The map μ from $\mathcal{T}_3^+(S)$ to the space $\mathcal{R}_3(S)$ of homomorphisms $\pi_1(S) \to PSL_3(\mathbb{R})$, considered up to a conjugation. The space $\mathcal{R}_3(S)$ possesses the Atyah-Hitchin Poisson structure. Since μ is a local diffeomorphism, it induces a Poisson structure on $\mathcal{T}_3^+(S)$.

3. The involution $\sigma : \mathcal{T}_3^+(S) \to \mathcal{T}_3^+(S)$, defined by the property that the convex domain corresponding to the point σx, where $x \in \mathcal{T}_3^+(S)$, is projectively dual to the convex domain corresponding to x. The representation $\mu(\sigma(x))$ is defined by composing $\mu(x)$ with the outer automorphism $g \to (g^t)^{-1}$ of the group PSL_3. The map σ preserves the Poisson structure.

4. The action of the mapping class group $\Gamma_S := \mathrm{Diff}(S)/\mathrm{Diff}^0(S)$ on $\mathcal{T}_3^+(S)$, also preserving the Poisson structure.

Before we proceed any further, let us introduce a toy model of the moduli space $\mathcal{T}_3^+(S)$, which not only contains the main features of the latter, but also plays a key role in its study.

2. The moduli spaces of pairs of convex polygons, one inscribed into the other. Let $\mathcal{P}_3^n$ be the space of pairs of convex n-gons in $\mathbb{RP}^2$, one inscribed into the other, and considered up to the action of $PSL_3(\mathbb{R})$.

One can think of this space as of a kind of discrete approximation to the space of parametrised closed convex curves in $\mathbb{RP}^2$. Indeed, fix a set R of n points on the standard circle S^1. Then for any convex curve $\gamma : S^1 \to \mathbb{RP}^2$ one can associate the convex polygon with vertices $\gamma(R)$ and the polygon with edges tangent to γ at $\gamma(R)$.

This space has a natural Poisson structure, and there are analogs of the maps μ,σ, i and the mapping class group action.

Namely, let F_3 be the space of flags in $\mathbb{R}^3$, or, equivalently the space of pairs (A, a), where $a \subset \mathbb{RP}^2$ is a line, and $A \in a$ is a point. There exists a natural map $\mu : \mathcal{P}_3^n \to F_3^n/PSL_3(\mathbb{R})$ which is the analog of the map μ for $\mathcal{T}_3^+(S)$. Its image is a connected component in the space of collections of flags in general position.

The explicit formulae given below provide a definition of the Poisson structure in this case.

The projective duality interchanges the inscribed and circumscribed polygons. It acts as an involution σ of $\mathcal{P}_3^n$.

Let $\mathcal{P}_2^n$ be the configuration space of n points on the oriented real projective line, whose cyclic order is compatible with the chosen orientation of $\mathbb{RP}^1$. Then there is a canonical embedding $i : \mathcal{P}_2^n \hookrightarrow \mathcal{P}_3^n$, which identifies $\mathcal{P}_2^n$ with the space of polygons inscribed into a conic. Indeed, for each such a polygon we assign the circumscribed polygon given by tangents to the conic at the vertices of the original polygon. The set of stable points of σ is precisely the image of i.

The role of mapping class group is played by the cyclic group shifting simultaneously the vertices of the two polygons.

One can unify these two moduli spaces by considering the moduli space $\mathcal{T}_3^+(\widehat{S})$ framed convex projective structures on surfaces with geodesic boundary, equipped with a finite (possible empty) collection of marked points on the boundary, see Section 2.7.

Our next goal is to introduce a set of global coordinates on $\mathcal{T}_3^+(S)$, describe it's natural Poisson structure in terms of these parametrization and give explicit formulae for the maps

i, μ, σ and the action of the mapping class group. Before addressing this problem, let us first do a simpler one, however in a way containing most of the tools used for the problem in question.

3. Parameterizations of the spaces $\mathcal{P}_3^n$. Cut the inscribed polygons into triangles and mark two distinct points on every edge of the triangulation except the edges of the polygon. Mark also one point inside each triangle.

Theorem 2.2 *There exists a canonical bijective correspondence between the space $\mathcal{P}_3^n$ and assignments of positive real numbers to the marked points.*

Proof. It is constructive: we are going to describe how to construct numbers from a pair of polygons and visa versa.

A small remark about notations. We denote points (resp. lines) by uppercase (resp. lowercase) letters. A triangle on $\mathbb{RP}^2$ is determined neither by vertices nor by sides, since there exists four triangles for any generic triple of vertices or sides. If the triangle is shown on a figure, it is clear which one corresponds to the vertices since only one of the four fits entirely into the drawing. If we want to indicate a triangle which does not fit, we add to its vertices a point which belongs to the interior triangle in braces. For example, the points A, B and C on fig. 1 are vertices of the triangles ABC, $ABC\{a \cap c\}$, $ABC\{a \cap b\}$ and $ABC\{b \cap c\}$.

Another convention is about the cross-ratio. We assume that the cross-ratio of four points x_1, x_2, x_3, x_4 on a line is the value at x_4 of a projective coordinate taking value ∞ at x_1, -1 at x_2, and 0 at x_3. So we employ the formula $\frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_2-x_4)}$ for the cross-ratio. Observe that the lines passing through a point form a projective line. So the cross-ratio of an ordered quadruple of lines passing through a point is defined.

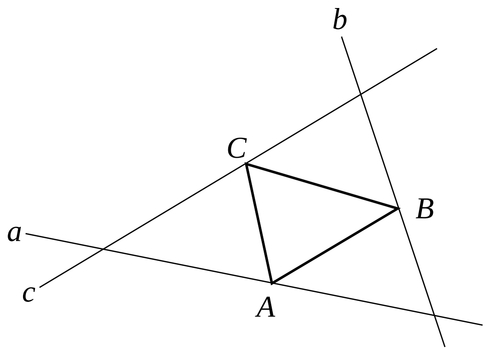

Figure 1: $\mathcal{P}_3^3$

Let us first consider the case of $\mathcal{P}_3^3$. It is the space of pairs of triangles abc and ABC (fig. 1), where the second is inscribed into the first and considered up to projective transformations. This space is one dimensional, and its invariant X is just the cross-ratio of the quadruple of lines $a, AB, A(b \cap c), AC$. This invariant is called the *triple ratio* ([8], Section 3) since it can also be computed as follows. Let V_3 be the three dimensional real vector space whose projectivisation is $\mathbb{RP}^2$. Choose linear functionals $f_a, f_b, f_c \in V^*$

defining the lines a, b, c. Choose non-zero vectors $\widetilde{A}, \widetilde{B}, \widetilde{C}$ projecting onto the points A, B, C. Then

$$X := \frac{f_a(\widetilde{B})f_b(\widetilde{C})f_c(\widetilde{A})}{f_a(\widetilde{C})f_b(\widetilde{A})f_c(\widetilde{B})}$$

It appears to be independent on the choices involved in the definition, and manifestly $\mathbb{Z}/3\mathbb{Z}$-invariant. Here is a yet another, the third definition of this invariant borrowed from ([9], Section 4.2). Every point (resp. line) of $\mathbb{RP}^2$ corresponds to a line (resp. plane) in V_3, and we shall denote them by the same letters. The line C belongs to the plane c and defines a linear map $C : c \cap a \to c \cap b$. Similarly there are linear maps $A : a \cap b \to a \cap c$ and $B : b \cap c \to b \cap a$. The composition of these three linear maps is the multiplication by the invariant X.

Lemma 2.3 *The triangle ABC is inscribed into the triangle abc if and only if X is positive.*

Proof. Here is an old-fashioned argument: a projective version of the classical Ceva theorem asserts that the lines $A(b \cap c)$, $B(c \cap a)$ and $C(a \cap b)$ intersect in a point if and only if the triple ratio is equal to one. Equivalently, the points $(a \cap c, A, a \cap b, BC \cap a)$ form an anharmonic quadruple. The lemma is proved.

Now consider the next case: $\mathcal{P}_3^4$. It is the space of pairs of quadrilaterals $abcd$ and $ABCD$, where the second is inscribed into the first, considered up to projective transformations. The space of such configurations has dimension four. Two parameters of a configuration are given by the triple ratio X of the triangles ABC inscribed into abc and the triple ratio Y of ACD inscribed into $acd\{B\}$ (fig.2). Another two parameters are given by the cross-ratios of quadruples of lines a, AB, AC, AD and c, CD, CA, CB, denoted by Z and W, respectively. Assign the coordinates X, Y, Z and W to the marked points as shown on the figure 2.

Lemma 2.4 *The convex quadrangle $ABCD$ is inscribed into the convex quadrangle $abcd$ if and only if X, Y, Z, W are positive.*

Proof. Indeed, we have $X, Y > 0$ by Lemma 2.3. Observe that the cross-ratio of four points on $\mathbb{RP}^1$ is positive if and only if the cyclic order of the points is compatible with the one provided by their location on $\mathbb{RP}^1$, i.e. with one of the two orientations of $\mathbb{RP}^1$. This immediately implies that $Z, W > 0$. On the other hand, given $X > 0$ we have, by Lemma 2.3, a unique projective equivalence class of a triangle ABC inscribed into a triangle abc. We use the coordinates Z, W to define the lines AD and CD, and the positivity of Z and W guarantees that these lines intersect in a point D located inside of the triangle abc, that is in the same connected component as the triangle ABC. Finally, Y is used to define the line d, and positivity of Y just means that it does not intersect the interior of the quadrangle $ABCD$. The lemma is proved.

Now let us consider the space $\mathcal{P}_3^n$ for an arbitrary $n > 2$. A point of this space is represented by a polygon $A_1 \ldots A_n$ inscribed into $a_1 \ldots a_n$. Cut the polygon $A_1 \ldots A_n$

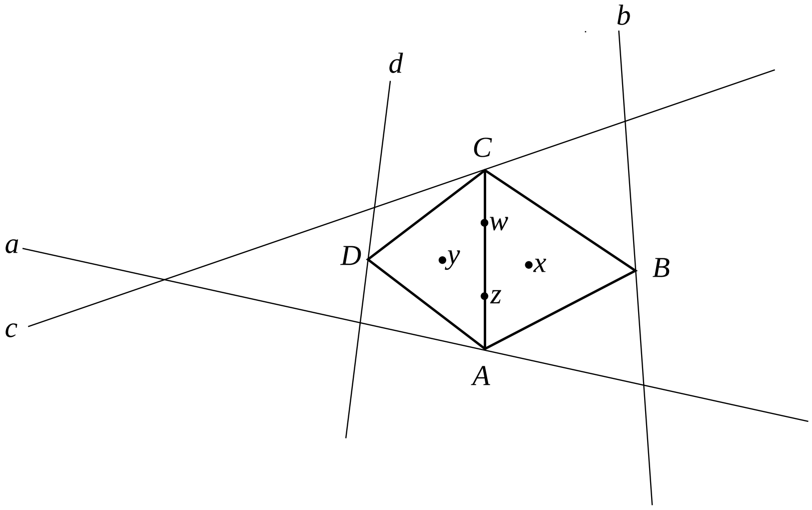

Figure 2: $\mathcal{P}_3^4$

into triangles. Each triangle $A_iA_jA_k$ of the triangulation is inscribed into a_i, a_ja_k, and we can assign their triple ratio to the marked point inside of $A_iA_jA_k$. Moreover for any pair of adjacent triangles $A_iA_jA_k$ and $A_jA_kA_l$ forming a quadrilateral $A_iA_jA_kA_l$ inscribed into the quadrilateral $a_ia_ja_ka_l$ one can also compute a pair of cross-ratios and assign it to two points on the diagonal A_jA_k. The cross-ratio of the lines passing through A_j being associated to the point closer to the point A_j.

The converse construction is straightforward. For every triangle of the triangulation one can construct a pair of triangles using number in the center as a parameter and using Lemma 2.3. Then we assemble them together using numbers on edges as gluing parameters and Lemma 2.4. The theorem is proved.

Now let us proceed to convex projective structures on surfaces.

4. A set of global coordinates on $\mathcal{T}_3^+(S)$. Let S be a Riemann surface of genus g with s boundary components. Assume, that $s \geq 1$ and moreover $s \geq 3$ if $g = 0$. Shrink all boundary components to points. Then the surface S can be cut into triangles with vertices at the shred boundary components. We call it *ideal triangulation* of S. Let us put two distinct marked points to each edge of the triangulation and one marked point to the center of every triangle.

Theorem 2.5 *Given an ideal triangulation T of S, there exists a canonical isomorphism*

$$\varphi_T : \mathcal{T}_3^+(S) \xrightarrow{\sim} \mathbb{R}_{>0}^{\{marked\ points\}}$$

Proof. It is quite similar to the proof for $\mathcal{P}_3^n$. Let us first construct a surface starting from a collection of positive real numbers on the triangulation. Consider the universal cover $\widetilde{S}$ of the surface and lift the triangulation together with the marked points and numbers from S to it. According to Theorem 2.2 to any finite polygon composed of the triangles of the arising infinite triangulated polygon we can associate a pair of polygons in $\mathbb{RP}^2$. Let U be the union of all inscribed polygons corresponding to such finite sub-polygons. Observe that it coincides with the intersection of all circumscribed polygons, and therefore is convex. The group $\pi_1(S)$ acts on $\widetilde{S}$ and hence on U by projective transformations. The desired projective surface is $U/\pi_1(S)$.

Now let us describe the inverse map, i.e. construct the numbers out of a given framed convex projective structure and a given triangulation. Take a triangle and send it to $\mathbb{RP}^2$ using a developing map φ. The vertices of the triangle correspond to boundary components. For a given boundary component C_i of S the choice of the developing map φ and the framing allows to assign a canonical flag (A, a) on $\mathbb{RP}^2$ invariant under the action of the monodromy operator around C_i. Indeed, if the boundary component C_i is non-degenerate, we take the line containing the interval $\varphi(C_i)$ for a, and one of the endpoints on this interval for A. The choice between the two endpoints is given by the framing, so that the interval is oriented out of the point A. If the boundary component is degenerate, the point A belongs to the limit of the image under φ of a point tending to S, and the flag is uniquely characterized by its invariance under the monodromy operator.

Assigning the flags to all three vertices of the triangle one gets a point of $\mathcal{P}_3^3$. The corresponding coordinate is associated to the central marked point of the original triangle. Similarly, taking two adjacent triangles of the triangulation one obtains the numbers for the marked points on edges.

These two constructions are evidently inverting each other. The theorem is proved.

5. Properties of the constructed coordinates for both $\mathcal{P}_3^3$ and $\mathcal{T}_3^+(S)$.

1. It turns out that the Poisson brackets between the coordinates are very simple, namely

$$\{X_i, X_j\} = 2\varepsilon_{ij} X_i X_j, \tag{1}$$

where ε_{ij} is a skew-symmetric integral valued function. To define the function ε_{ij} consider the graph with vertices in marked points and oriented edges connecting them as shown on fig 3. (We have shown edges connecting marked points belonging to one triangle only. Points of other triangles are connected by arrows in the same way. Points connected by edges without arrows are not taken into account.) Then

$$\varepsilon_{ij} = (\text{number of arrows from } i \text{ to } j) - (\text{number of arrows from } j \text{ to } i)$$

The proof of this formula amounts to a long computation. However it is an easy exercises

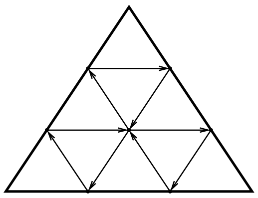

Figure 3: Poisson structure tensor.

to verify that this bracket does not depend on the triangulation, and therefore can be taken just as a definition. For $\mathcal{P}_3^n$ it is the only natural way to define the Poisson structure known to us.

2. Once we have positive numbers assigned to marked points, the construction of the corresponding projective surface is explicit. In particular one can compute the monodromy group of the corresponding projective structure or, in other words, the image of the map μ. To describe the answer we use the following picture. Starting from the triangulation, construct a graph embedded into the surface by drawing small edges transversal to each side of the triangles and inside each triangle connect the ends of edges pairwise by three more edges, as shown on fig. 4. Orient the edges of the triangles in counterclockwise direction and the other edges in the arbitrary way. Let

$$T(X) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ X & 1+X & 1 \end{pmatrix} \text{ and } E(Z,W) = \begin{pmatrix} 0 & 0 & Z^{-1} \\ 0 & -1 & 0 \\ W & 0 & 0 \end{pmatrix}.$$

Assign the matrix $T(X)$ to the edges of each triangle with X assigned to its center. And assign the matrix $E(Z,W)$ to the edge connecting two triangles, where W (resp. Z) is the number to the left (resp. right) from this edge (fig.4). Then for any closed path on the graph one can assign an element of $PSL_3(\mathbb{R})$ by multiplying the group elements (or their inverses if the path goes along the edge against its orientation) assigned to edges the path passes along. The image of the fundamental group of the graph is just the desired monodromy group. The proof of this statement is also constructive. Once we have a

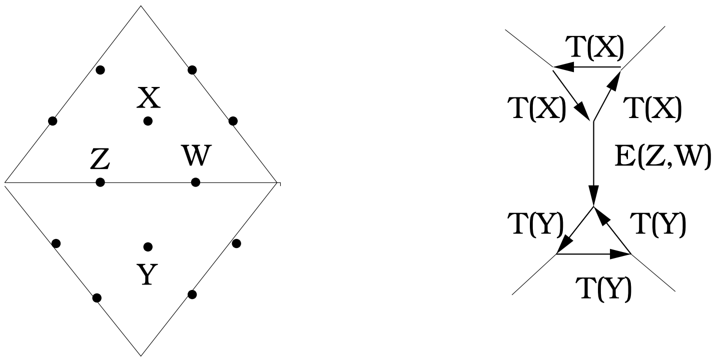

Figure 4: Construction of the monodromy group.

configuration of flags from $\mathcal{P}_3^3$ with triple ratio X, one can define a projective coordinate system on $\mathbb{RP}^2$. Namely, take the one where the point $b \cap c$ has coordinates $[0:1:0]^t$, the point A has coordinates $[1:-1:1]^t$, C – $[1:0:0]^t$ and B – $[0:0:1]^t$ (fig.1). The line a has coordinates$[1 : 1+X : X]$. The cyclic permutation of the flags induces the coordinate change given by the matrix $T(X)$. Similarly if we have a quadruple of flags $F_1F_2F_3F_4$ with two cross-ratios Z and W, then the coordinate system related to the triple $F_2F_4F_1$ is obtained from the coordinate system related to the triple $F_4F_2F_3$ by the coordinate change $E(Z,W)$.

3. The involution σ acts in a very simple way, where $Z' = \frac{W(1+Y)}{Y(1+X)}$ and $W' = \frac{Z(1+X)}{X(1+Y)}$:

In particular a point of $\mathcal{T}_3^+(S)$ is stable under σ if the two coordinates on each edge coincide, and the coordinates in the center of each triangle are equal to one. Taking into

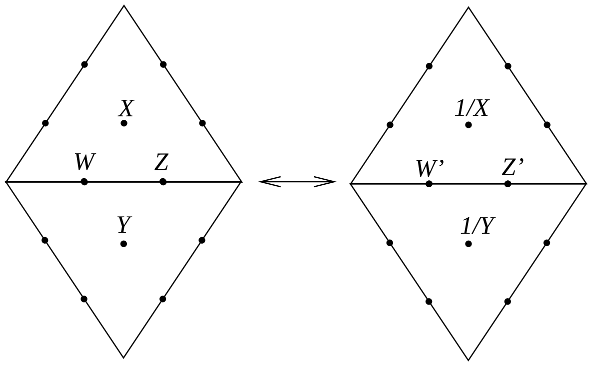

Figure 5: Involution σ.

account that the set of σ-stable points is just the ordinary Teichmüller space, one obtains its parametrisation. Actually it coincides with the one described in [2].

4. Each triangulation of S provides its own coordinate system and in general the transition from one such system to another one is given by complicated rational maps. However any change of triangulation may be decomposed into a sequence of elementary changes — the so called *flips*. A flip removes an edge of the triangulation and inserts another one into the arising quadrilateral as shown on fig.6. This figure shows also how the numbers at the marked points change under the flip. Observe that these formulae allow in particular to pass from one triangulation to the same one, but moved by a nontrivial element of the mapping class group of S, and thus give explicit formulae for the mapping class group action. Here

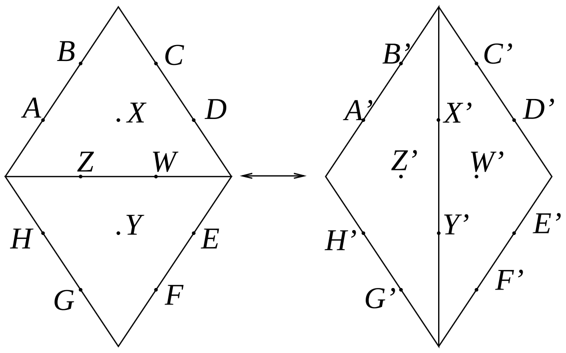

Figure 6: Flip.

$$A' = A(1+Z), \qquad D' = D\frac{W}{1+W}, \qquad E' = E(1+W), \qquad H' = H\frac{Z}{1+Z},$$

$$B' = B\frac{1+Z+ZX+ZXW}{(1+Z)}, \qquad C' = C\frac{(1+W)XZ}{1+Z+ZX+ZXW}$$

$$F' = F\frac{1+W+WY+WYZ}{(1+W)}, \qquad G' = G\frac{(1+Z)YW}{1+W+WY+WYZ},$$

$$X' = \frac{1+Z}{XZ(1+W)}, \quad Y' = \frac{1+W}{YW(1+Z)},$$
$$Z' = X\frac{1+W+WY+WYZ}{1+Z+ZX+ZXW}, \quad W' = Y\frac{1+Z+ZX+ZXW}{1+W+WY+WYZ}.$$
The formulae can be derived directly, or more simply as in Section 11 in [3]. We will return to discussion of the structure of these formulas in Section 2.5 below.

6. Convex projective structures on surfaces with piece-wise geodesic boundary. A convex projective structure on a surface has *piece-wise geodesic* boundary if a vicinity of every boundary point is projectively isomorphic to a vicinity of a boundary point of a half plane or a vertex of an angle. In the latter case it makes sense to consider a line passing through the vertex "outside" of the surface, see punctured lines on Picture 7.

Let us introduce *framed convex projective structure with piece-wise geodesic boundary.* Let $\widehat{S}$ be a pair consisting of an oriented surface S with the boundary ∂S, and a finite (possibly empty) collection of *distinguished* points $\{x_1, ..., x_k\}$ on the boundary. We define *punctured boundary* of $\widehat{S}$ by
$$\partial\widehat{S} := \partial S - \{x_1, ..., x_k\}$$
So the connected components of $\partial\widehat{S}$ are either circles or arcs.

Definition 2.6 *A framed convex projective structure with piece-wise geodesic boundary on $\widehat{S}$ is a convex projective structure on S with the following data at the boundary:*

i) Each connected component of the punctured boundary $\partial\widehat{S}$ is a geodesic interval. Moreover at each point x_i we choose a line passing through x_i outside of S, as on Figure 7.

ii) If a boundary component does not contain distinguished points, we choose its orientation.

The space $\mathcal{T}_3^+(\widehat{S})$ parametrises framed convex projective structures with piece-wise geodesic boundary on $\widehat{S}$.

Remark. Sometimes it is useful to employ a different definition of the framing. Namely, instead of choosing lines through the distinguished points x_i we choose a point p_i inside of each of the geodesic segments bounded by the distinguished points. $\mathcal{T}_3^+(\widehat{S})$ is the moduli space of each of these two structures on S. Indeed, the equivalence between the two definitions is seen as follows. Consider the convex hull of the points p_i. We get a surface S' inscribed into S, with the induced convex projective structure. Conversely, given S' we reconstruct S by taking the circumscribed surface.

The convexity implies that each boundary component develops into a convex infinite polygon connecting two points in $\mathbb{RP}^2$, stabilized by the monodromy around the component. Moreover at each vertex of this polygon there is a line segment located outside of $\widetilde{S}$.

Both spaces $\mathcal{T}_3^+(S)$ and $\mathcal{P}_3^n$ are particular cases of the moduli space $\mathcal{T}_3^+(\widehat{S})$. Indeed, $\mathcal{T}_3^+(S)$ is obtained when the set $\{p_1, ..., p_k\}$ is empty, and we get $\mathcal{P}_3^n$ when S is a disc: in

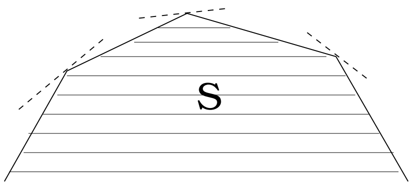

Figure 7: Framed piece-wise geodesic boundary of a convex projective structure on S

this case S is a convex polygon, serving as the inscribed polygon, and the circumscribed polygon is given by the chosen lines.

To introduce the coordinates let us shrink the boundary components without marked points into punctures, and take a triangulation T of S with vertices at the distinguished points on the boundary and at the shred boundary components. We call it *ideal triangulation*. The interior of each side of such a triangulation is either inside of S, or at the boundary. We assign the coordinates to the centers of the triangles and pairs of marked points on the internal sides of the triangulations. Namely, going to the universal cover $\widetilde{S}$ of S, we get a triangulation $\widetilde{T}$ there. Then, by the very definition, for every vertex $\widetilde{p}$ of the triangulation $\widetilde{T}$ there is a flag given at $\widetilde{p}$. We use these flags just as above.

The space $\mathcal{T}_3^+(\widehat{S})$ has the same features -4. as $\mathcal{T}_3^+(S)$. The only thing deserving a comment is construction of a canonical embedding $i : \mathcal{T}_2^+(\widehat{S}) \hookrightarrow \mathcal{T}_3^+(\widehat{S})$. We define the Teichmüller space $\mathcal{T}_2^+(\widehat{S})$ as the space of pairs (a complete hyperbolic metrics on S, a distinguished collection of points $\{x_1, ..., x_k\}$ located at the absolute of S). A point of the absolute of S can be thought of as a $\pi_1(S)$-orbit on the absolute of the hyperbolic plane which is at the infinity of an end of S. Thus going to the universal cover of S in the Klein model we get a finite collection of distinguished points at every arc of the absolute corresponding to an end of S. It remains to make, in a $\pi_1(S)$-equivariant way, finite geodesic polygons with vertices at these points, and add the tangent lines to the absolute at the vertices of these polygons. Taking the quotient by the action of $\pi_1(S)$ we get a framed convex projective structure on S with piece-wise geodesic boundary.

7. Comparing the moduli space of local systems and of convex projective structures. Let $G := PGL_3$ and let $\mathcal{B}$ be the corresponding flag variety. The set of its real points of was denoted by F_3 above. For a G-local system $\mathcal{L}$ on S let $\mathcal{L}_{\mathcal{B}} := \mathcal{L} \times_G \mathcal{B}$ be the associated local system of flags. Recall (Section 2 of [3]) that a *framed G-local system* on $\widehat{S}$ is a pair $(\mathcal{L}, \beta)$ consisting of a G-local system on S and a flat section β of the restriction of $\mathcal{L}_{\mathcal{B}}$ to the punctured boundary $\partial\widehat{S}$. The corresponding moduli space is denoted by $\mathcal{X}_{G,\widehat{S}}$.

Let us shrink all holes on S without distinguished points to punctures. An *ideal triangulation* T of $\widehat{S}$ is a triangulation of S with vertices either at punctures or at the boundary components, so that each connected component of $\partial\widehat{S}$ hosts exactly one vertex of the triangulation. Repeating the construction of Sections 2.4-2.5 we get a set of coordinate systems on $\mathcal{X}_{G,\widehat{S}}$ parametrized by ideal triangulations of $\widehat{S}$. Precisely, given ideal triangulations T there is an open embedding

$$\varphi_T : \mathbb{G}_m^{\{\text{internal edges of } T\}} \hookrightarrow \mathcal{X}_{G,\widehat{S}}$$

Here $\mathbb{G}_m$ stays for the multiplicative group understood as an algebraic group. So the set of its complex points is $\mathbb{C}^*$. The natural coordinate on $\mathbb{G}_m$ provides a rational coordinate function on the moduli space $\mathcal{X}_{G,\widehat{S}}$. Thus the map φ_T provides a (rational) coordinate system on the moduli space. The transition functions between the two coordinate systems corresponding to triangulations related by a flip are given by the formulas written in the very end of Section 2.6. Since these formulas are subtraction free, the real locus of $\mathcal{X}_{G,\widehat{S}}$ contains a well-defined subset

$$\mathcal{X}^+_{G,\widehat{S}} := \varphi_T\left(\mathbb{R}_{>0}^{\{\text{internal edges of } T\}}\right) \hookrightarrow \mathcal{X}_{G,\widehat{S}}(\mathbb{R})$$

Theorem 2.7 *There is a canonical identification* $\mathcal{X}^+_{PGL_3,\widehat{S}} = \mathcal{T}_3^+(\widehat{S})$.

In other words, $\mathcal{T}_3^+(\widehat{S})$ is the subset of the real locus of the moduli space $\mathcal{X}_{PGL_3,\widehat{S}}$ consisting of the points which have positive real coordinates in the coordinate system corresponding to one (and hence any) ideal triangulation of $\widehat{S}$.

Proof. We will assume that $\mathcal{T}_3^+(\widehat{S})$ parametrises the structures described in the Remark after Definition 2.6. (If we adopt Definition 2.6, the distinguished points will play different role in the definition of these moduli spaces).

Let us define a canonical embedding $\mathcal{T}_3^+(\widehat{S}) \to \mathcal{X}^+_{G,\widehat{S}}$. The local system $\mathcal{L}_p$ corresponding to a point $p \in \mathcal{T}_3^+(\widehat{S})$ is the one corresponding to the representation $\mu(p)$. Intrinsically, it is the local system of projective frames on S. Let us define a framing on it. Let C_i be a boundary component of S. If it does not contain the distinguished points, the corresponding component of the framing is provided by the flag (a, A) defined in the proof of Theorem 2.5. If C_i contains distinguished points, we use the flag (p, L) where p is the chosen point on the boundary geodesic interval L. The restriction of the canonical coordinates on the moduli space $\mathcal{X}_{PGL_3,\widehat{S}}$ to the space of convex projective structures gives, by the very definition, the coordinates on the latter space defined above. Now the proof follows immediately from the results of Sections 2.3-2.5. The theorem is proved.

8. Laurent positivity properties of the monodromy representations. Consider the universal PGL_3-local system on the space $S \times \mathcal{X}_{PGL_3,S}$. Its fiber over $S \times p$ is the local system corresponding to the point p of the moduli space $\mathcal{X}_{PGL_3,S}$. Our next goal is to study its monodromy representation.

Let $\mathbb{F}_S$ be the field of rational functions on the moduli space $\mathcal{X}_{PGL_3,S}$. The monodromy of the universal local system around a loop on S is a conjugacy class in $PGL_3(\mathbb{F}_S)$.

Recall that, given an ideal triangulation T of S, we defined canonical coordinates $\{X_i^T\}$ on $\mathcal{X}_{PSL_3,S}$ corresponding to T. Given a coordinate system $\{X_i\}$, a *positive rational function in* $\{X_i\}$ is a function which can be presented as a ratio of two polynomials in $\{X_i\}$ with positive integral coefficients. For instance $1 - x + x^2$ is a positive rational function since it is equal to $(1+x^3)/(1+x)$. A matrix is *totally positive* if all its minors are non-zero positive rational functions in $\{X_i^T\}$. Similarly we define upper/lower triangular totally positive matrices: any minor, which is not identically zero for generic matrix in question, is a positive rational functions.

Theorem 2.8 *Let T be an ideal triangulation of S. Then monodromy of the universal PGL_3-local system on $S \times \mathcal{X}_{PGL_3,S}$ around any non-boundary loop on S is conjugated in $PGL_3(\mathbb{F}_S)$ to a totally positive matrix in the canonical coordinates assigned to T.*

The monodromy around a boundary component is conjugated to an upper/lower triangular totally positive matrix.

Proof. Recall the trivalent graph Γ'_T on Figure 4, defined starting from an ideal triangulation T of S. Let us call its edges forming the little triangles by t-edges, and the edges dual to the edges of the triangulation T by e-edges. The matrices $T(*)$ and $E(*)$ assigned in Section 5.2 to the t- and e-edges of this graph provide an explicit construction of the universal local system. Indeed, since $T(X)^3$ is the identity in PGL_3, we get a universal local system on the dual graph to T. This graph is homotopy equivalent to S.

Given a loop on S, we shrink it to a loop on Γ'_T. We may assume that this loop contains no consecutive t-edges: indeed, a composition of two t-edges is a t-edge. Thus we may choose an initial vertex on the loop so that the edges have the pattern *et...etet*. Therefore the monodromy is computed as a product of matrices of type ET or ET^{-1}. Observe that

$$E(Z,W)T(X) = \begin{pmatrix} Z^{-1}X & Z^{-1}(1+X) & Z^{-1} \\ 0 & 1 & 1 \\ 0 & 0 & W^{-1} \end{pmatrix} \quad \text{and} \tag{2}$$

$$E(Z,W)T(X)^{-1} = \begin{pmatrix} Z^{-1} & 0 & 0 \\ 1 & 1 & 0 \\ W & W(1+X^{-1}) & WX^{-1} \end{pmatrix} \quad . \tag{3}$$

These matrices are upper/lower triangular totally positive integral Laurent matrices in the coordinates X, Y, Z, W. A loop on S contains matrices of just one kind if and only if it is the loop around a boundary component. Therefore the monodromy around any non-boundary loop is obtained as a product of both lower and upper triangular matrices. It remains to use the following fact: if M_i is either lower or upper triangular totally positive matrix, and there are both upper and lower triangular matrices among $M_1, ..., M_k$, then the product $M_1...M_k$ is a totally positive matrix. The theorem is proved.

We say that the monodromy of a $PGL_3(\mathbb{R})$-local system is *regular hyperbolic* if the monodromy around a non-boundary loop has distinct real eigenvalues, and the monodromy around a boundary component is conjugated to a real totally positive upper triangular matrix.

Corollary 2.9 *The monodromy of a convex projective structure with geodesic boundary on S is faithful and regular hyperbolic.*

Proof. Follows immediately from Theorem 2.8 and the Gantmacher-Krein theorem [6] claiming that the eigenvalues of a totally positive matrix are distinct real numbers. The corollary is proved.

Remark. Both Theorem 2.8 and Corollary 2.9 can be generalized to the case when G is an arbitrary split reductive group with trivial center, see [3]. When S have no holes and $G = PSL_m(\mathbb{R})$, a statement similar to Corollary 2.9 was proved in [13].

A Laurent polynomial in $\{X_i^T\}$ is *positive integral* if its coefficients are positive integers. A rational function on the moduli space $\mathcal{X}_{PGL_3,S}$ is called a *good positive Laurent polynomial* if it is a positive integral Laurent polynomial in every canonical coordinate system on $\mathcal{X}_{PGL_3,S}$.

Corollary 2.10 *The trace of the n-th power of the monodromy of the universal PGL_3-local system on S around any loop on S is a good positive Laurent polynomial on $\mathcal{X}_{PGL_3,S}$.*

Proof. A product of matrices with positive intergal Laurent coefficients is again a matrix of this type. So the corollary follows from the explicit construction of the universal PSL_3-local system given in the proof of Theorem 2.8, and formulas (2)-(3).

A good positive Laurent polynomial on $\mathcal{X}_{PGL_3,S}$ is *indecomposable* if it can not be presented as a sum of two non-zero good positive Laurent polynomials on $\mathcal{X}_{PGL_3,S}$.

Conjecture 2.11 *The trace of the n-th power of the monodromy of the universal PGL_3-local system on S around any loop on S is indecomposable.*

3 The universal higher Teichmüller space for $PSL_3(\mathbb{R})$

1. The universal higher Teichmüller space.

Definition 3.1 *i) A set C is* cyclically ordered *if any its finite subset is cyclically ordered, and any inclusion of finite subsets preserves the cyclic order.*

ii) A map β from a cyclically ordered set C to the flag variety F_3 is positive *if it maps every cyclically ordered quadruple (a, b, c, d) to a positive quadruple of flags $(\beta(a), \beta(b), \beta(c), \beta(d))$.*

A positive map is necessarily injective. So, abusing terminology, we define *positive subset* $C \subset F_3$ as the image of a positive map, keeping in mind a cyclic structure on C.

The set of positive n-tuples of flags in F_3 is invariant under the operation of reversing the cyclic order to the opposite one. So it has not only cyclic, but also dihedral symmetry.

Consider a cyclic order on $\mathbb{P}^1(\mathbb{Q})$ provided by the embedding $\mathbb{P}^1(\mathbb{Q}) \hookrightarrow \mathbb{P}^1(\mathbb{R})$.

Definition 3.2 *The universal higher Teichmüller space $\mathcal{T}_3^+$ is the quotient of the space of positive maps $\beta : \mathbb{P}^1(\mathbb{Q}) \longrightarrow F_3$ by the action of the group $PSL_3(\mathbb{R})$.*

Canonical coordinates on $\mathcal{T}_3^+$. Let us choose three distinct points on $\mathbb{P}^1(\mathbb{Q})$, called $0, 1, \infty$. Recall the Farey triangulation, understood as a triangulation of the hyperbolic disc with a distinguished oriented edge. Then we have canonical identifications

$$\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \infty = \{\text{vertices of the Farey triangulation}\} \qquad (4)$$

The distinguished oriented edge goes from 0 to ∞. Consider the infinite set

$$I_3 := \{\text{pairs of points on each edge of the Farey triangulation}\} \cup$$
$$\{\text{(centers of the) triangles of the Farey triangulation}\}$$

A point of $\mathcal{T}_3^+$ gives rise to a positive map

$$\{\text{vertices of the Farey triangulation}\} \longrightarrow F_3$$

considered up to the action of $PGL_3(\mathbb{R})$. We assign to every triple of flags at the vertices of a Farey triangle their triple ratio, and to every quadruple of flags at the vertices of a Farey quadrilateral the related two cross-ratios, pictured at the diagonal. We get a canonical map

$$\varphi_3 : \mathcal{T}_3^+ \longrightarrow \mathbb{R}_{>0}^{I_3} \tag{5}$$

Theorem 3.3 *The map (5) is an isomorphism.*

Proof. It is completely similar to the one of Theorem 2.2, and thus is omitted.

The right hand side in (5) has a natural quadratic Poisson structure given by the formula (1). Using the isomorphism φ_3, we transform it to a Poisson structure on $\mathcal{T}_3^+$.

Here is another way to produce points of $\mathcal{T}_3^+$.

2. Convex curves on $\mathbb{RP}^2$ and positive curves in the flag variety F_3. Let K be a continuous convex curve in $\mathbb{RP}^2$. Then at every point $p \in K$ there is a nonempty collection of lines passing through p and located outside of K. We call them *osculating lines at p*. A *regular convex* curve in $\mathbb{RP}^2$ is a continuous convex curve which has exactly one osculating line at each point. Assigning to a point p of a regular convex curve K the unique osculating line at p we get the *osculating curve* $\widetilde{K}$ in the flag variety F_3. It is continuous. Moreover, thanks to the results of Section 2.3, it is positive, i.e. the map $K \to \widetilde{K} \subset F_3$ is positive. It follows that K is a C^1–smooth convex curve in $\mathbb{RP}^2$, and the

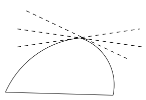

Figure 8: Osculating lines to a convex curve.

osculating curve $\widetilde{K}$ is given by the flags (p, T_pK) where p runs through K. It turns out that this construction gives all positive continuous curves in the flag variety F_3. Recall the double bundle

$$\mathbb{RP}^2 \xleftarrow{p} F_3 \xrightarrow{\widehat{p}} \widehat{\mathbb{RP}}^2$$

where $\widehat{\mathbb{RP}}^2$ is the set of lines in $\mathbb{RP}^2$. A subset of $\mathbb{RP}^2$ is *convex* if no line contains its three distinct points. (warning: a convex curve is a convex subset, but a convex domain is not).

Proposition 3.4 *i) If C a positive subset of F_3, then $p(C)$ and $\widehat{p}(C)$ are convex.*

ii) If C is a continuous positive curve in F_3, then $p(C)$ is a regular convex curve. This gives rise to a bijective correspondence

$$\textit{continuous positive curves in } F_3 <=> \textit{regular convex curves in } \mathbb{RP}^2. \qquad (6)$$

Proof. i) Follows immediately from the results of Section 2.3.

ii) $p(C)$ is continuous and, by i), convex. It comes equipped with a continuous family of osculating lines. It follows from this that $p(C)$ is regular. This gives the arrow => in (6). The osculating curve provide the opposite arrow. Since a convex curve can not have two different continuous families of osculating lines, we got a bijection. The proposition is proved.

So identifying $P^1(\mathbb{Q})$ with a subset of a regular convex curve K respecting the cyclic orders we produce a point of $\mathcal{T}_3^+$.

The Thompson group $\mathbb{T}$. It is the group of all piecewise linear automorphisms of $\mathbb{P}^1(\mathbb{Q})$. By definition, for every $g \in \mathbb{T}$ there exists a decomposition of $\mathbb{P}^1(\mathbb{Q})$ into a union of finite number of segments, which may overlap only at the ends, so that the restriction of g to each segment is given by an element of $PSL_2(\mathbb{Z})$. The Thompson group acts on $\mathcal{T}_3^+$ in an obvious way. Here is another way to look at it. The Thompson group contains the following elements, called flips: Given an edge E of a triangulation T with a distinguished oriented edge, we do a flip at E (as on the Figure 6), obtaining a new triangulation T' with a distinguished oriented edge. Observe that the sets of ends of the trivalent trees dual to the triangulations T and T' are identified. On the other hand, there exists unique isomorphism of the plane trees T and T' which identifies their distinguished oriented edges. It provides a map of the ends of these trees, and hence an automorphism of $P^1(\mathbb{Q})$, which is easily seen to be piecewise linear. The Thompson group is generated by flips ([11]). So the formulas in the end of Section 2.5 allow to write the action of the Thompson group explicitly in our coordinates.

Proposition 3.5 *The action of the Thompson group preserves the Poisson bracket on $\mathcal{T}_3^+$.*

Proof. Each flip preserves the Poisson bracket. The proposition follows.

Let $\mathcal{H}$ be the hyperbolic plane. For a torsion free subgroup $\Delta \subset PSL_2(\mathbb{Z})$, set $S_\Delta := \mathcal{H}/\Delta$.

Proposition 3.6 *The Teichmüller space $\mathcal{T}_3^+(S_\Delta)$ is embedded into the universal Teichmüller space $\mathcal{T}_3^+$ as the subspace of Δ-invariants:*

$$\mathcal{T}_3^+(S_\Delta) = (\mathcal{T}_3^+)^\Delta \qquad (7)$$

This isomorphism respects the Poisson brackets.

Demonstration. The surface S_Δ has a natural triangulation T_Δ, the image of the Farey triangulation under the projection $\pi_\Delta : \mathcal{H} \to S_\Delta$. So according to Theorem 2.2 the left hand side in (7) is identified with the positive valued functions on I_3/Δ, that is with Δ-invariant positive valued functions on I_3. It remains to use Theorem 3.3. The claim about the Poisson structures follows from the very definitions. The proposition is proved.

Remark. The mapping class group of S_Δ is not embedded into the Thompson group, unless we want to replace the latter by a bigger group. Indeed, a flip at an edge E on the surface S_Δ should correspond to the infinite composition of flips at the edges of the Farey triangulations projected onto E. Nevertheless the Thompson group plays the role of the mapping class group for the universal Teichmüller space.

The universal Teichmüller space and the SL_3 Gelfand-Dikii Poisson brackets. The functional space of osculating curves to smooth curves has a natural Poisson structure, the SL_3 Gelfand-Dikii bracket. One can show that its restriction to the subspace provided by the regular convex curves is compatible with our Poisson bracket.

3. Quantum universal higher Teichmüller space $\mathcal{T}_3^q$. Let T_3^q be a non-commutative algebra, (a quantum torus), generated by the elements X_i, where $i \in I_3$, subject to the relations

$$q^{-\varepsilon_{ij}} X_i X_j = q^{-\varepsilon_{ji}} X_j X_i, \quad i, j \in I_3 \tag{8}$$

Here q is a formal variable, so it is an algebra over $\mathbb{Z}[q, q^{-1}]$. The algebra T_3^q is equipped with an antiautomorphism $*$, acting on the generators by $*(X_i) = X_i, *(q) = q^{-1}$. Let us denote by $\mathrm{Frac}(T_3^q)$ its non-commutative fraction field.

Recall that the Thompson group $\mathbb{T}$ is generated by flips at edges of the Farey triangulation. Given such an edge E, let us define an automorphism

$$\varphi_E : \mathrm{Frac}(T_3) \longrightarrow \mathrm{Frac}(T_3) \qquad \text{where } T_3 := T_3^1$$

Consider the 4-gon of the Farey triangulation obtained by gluing the two triangles sharing the edge E — see fig. 6, where E is the horizontal edge. Let $S_E \subset I_3$ be the 12-element set formed by the pairs of marked points on the four sides and two diagonals of the 4-gon. On fig. 6 these are the points labeled by $A, B, ..., Z, W$. We set

$$\varphi_E(X_i) := \begin{cases} X_i' & i \in S_E \\ X_i & i \notin S_E \end{cases}$$

where X_i' is computed using the formulas after fig. 6, so for instance if $X_i = A$, then $X_i' = A'$ and so on.

Proposition 3.7 *The automorphisms φ_E give rise to an action of the Thompson group $\mathbb{T}$ by automorphisms of the field $\mathrm{Frac}(T_3)$ preserving the Poisson bracket.*

Proof. It is known that the only relations between the flips f_E are the following ones:

i)$f_E^2 = \mathrm{Id}$, ii)$f_{E_1} f_{E_2} = f_{E_2} f_{E_1}$ if E_1 and E_2 are disjoint, iii)$(f_{E_5} f_{E_4} f_{E_3} f_{E_2} f_{E_1})^2 = \mathrm{Id}$

where the sequence of flips $f_{E_1}, ..., f_{E_5}$ is shown on fig.8. The first two relations are

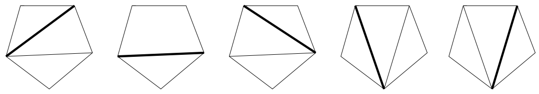

Figure 9: Pentagon relation.

obviously valid. The pentagon relation iii) is clear from the geometric origin of the formula. The proposition is proved.

To introduce the action of the Thompson group on the non-commutative field $\mathrm{Frac}(T_3^q)$, let us recall some facts about the quantum cluster ensembles from [4].

The $\mathcal{X}$-space of the quantum cluster ensemble. We start from the quasiclassical case. Let I be a set, possibly infinite. Consider the following data:

i) A skew-symmetric function $\varepsilon_{ij} : I \times I \to \mathbb{Z}$.

ii) A Poisson algebra $T_\varepsilon := \mathbb{Q}[X_i]$, $i \in I$, with the Poisson bracket $\{X_i, X_j\} = \varepsilon_{ij} X_i X_j$.

We assume that for every $i \in I$, the set of j's such that $\varepsilon_{ij} \neq 0$ is finite.

Consider the homomorphism $\mu_k : \mathrm{Frac}(T_{\varepsilon'}) \longrightarrow \mathrm{Frac}(T_\varepsilon)$ defined on the generators by

$$\mu_k(X_i') := \begin{cases} X_i(1+X_k)^{-\varepsilon_{ik}} & \text{if } k \neq i,\ \varepsilon_{ik} \leq 0 \\ X_i(1+X_k^{-1})^{-\varepsilon_{ik}} & \text{if } \varepsilon_{ik} > 0 \\ X_k^{-1} & \text{if } k = i \end{cases}$$

Lemma 3.8 *The map μ_k preserves the Poisson bracket if and only if the functions ε'_{ij} and ε_{ij} are related by the following formula:*

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \textit{if } k \in \{i,j\} \\ \frac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|}{2} & \textit{if } k \notin \{i,j\} \end{cases} \tag{9}$$

Proof. Straitforward.

Formula (9) appeared first in [5] in the definition of cluster algebra.

Now let us consider the the quantum case. Consider the following data:

i) A skew-symmetric function $\varepsilon_{ij} : I \times I \to \mathbb{Z}$.

ii) The non-commutative field of fractions $\mathrm{Frac}(T_\varepsilon^q)$ of the algebra generated by the elements X_i, $i \in I$, subject to the relations (8).

A *mutation in the direction* $k \in I$ changes the function ε_{ij} to a new one ε'_{ij} given by formula (9). It was proved in Section 3 of [4] that there is a $*$-algebra homomorphism

$$\mu_k : \mathrm{Frac}(T^q_{\varepsilon'}) \longrightarrow \mathrm{Frac}(T^q_\varepsilon)$$

acting on the generators by

$$\mu_k(X_i) = \begin{cases} X_i G_{|\varepsilon_{ik}|}(q; X_k) & \text{if } k \neq i,\ \varepsilon_{ik} \leq 0 \\ X_i G_{|\varepsilon_{ik}|}(q; X_k^{-1})^{-1} & \text{if } \varepsilon_{ik} > 0 \\ X_k^{-1} & \text{if } k = i \end{cases}$$

where

$$G_a(q; X) := \begin{cases} \prod_{i=1}^a (1 + q^{2i-1}X) & a > 0 \\ 1 & a = 0 \end{cases}$$

Our goal is the following result.

Theorem 3.9 *The Thompson group* $\mathbb{T}$ *acts by* $*$*-automorphisms of the non-commutative field* $\mathrm{Frac}(T_3^q)$*, so that specializing* $q = 1$ *we recover the action from Proposition 3.7.*

Proof. To get the explicit formulae for the action of flips, we will apply the above construction to the situation when $I = I_3$, and the function ε_{ij} was defined in Section 2.5.

We define a flip

$$\varphi_E : \mathrm{Frac}(T_3^q) \longrightarrow \mathrm{Frac}(T_3^q)$$

as the composition of the mutations in the directions of Z, W, X, Y on fig.6. An easy computation, presented in the Appendix below in the quantum form, shows that in the case $q = 1$ this leads to the formulas after fig.6. In the general case one needs to check the relations for the flips. The first two of them are obvious. The pentagon relation is checked by a tedious computation. In fact $f_{E_5}f_{E_4}f_{E_3}f_{E_2}f_{E_1}$ differs from the identity only by an involution of the seven variables assigned to the marked points inside of the pentagon. A more conceptual (and general) proof of this relation will be discussed somewhere else. The theorem is proved.

Summarizing, we can reformulate all this using the cluster ensemble language as follows:

Theorem 3.10 *The universal higher Teichmüller space* $\mathcal{T}_3^+$ *is the set of the positive real points of the* $\mathcal{X}$*-space of the cluster ensemble related to the pair*

$$\{\text{the set } I_3, \text{ the cluster function } \varepsilon_{ij} \text{ defined in Section 2.5}\}$$

The Thompson group is a subgroup of the mapping class group of that cluster ensemble.

The quantum universal higher Teichmüller space and the W_3*-algebra.* We suggest that the action of the Thompson group by birational $*$-automorphisms of the quantum universal Teichmüller space $\mathcal{T}_3^q$ should be considered as an incarnation of the W-algebra corresponding to SL_3.

4. Quantum higher Teichmüller space $\mathcal{T}_3^q(S)$. Let us present a surface S as the quotient $S = S_\Delta := \mathcal{H}/\Delta$. We set

$$\mathcal{X}_3^q(S_\Delta) := \mathrm{Frac}(T_3^q)^\Delta \tag{10}$$

The mapping class group of S acts by positive birational $*$-automorphisms of the $*$-algebra (10).

5. Appendix: Formulas for a quantum flip. Performing mutations at Z and W we get

$$B_1 = B, \quad C_1 = C, \quad F_1 = F, \quad G_1 = G, \quad Z_1 = Z^{-1}, \quad W_1 = W^{-1}$$

$$A_1 = A(1+qZ), \quad D_1 = D(1+qW^{-1})^{-1}, \quad H_1 = H(1+qZ^{-1})^{-1}, \quad E_1 = E(1+qW),$$

$$X_1 = X(1+qZ^{-1})^{-1}(1+qW), \quad Y_1 = Y(1+qZ)(1+qW^{-1})^{-1}$$

Mutations of the function ε_{ij} are illustrated on the picture, where the centers of mutations are shown by little circles.

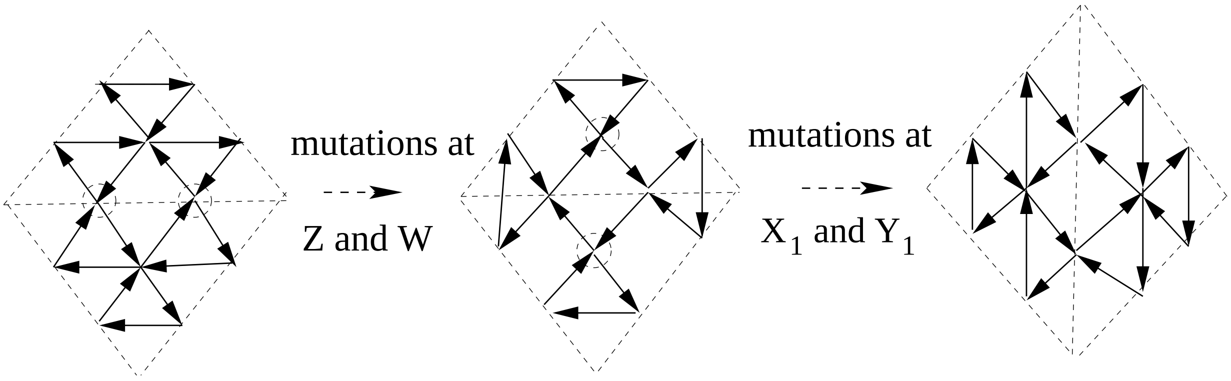

Figure 10: Mutations of the function ε_{ij}

Performing mutations at X_1 and Y_1 we arrive at the final formulas for the quantum flip:

$$A' = A(1+qZ), \quad D' = D(1+qW^{-1})^{-1}, \quad E' = E(1+qW), \quad H' = H(1+qZ^{-1})^{-1},$$

$$X' = (1+qW)^{-1}(1+qZ^{-1})X^{-1}, \quad Y' = (1+qW^{-1})(1+qZ)^{-1}Y^{-1},$$

$$B' = B(1+qZ)^{-1}(1+qZ+q^2ZX+q^3ZXW)$$

$$F' = F(1+qW)^{-1}(1+qW+q^2WY+q^3WYZ)$$

$$C' = CZX(1+qW)\Big(1+q^{-1}Z+q^{-2}XZ+q^{-3}WXZ\Big)^{-1}$$

$$G' = GWY(1+qZ)(1+q^{-1}W+q^{-2}YW+q^{-3}ZYW)^{-1}$$

$$Z' = X(1+qW)\Big(1+q^{-1}Z+q^{-2}XZ+q^{-3}WXZ\Big)^{-1}(1+qW)^{-1}(1+qW+q^2WY+q^3WYZ)$$

$$W' = W^{-1}(1+qZ)^{-1}(1+qZ+q^2ZX+q^3ZXW)WY(1+qZ)\Big(1+q^{-1}W+q^{-2}YW+q^{-3}ZYW\Big)^{-1}$$

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