Representation theory of (modified) Reflection Equation Algebra of $GL(m|n)$ type

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Abstract

Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a Hecke type solution of the quantum Yang-Baxter equation (a Hecke symmetry). Then, the Hilbert-Poincré series of the associated R-exterior algebra of the space V is a ratio of two polynomials of a degree m (numerator) and n (denominator).

Assuming R to be skew-invertible, we define a rigid quasitensor category $\mathrm{SW}(V_{(m|n)})$ of vector spaces, generated by the space V and its dual V^*, and compute certain numerical characteristics of its objects. Besides, we introduce a braided bialgebra structure in the modified Reflection Equation Algebra, associated to R, and equip objects of the category $\mathrm{SW}(V_{(m|n)})$ with an action of this algebra. In the case related to the quantum group $U_q(sl(m))$, we consider the Poisson counterpart of the modified Reflection Equation Algebra and compute the semiclassical term of the pairing, defined via the categorical (or quantum) trace.

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1 Introduction

Reflection Equation Algebra (REA) is a very useful tool of the theory of integrable systems with boundaries. It inherits its name from an equation describing the factorized scattering on a half-line (cf. [C], where the REA depending on a spectral parameter was first introduced).

Nowadays, different types of the REA are known (cf. [KS]), which have applications in mathematical physics and non-commutative geometry.

The REA related to the Drinfeld-Jimbo Quantum Group (QG) $U_q(sl(m))$ appears in constructing a q-analog of differential calculus on the groups $GL(m)$ and $SL(m)$, where it was treated to be a q-analog of the exponential of vector fields (cf. [FP]).

In the case related to the QG $U_q(\mathfrak{g})$, an appropriate quotient of the REA can be treated as a deformation of the coordinate ring $\mathbb{K}[G]$ where G is the Lie group, corresponding to a classical Lie

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algebra $\mathfrak{g}$. The Poisson bracket corresponding to this deformation was introduced by M.Semenov-Tian-Shansky[1].

Though the best known REA is related to the QG $U_q(\mathfrak{g})$, such an algebra can be associated to any braiding $R : V^{\otimes 2} \to V^{\otimes 2}$, where V is a finite dimensional linear space over the ground field[2] $\mathbb{K}$ and R is an invertible solution of the quantum Yang-Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}\,. \tag{1.1}$$

Here the indices of R relate to the space (or spaces) in which the operator is applied. Thus, R_{12} and R_{23} are the following operators in the space $V^{\otimes 3}$: $R_{12} = R \otimes I$, $R_{23} = R \otimes I$.

In the present paper we shall deal with a *Hecke type* solutions of the Yang-Baxter equation (1.1) which satisfy the following condition

$$(R - q\,I)(R + q^{-1}\,I) = 0\,, \tag{1.2}$$

where the nonzero parameter $q \in \mathbb{K}$ is assumed to be *generic*. By definition, this means, that the values of q do not belong to a countable set of the roots of unity: $q^k \neq 1$, $k = 2, 3, ...$ (whereas the value $q = 1$ is not excluded). Consequently,

$$k_q := \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0, \qquad \forall\, k \in \mathbb{N}\,,$$

k_q being a a q-analog an integer k. In what follows, a braiding satisfying relation (1.2) will be called *a Hecke symmetry*.

Especially, we shall be interested in families of Hecke symmetries R_q analytically depending on the parameter q in a neighbourhood of $1 \in \mathbb{K}$ in such a way, that for $q = 1$ the symmetry $R = R_1$ is involutive: $R^2 = I$.

The well known example of such a family is the $U_q(sl(m))$ Drinfeld-Jimbo braidings

$$R_q = \sum_{i,j=1}^{m} q^{\delta_{ij}}\, h_i^j \otimes h_j^i + \sum_{i<j}^{m} (q - q^{-1})\, h_i^i \otimes h_j^j \tag{1.3}$$

where the elements h_i^j form the natural basis in the space of left endomorphisms of V, that is $h_i^j(x_k) = \delta_k^j\, x_i$ in a fixed basis $\{x_k\}$ of the space V. Note that for $q = 1$ the above braiding R equals the usual flip P.

The Hecke symmetry (1.3) and all related objects will be called *standard*. However, a big amount of Hecke symmetries different from the standard one is known, even those which are not deformations of the usual flip (cf. [G3]).

Let us consider the REA corresponding to the standard $U_q(sl(m))$ Hecke symmetry (1.3) in more detail. This algebra possesses some very important properties, in contrast with the REA related to other quantum groups $U_q(\mathfrak{g})$, $\mathfrak{g} \neq sl(m)$.

First of all, it is a q-deformation of the commutative algebra $\mathrm{Sym}\,(gl(m)) = \mathbb{K}[gl(m)^*]$ (so, we get a deformation algebra without any additional quotation). Second, by a linear shift of REA generators (proportional to a parameter $\hbar$), we come to quadratic-linear commutation relations for the shifted generators. In this basis the REA can be treated as a "double deformation" of the initial commutative algebra $\mathbb{K}[gl(m)^*]$. We shall refer to this form of the REA as *modified Reflection*

[1]Note that on any classical Lie group G there exists another Poisson bracket due to E.Sklyanin. Its quantum analog is an appropriate quotient of the so-called RTT algebra (cf. [FRT]). These two quantum analogs of the space $\mathbb{K}[G]$ are related by a transmutation procedure introduced by S. Majid (cf. [M] and references therein). Nowadays, there exits their universal treatment based on pairs of so-called compatible braidings (cf. [IOP, GPS1, GPS2]).

[2]Mainly we are dealing with $\mathbb{K} = \mathbb{C}$ but sometimes $\mathbb{K} = \mathbb{R}$ is allowed.

Equation Algebra (mREA) and shall denote it as $\mathcal{L}(R_q,\hbar)$. By specializing at $\hbar=0$ we return to the (non-modified) REA $\mathcal{L}(R_q)$.

The specialization of the algebra $\mathcal{L}(R_q,\hbar)$ at $q=1$ gives the enveloping algebra $U(gl(m)_\hbar)$ where the notation $\mathfrak{g}_\hbar$ means that the bracket $[\,,\,]$ of a Lie algebra $\mathfrak{g}$ is replaced by $\hbar[\,,\,]$. (Note, that this fact was observed in [IP].) The commutative algebra $\mathbb{K}[gl(m)^*]$ is obtained by the double specialization of the algebra $\mathcal{L}(R_q,\hbar)$ at $\hbar=0$ and $q=1$.

Being supplied with the $U_q(sl(m))$-module structure, the algebra $\mathcal{L}(R_q,\hbar)$ (as well as $\mathcal{L}(R_q)$) is $U_q(sl(m))$-equivariant (or covariant). This means that

$$M(x\cdot y)=M_{(1)}(x)\cdot M_{(2)}(y)\,,\quad \forall\, M\in U_q(sl(m))\,,\quad \forall\, x,y\in \mathcal{L}(R_q,\hbar)$$

where we use the Sweedler's notation for the quantum group coproduct $\Delta(M)=M_{(1)}\otimes M_{(2)}$.

The Poisson counterpart of the above double deformation of the algebra $\mathbb{K}[gl(m)^*]$ is a Poisson pencil

$$\{\,,\,\}_{PL,r}=a\,\{\,,\,\}_{PL}+b\,\{\,,\,\}_r\,,\quad a,b\in\mathbb{K} \tag{1.4}$$

where $\{\,,\,\}_{PL}$ is the linear Poisson-Lie bracket related to the Lie algebra $gl(m)$ and $\{\,,\,\}_r$ is a natural extension of the Semenov-Tian-Shansky bracket on the linear space $gl(m)^*$.

We consider these Poisson structures and briefly discuss their role in defining a "quantum orbit" $\mathcal{O}\subset gl(m)^*$ in section 7. Taking the two-dimensional sphere as an example, we suggest a method of constructing such quantum orbits. In contrast with other definitions of quantum homogeneous spaces, our quantum orbits are some quotients of the algebra $\mathcal{L}(R_q,\hbar)$. They look like the "fuzzy sphere"

$$\mathcal{SL}^c(\hbar)=U(su(2)_\hbar)/\langle C-c\rangle$$

where C is the quadratic Casimir element. As is known, there exists a discrete series of numbers $c_k\in\mathbb{K}$ such that any algebra $\mathcal{SL}^{c_k}(\hbar)$ has a finite dimensional representation in a linear space V_k and the corresponding map $\mathcal{SL}^{c_k}(\hbar)\to\mathrm{End}(V_k)$ is an $su(2)$-morphism.

A similar statement is valid for the aforementioned quotients of the algebra $\mathcal{L}(R_q,\hbar)$. However, the corresponding spaces V_k become objects of a quasitensor category. In such a category, an object is characterized by its quantum dimension which is defined via the categorical (quantum) trace. A deformation of the usual trace is one of the main features of our approach to the quantum homogeneous spaces. In section 7 we describe the semiclassical term of the paring defined via the quantum trace in the case of the standard Hecke symmetry.

In a similar way we treat other quasitensor categories generated by skew-invertible Hecke symmetries. Roughly speaking, we are dealing with three problems in the present paper. The first problem is the classification of all (skew-invertible) Hecke symmetries R. One of the main tools for studding this problem is the Hilbert-Poincaré (HP) series $P_-(t)$ corresponding to the "R-exterior algebra" of the space V (its definition is presented in section 3). Though a classification of all possible forms of the HP series $P_-(t)$ has not been found yet, it is known that the HP series $P_-(t)$ of any Hecke symmetry is a rational function[3] [H, D]. The ordered pair of integers $(m|n)$, where m (resp., n) is the degree of the numerator $N(t)$ (resp., denominator $D(t)$) of $P_-(t)$, plays an important role in the sequel and will be called the *bi-rank* of the Hecke symmetry R (or of the corresponding space V). This pair enters our notation of the quasitensor Schur-Weyl category $\mathrm{SW}(V_{(m|n)})$ generated by V.

Constructing the category $\mathrm{SW}(V_{(m|n)})$ is the second problem we are dealing with in this paper. The objects of the category are direct sums of vector spaces $V_\lambda\otimes V_\mu^*$. Here V is the basic vector

[3]The HP series corresponding to a skew-invertible Hecke symmetry is described in section 3. When $P_-(t)$ is a polynomial (in this case we say that R is even) it can drastically differ from the classical one $(1+t)^n$, $n=\dim V$. Thus, in [G3] all skew-invertible Hecke symmetries with $P_-(t)=1+nt+t^2$ were classified. Besides, suggested in [G3] was a way of "gluing" such symmetries which gives rise to skew-invertible Hecke symmetries with other non-standard HP series.

space equipped with a skew-invertible Hecke symmetry R, V^* is its dual, and λ and μ stand for arbitrary partitions (Young diagrams) of positive integers. The map $V \to V_\lambda$ is nothing but the Schur functor corresponding to the Hecke symmetry R (for its classical version cf. [FH]). The map $V^* \to V^*_\mu$ can be defined in a similar way. Note, that $\mathrm{SW}(V_{(m|n)})$ is a monoidal quasitensor rigid category (as defined in [CP]) but it is not abelian.

We compute some numerical characteristics of objects of this category. Namely, we are interested in their dimensions (classical and quantum). In contrast with the classical dimensions, which essentially depend on a concrete form of the initial Hecke symmetry and is expressed via the roots of above polynomials $N(t)$ and $D(t)$, the quantum dimensions depend only on the bi-rank $(m|n)$. Moreover, in a sense, the category $\mathrm{SW}(V_{(m|n)})$ looks like the tensor category of $U(gl(m|n))$-modules.

The third problem elaborated below is in constructing the representations of the mREA $\mathcal{L}(R_q, \hbar)$ in the category $\mathrm{SW}(V_{(m|n)})$. Since for $q \neq 1$ the algebra $\mathcal{L}(R_q, \hbar)$ is isomorphic to the non-modified REA $\mathcal{L}(R_q)$ (in fact, we have the same algebra written in two different basises), we automatically get a representation category of the latter algebra[4]. Note, that certain representations of the REA have already known, mainly for the even case (the bi-rank $(m|0)$) [K, Mu1, GS2, S]. By contrast with those papers, we here consider the mREA $\mathcal{L}(R_q, \hbar)$ connected with a general type skew-invertible Hecke symmetry R of the bi-rank $(m|n)$ and equip objects of the category $\mathrm{SW}(V_{(m|n)})$ with the $\mathcal{L}(R_q, \hbar)$-module structure. Note, that all the corresponding representations are equivariant (see section 6).

A particular example we are interested in is the "adjoint" representation. By this we mean a representation ρ_{ad} of the mREA $\mathcal{L}(R_q, \hbar)$ in the linear span of its generators. In the case, when a Hecke symmetry is a super-flip in a Z_2-graded linear space V

$$R: \; V^{\otimes 2} \to V^{\otimes 2}, \qquad R(x \otimes y) = (-1)^{\overline{x}\,\overline{y}} y \otimes x\,,$$

where x and y are homogenous elements of V and $\bar{z}$ denotes the parity (grading) of a homogeneous element z, the mREA becomes the enveloping algebra $U(gl(m|n))$ and the representation ρ_{ad} coincides with the usual adjoint one. This is one of the reasons why we treat the mREA $\mathcal{L}(R_q, \hbar)$ as a suitable analog of the enveloping algebra. Moreover, in the case of involutive skew-invertible Hecke symmetry, the corresponding mREA becomes the enveloping algebra of a generalized Lie algebra $\mathrm{End}(V)$ as is explained in section 5. Such type algebras were introduced in [G1].

The other property that makes the mREA to be similar to the enveloping algebra of a generalized Lie algebra (in particular, a super Lie algebra) is its braided bialgebra structure. Such a structure is determined by a coproduct Δ and a counit ε. On the generators of mREA (organized into a matrix L (see section 6)) the coproduct reads

$$\Delta(L) = L \otimes 1 + 1 \otimes L - (q - q^{-1}) L \otimes L$$

and coincides with the coproduct of the enveloping algebra of the (generalized) Lie algebra at $q = 1$. Note, that though we do not define an antipode in the algebra $\mathcal{L}(R_q, \hbar)$, the category $\mathrm{SW}(V_{(m|n)})$ of its representations is closed.

In addition to the $\mathcal{L}(R_q, \hbar)$-module structure, the objects of the Schur-Weyl category, corresponding to the standard Hecke symmetry (1.3), can be equipped with the action of the QG $U_q(sl(m))$. Besides, the q-analogs of super-groups (cf. [KT]) can also be represented in the corresponding Schur-Weyl category. (Suggested in [Z] is another way of constructing the representations of q-deformed algebras $U(gl(m|n)$ which is based on the triangular decomposition.) Nevertheless, in general we know no explicit construction of the QG type algebra for a skew-invertible Hecke symmetry[5] whereas the mREA can be defined for any skew-invertible Hecke symmetry.

[4]Since for $q = 1$ the isomorphism $\mathcal{L}(R_q, \hbar) \cong \mathcal{L}(R_q)$ breaks, we prefer to consider these algebras separately and use different names for them.

[5]An attempt of explicit description of such an object for some even non-quasiclassical Hecke symmetries was undertaken in [AG].

The mREA has one more advantage as compared with the QG or their super-analogs. It is a more convenient tool for the explicit construction of projective modules over quantum orbits in the frameworks of approach suggested in [GS1, GS3]. We plan to turn to these objects in a general (not necessarily even) case in our subsequent publications.

To complete the Introduction, we would like to emphasize a difference between the Hecke type braidings and the Birman-Murakami-Wenzl ones (in particular, those coming from the QG of B_n, C_n and D_n series). In the latter case it is not difficult to define a "braided Lie bracket" in the space $\mathrm{End}(V)$ (cf. [DGG]) and introduce the corresponding "enveloping algebra". But this "enveloping algebra" is not a deformation of its classical counterpart and therefore is not an interesting object from our viewpoint.

The paper is organized as follows. In the next section we reproduce some elements of R-technique which form the base of subsequent computations of some interesting numerical characteristics of objects involved (the most cumbersome part of the computations is placed in Appendix). Section 3 is devoted to the classification of (skew-invertible) Hecke symmetries. In section 4 we construct the Schur-Weyl category $\mathrm{SW}(V_{(m|n)})$ generated by the space V. Our main object, the mREA $\mathcal{L}(R_q,\hbar)$, is introduced in section 5 where we also study its deformation properties. In section 6 we equip the mREA with a braided bialgebra structure which allows us to define an equivariant action of the algebra $\mathcal{L}(R_q,\hbar)$ on each object of the category $\mathrm{SW}(V_{(m|n)})$. There we also present our viewpoint on definition of braided (quantum) Lie algebras. Section 7 is devoted to study of some semiclassical structures.

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2 Elements of R-technique

By R-technique we mean computational methods based on general properties of braidings (in particular, Hecke symmetries) regardless of their concrete form. We are mostly interested in the so-called skew-invertible braidings since they enable us to define numerical characteristics of Hecke symmetries and related objects.

A braiding R (see (1.1)) is called *skew-invertible* if there exists an endomorphism $\Psi : V^{\otimes 2} \to V^{\otimes 2}$ such that

$$\mathrm{Tr}_{(2)}\, R_{12}\, \Psi_{23} = P_{13} = \mathrm{Tr}_{(2)}\, \Psi_{12}\, R_{23} \tag{2.1}$$

where the symbol $\mathrm{Tr}_{(2)}$ means calculating trace in the second factor of the tensor product $V^{\otimes 3}$. Hereafter P stands for the usual flip $P(x\otimes y) = y\otimes x$.

Fixing basises $\{x_i\}$ and $\{x_i\otimes x_j\}$ in V and $V^{\otimes 2}$ respectively, we can identify R (resp. Ψ) with a matrix $\|R_{ij}^{kl}\|$ (resp. $\|\Psi_{ij}^{kl}\|$):

$$R(x_i\otimes x_j) = x_k\otimes x_l\, R_{ij}^{kl} \tag{2.2}$$

where the upper indices mark the rows of the matrix and from now on the summation over the repeated indices is understood.

Being written in terms of matrices, relation (2.1) reads

$$R_{jb}^{ia}\, \Psi_{ak}^{bl} = \delta_k^i \delta_j^l = \Psi_{jb}^{ia}\, R_{ak}^{bl}\,.$$

With the use of Ψ we can define two endomorphisms B and C of the space V

$$B(x_i) = x_j B_i^j, \qquad C(x_i) = x_j C_i^j,$$

where

$$B_i^{\,j} := \Psi_{ki}^{kj}, \qquad C_i^{\,j} := \Psi_{ik}^{jk}, \tag{2.3}$$

that is

$$B := \mathrm{Tr}_{(1)}\,\Psi, \qquad C := \mathrm{Tr}_{(2)}\,\Psi.$$

If the operator B (or C) is invertible, then the corresponding braiding R will be called *strictly* skew-invertible. As was shown in [O], R is strictly skew-invertible iff R^{-1} is skew-invertible and, besides, the invertibility of B leads to the invertibility of C and vice versa.

The well known important example of a strictly skew-invertible braiding is the super-flip R on a super-space $V = V_0 \oplus V_1$, where V_0 and V_1 are respectively the even and odd components of V. In this case the operators B and C are called *the parity operators* and their explicit form is as follows

$$B(z) = C(z) = z_0 - z_1, \quad \forall\, z \in V,$$

where $z_0(z_1)$ is the even (odd) component of $z = z_0 + z_1$.

Let R be a skew-invertible braiding. Listed below are some useful properties of the corresponding endomorphisms Ψ, B and C.

1. $$\mathrm{Tr}\,B = \mathrm{Tr}\,C,$$

   $$\mathrm{Tr}_{(2)}\,B_2\,R_{21} = \mathrm{Tr}_{(2)}\,C_2\,R_{12} = I_1, \tag{2.4}$$

   where I is the identical automorphism of V. These relations directly follow from definitions (2.1) and (2.3).

2. The endomorphisms B and C commute and their product is a scalar operator

   $$B\,C = C\,B = \nu\, I, \tag{2.5}$$

   where a numeric factor ν is nonzero iff the braiding R is strictly skew-invertible (in particular, if R is a skew-invertible Hecke symmetry).

3. The matrix elements of B and C realize a one-dimensional representation of the so-called *RTT* algebra, associated with R (cf. [FRT]), that is

   $$R_{12}B_1B_2 = B_1B_2R_{12}, \qquad R_{12}C_1C_2 = C_1C_2R_{12}\,. \tag{2.6}$$

   As a direct consequence of the above relations, we have

   $$\mathrm{Tr}_{(12)}(B_1B_2R_{12}X_{12}R_{12}^{-1}) = \mathrm{Tr}_{(12)}(B_1B_2R_{12}^{-1}X_{12}R_{12}) = \mathrm{Tr}_{(12)}(B_1B_2X_{12}),$$
   $$\mathrm{Tr}_{(12)}(C_1C_2R_{12}X_{12}R_{12}^{-1}) = \mathrm{Tr}_{(12)}(C_1C_2R_{12}^{-1}X_{12}R_{12}) = \mathrm{Tr}_{(12)}(C_1C_2X_{12})$$

   where $X \in \mathrm{End}(V^{\otimes 2})$ is an arbitrary endomorphism and $\mathrm{Tr}_{(12)}(\ldots) = \mathrm{Tr}_{(1)}(\mathrm{Tr}_{(2)}(\ldots))$.

4. The following important relations were proved in [OP2, S]

   $$\begin{aligned} B_1\Psi_{12} &= R_{21}^{-1}B_2, \qquad & \Psi_{12}B_1 &= B_2R_{21}^{-1},\\ C_2\Psi_{12} &= R_{21}^{-1}C_1, \qquad & \Psi_{12}C_2 &= C_1\,R_{21}^{-1}, \end{aligned} \tag{2.7}$$

   where $R_{21} = PR_{12}P$. In case $\nu \neq 0$, only one of the lines above is independent due to relation (2.5).

   Therefore, for an arbitrary endomorphism $X \in \mathrm{End}(V)$ we obtain

   $$\begin{aligned} &\mathrm{Tr}_{(1)}(B_1R_{12}X_2R_{12}^{-1}) = \mathrm{Tr}_{(1)}(B_1R_{12}^{-1}X_2R_{12}) = \mathrm{Tr}(BX)\,I_2,\\ &\mathrm{Tr}_{(2)}(C_2R_{12}X_1R_{12}^{-1}) = \mathrm{Tr}_{(2)}(C_2R_{12}^{-1}X_1R_{12}) = \mathrm{Tr}(CX)\,I_1. \end{aligned} \tag{2.8}$$

This completes the list of technical facts to be used in the text below.

3 The general form of a Hecke symmetry

In this section we study the classification problem of (skew-invertible) Hecke symmetries. Our presentation is based on the theory of the A_{k-1} series Hecke algebras and their R-matrix representations. As a review of the subject we can recommend the work [OP1]. Some necessary facts of the mentioned theory are given in Appendix for the reader's convenience.

Given a Hecke symmetry $R : V^{\otimes 2} \to V^{\otimes 2}$, we consider the R-symmetric $\Lambda_+(V)$ and the R-skew-symmetric $\Lambda_-(V)$ algebras of the space V, which by definition are the following quotients

$$\Lambda_\pm(V) := T(V)/\langle(\mathrm{Im}(q^{\pm 1}\, I_{12} \mp R_{12})\rangle, \qquad I_{12} = I \otimes I. \tag{3.1}$$

Hereafter $T(V)$ stands for the free tensor algebra of the space V and $\langle J\rangle$ denotes the two-sided ideal generated in this algebra by a subset $J \subset T(V)$.

Then, we consider the Hilbert-Poincaré (HP) series of the algebras $\Lambda_\pm(V)$

$$P_\pm(t) := \sum_{k\ge 0} t^k \dim\, \Lambda^k_\pm(V), \tag{3.2}$$

where $\Lambda^k_\pm(V) \subset \Lambda_\pm(V)$ is the homogenous component of degree k.

The following proposition plays the decisive role in the classification of all possible forms of the Hecke symmetries.

Proposition 1 *Consider an arbitrary Hecke symmetry R, satisfying* (1.1) *and* (1.2) *at a generic value of the parameter q. Then the following properties hold true.*

1. *The HP series $P_\pm(t)$ obey the relation*

$$P_+(t)\, P_-(-t) = 1.$$

2. *The HP series $P_-(t)$ (and hence $P_+(t)$) is a rational function of the form:*

$$P_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\, t + \ldots + a_m\, t^m}{1 - b_1\, t + \ldots + (-1)^n\, b_n\, t^n} = \frac{\prod_{i=1}^m (1 + x_i t)}{\prod_{j=1}^n (1 - y_j t)}\,, \tag{3.3}$$

where the coefficients a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are mutually prime, and all real numbers x_i and y_i are positive.

3. *If, in addition, the Hecke symmetry is skew-invertible, then the polynomials $N(t)$ and $D(-t)$ are reciprocal*[6]*.*

The first item of the above list was proved in [G2], the second and the third ones — in [H, Da] and [DH].

Definition 2 Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a skew-invertible Hecke symmetry and let m (resp. n) be the degree of the numerator $N(t)$ (resp. the denominator $D(t)$) of the HP series $P_-(t)$. The ordered pair of integers $(m|n)$ will be called *the bi-rank* of R. If $n = 0$ (resp. $m = 0$), the Hecke symmetry will be called *even* (reps. *odd*). Otherwise we shall say that R is of the general type.

Remark 3 In the sense of the above definition, any skew-invertible Hecke symmetry is a generalization of the super-flip for which $P_-(t) = (1+t)^m(1-t)^{-n}$, where $m = \dim\, V_0$, $n = \dim\, V_1$. Such a treatment of Hecke symmetries is also motivated by similarity of the corresponding Schur-Weyl categories (see below).

[6]Recall, that a polynomial $p(t) = c_0 + c_1 t + \ldots + c_n t^n$ with real coefficients c_i is called *reciprocal* if $p(t) = t^n p(t^{-1})$ or, equivalently, $c_i = c_{n-i}$, $0 \le i \le n$.

Now we obtain some important consequences of Proposition 1. Let R be a Hecke symmetry of the bi-rank $(m|n)$. As is known, the Hecke symmetry R allows us to define a representations ρ_R of the A_{k-1} series Hecke algebras $H_k(q)$, $k \geq 2$, in homogeneous components $V^{\otimes p} \subset T(V)$, $\forall\, p \geq k$

$$\rho_R: \ H_k(q) \to \mathrm{End}(V^{\otimes p}), \quad p \geq k\,.$$

Explicitly, these representations are given in (A.3) of Appendix.

Under the presentation ρ_R, the primitive idempotents $e_a^\lambda \in H_k(q)$, $\lambda \vdash k$, convert to the projection operators

$$E_a^\lambda(R) = \rho_R(e_a^\lambda) \in \mathrm{End}(V^{\otimes p}), \quad p \geq k\,, \tag{3.4}$$

where the index a enumerates the standard Young tableaux (λ, a), which can be constructed for a given partition $\lambda \vdash k$. The total number of the standard Young tableaux corresponding to the partition λ shall be denoted as d_λ.

Under the action of these projectors the spaces $V^{\otimes p}$, $p \geq 2$, are expanded into the direct sum

$$V^{\otimes p} = \bigoplus_{\lambda \vdash p} \bigoplus_{a=1}^{d_\lambda} V_{(\lambda,a)}, \qquad V_{(\lambda,a)} = \mathrm{Im}(E_a^\lambda). \tag{3.5}$$

Due to relation (A.2), the projectors E_a^λ with different a are connected by invertible transformations and, therefore, all spaces $V_{(\lambda,a)}$ with fixed λ and different a are isomorphic.

At a generic value of q, the Hecke algebra $H_k(q)$ is known to be isomorphic to the group algebra $\mathbb{K}[\mathfrak{S}_k]$ [We]. Basing on this fact, we can prove the following result [GLS1, H]

$$V_{(\lambda,a)} \otimes V_{(\mu,b)} = \bigoplus_{\nu} \bigoplus_{d_{ab} \in I_{ab}} V_{(\nu, d_{ab})} \cong \bigoplus_{\nu} c_{\lambda\mu}^\nu\, V_{(\nu,d_0)}\,, \qquad \lambda \vdash p,\, \mu \vdash k\,, \nu \vdash (p+k)\,, \tag{3.6}$$

where the integers $c_{\lambda\mu}^\nu$ are the Littlwood-Richardson coefficients, the tableau index d_{ab} takes the values form a subset $I_{ab} \subset \{1, 2, \dots, d_\nu\}$, which depends on the values of the indices a and b. The number d_0 in the last equality stands for the index of an arbitrary fixed tableau from the set (ν, d), $1 \leq d \leq d_\nu$. This equality have the following meaning. Though the summands $V_{(\nu,d_{ab})}$ do depend on the values of a and b, the *total* number of these summands (the cardinality of I_{ab}) depends only on the partitions λ, μ and ν and is equal to the Littlewood-Richardson coefficient $c_{\lambda\mu}^\nu$. Therefore, due to isomorphism $V_{(\nu,d_{ab})} \cong V_{(\nu,d_0)}$, we can replace the sum over d_{ab} by the space $V_{(\nu,d_0)}$ with the corresponding multiplicity $c_{\lambda\mu}^\nu$ (cf. [GLS1]).

A particular example of the spaces $V_{(\lambda,a)}$ is the homogeneous components $\Lambda_+^k(V)$ and $\Lambda_-^k(V)$ of the algebras $\Lambda_\pm(V)$ (3.1). They are images of the projectors $E^{(k)}$ and $E^{(1^k)}$, corresponding to one-row and one-column partitions (k) and (1^k) respectively. This important fact allows us to calculate the dimensions (over the ground field $\mathbb{K}$) of all spaces $V_{(\lambda,a)}$, provided that the Poincaré series $P_-(t)$ is known. Since all the spaces $V_{(\lambda,a)}$ corresponding to the same partition λ are isomorphic, we shall denote their $\mathbb{K}$-dimensions by the symbol $\dim V_\lambda$.

In the sequel, the following corollary of Proposition 1 will be useful.

Corollary 4 *Let R be a Hecke symmetry of the bi-rank $(m|n)$, the Poincaré series of $\Lambda_-(V)$ being given by (3.3). Then for the partitions (k) and (1^k), $k \in \mathbb{N}$, the dimensions of the spaces $V_{(k)}$ and $V_{(1^k)}$ is determined by the formulae*

$$\dim V_{(k)} \;=\; s_{(k)}(x|y) := \sum_{i=0}^{k} h_i(x) e_{k-i}(y), \tag{3.7}$$

$$\dim V_{(1^k)} \;=\; s_{(1^k)}(x|y) := \sum_{i=0}^{k} e_i(x) h_{k-i}(y), \tag{3.8}$$

where h_i and e_i are respectively the complete symmetric and elementary symmetric functions of their arguments.

Proof. We prove only the first of the above formulae since the second one can be proved in the same way. Since $V_{(k)} = \Lambda_+^k(V)$, the dimension of $V_{(k)}$ can be found as an appropriate derivative of the Poincaré series $P_+(t)$

$$\dim V_{(k)} = \frac{1}{k!}\,\frac{d^k}{dt^k}P_+(t)_{|t=0}.$$

With the use of $P_+(t)P_-(-t) = 1$ (see Proposition 1) and relation (3.3) we can present $P_+(t)$ in the form

$$P_+(t) = \prod_{i=1}^{n}(1+y_it)\prod_{j=1}^{m}\frac{1}{(1-x_jt)} = \mathcal{E}(y|t)\mathcal{H}(x|t),$$

where $\mathcal{E}(\cdot)$ and $\mathcal{H}(\cdot)$ stands for the generating functions of the elementary and complete symmetric functions in the finite set of variables [Mac]:

$$e_k(y) = \sum_{1\le i_1<\ldots<i_k\le n} y_{i_1}\ldots y_{i_k} = \frac{1}{k!}\,\frac{d^k}{dt^k}\mathcal{E}(y|t)_{|t=0}$$

$$h_k(x) = \sum_{1\le j_1\le\ldots\le j_k\le m} x_{j_1}\ldots x_{j_k} = \frac{1}{k!}\,\frac{d^k}{dt^k}\mathcal{H}(x|t)_{|t=0}\,.$$

Calculating the k-th derivative of $P_+(t)$ at $t=0$ we get the answer (3.7). ■

Note, that polynomials $s_{(k)}(x|y)$ and $s_{(1^k)}(x|y)$ defined in (3.7) and (3.8) belong to the class of super-symmetric polynomials in $\{x_i\}$ and $\{y_j\}$. By definition [St], a polynomial $p(u|v)$ in two sets of variables is called *super-symmetric* if it is symmetric with respect to any permutation of arguments $\{u_i\}$ as well as of arguments $\{v_j\}$ and, additionally, on setting $u_1 = v_1 = t$ in $p(u|v)$ one gets the result independent of t. Evidently, the polynomials in question satisfy this definition if we set, for example, $u = x$, $v = -y$.

Actually, the the set of polynomials $s_{(k)}(x|y)$ (respectively $s_{(1^k)}(x|y)$) $k\in\mathbb{N}$ are super-symmetric analogs of complete symmetric (respectively elementary symmetric) functions in finite numbers of variables. In particular, they generate the whole ring of super-symmetric polynomials in variables $\{x_i\}$ and $\{y_j\}$. The $\mathbb{Z}$-basis of this ring is formed by the Schur super-symmetric functions $s_\lambda(x|y)$ which can be expressed in terms of $s_{(k)}$ (or $s_{(1^k)}$) through Jacobi-Trudi relations [Mac]. The Schur super-symmetric functions determine the value of dimensions $\dim V_\lambda$. In order to formulate the corresponding result we need one more definition.

Definition 5 ([BR]) Given two arbitrary integers $m\ge 0$ and $n\ge 0$, consider a partition $\lambda = (\lambda_1,\lambda_2,\ldots)$, satisfying the following restriction $\lambda_{m+1}\le n$. The (infinite) set of all such partitions shall be denoted as $\mathsf{H}(m,n)$ and any partition $\lambda\in\mathsf{H}(m,n)$ will be called a hook partition of the type $\mathsf{H}(m,n)$.

Proposition 6 ([H]) *Let R be a Hecke symmetry of the bi-rank $(m|n)$. Then the dimensions $\dim V_\lambda$ of spaces in decomposition (3.5) are determined by the rules:*

1. *For any $\lambda = (\lambda_1,\ldots,\lambda_k)\in\mathsf{H}(m,n)$ the dimension $\dim V_\lambda\ne 0$ and is given by the formula*

$$\dim V_\lambda = s_\lambda(x|y)\,. \tag{3.9}$$

   Here

$$s_\lambda(x|y) = \det\|s_{(\lambda_i-i+j)}(x|y)\|_{1\le i,j\le k}\,.$$

   where $s_{(k)}(x|y)$ is defined in (3.7) for $k\ge 0$ and $s_{(k)} := 0$ for $k<0$.

2. *For arbitrary partition λ we have*

$$\dim V_\lambda = 0 \;\Leftrightarrow\; \lambda\notin\mathsf{H}(m,n).$$

Proof. Taking into account that

$$\dim(U \otimes W) = \dim U \dim W, \qquad \dim(U \oplus W) = \dim U + \dim W$$

and calculating dimensions of the spaces in the both sides of (3.6) we find

$$\dim V_\lambda \dim V_\mu = \sum_\nu c^\nu_{\lambda\mu} \dim V_\nu.$$

Now the result (3.9) is a direct consequence of an inductive procedure based on Corollary 4 (cf., for example, [GPS2]).

The second claim can be deduced from the properties of the Schur functions $s_\lambda(x|y)$ established in [BR] (also cf. [H]). ∎

To end the section, we present one more important numerical characteristic of the Hecke symmetry which can be expressed in terms of its bi-rank.

Proposition 7 *Let R be a skew-invertible Hecke symmetry with the bi-rank $(m|n)$. Then*

$$Tr\, B = \mathrm{Tr}\, C = q^{n-m}(m-n)_q. \tag{3.10}$$

The proof of the theorem is rather technical and is placed into the Appendix.

Corollary 8 *For a skew-invertible Hecke symmetry with the bi-rank $(m|n)$, the factor ν in (2.5) equals $q^{2(n-m)}$ that is*

$$BC = CB = q^{2(n-m)} I\,.$$

Proof. First, observe, that if R is a skew-invertible Hecke symmetry, then the same is true for the operator $R_{21} = PR_{12}P$, and therefore

$$R^{-1}_{21} = R_{21} - (q - q^{-1})\, I_{21}.$$

Applying $\mathrm{Tr}_{(2)}$ to the first formula in (2.7), we have

$$\begin{aligned} B_1 C_1 \;&=\; \mathrm{Tr}_{(2)}\,(B_1 \Psi_{12}) = \mathrm{Tr}_{(2)}(R^{-1}_{21} B_2) \\ &=\; \mathrm{Tr}_{(2)}((R_{21} - (q - q^{-1})\, I_{21}) B_2) = I_1 - (q - q^{-1})\, I_1 \mathrm{Tr}(B) = q^{2(n-m)}\, I_1. \end{aligned}$$ ∎

4 The quasitensor category $\mathrm{SW}(V_{(m|n)})$

Our next goal is to construct the quasitensor *Schur-Weyl category* $\mathrm{SW}(V_{(m|n)})$ of vector spaces, generated by the space V equipped with a skew-invertible Hecke symmetry R of the bi-rank $(m|n)$. The objects of this category possess the module structure over the *reflection equation algebra*, which will be considered in detail in the next sections.

In constructing the above mentioned category we shall proceed analogously to the paper [GLS1], where such a category was constructed for an even Hecke symmetry of the bi-rank $(m|0)$. A peculiarity of the even case is that the space V^*, dual to V, can be identified with a specific object $V_{(1^{m-1})}$ (see (3.5) for the definition of V_λ) of the category. This property ensures the category constructed in [GLS1] to be rigid[7].

It is not so in the case of general bi-rank $(m|n)$, and we have to properly enlarge the category by adding the dual spaces to all objects. This requires, in turn, the consistent extending of the

[7]Recall, that a (quasi)tensor category of vector spaces is *rigid* if to any of its objects U there corresponds a dual object U^* such that the maps $U \otimes U^* \to \mathbb{K}$ and $U^* \otimes U \to \mathbb{K}$ are categorical morphisms.

categorical braidings to the dual objects and defining the invariant pairings. In the present section we elaborate these problems in detail.

So, let R be a skew-invertible Hecke symmetry of the bi-rank $(m|n)$, which, upon fixing a basis $\{x_i\}_{1\le i\le N}$ of the space V, $\dim V = N$, is represented by the matrix (2.2). Also, we introduce the dual vector space V^* and choose the basis $\{x^i\}_{1\le i\le N}$ in V^* dual to $\{x_i\}$ with respect to the nondegenerate bilinear form

$$\langle\, ,\rangle_r : \; V\otimes V^* \to \mathbb{K}\,, \qquad \langle x_i, x^j\rangle_r = \delta_i^j\,. \tag{4.1}$$

The subscript r (right) refers to the order of arguments in the form $\langle\, ,\, \rangle_r$: the vectors of the dual space V^* stand on the right of the vectors of V.

By definition, the dual space to the tensor product $U\otimes W$ is $W^*\otimes U^*$:

$$\langle U\otimes W, W^*\otimes U^*\rangle_r := \langle W, W^*\rangle_r \langle U, U^*\rangle_r\,.$$

As a consequence, the numbering of components in a tensor power $V^{*\otimes k}$ is reverse to that in a tensor power $V^{\otimes k}$

$$V^{*\otimes k} := V_k^*\otimes \dots V_2^*\otimes V_1^*\,, \qquad V^{\otimes k} := V_1\otimes V_2\otimes\dots\otimes V_k\,.$$

This should be always kept in mind when working with operators marked by numbers of spaces where these operators act (like in formulae (1.1) and all other similar expressions).

Extend now the braiding (2.2) onto the space $V^*\otimes V^*$. Below we show that, requiring a consistence of the extended braiding and an invariance of the pairing (4.1), we have the only choice

$$R(x^i\otimes x^j) = x^r\otimes x^s\, R_{sr}^{ji}\,. \tag{4.2}$$

Therefore, by analogy with the construction of section 3, we can define the representations of the Hecke algebras $H_k(q)$, $k\in\mathbb{K}$, in tensor powers $V^{*\otimes k}$, construct the projectors E_a^λ and introduce subspaces $V^*_{(\lambda,a)}\subset V^{*\otimes k}$ as images of the corresponding projectors (see (3.4)–(3.6)). Taking into account the above remark on the numbering of tensor product components, one can show that for any Young tableau (λ, a) there exists the only tableau (λ, a') such that the spaces $V_{(\lambda,a)}$ and $V^*_{(\lambda,a')}$ are dual with respect to the form (4.1).

By definition, the class of objects of the category $\mathrm{SW}(V_{(m|n)})$ consists of all the the direct sums of spaces $V_\lambda\otimes V_\mu^*$ and $V_\mu^*\otimes V_\lambda$, where λ and μ are partitions of non-negative integers. The zero partition corresponds to the basic space $V_0 := V$ or to its dual space $V_0^* := V^*$. The ground field $\mathbb{K}$ will be treated as the unit object of the category $\mathrm{SW}(V_{(m|n)})$

$$\mathbb{K}\otimes V = V = V\otimes\mathbb{K}.$$

Define now the class of morphisms of $\mathrm{SW}(V_{(m|n)})$. First of all, we should define the set of braiding morphisms $R_{U,W}$ realizing the isomorphisms $U\otimes W\cong W\otimes U$ for any two objects U and W. The braidings R_{V_λ,V_μ} and $R_{V^*_\lambda,V^*_\mu}$ are completely determined by $R_{V,V}$ and R_{V^*,V^*} given by (2.2) and (4.2). Therefore, we only need for the consistent definition of R_{V,V^*} and $R_{V^*,V}$, since the braidings R_{V_λ,V^*_μ} and $R_{V^*_\lambda,V_\mu}$ can be then constructed by standard methods (cf, for example, [GLS1]). The consistence condition is in the following requirement. Having defined the four braidings mentioned above, we get a linear operator on $(V\oplus V^*)^{\otimes 2}$. Our definitions are consistent if this operator satisfies the Yang-Baxter equation. This problem is solved by the proposition below, whose main idea belongs to V.Lyubashenko (cf. [LS] and the references therein).

Proposition 9 *Let Ψ be the skew-inverse operator* (2.1) *of the skew-invertible braiding R. Consider an extension of R up to the linear operator*

$$R : (V\oplus V^*)^{\otimes 2}\to (V\oplus V^*)^{\otimes 2}$$

(we keep the same notation for the extended operator) in accordance with the formulae

$$
\begin{array}{lcl}
V\otimes V^* \to V^*\otimes V & : & R(x_i\otimes x^j) = x^k\otimes x_l\,(R^{-1})^{lj}_{ki}\,,\\
V^*\otimes V \to V\otimes V^* & : & R(x^j\otimes x_i) = x_k\otimes x^l\,\Psi^{kj}_{li}\,,\\
V^*\otimes V^* \to V^*\otimes V^* & : & R(x^i\otimes x^j) = x^k\otimes x^l\,R^{ji}_{lk}\,,\\
V\otimes V \to V\otimes V & : & R(x_i\otimes x_j) = x_k\otimes x_l\,R^{kl}_{ij}\,.
\end{array}
\tag{4.3}
$$

Then the extended operator R is a braiding, i.e. it satisfies the Yang-Baxter equation (1.1) *on the space $(V\oplus V^*)^{\otimes 3}$.*

Proof. Due to R is a linear operator, it suffices to prove the proposition on the basis vectors of the space $(V\oplus V^*)^{\otimes 3}$. This space splits into the direct sum of eight subspaces (from $V\otimes V\otimes V$ till $V^*\otimes V^*\otimes V^*$) and the verification of the statement of proposition on each of these subspaces is a matter of straightforward calculations based on formulae (4.3). ■

At the second step of our construction, we assume that a linear combination, the product, the direct sum and the tensor product of a finite family of categorical morphisms is a morphism too.

Then, following [T], we shall require the morphisms to be natural (or functorial). This means that

$$(g\otimes f)\circ R_{U,W} = R_{U',W'}\circ(f\otimes g)$$

where $f: U\to U'$ and $g: W\to W'$ are two categorical morphisms. As a consequence, we get the necessary condition for a map $f: U\to U'$ to be a categorical morphism:

$$(\mathrm{id}_W\otimes f)\circ R_{U,W} = R_{U',W}\circ(f\otimes \mathrm{id_W}),\qquad (f\otimes \mathrm{id}_W)\circ R_{W,U} = R_{W,U'}\circ(\mathrm{id}_W\otimes f). \tag{4.4}$$

A map f satisfying this condition will be called *R-invariant*. Thus, any categorical morphism must be R-invariant.

Proposition 10 *Provided R satisfies* (4.3)*, the following claims holds true:*

1. *The pairing* (4.1) *is R-invariant.*
2. *The linear map $\pi_r: \mathbb{K}\to V^*\otimes V$ generated by*

$$1 \stackrel{\pi_r}{\mapsto} \sum_{i=1}^{N} x^i\otimes x_i, \tag{4.5}$$

is also R-invariant.

Proof. In proving the claim 1, one can confine oneself to considering the simplest case of the formula (4.4) when $W = V$ or $W = V^*$. This is a consequence of the structure of objects of the category $\mathrm{SW}(V_{(m|n)})$. In other words, we have to show the commutativity of the diagram

$$
\begin{array}{ccc}
(V\otimes V^*)\otimes V^{\#} & \stackrel{(4.3)}{\longleftrightarrow} & V^{\#}\otimes(V\otimes V^*)\\
\langle\,,\rangle_r\otimes\mathrm{id}\ \downarrow & & \downarrow\ \mathrm{id}\otimes\langle\,,\rangle_r\\
\mathbb{K}\otimes V^{\#} & = & V^{\#}\otimes\mathbb{K}
\end{array}
\tag{4.6}
$$

where $V^{\#}$ stands for V or V^*. The commutativity of the diagram immediately follows from formulae (4.3), definitions (4.1), (2.1) and the definition of the inverse matrix R^{-1}.

Next, the same reasoning shows, that the claim 2 is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} (V^*\otimes V)\otimes V^{\#} & \stackrel{(4.3)}{\longleftrightarrow} & V^{\#}\otimes (V^*\otimes V) \\ \pi_r\otimes \mathrm{id}\ \uparrow & & \uparrow\ \mathrm{id}\otimes\pi_r \\ \mathbb{K}\otimes V^{\#} & = & V^{\#}\otimes\mathbb{K} \end{array}$$

which can be proved by analogy with the previous case. ■

Remark 11 Note, that the R-invariance of the maps (4.1) and (4.5) is a motivation of the extension (4.3) of the initial braiding R. It can be shown that such an extension is unique.

In what follows, besides the right form (4.1), we shall also need a *left* nondegenerated bilinear form

$$\langle\,,\rangle_l:\ V^*\otimes V\to\mathbb{K},$$

with the additional requirement that the above pairing would be R-invariant. This requirement prevents us from setting $\langle x^i, x_j\rangle_l = \delta^i_j$, since it is not an R-invariant pairing (direct consequence of (4.3)).

Let us choose the form $\langle\,,\rangle_l$ in such a way that the following diagram would be commutative

$$\begin{array}{ccc} V^*\otimes V & \stackrel{(4.3)}{\longrightarrow} & V\otimes V^* \\ \langle\,,\rangle_l\ \downarrow & & \downarrow\ \langle\,,\rangle_r \\ \mathbb{K} & = & \mathbb{K}\,. \end{array} \qquad (4.7)$$

Having done a simple calculation on the base of (4.7), we get the following explicit expression

$$\langle x^i, x_j\rangle_l = B^i_{\ j}, \qquad (4.8)$$

where the matrix $\|B^i_{\ j}\|$ is defined in (2.3). Such a choice guarantees the R-invariance of the left pairing $\langle\,,\rangle_l$. The commutativity of the corresponding diagram (analogous to (4.6)) can be easily verified with the use of (4.3) and (2.7).

Remark 12 Note, that the backward diagram

$$\begin{array}{ccc} V^*\otimes V & \stackrel{(4.3)}{\longleftarrow} & V\otimes V^* \\ \langle\,,\rangle_l\ \downarrow & & \downarrow\ \langle\,,\rangle_r \\ \mathbb{K} & = & \mathbb{K}\,. \end{array} \qquad (4.9)$$

is *not* commutative with the definition (4.8). In a tensor category one can define the left pairing in such a way that both diagrams (4.7) and (4.9) would be commutative, while in a quasitensor category it is impossible. This is a consequence of the fact that the braiding R is not involutive: $R^2\neq I$.

In principle, we could demand the commutativity of the above diagram instead of diagram (4.7). In this case in the right hand side of (4.8) we would obtain an additional factor $q^{2(m-n)}$. Actually, both variants are equivalent and choosing between them is a matter of taste.

Now we can find another basis $\{{}^i\!x\}_{1\le i\le N}$ of V^* which is dual to the basis $\{x_i\}_{1\le i\le n}$ with respect to the left form

$$ {}^i\!x := q^{2(m-n)}\, C^i_{\,j}\, x^j, \quad \Rightarrow \quad \langle {}^i\!x, x_j\rangle_l = \delta^i_j. \tag{4.10}$$

The normalizing factor in the definition of the basis vector ${}^i\!x$ is chosen in accordance with Corollary 8.

So, we have two R-invariant bilinear forms and two basis sets $\{x^i\}$ and $\{{}^i\!x\}$ in the space V^* which are dual to the basis $\{x_i\}$ of the space V with respect to the right and left forms correspondingly (see (4.1) and (4.10)). Due to this reason, we shall refer to $\{x^i\}$ (respectively $\{{}^i\!x\}$) as the *right* (respectively the *left*) basis of V^*.

With the use of (2.7) one can rewrite the formulae (4.3) in terms of the left basis.

Corollary 13 *In terms of the left basis $\{{}^i\!x\}_{1\le i\le N}$ of the space V^*, the extension of the braiding R defined by* (4.3) *has the following form*

$$\begin{aligned}
R(x_i\otimes {}^j\!x) &= {}^k\!x\otimes x_l\, \Psi^{jl}_{ik}\,,\\
R({}^j\!x\otimes x_i) &= x_k\otimes {}^l\!x\, (R^{-1})^{jk}_{il}\,,\\
R({}^i\!x\otimes {}^j\!x) &= {}^k\!x\otimes {}^l\!x\, R^{ji}_{lk}\,,\\
R(x_i\otimes x_j) &= x_k\otimes x_l\, R^{kl}_{ij}\,.
\end{aligned} \tag{4.3$'$}$$

Besides, the linear map $\pi_l:\mathbb{K}\to V\otimes V^*$ generated by

$$1 \stackrel{\pi_l}{\mapsto} \sum_{i=1}^N x_i\otimes {}^i\!x \tag{4.11}$$

is R-invariant.

Proof. We prove the first formula in the above list (4.3$'$), the other ones are proved in the same way. Taking into account the definition of the left basis (4.10) and the first formula in (4.3), we get (recall the summation over the repeated indices)

$$R(x_i\otimes {}^j\!x) = x^u\otimes x_l\, q^{2(m-n)}\, C^j_{\,s}(R^{-1})^{ls}_{ui} = {}^k\!x\otimes x_l\, q^{2(m-n)} C^j_{\,s}(R^{-1})^{ls}_{ui} B^u_{\,k},$$

where in the last equality we come back from the right basis to the left one by the formula inverse to (4.10)

$$x^u = B^u_{\,k}\, {}^k\!x.$$

Then, from the second line formulae of (2.7) and Corollary 8 we deduce

$$q^{2(m-n)} C_1 R^{-1}_{21} B_2 = \Psi_{12}$$

which allows us to make the following substitution in the above line of transformations

$$q^{2(m-n)} C^j_{\,s}(R^{-1})^{ls}_{ui} B^u_{\,k} = \Psi^{jl}_{ik}.$$

So, we finally get

$$R(x_i\otimes {}^j\!x) = {}^k\!x\otimes x_l\, \Psi^{jl}_{ik},$$

that is the first line formula in (4.3$'$).

As for the R-invariance of the map π_l, it can be proved by straightforward calculations on the base of (4.3) or (4.3$'$) in the same way as it was done in proving of (4.5). ■

Now we are able to define the categorical morphisms of $\mathrm{SW}(V_{(m|n)})$. Together with the identical map, the list of the morphisms includes (4.1), (4.5), (4.8), (4.11) and all maps (4.3) (or equivalently (4.3$'$)). Besides, as we have already mentioned above, any linear combination, the product (successive application), the tensor product or the direct sum of categorical morphisms is also a categorical morphism.

Remark 14 Given a particular braiding R, one can in principle compose a bigger list of R-invariant maps than that mentioned above. Thus, for a super-space $V = V_0 \oplus V_1$ the projections $V \to V_0$ and $V \to V_1$ are R-invariant maps.

In what follows we shall be especially interested in the objects $V^* \otimes V$ and $V \otimes V^*$ which are isomorphic to the space $\mathrm{End}(V)$ of endomorphisms of the space V. Upon fixing the basis $\{x_i\}$ in the space V, we come to the standard basis $\{h_i^j\} = x_i \otimes {}^j x$ in the space $V \otimes V^*$. Defining the action of an element $v \otimes v^* \in V \otimes V^*$ on a vector $u \in V$ by the usual rule

$$(v \otimes v^*)(u) := v\langle v^*, u\rangle_l$$

we get the action

$$h_i^j(x_k) = \delta_k^j\, x_i$$

and the multiplication table of elements h_i^j treated as endomorphisms of the space V

$$h_i^j \circ h_k^s = \delta_k^j\, h_i^s.$$

Having fixed the right basis $\{x^i\}$ in V^*, we come to another basis $l_i^j = x_i \otimes x^j$ of $V \otimes V^*$ with the properties (see (4.8))

$$l_i^i(x_k) = B^j_{\ k}\, x_i, \qquad l_i^j \circ l_k^s = B^j_{\ k}\, l_i^s\,. \tag{4.12}$$

Taking into account (4.10), we find the connection of the two basis sets

$$h_i^j = q^{2(m-n)}\, C^j_{\ k} l_i^k\,. \tag{4.13}$$

Introduce now the linear map $\mathrm{Tr}_R : \mathrm{End}(V) \to \mathbb{K}$ by means of the categorical morphism (4.1)

$$\mathrm{Tr}_R(l_j^i) = \langle x_j, x^i\rangle_r = \delta_j^i. \tag{4.14}$$

This map will be called the *R-trace* in what follows. In virtue of (4.13), the R-trace of an operator $\mathsf{F} \in \mathrm{End}(V)$ is given by

$$\mathrm{Tr}_R(\mathsf{F}) = q^{2(m-n)}\, Tr(F \cdot C)\,, \tag{4.15}$$

where F is the matrix of the operator F with respect to basis $\{x_i\}$.

To complete the section, we calculate the R-dimension of the objects V_λ of our category. By definition, the R-dimension of an object $V_\lambda \subset V^{\otimes k}$, $\lambda \vdash k$, is given by

$$\dim_R V_\lambda := \mathrm{Tr}_R(\mathrm{id}_{V_\lambda}) = q^{2k(m-n)} Tr_{(1\dots k)}(C_1 \dots C_k E_a^\lambda)\,. \tag{4.16}$$

Basing on (A.2), one can prove, that the above definition does not depend on the value of a. Besides, the R-dimension is an additive-multiplicative functional as the classical dimension

$$\dim_R(U \otimes W) = \dim_R U \dim_R W\,, \qquad \dim_R(U \oplus W) = \dim_R U + \dim_R W\,.$$

Let us introduce the R-analogs $Q_\pm(t)$ of the HP series $P_\pm(t)$ (3.2) by the relation

$$Q_\pm(t) = \sum_{k \geq 0} t^k\, \dim_R \Lambda_\pm^k(V)\,.$$

Then the following proposition holds true.

Proposition 15 *Given a skew-invertible Hecke symmetry with a bi-rank $(m|n)$, we find the following properties of the series $Q_\pm$:*

1. *if $m - n = 0$, then $\dim_R V_\lambda = 0$ for any $\lambda \neq 0$ and therefore $Q_+(t) = Q_-(t) = 1$.*

2. If $m-n>0$, then

$$\dim_R V_\lambda = \dim_R {V_\lambda}^* = s_\lambda(q^{m-n-1}, q^{m-n-3}, ..., q^{1-m+n}),$$

and therefore

$$Q_-(t) = \sum_{k=0}^{m-n} \binom{m-n}{k}_q t^k\,, \qquad \binom{p}{k}_q := \frac{p_q(p-1)_q\dots(p-k+1)_q}{k_q(k-1)_q\dots 2_q 1_q}\,.$$

3. If $m-n<0$, then

$$\dim_R V_\lambda = \dim_R {V_\lambda}^* = s_{\lambda^*}(q^{n-m-1}, q^{n-m-3}, ..., q^{1-n+m}),$$

where λ^* is the conjugate partition, and therefore

$$Q_+(t) = \sum_{k=0}^{n-m} \binom{n-m}{k}_q t^k\,.$$

Proof. The proposition is proved by the direct calculations on the base of definition (4.16). The calculation are analogous to those in the even case (cf., for example, [GLS1]). ■

Emphasize, that $Q_\pm(t)$ depend only on the bi-rank of a given Hecke symmetry R, whereas the corresponding HP series $P_\pm(t)$, besides the bi-rank, essentially depend on a concrete form of R.

5 mREA: definition and deformation properties

If R is an involutive ($R^2 = I$) skew-invertible symmetry, then the space $\mathrm{End}(V)$ can be endowed with the structure of *a generalized Lie algebra* (cf. [G1, G3]). The corresponding enveloping algebra $U_R(\mathrm{End}(V))$ is defined as the following quotient

$$U_R(\mathrm{End}(V)) = T(\mathrm{End}(V))/\langle \mathcal{J}_R\rangle \tag{5.1}$$

where $\langle \mathcal{J}_R\rangle$ is a two sided ideal of the free tensor algebra $T(\mathrm{End}(V))$ generated by the subset $\mathcal{J}_R \subset T(\mathrm{End}(V))$ of the following form

$$\mathcal{J}_R = \{X\otimes Y - R_{\mathrm{End}}(X\otimes Y) - X\circ Y + \circ R_{\mathrm{End}}(X\otimes Y)\mid \forall X, Y\in \mathrm{End}(V)\}\,. \tag{5.2}$$

Here $\circ$ is the product in $\mathrm{End}(V)$ considered as an associative algebra of linear operators on V and the linear operator $R_{\mathrm{End}} : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2}$ is an extension of the braiding R to the space $\mathrm{End}(V)^{\otimes 2}$. Its explicit form can be obtained with the use of (4.3). Namely, choosing the basis $l_j^i = x_j \otimes x^i$ in the space $\mathrm{End}(V)$ and applying the corresponding formulae from the list (4.3) we find

$$R_{\mathrm{End}}(l_j^i\otimes l_s^k) = l_{b_1}^{a_1}\otimes l_{b_2}^{a_2}\,(R^{-1})_{a_1c_2}^{b_2c_1}\,R_{jr_1}^{b_1c_2}\,R_{a_2c_1}^{kr_2}\,\Psi_{r_2s}^{r_1i}\,. \tag{5.3}$$

In order to present this formula in more transparent form we introduce the matrix notations which will be useful in what follows. Define the $N\times N$ (recall, that $N = \dim V$) matrix L with the matrix elements

$$L_i^{\,j} = l_i^j, \tag{5.4}$$

where the first (lower) index numerate rows and the second (upper) one numerates columns of L. Then, introducing the matrix $\bar R$ transposed to R

$$\bar R_{i_1i_2}^{\,j_1j_2} = R_{\,i_1i_2}^{j_1j_2}\,,$$

we denote

$$L_{\bar 1} = L\otimes I, \quad L_{\bar 2} = \bar R_{12} L_{\bar 1} \bar R_{12}^{-1}. \tag{5.5}$$

Now, multiplying the both sides of (5.3) by R and R^{-1} and taking into account the definition of Ψ (2.1), we represent formula (5.3) in the equivalent form

$$R_{\mathrm{End}}(L_{\bar 1}\otimes L_{\bar 2}) = L_{\bar 2}\otimes L_{\bar 1}\,, \tag{5.6}$$

where the summation over the corresponding matrix indices is understood.

Note, that the direct generalization of (5.1)–(5.2) with (5.6) from the involutive symmetry to the Hecke symmetry case leads to an algebra which possesses bad deformational properties and a poor representation theory. Fortunately, for any skew-invertible Hecke symmetry R there exists another generalization of the enveloping algebra $U_R(\mathrm{End}(V))$ (5.1) which has good deformational properties (see Proposition 20 below) and coincides with the enveloping algebra $U_R(\mathrm{End}(V))$ when R is involutive.

Definition 16 The associative algebra generated by the unit element $e_{\mathcal L}$ and the indeterminates l_i^j, $1\le i,j\le N$ subject to the system of relations

$$R^{kl}_{ij}\, l_k^m\, R^{pq}_{ml}\, l_p^r - l_i^a\, R^{bc}_{aj}\, l_b^d\, R^{rq}_{dc} - \hbar(R^{aq}_{ij}\, l_a^r - l_i^b\, R^{rq}_{bj}) = 0 \tag{5.7}$$

will be called the reflection equation algebra (REA) and denoted as $\mathcal L(R_q)$ if $\hbar = 0$, and it will be called the modified reflection equation algebra (mREA) and denoted as $\mathcal L(R_q,\hbar)$ if $\hbar\neq 0$.

The defining relations (5.7) can be presented in a compact form in terms of the matrix L (5.4) and transposed matrix $\bar R$

$$\bar R_{12} L_1 \bar R_{12} L_1 - L_1 \bar R_{12} L_1 \bar R_{12} - \hbar(\bar R_{12}\, L_1 - L_1\, \bar R_{12}) = 0\,. \tag{5.8}$$

Remark 17 Note, that upon making the linear transformation of generators $l_i^j\mapsto m_i^j$ (at $q\neq\pm 1$)

$$M = I e_{\mathcal L} - \omega\hbar^{-1} L, \qquad \omega = q-q^{-1}\,, \quad M = \|m_i^j\|\,,$$

we come to the following form of commutation relations (5.8)

$$\bar R_{12} M_1 \bar R_{12} M_1 - M_1 \bar R_{12} M_1 \bar R_{12} = 0\,. \tag{5.9}$$

This means, that the algebras $\mathcal L(R_q,\hbar)$ and $\mathcal L(R_q)$ are isomorphic at $q\neq\pm 1$. The basis of mREA generators with commutation relations (5.8) is more suitable for treating this algebra as an analog of universal enveloping algebra $U(gl(m|n))$.

We should prove that the commutation relations (5.8) are consistent with the structure of the category $\mathrm{SW}(V_{(m|n)})$ in the following sense. We treat the mREA as a quotient of the tensor algebra $T(V\otimes V^*)$ over the two sided ideal, generated by relations (5.8) or, equivalently, by (5.9). These relations are consistent with the structure of the category if the corresponding two sided ideal is invariant with respect to braidings of the category, or, in other words, if the mREA commutation relations are *R-invariant.*

Proposition 18 *The commutation relations* (5.8) *are R-invariant.*

Proof. To prove the proposition, it is sufficient to show, that the commutation relations (5.8) are preserved when commuting with V or V^* with respect to the braidings of the category $\mathrm{SW}(V_{(m|n)})$. This can be done by straightforward calculations on the base of formulae (4.3) and $l_i^j = x_i\otimes x^j$. To simplify the calculations, working with generators m_i^j (5.9) is more convenient.

For example, taking a basis vector $x_i \in V$ we get with the use of (4.3)

$$\mathsf{R}(x_{i_1} \otimes m_{i_2}^{j_2}) = \bar{R}_{i_1 i_2}^{a_1 a_2} m_{a_1}^{b_1} (\bar{R}^{-1})_{b_1 a_2}^{c_1 j_2} \otimes x_{c_1} \quad \text{or} \quad \mathsf{R}(x_1 \otimes M_2) = \bar{R}_{12} M_1 \bar{R}_{12}^{-1} \otimes x_1\,,$$

where R is a general notation for the corresponding braiding, $\mathsf{R} = R_{V, V\otimes V^*}$ in the above formulae. Now we can directly get the desired result

$$x_1 \otimes (\bar{R}_{23} M_2 \bar{R}_{23} M_2 - M_2 \bar{R}_{23} M_2 \bar{R}_{23}) \overset{\mathsf{R}}{\to}$$
$$\bar{R}_{12}\bar{R}_{23}(\bar{R}_{12} M_1 \bar{R}_{12} M_1 - M_1 \bar{R}_{12} M_1 \bar{R}_{12}) \bar{R}_{23}^{-1} \bar{R}_{12}^{-1} \otimes x_1\,.$$

The commutativity with V^* is verified analogously. ■

Proposition 19 *Let R be an involutive skew-invertible symmetry. Then the commutative relations among the generators $\{l_i^j\}$ of the algebra $U_R(\mathrm{End}(V))$ (5.1) are equivalent to (5.8) with $\hbar = 1$. Therefore, according to Definition 16, the algebra (5.1) coincides with mREA $\mathcal{L}_q(R, 1)$.*

Proof. In involutive case $R = R^{-1}$ by definition. Therefore, the matrix $L_{\overline{2}}$ defined in (5.5) can be written as $L_{\overline{2}} = \bar{R}_{12} L_1 \bar{R}_{12}$. This leads to the following action of R_{End} (5.6)

$$R_{\mathrm{End}}(L_1 \otimes \bar{R}_{12} L_1 \bar{R}_{12}) = \bar{R}_{12} L_1 \bar{R}_{12} \otimes L_1.$$

Now, by setting in (5.2) $X = L_1$, $Y = \bar{R}_{12} L_1 \bar{R}_{12}$, and on taking into account the multiplication table for the generators l_i^j (4.12), we get

$$X \circ Y = L_1 R_{12}\,, \qquad \circ R_{\mathrm{End}}(X \otimes Y) = R_{12} L_1\,,$$

Together with the above form of action of R_{End} this allows us to represent the set $\mathcal{J}_R$ (5.2) in the form (5.8) with $\hbar = 1$. ■

The main deformation property of the mREA is given by the following proposition.

Proposition 20 *Let R be a skew-invertible involutive Hecke symmetry: $R^2 = I$ and $U \subset \mathbb{K}$ be a neighbourhood of $1 \in \mathbb{K}$. Consider a family of skew-invertible Hecke symmetries R_q, analytically depending on $q \in U$ and satisfying the condition $R_1 = R$. Denote the homogeneous component of $\mathcal{L}(R_q)$ of the k-th order by $\mathcal{L}^{(k)}(R_q)$. Then, provided q is generic, the following claims hold true.*

1. $$\dim\, \mathcal{L}^{(k)}(R_q) = \dim\, \mathcal{L}^{(k)}(R), \qquad \forall\, k \ge 0\,.$$

2. $$\mathrm{Gr}\, \mathcal{L}(R_q, \hbar) \cong \mathcal{L}(R_q)\,,$$

   where $\mathrm{Gr}\, \mathcal{L}(R_q, \hbar)$ is the graded algebra associated to the filtrated algebra $\mathcal{L}(R_q, \hbar)$.

Proof. The verification of the item 1 is based on the following observations. Below we construct a projector $(\mathrm{Span}(l_i^j))^{\otimes 3} \to \mathcal{L}^{(3)}(R_q)$. The explicit form of the projector allows us to conclude that its rank is constant for generic $q \in U$. Therefore

$$\dim\, \mathcal{L}^{(3)}(R_q) = \dim\, \mathcal{L}^{(3)}(R). \tag{5.10}$$

For an involutive R, the algebra $\mathcal{L}(R)$ is the symmetric algebra of the linear space $\mathrm{Span}(l_i^j)$ equipped with the involutive braiding R_{End}. This algebra is a Koszul one. (For the definition of this notion the reader is referred to [PP].) The Koszul property of $\mathcal{L}(R)$ follows easily from exactness of the second kind Koszul complex constructed in [G3].

Now we apply the result of [PP] (generalizing [Dr]) saying that the Koszul property of $\mathcal{L}(R)$ and relation (5.10) implies the claim 1 of our proposition. Moreover, it can be shown, that for generic $q \in U$ the algebra $\mathcal{L}(R_q)$ is also a Kozsul algebra.

In order to prove the second claim of the proposition, we consider the map [,] sending the l.h.s. of (5.7) to its r.h.s. As was shown in [G4], this map satisfies the Jacobi relation in form of [PP]. Then by virtue of a generalization of the PBW theorem, given in [PP] (cf., also, [BG]), we arrive at the claim 2. ■

Remark 21 Note, that skew-invertible Hecke symmetries with non-classical HP series $P_-(t)$, constructed by methods of [G3], analytically depend on q in a neighbourhood of 1.

Now, we pass to construction of the projector mentioned in the proof of proposition 20. Rewrite REA $\mathcal{L}(R_q)$ ((5.8) at $\hbar = 0$) in the equivalent form

$$\bar{R}_{12} L_{\bar{1}} L_{\bar{2}} - L_{\bar{1}} L_{\bar{2}} \bar{R}_{12} = 0. \tag{5.11}$$

Consider the unital associative algebra $\mathfrak{L}$ over $\mathbb{K}$ freely generated by N^2 generators l_i^j

$$\mathfrak{L} = \mathbb{K}\langle l_i^j \rangle \quad 1 \le i, j \le N.$$

The algebra $\mathcal{L}(R_q)$ is the quotient of $\mathfrak{L}$ over the two sided ideal $\langle \mathcal{I}_- \rangle$ generated by the left hand side of (5.11)

$$\mathcal{L}(R_q) = \mathfrak{L}/\langle \mathcal{I}_- \rangle, \qquad \mathcal{I}_- = L_{\overline{1}} L_{\overline{2}} - \bar{R}_{12} L_{\overline{1}} L_{\overline{2}} \bar{R}_{12}^{-1}\,. \tag{5.12}$$

As a vector space, the algebra $\mathfrak{L}$ can be decomposed into the direct sum of homogeneous components

$$\mathfrak{L} = \bigoplus_{k \ge 0} \mathfrak{L}_k, \quad \mathfrak{L}_0 \cong \mathbb{K},$$

where each $\mathfrak{L}_k$ is a linear span of the k-th order monomials in generators l_i^j. The following basis turns out to be convenient to use

$$\mathfrak{L}_k = \mathrm{Span}[L_{\overline{1}} L_{\overline{2}} \dots L_{\overline{k}}]. \tag{5.13}$$

This notation means that $\mathcal{L}_k$ is spanned by the matrix elements of the right hand side matrix. The matrices $L_{\overline{m}}$ are defined by the recurrent rule

$$L_{\overline{1}} = L \otimes I, \qquad L_{\overline{k+1}} = \bar{R}_k L_{\overline{k}} \bar{R}_k^{-1}, \quad k \ge 1, \tag{5.14}$$

where the shorthand notation $R_k := R_{k\,k+1}$ will be systematically used below.

Remark 22 Note, that due to definitions (5.11), (5.14), and the Yang-Baxter equation for R the following relation holds

$$\bar{R}_k L_{\overline{k}} L_{\overline{k+1}} = L_{\overline{k}} L_{\overline{k+1}} \bar{R}_k \quad \forall\, k \ge 1. \tag{5.15}$$

Relation (5.15) is typical for the so-called quantum matrix algebras, REA being a particular case of them. For detailed treatment of the question the reader is referred to [IOP].

For the algebra $\mathcal{L}(R_q)$ we have an analogous vector space decomposition

$$\mathcal{L}(R_q) = \bigoplus_{k \ge 0} \mathcal{L}_k, \quad \mathcal{L}_0 \cong \mathbb{K}, \quad \mathcal{L}_k \subset \mathfrak{L}_k.$$

Let us try to describe the subspaces $\mathcal{L}_k$ explicitly. In other words, we should find a series of projector operators $\mathcal{S}_k : \mathfrak{L}_k \to \mathfrak{L}_k$ with the property

$$\mathrm{Im}\, \mathcal{S}_k = \mathcal{L}_k \subset \mathfrak{L}_3\,.$$

Here we construct such projectors for the second and third order components $\mathcal{L}_k$, $k = 2, 3$.

Introduce a linear operator $Q : \mathfrak{L}_2 \to \mathfrak{L}_2$ by the formula

$$Q(L_{\overline{1}} L_{\overline{2}}) := \bar{R}_1 L_{\overline{1}} L_{\overline{2}} \bar{R}_1^{-1} \tag{5.16}$$

or symbolically $Q = \bar{R}_1 \circ \bar{R}_1^{-1}$. Taking into account the Yang-Baxter equation for $\bar{R}$ we can easily obtain that Q also satisfies the Yang-Baxter equation

$$Q_1 Q_2 Q_1 = Q_2 Q_1 Q_2 \,, \tag{5.17}$$

where $Q_1 = Q \otimes \mathrm{id}$ and $Q_2 = \mathrm{id} \otimes Q$ are obvious extensions of the operator Q on the space $\mathfrak{L}_3$. Moreover, using the fact that R is a Hecke symmetry, one can find a minimal polynomial of the operator Q

$$(Q + q^2\,\mathbf{I})(Q + q^{-2}\,\mathbf{I})(Q - \mathbf{I}) = 0, \quad \mathbf{I} := I \circ I \tag{5.18}$$

As follows from (5.18), the operator Q possesses three eigenvalues on the space $\mathfrak{L}_2$. Introducing evident notations, we get the vector space decomposition

$$\mathfrak{L}_2 = \mathfrak{L}^{(-q^2)} \oplus \mathfrak{L}^{(1)} \oplus \mathfrak{L}^{(-q^{-2})} \,.$$

With the help of the Hecke condition (1.2) one can find the expressions for the corresponding projectors

$$\begin{aligned}
\mathcal{P}^{(-q^2)} &= P_+(R) \circ P_-(R)\\
\mathcal{P}^{(-q^{-2})} &= P_-(R) \circ P_+(R)\\
\mathcal{P}^{(1)} &= P_+(R) \circ P_+(R) + P_-(R) \circ P_-(R)\,,
\end{aligned} \tag{5.19}$$

where

$$P_\pm(R) = \frac{q^{\mp 1} I \pm \bar{R}}{2_q}.$$

Indeed, the direct calculation shows that

$$Q\mathcal{P}^{(a)} = \mathcal{P}^{(a)}Q = a\,\mathcal{P}^{(a)}, \quad a = -q^{\pm 2},\, 1$$

and, on the other hand, the operators $\mathcal{P}^{(a)}$ form the full set of orthonormal projectors on $\mathfrak{L}_2$

$$\mathcal{P}^{(a)}\mathcal{P}^{(b)} = \delta^{ab}\,\mathcal{P}^{(a)}, \quad \mathcal{P}^{(-q^2)} + \mathcal{P}^{(1)} + \mathcal{P}^{(-q^{-2})} = \mathbf{I}.$$

Now we take into account that (5.11) is equivalent to

$$(Q - \mathbf{I})(L_{\overline{1}} L_{\overline{2}}) = 0.$$

This means that the second order component $\mathcal{L}_2$ of the REA $\mathcal{L}(R_q)$ coincides (as a vector space) with the subspace $\mathfrak{L}^{(1)} \subset \mathfrak{L}_2$. Introduce a couple of the orthonormal projection operators on $\mathfrak{L}_2$

$$\mathcal{S} := \mathcal{P}^{(1)}, \quad \mathcal{A} := \mathcal{P}^{(-q^2)} + \mathcal{P}^{(-q^{-2})}, \quad \mathcal{S}\mathcal{A} = \mathcal{A}\mathcal{S} = 0, \quad \mathcal{S} + \mathcal{A} = \mathbf{I}.$$

These operators can be expressed in terms of Q

$$\begin{aligned}
\mathcal{S} &= \frac{1}{2_q^2}\left((q^2 + q^{-2})\,\mathbf{I} + Q + Q^{-1}\right),\\
\mathcal{A} &= \frac{1}{2_q^2}\left(2\,\mathbf{I} - Q - Q^{-1}\right),
\end{aligned} \tag{5.20}$$

where the inverse operator Q^{-1} can be obtained from (5.18)

$$Q^{-1} = Q^2 + (q^2 - 1 + q^{-2})Q - (q^2 - 1 + q^{-2})\mathbf{I}\,.$$

One can show that

$$\mathrm{Span}(\mathcal{I}_-) = \mathrm{Im}\,\mathcal{A}$$

as vector subspaces in $\mathfrak{L}_2$, where $\mathcal{I}_-$ is defined in (5.12).

The above considerations proves the following proposition.

Proposition 23 *The second order homogeneous component $\mathcal{L}_2$ of the REA $\mathcal{L}(R_q)$ coincides with the image of the q-symmetrizer $\mathcal{S}$*

$$\mathcal{L}_2 = \mathrm{Im}\,\mathcal{S} = \mathfrak{L}_2/\mathrm{Im}\,\mathcal{A}. \tag{5.21}$$

Let us pass to the third order homogeneous component $\mathfrak{L}_3$ and find the projector on the corresponding component $\mathcal{L}_3 \subset \mathfrak{L}_3$ of the REA $\mathcal{L}(R_q)$.

Extend the projectors $\mathcal{S}$ and $\mathcal{A}$ onto the subspace $\mathfrak{L}_3$. For the projector $\mathcal{S}$ the extension is given by the two operators $\mathcal{S}_1$ and $\mathcal{S}_2$ in accordance with the rule (see definition (5.19))

$$\mathcal{S}_1 := \mathcal{P}^{(1)}(R_1), \qquad \mathcal{S}_2 := \mathcal{P}^{(1)}(R_2)\,,$$

which means that

$$\mathcal{S}_1(xyz) := (\mathcal{S}(xy))z, \qquad \mathcal{S}_2(xyz) := x(\mathcal{S}(yz)), \qquad \forall\, xyz \in \mathfrak{L}_3\,.$$

For $\mathcal{A}$ the formulae are analogous.

At this point we can see an advantage of using the basis (5.13). Indeed, the *quadratic* homogeneous component $\mathcal{L}_2$ can be embedded into $\mathfrak{L}_3$ in different ways, the following two being the most important in the sequel

$$\mathcal{L}_2 \cdot \mathfrak{L}_1 \subset \mathfrak{L}_3 \quad \text{and} \quad \mathfrak{L}_1 \cdot \mathcal{L}_2 \subset \mathfrak{L}_3\,.$$

As follows from (5.11), (5.15) and proposition 23, these embeddings can be identified with the images of the operators $\mathcal{S}_1$ and $\mathcal{S}_2$

$$\mathcal{L}_2 \cdot \mathfrak{L}_1 = \mathcal{S}_1(\mathfrak{L}_3)\,, \qquad \mathfrak{L}_1 \cdot \mathcal{L}_2 = \mathcal{S}_2(\mathfrak{L}_3).$$

The following technical lemma plays the crucial role in the further consideration.

Lemma 24 *The q-symmetrizer $\mathcal{S}$ obeys the following fifth order relation on the subspace $\mathfrak{L}_3$:*

$$\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1 - a\,\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1 + b\,\mathcal{S}_1 = \mathcal{S}_2\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2 - a\,\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2 + b\,\mathcal{S}_2 \tag{5.22}$$

where

$$a = (q^4 + q^2 + 4 + q^{-2} + q^{-4})/2_q^4 \quad b = 4_q^2/2_q^8\,. \tag{5.23}$$

Proof. The lemma is proved by a direct calculation. The calculation can be considerably simplified if for $\mathcal{S}$ one uses expression (5.20) instead of the initial definition (5.19). ■

Consider the operator $\mathcal{S}^{(3)} : \mathfrak{L}_3 \to \mathfrak{L}_3$ defined by

$$\mathcal{S}^{(3)} = \frac{2_q^6}{4\cdot 3_q^2}\,(\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1 - a\,\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1 + b\,\mathcal{S}_1)\,, \tag{5.24}$$

with a and b given in (5.23). Due to (5.22) there exist the equivalent form of the above operator

$$\mathcal{S}^{(3)} = \frac{2_q^6}{4\cdot 3_q^2}\,(\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2 - a\,\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2 + b\,\mathcal{S}_2)\,. \tag{5.25}$$

In fact, the operator $\mathcal{S}^{(3)}$ is the projector on $\mathcal{L}_3 \subset \mathfrak{L}_3$ we are looking for.

Proposition 25 *The third order homogeneous component $\mathcal{L}_3$ of the REA $\mathcal{L}(R_q)$ is the image of the projecting operator $\mathcal{S}^{(3)}$ under its action on $\mathfrak{L}_3$*

$$\mathcal{L}_3 = \mathrm{Im}\,\mathcal{S}^{(3)}\,.$$

Proof. The fact that $(\mathcal{S}^{(3)})^2 = \mathcal{S}^{(3)}$ can be verified by a straightforward calculation.

Consider now the projection of the relation (5.12) onto the third order homogeneous component:

$$\mathcal{L}_3 = \mathfrak{L}_3/\langle \mathcal{I}_- \rangle_3, \qquad \langle \mathcal{I}_- \rangle_3 = \mathfrak{L}_1 \cdot \operatorname{Im} \mathcal{A}_2 \cup \operatorname{Im} \mathcal{A}_1 \cdot \mathfrak{L}_1 . \tag{5.26}$$

As can be seen from (5.24) and (5.25), $\langle \mathcal{I}_- \rangle_3 \subseteq \operatorname{Ker} \mathcal{S}^{(3)}$ and, therefore, $\operatorname{Im} \mathcal{S}^{(3)} \subseteq \mathcal{L}_3$.

¿From the other side, since $\mathcal{S} + \mathcal{A} = \mathbf{I}$, the subspace $\mathcal{L}_3$ given in (5.26) can be presented as

$$\mathcal{L}_3 = \mathfrak{L}_1 \cdot \operatorname{Im} \mathcal{S}_2 \cap \operatorname{Im} \mathcal{S}_1 \cdot \mathfrak{L}_1 .$$

Comparing this form of $\mathcal{L}_3$ with the structure of $\mathcal{S}^{(3)}$ given in (5.24) and (5.25), we get $\operatorname{Im} \mathcal{S}^{(3)} \supseteq \mathcal{L}_3$. This relation together with the opposite inclusion obtained above completes the proof. ■

6 The braided bialgebra structure and representation theory

In this section we shall consider finite dimensional representations of the mREA (5.8) in the category $\mathrm{SW}(V_{(m|n)})$. Note, that the class of finite dimensional representations of mREA is wider, for instance, it includes a large number of one-dimensional representations. For the particular case of the $U_q(sl(m))$ R-matrix all such a representations were classified in [Mu1]. Besides, the finite dimensional representations of mREA can be constructed on the base of the $U_q(sl(m))$ representations, since the REA $\mathcal{L}(R_q)$ can be embedded (as an algebra) into the quantum group in this particular case.

The representation theory developed below does not depend on a particular choice of R-matrix and works well in the general situation, when the quantum group does not exist. Besides, an important property of the suggested theory is the equivariance of the representations we are dealing with. By definition, a representation ρ_U of mREA in a space U is called *equivariant* if the map

$$\mathrm{End}(V) \to \mathrm{End}(U): \quad l_i^j \mapsto \rho_U(l_i^j)$$

is a morphism of the category $\mathrm{SW}(V_{(m|n)})$.

This property has an important consequence, which will be used below. Namely, given an mREA module W with equivariant representation $\rho_W : \mathcal{L}(R_q, 1) \to \mathrm{End}(W)$, the diagram

$$\begin{array}{ccc} U \otimes (A \otimes W) & \stackrel{\mathsf{R}}{\longleftrightarrow} & (A \otimes W) \otimes U \\ \downarrow \mathrm{id} \otimes \rho_W & & \rho_W \otimes \mathrm{id} \downarrow \\ U \otimes W & \stackrel{\mathsf{R}}{\longleftrightarrow} & W \otimes U \end{array} \tag{6.1}$$

is commutative for any object U of the category $\mathrm{SW}(V_{(m|n)})$ and for any subspace $A \subset \mathcal{L}(R_q, 1)$. The equivariance condition allows us to define an mREA representation in the tensor product of mREA modules.

For the particular case of an even Hecke symmetry of the rank $(m|0)$, the equivariant representation theory of the associated mREA turns out to be similar to the representation theory of the algebra $U(sl(m))$. At the beginning of the section 4 we mentioned the specific peculiarity of the even case. Namely, in the corresponding category of the mREA representations, the space V^* can be identified with the object $V_{(1^{m-1})}$ and for constructing the complete representation theory it suffices, in fact, to define the mREA-module structure on the space V and on its tensor powers $V^{\otimes k}$. Any tensor product $V^{\otimes k}$ is a reducible mREA-module and expands into the direct sum (3.5) of mREA-invariant subspaces V_λ (for details cf. [GS2, S]).

However, the construction of cited papers is insufficient for the treatment of the general case of the bi-rank $(m|n)$. The matter is that in general case we have to construct representations in the tensor products $V^{*\otimes k}$ independently of those in tensor products $V^{\otimes k}$ and the central problem here consists in extending the mREA-module structure on the tensor product of modules $V_\lambda \otimes V^*_\mu$.

In the present section we suggest the regular procedure for constructing the mREA representations which works well independently of the bi-rank of the Hecke symmetry. Our construction is based on the braided bialgebra structure in the mREA. Throughout this section we set $\hbar = 1$ in the mREA commutation relations (5.8).

The main component of the braided bialgebra is the coproduct Δ, which is a homomorphism of the mREA $\mathcal{L}(R_q, 1)$ into an associative braided algebra $\mathbf{L}(R_q)$ which is defined as follows.

- As a vector space over the field $\mathbb{K}$ the algebra $\mathbf{L}(R_q)$ is isomorphic to the tensor product of two copies of mREA
$$\mathbf{L}(R_q) = \mathcal{L}(R_q, 1) \otimes \mathcal{L}(R_q, 1)\,.$$
- The product $\star : \mathbf{L}(R_q)^{\otimes 2} \to \mathbf{L}(R_q)$ is defined by the rule
$$(a_1 \otimes b_1) \star (a_2 \otimes b_2) := a_1 a_2' \otimes b_1' b_2\,, \qquad a_i \otimes b_i \in \mathbf{L}(R_q)\,, \tag{6.2}$$
where $a_1 a_2'$ and $b_1 b_2'$ are the usual product of elements of mREA, while a_1' and b_1' result from acting the braiding $R_{\rm End}$ (see (5.6)) on the tensor product $b_1 \otimes a_2$
$$a_2' \otimes b_1' := R_{\rm End}(b_1 \otimes a_2)\,. \tag{6.3}$$

We should verify that product (6.2) is indeed associative. For this aim, we need the following lemma.

Lemma 26 *Consider copies of the matrix L defined in* (5.14)*. Then the following relation holds*
$$R_{\rm End}(L_{\overline{k}} \otimes L_{\overline{p}}) = L_{\overline{p}} \otimes L_{\overline{k}}\,, \qquad \forall\, k < p,\ k, p \in \mathbb{N}\,. \tag{6.4}$$

Proof. The proof consists in a straightforward calculation on the base of relation (5.6) rewritten in a slightly modified form
$$R_{\rm End}(L_1 \bar R_{12} \otimes L_1) = \bar R_{12} L_1 \bar R_{12}^{-1} \otimes L_1 \bar R_{12}\,,$$
and on the Yang-Baxter equation (1.1) which allows one to interchange the chains of R-matrices forming the copies $L_{\overline{k}}$ and $L_{\overline{p}}$. ■

Now the associativity of (6.2) can be easily proved for the elements $X^i_{r,s} \in \mathbf{L}(R_q)$ whose components are homogeneous monomials in generators of $\mathcal{L}(R_q, 1)$:
$$X^i_{r,s} := L_{\overline{i}} \dots L_{\overline{i+r-1}} \otimes L_{\overline{i+r}} \dots L_{\overline{i+r+s-1}}\,,$$
where we represent the homogeneous components of mREA as a linear span of elements similar to those (5.13).

Note, that for any triple $X^{i_1}_{r_1,s_1}$, $X^{i_2}_{r_2,s_2}$ and $X^{i_3}_{r_3,s_3}$ we can always take
$$i_3 \ge i_2 + r_2 + s_2 \ge i_1 + r_1 + s_1$$
and then the associativity condition
$$(X^{i_1}_{r_1,s_1} \star X^{i_2}_{r_2,s_2}) \star X^{i_3}_{r_3,s_3} = X^{i_1}_{r_1,s_1} \star (X^{i_2}_{r_2,s_2} \star X^{i_3}_{r_3,s_3})$$
is an immediate consequence of lemma 26. Since any element of $\mathbf{L}(R_q)$ can be presented as a linear combination of some $X^{(i)}_{r,s}$ we conclude that (6.2) defines an associative product in $\mathbf{L}(R_q)$.

Note, that the mREA are isomorphic to two subalgebras of $\mathbf{L}(R_q)$ by means of the following embeddings

$$a \mapsto e_{\mathcal{L}} \otimes a \quad \text{or} \quad a \mapsto a \otimes e_{\mathcal{L}}\,,$$

where $e_{\mathcal{L}}$ is the unit of mREA $\mathcal{L}(R_q, 1)$. This can be easily obtained from the fact that the unit element $e_{\mathcal{L}}$ trivially commutes with any $a \in \mathcal{L}(R_q, 1)$ with respect to the braiding R_{End}. As a consequence we have

$$(e_{\mathcal{L}} \otimes a_1) \star (e_{\mathcal{L}} \otimes a_1) = (e_{\mathcal{L}} \otimes a_1 a_2) \quad \text{and} \quad (a_1 \otimes e_{\mathcal{L}}) \star (a_2 \otimes e_{\mathcal{L}}) = (a_1 a_2 \otimes e_{\mathcal{L}})\,.$$

Define a linear map $\Delta : \mathcal{L}(R_q, 1) \to \mathbf{L}(R_q)$ by the following rules:

$$\begin{aligned}
&\Delta(e_{\mathcal{L}}) := e_{\mathcal{L}} \otimes e_{\mathcal{L}} \\
&\Delta(l_i^j) := l_i^j \otimes e_{\mathcal{L}} + e_{\mathcal{L}} \otimes l_i^j - (q - q^{-1}) \textstyle\sum_k l_i^k \otimes l_k^j \\
&\Delta(ab) := \Delta(a) \star \Delta(b) \qquad \forall\, a, b \in \mathcal{L}(R_q, 1)\,.
\end{aligned} \tag{6.5}$$

In addition to (6.5), we introduce a linear map $\varepsilon : \mathcal{L}(R_q, 1) \to \mathbb{K}$

$$\begin{aligned}
&\varepsilon(e_{\mathcal{L}}) := 1 \\
&\varepsilon(l_i^j) := 0 \\
&\varepsilon(ab) := \varepsilon(a)\varepsilon(b) \qquad \forall\, a, b \in \mathcal{L}(R_q, 1)\,.
\end{aligned} \tag{6.6}$$

The proposition below establishes an important property of the maps Δ and ε.

Proposition 27 *The maps Δ and ε given in* (6.5) *and* (6.6) *are respectively the coproduct and counit of the braided bialgebra structure on the mREA $\mathcal{L}(R_q, 1)$.*

Proof. First, we prove that the map Δ defines the algebra homomorphism $\mathcal{L}(R_q, 1) \to \mathbf{L}(R_q)$. It is convenient to work with the generators m_i^j introduced in Remark 17. In terms of these generators the map Δ reads

$$\Delta(m_i^j) = \sum_s m_i^s \otimes m_s^j\,, \tag{6.7}$$

or, in the matrix form

$$\Delta(M_{\overline{k}}) = M_{\overline{k}} \otimes M_{\overline{k}} \qquad \forall\, k \geq 1\,.$$

Taking into account definitions (6.2) and (5.6) we find

$$\Delta(M_{\overline{1}} M_{\overline{2}}) = \Delta(M_{\overline{1}}) \star \Delta(M_{\overline{2}}) = M_{\overline{1}} M_{\overline{2}} \otimes M_{\overline{1}} M_{\overline{2}}\,.$$

Comparing this with (5.9) we finally get

$$R_{12} \Delta(M_{\overline{1}} M_{\overline{2}}) = \Delta(M_{\overline{1}} M_{\overline{2}}) R_{12}$$

which means that the map Δ is an algebra homomorphism. Note, that braided coproduct (6.7) was suggested in [M].

The interrelation of ε and Δ

$$(\mathrm{id} \otimes \varepsilon)\, \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id})\, \Delta$$

is verified trivially. ■

Consider now the question on representations of $\mathbf{L}(R_q)$ which can be constructed with the use of mREA representations. Given two equivariant mREA modules U and W with representations

$\rho_U : \mathcal{L}(R_q, 1) \to \mathrm{End}(U)$ and $\rho_W : \mathcal{L}(R_q, 1) \to \mathrm{End}(W)$, we construct the map $\rho_{U\otimes W} : \mathbf{L}(R_q) \to \mathrm{End}(U \otimes W)$ by the following rule

$$\rho_{U\otimes W}(a\otimes b) \triangleright (u\otimes w) = (\rho_U(a)\triangleright u')\otimes(\rho_W(b')\triangleright w)\,, \qquad a\otimes b \in \mathbf{L}(R_q)\,, \tag{6.8}$$

where the symbol $\triangleright$ stands for the action of the corresponding operator, the element(s) b' and vector(s) u' result from the action of the corresponding braiding (depending on b and u) of the category $\mathrm{SW}(V_{(m|n)})$

$$u'\otimes b' := \mathsf{R}(b\otimes u)\,.$$

In virtue of Proposition 18 the definition (6.8) is selfconsistent since the map $b \mapsto \rho_W(b')$ is also a representation of mREA $\mathcal{L}(R_q, 1)$.

Proposition 28 *The action* (6.8) *defines a representation of the algebra* $\mathbf{L}(R_q)$.

Proof. Consider two arbitrary elements of the algebra $\mathbf{L}(R_q)$

$$X_i = (a_i\otimes b_i)\in \mathbf{L}(R_q), \quad i = 1, 2\,.$$

We have to prove that $\forall\, u\otimes w\in U\otimes W$ the following relation holds

$$\rho_{U\otimes W}(X_1\star X_2)\triangleright(u\otimes w) = \rho_{U\otimes W}(X_1)\triangleright(\rho_{U\otimes W}(X_2)\triangleright(u\otimes w))\,. \tag{6.9}$$

The left and right hand sides of (6.9) are actually the maps sending the element

$$(a_1\otimes b_1)\otimes(a_2\otimes b_2)\otimes(u\otimes w)$$

into a vector of the space $U\otimes W$. We shall prove that the results of applying these maps to the above element are the same.

Let us introduce the shorthand notations

$$\mathsf{R}(b_1\otimes a_2) = a_2'\otimes b_1'\,, \quad \mathsf{R}(b_2\otimes u) = u'\otimes b_2'\,, \quad \mathsf{R}(b_1'\otimes u') = u''\otimes b_1''\,.$$

The definitions (6.2) and (6.8) allow us to represent the left hand side of (6.9) as the composition of the following morphisms

$$\begin{aligned}(a_1\otimes b_1)\otimes(a_2\otimes b_2)\otimes(u\otimes w) &\mapsto (a_1\otimes a_2')\otimes(b_1'\otimes b_2)\otimes(u\otimes w)\mapsto(a_1a_2'\otimes b_1'b_2)\otimes(u\otimes w)\\ &\mapsto (a_1a_2'\otimes u'')\otimes(b_1''b_2'\otimes w)\mapsto(\rho_U(a_1a_2')\triangleright u'')\otimes(\rho_W(b_1''b_2')\triangleright w)\,.\end{aligned}$$

Now, take into account the equivariance condition (6.1) for the representations ρ_U and ρ_W. This condition means that under the action of the categorical braidings a vector $\rho_U(a)\triangleright u$ commutes with any object in the same manner as the element $a\otimes u$ does. Therefore the right hand side of (6.9) can be represented as the composition of the maps

$$\begin{aligned}(a_1\otimes b_1)\otimes(a_2\otimes b_2)\otimes(u\otimes w) &\mapsto (a_1\otimes b_1)\otimes(a_2\otimes u')\otimes(b_2'\otimes w)\\ &\mapsto (a_1\otimes b_1)\otimes(\rho_U(a_2)\triangleright u'\otimes\rho_W(b_2')\triangleright w)\mapsto(a_1\otimes\rho_U(a_2')\triangleright u'')\otimes(b_1''\otimes\rho_W(b_2')\triangleright w)\\ &\mapsto (\rho_U(a_1)\triangleright\rho_U(a_2')\triangleright u'')\otimes(\rho_W(b_1'')\triangleright\rho_W(b_2')\triangleright w)\mapsto(\rho_U(a_1a_2')\triangleright u'')\otimes(\rho_W(b_1''b_2')\triangleright w)\end{aligned}$$

So, having started from the same initial element, the maps in the left and right hand sides of (6.9) give the same resulting vector in $U\otimes W$. Therefore, these maps are identical. ■

Corollary 29 *Let U and W be two $\mathcal{L}(R_q, 1)$-modules with equivariant representations ρ_U and ρ_W. Then equivariant representation $\mathcal{L}(R_q, 1) \to \mathrm{End}(U\otimes W)$ is given by the rule*

$$a\mapsto \rho_{U\otimes W}(\Delta(a))\,, \qquad \forall\, a\in\mathcal{L}(R_q, 1)\,, \tag{6.10}$$

where the coproduct Δ and the map $\rho_{U\otimes W}$ are given respectively by formulae (6.5) *and* (6.8).

Proof. This corollary is a direct consequence of Propositions 27 and 28. ■

As was mentioned at the beginning of this section, the equivariant representations of mREA in spaces V_λ, $\lambda \vdash k \in \mathbb{N}$, were constructed in [GS2, S]. By the same method we can define the mREA representations in spaces V_λ^*. Then, the formula (6.10) allows us to define the mREA-module structure on any object of the category $\mathrm{SW}(V_{(m|n)})$.

To complete the picture, we shortly outline the main ideas of [GS2, S], and, besides, prove the equivariance of representation in the space V^*. Contrary to the even case, for the Hecke symmetry of the general bi-rank the equivariance of representations in the dual spaces V_λ^* should be established independently of that in the spaces V_λ.

The basic left representation of mREA $\mathcal{L}(R_q, 1)$ in the space V is defined in terms of matrix elements of the operator B

$$\rho_1(l_i^j) \triangleright x_k = B^j_k x_i. \tag{6.11}$$

As was shown in [S], the map $\rho_2 : \mathcal{L}(R_q, 1) \to \mathrm{End}(V^{\otimes 2})$ defined by

$$\rho_2(l_i^j) \triangleright (x_{k_1} \otimes x_{k_2}) = (\rho_1(l_i^j) \triangleright x_{k_1}) \otimes x_{k_2} + \Big(R^{-1} \circ (\rho_1(l_i^j) \otimes I) \circ R^{-1}\Big) \triangleright (x_{k_1} \otimes x_{k_2})$$

is a representation. An extension of the basic representation up to the higher representations $\rho_p : \mathcal{L}(R_q, 1) \to \mathrm{End}(V^{\otimes p})$, $p \geq 3$, is defined in a similar way. It can be shown by direct calculations, that these extensions coincide with the universal recipe (6.10).

The representations of the above form are completely reducible — the space $V^{\otimes p}$ expands into the direct sum of invariant subspaces V_λ labelled by partitions $\lambda \vdash p$. The restriction of the representation ρ_p to subspaces V_λ is obtained by the action of orthogonal projectors E_a^λ (see (3.4) and (3.5))

$$\rho_{\lambda,a} = E_a^\lambda \circ \rho_p \circ E_a^\lambda\,, \tag{6.12}$$

the modules with different a being equivalent.

The basic representation $\rho_1^* : \mathcal{L}(R_q, 1) \to \mathrm{End}(V^*)$ is given by

$$\rho_1^*(l_i^j) \triangleright x^k = -x^r\, \bar{R}_{ri}^{\,kj}\,. \tag{6.13}$$

To prove the equivariance of this representation, we need the following lemma.

Lemma 30 *Let R be a skew-invertible Hecke symmetry. Then the map*

$$V \otimes V^* \to V^* \otimes V \ : \quad x_i \otimes x^j \mapsto x^k \otimes x_l R^{lj}_{ki} \tag{6.14}$$

is a categorical morphism.

Proof. We use the fact that for a Hecke symmetry $R = R^{-1} + (q - q^{-1})I$ (see (1.2)). Substituting this into (6.14)

$$x_i \otimes x^j \mapsto x^k \otimes x_l\,(R^{-1})^{lj}_{ki} + (q - q^{-1})\,\delta_i^j\, x^k \otimes x_k$$

we find that the map in question is a linear combination of a categorical morphism from the list (4.3) and of the map

$$x_i \otimes x^j \mapsto \delta_i^j\, x^k \otimes x_k.$$

It can be presented as a composition of categorical morphisms

$$x_i \otimes x^j \overset{\langle\,,\rangle_r}{\mapsto} \delta_i^j\, 1 \overset{\pi_r}{\mapsto} \delta_i^j\, x^k \otimes x_k,$$

where the categorical morphisms (4.1) and (4.5) were used in the consecutive order.

So, we conclude that the initial map (6.14) is a linear combination of categorical morphism, therefore it is a categorical morphism by definition. ■

Proposition 31 *The representation* (6.13) *of the algebra* $\mathcal{L}(R_q, 1)$ *in the space* V^* *is equivariant.*

Proof. To prove the equivariance of ρ_1^*, we are to show that the map $\rho_1^* : \mathcal{L}(R_q, 1) \to \mathrm{End}(V^*)$ is a categorical morphism.

Identifying l_i^j with $x_i \otimes x^j$, we can treat any left equivariant action of l_i^j on the basis vector $x^k \in V^*$ as a categorical morphism

$$V \otimes V^* \otimes V^* \to V^*.$$

Construct such an action as the following composition of morphisms

$$V \otimes V^* \otimes V^* \stackrel{(6.14)\otimes I}{\longrightarrow} V^* \otimes V \otimes V^* \stackrel{I\otimes\langle\,,\,\rangle_r}{\longrightarrow} V^* \otimes \mathbb{K} \cong V^*$$

which gives explicitly

$$l_i^j \triangleright x^k = R^{kj}_{\;ri}\, x^r = x^r\, \bar{R}^{\;kj}_{ri}.$$

Up to a sign, this categorical morphism coincides with the left representation (6.13). ∎

Let us consider a particular example of the "adjoint" mREA representation acting in the linear span of generators l_i^j. Since

$$\mathrm{Span}(l_i^j) \cong V \otimes V^*,$$

the representation involved is constructed by the general formula (6.10) which now reads

$$l_i^j \mapsto \rho_{V\otimes V^*}(\Delta(l_i^j)),$$

where, as $\rho_V(l_i^j)$ and $\rho_{V^*}(l_i^j)$, we should take the basic representations (6.11) and (6.13) respectively. Omitting straightforward calculations, we write the final result in the compact matrix form

$$\rho_{V\otimes V^*}(L_{\overline{1}}) \triangleright L_{\overline{2}} = L_1 R_{12} - R_{12} L_1. \tag{6.15}$$

Applying the coproduct Δ (6.5), we can extend this representation onto any homogeneous component of the mREA.

Note, that above action (6.15) is an L-linear part of the defining commutation relations of the mREA (5.8), if we rewrite them in the equivalent form

$$L_{\overline{1}} L_{\overline{2}} - R_{12}^{-1} L_{\overline{1}} L_{\overline{2}} R_{12} = L_1 R_{12} - R_{12} L_1 \,.$$

In this sense the action (6.15) is similar to the adjoint action of a Lie algebra $\mathfrak{g}$ on its universal enveloping algebra $U(\mathfrak{g})$, which is also determined by the linear part of the Lie bracket and then is extended from the Lie algebra onto higher components of $U(\mathfrak{g})$ by means of the standard coproduct operation.

To end the section, consider the question on the "sl-reduction", that is, the passing from mREA $\mathcal{L}(R_q, 1)$ to the quotient algebra

$$\mathcal{SL}(R_q) := \mathcal{L}(R_q, 1)/\langle \mathrm{Tr}_R L\rangle\,, \qquad \mathrm{Tr}_R L := Tr(CL). \tag{6.16}$$

The elements $\ell := \mathrm{Tr}_R L$ is central in mREA, which can be easily proved by calculating the R-trace in the second space from the matrix relation (5.8). In so doing, the formulae (2.8) is useful.

To describe the quotient algebra $\mathcal{SL}(R_q)$ explicitly, we pass to the set of new generators $\{f_i^j, \ell\}$, connected with the initial one by the linear transformation:

$$l_i^j = f_i^j + (Tr(C))^{-1}\delta_i^j\,\ell \quad \text{or} \quad L = F + (Tr(C))^{-1} I\,\ell, \tag{6.17}$$

where $F = \|f_i^j\|$. Obviously, $\mathrm{Tr}_R F = 0$. Note, that in virtue of (3.10), the above shift is possible iff $p \neq r$.

In terms of new generators, the commutation relations of mREA read

$$\begin{cases} \bar R_{12}F_1\bar R_{12}F_1 - F_1\bar R_{12}F_1\bar R_{12} = (e_{\mathcal{L}} - \dfrac{\omega}{Tr(C)}\,\ell)(\bar R_{12}F_1 - F_1\bar R_{12}) \\ \ell\, F = F\,\ell\,, \qquad \mathrm{Tr}_R F = 0\,, \end{cases}$$

where $\omega = q - q^{-1}$. Now, the quotient (6.16) is easy to describe. The matrix $F = \|f_i^j\|$ of $\mathcal{SL}(R_q)$ generators satisfy the same commutation relations (5.8) as the matrix L

$$\bar R_{12}F_1\bar R_{12}F_1 - F_1\bar R_{12}F_1\bar R_{12} = \bar R_{12}F_1 - F_1\bar R_{12}\,, \quad \mathrm{Tr}_R F = 0\,, \tag{6.18}$$

but the generators f_i^j are linear dependent due to relation $\mathrm{Tr}_R F = Tr(CF) = 0$.

It is not difficult to rewrite the representation (6.15) in terms of generators f_i^j and ℓ. Taking into account relation (6.17), we find after a short calculation

$$\begin{aligned} &\rho_{V\otimes V^*}(\ell)\triangleright \ell = 0, \qquad \rho_{V\otimes V^*}(F_1)\triangleright \ell = 0\,, \\ &\rho_{V\otimes V^*}(\ell)\triangleright F_1 = -\omega\, Tr(C)\, F_1 \\ &\rho_{V\otimes V^*}(F_{\overline{1}})\triangleright F_{\overline{2}} = F_1\bar R_{12} - \bar R_{12}F_1 + \omega \bar R_{12}F_1\bar R_{12}^{-1}\,. \end{aligned} \tag{6.19}$$

Note, that relation (6.19) defines the "adjoint" representation of the quotient algebra $\mathcal{SL}(R_q)$, but, contrary to the mREA $\mathcal{L}(R_q, 1)$, this representation is not given by the linear part or quadratic-linear commutation relations (6.18).

Generally, given a representation $\rho : \mathcal{L}(R_q, 1) \to \mathrm{End}(U)$ such that the element ℓ is a multiple of the unit operator (for example, an irreducible representation)

$$\rho(\ell) = \chi\, I_U\,, \qquad \chi \in \mathbb{K}$$

we can construct the corresponding representation $\tilde\rho : \mathcal{SL}(R_q) \to \mathrm{End}(U)$ by the formula [S]

$$\tilde\rho(f_i^j) = \frac{1}{\xi}\left(\rho(l_i^j) - (Tr(C))^{-1}\rho(\ell)\,\delta_i^j\right)\,, \qquad \xi = 1 - (q-q^{-1})(Tr(C))^{-1}\chi\,. \tag{6.20}$$

At last, we note, that REA (5.9) admits a series of automorhisms $M \mapsto zM$ with nonzero $z \in \mathbb{K}$. At the level of mREA representations these automorphisms read (recall, that $\hbar = 1$)

$$\rho_U(l_i^j) \mapsto \rho_U^z(l_i^j) = z\rho_U(l_i^j) + \delta_i^j(1-z)(q-q^{-1})^{-1}\, I_U\,.$$

Basing on (6.20), one can show that the corresponding representation $\tilde\rho_U$ of the algebra $\mathcal{SL}(R_q)$ constructed from ρ_U^z does not depend on z, in other words, the whole class of mREA representations ρ_U^z connected by the above automorphism gives the same representation of the quotient algebra $\mathcal{SL}(R_q)$.

Remark 32 In this connection we would like to discuss the problem of a suitable definition of braided (quantum, generalized) Lie algebras. For the first time such an object was introduced in [G1] as a data ($\mathfrak{g}$, σ, $[\,,\,]$) where $\mathfrak{g}$ is a vector space, $\sigma : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 2}$ is an involutive symmetry, and $[\,,\,] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ is an operator ("braided Lie bracket") such that

1. $[\,,\,]\sigma = -[\,,\,]$
2. $\sigma[\,,\,]_{23} = [\,,\,]_{12}\sigma_{23}\sigma_{12}$
3. $[\,,\,][\,,\,]_{23}(I + \sigma_{12}\sigma_{23} + \sigma_{23}\sigma_{12}) = 0.$

Remark that the third relation can be presented as follows

$$[\,,\,][\,,\,]_{12} = [\,,\,][\,,\,]_{23}(I - \sigma_{12}) \tag{6.21}$$

A typical example is

$$\mathfrak{g} = \mathrm{End}(V)\,, \quad \sigma = R_{\mathrm{End}}\,, \quad [\,,\,] = \circ(I - \sigma)$$

(in the setting of the section 5). The other example can be obtained by restricting the above operators onto the subspace of traceless elements of the algebra $\mathrm{End}(V)$. The enveloping algebras of the both braided Lie algebras can be defined by (5.1).

Now, observe that relation (6.21) takes the form (6.15) if we put

$$\mathfrak{g} = \mathrm{Span}(l_i^j)\,, \quad \sigma(L_{\overline{1}}L_{\overline{2}}) = R_{12}^{-1}L_{\overline{1}}L_{\overline{2}}R_{12}\,, \quad [L_{\overline{1}}, L_{\overline{2}}] = L_{\overline{1}}R_{12} - R_{12}L_{\overline{1}}\,. \tag{6.22}$$

So, if we define a braided Lie algebra with such $\mathfrak{g}$, σ and $[\,,\,]$, the third axiom of the above list (in the form (6.21) will be satisfied. By contrast, the relations from the items 1 and 2 fail and must be modified. Thus, an analog of the item 1 can be presented in the form

$$[\,,\,]\mathcal{S} = 0\,,$$

where $\mathcal{S}$ is defined in (5.20). The verification of this relation is straightforward and is left to the reader. In the item 2, the map σ must be replaced by R_{End}. This is consequence of the fact that the bracket $[\,,\,]$ in (6.22) is a categorical morphism. But if we restrict ourselves to the traceless part of the space $\mathfrak{g}$, the relation (6.21) fails too.

So, it is somewhat contradictory to define a braided Lie algebra in the space $\mathrm{End}(V)$ (where the space V is equipped with a skew-invertible Hecke symmetry) with the use of the above three axioms. However, in many papers (cf. [Wo, GM]) the braided (quantum) Lie algebras related to non-involutive braidings are introduced via just these axioms or their slight modifications. Taking in consideration our observation (see Introduction) on "braided Lie algebras" related to braiding of the Birman-Murakami-Wenzl type, we can conclude that there is no "braided Lie algebras" satisfying the above list of axioms and, at the same time, possessing the good deformation property of their enveloping algebras.

7 Quantization with a deformed trace

In this section we consider the semiclassical structures arising from the mREA $\mathcal{L}(R_q, \hbar)$ (5.8), provided that R is the standard $U_q(sl(m))$ Hecke symmetry (1.3). In this case the mREA $\mathcal{L}(R_q, \hbar)$ is treated as a two-parameter deformation of the commutative algebra $\mathbb{K}[gl(m)^*]$. We clarify the role of the corresponding Poisson brackets in defining the quantum homogeneous spaces. At the end of the section we study the infinitesimal counterpart of the deformed R-trace.

Given the Hecke symmetry R (1.3), we can find the Poisson pencil which is the semiclassical counterpart of the two-parameter algebra $\mathcal{L}(R_q, \hbar)$ by a straightforward calculation. Indeed, setting $q = 1$ in (1.3), we pass from $\mathcal{L}(R_q, \hbar)$ to the algebra $U(gl(m)_\hbar)$. Therefore, the Poisson bracket corresponding to the deformation described by the parameter $\hbar$ is the linear Poisson-Lie bracket on the space of functions on $gl(m)^*$.

In order to find the second generating bracket of the Poisson pencil, we put $\hbar = 0$ in (5.8) coming thereby to the non-modified REA. Introducing the matrix $\mathcal{R} = RP$

$$\mathcal{R} = \sum_{i,j}^{m} q^{\delta_{ij}} h_i^i \otimes h_j^j + (q - q^{-1}) \sum_{i<j}^{m} h_i^j \otimes h_j^i\,,$$

we transform the commutation relations of the REA into the form

$$\bar{\mathcal{R}}_{12}\, L_1\, \bar{\mathcal{R}}_{21}\, L_2 - L_2\, \bar{\mathcal{R}}_{12}\, L_1\, \bar{\mathcal{R}}_{21} = 0\,, \tag{7.1}$$

we recall that the bar over the symbol of a matrix means transposition.

On setting $q = e^{\nu}$, $\nu \in \mathbb{K}$, and noting, that $\mathcal{R} = I$ at $q = 1$, we come to the following expansion of $\mathcal{R}$ into a series in ν: $\mathcal{R} = 1 + \nu\, \mathsf{r} + O(\nu^2)$, where

$$\mathsf{r} = \sum_{i=1}^{m} h_i^i \otimes h_i^i + 2\sum_{i<j}^{m} h_i^j \otimes h_j^i. \tag{7.2}$$

is the classical $sl(m)$ r-matrix

$$[\mathsf{r}_{12}, \mathsf{r}_{13}] + [\mathsf{r}_{12}, \mathsf{r}_{23}] + [\mathsf{r}_{13}, \mathsf{r}_{23}] = 0.$$

Now, the part of the commutation relation (7.1) which is linear in ν represent the second Poisson bracket in $\mathbb{K}[gl(m)^*]$

$$\{L_1, L_2\}_r = L_2 L_1 \bar{\mathsf{r}}_{21} - \bar{\mathsf{r}}_{12} L_1 L_2 + L_2 \bar{\mathsf{r}}_{12} L_1 - L_1 \bar{\mathsf{r}}_{21} L_2\,. \tag{7.3}$$

This formula is defined on the matrix elements l_i^j of the matrix L which form a basis of linear functions on the space $gl(m)^*$. The extension of bracket (7.3) from the generators l_i^j to arbitrary functions (polynomials in generators) is described in terms of vector fields on $gl(m)^*$. In order to obtain such an extension, we introduce the matrices

$$\mathsf{r}_{\pm} = \frac{1}{2}\,(\mathsf{r}_{12} \pm \mathsf{r}_{21}).$$

As directly follows from (7.2), the above matrices are images of

$$r_- = \sum_{i<j}^{m} (e_i^j \otimes e_j^i - e_j^i \otimes e_i^j) := \sum_{i<j}^{m} e_i^j \wedge e_j^i\,, \qquad r_+ = \sum_{i,j}^{m} e_i^j \otimes e_j^i\,, \qquad r_{\pm} \in gl(m)^{\otimes 2} \tag{7.4}$$

under the fundamental vector representation $e_i^j \mapsto h_i^j$, the elements e_i^j being the standard basis of $gl(m)$

$$[e_i^j, e_k^r] = \delta_k^j\, e_i^r - \delta_i^r\, e_k^j.$$

Consider now the actions of $gl(m)$ on $\mathbb{K}[gl(m)^*]$ by the left, right and adjoint vector fields

$$e_i^j \triangleright l_k^s := \delta_k^j\, l_i^s\,, \qquad l_k^s \triangleleft e_i^j := \delta_i^s\, l_k^j\,, \qquad \mathrm{ad}\, e_i^j(l_k^s) = e_i^j \triangleright l_k^s - l_k^s \triangleleft e_i^j\,, \tag{7.5}$$

which are extended on any polynomial from $\mathbb{K}[gl(m)^*]$ by the Leibnitz rule. Then, taking into account the above definitions of $r_{\pm}$ we can rewrite (7.3) in the general form

$$\{f, g\}_r = \circ\, r_+^{\mathrm{l,r}}(f \otimes g) - \circ\, r_+^{\mathrm{r,l}}(f \otimes g) - \circ\, r_-^{\mathrm{ad,ad}}(f \otimes g), \qquad \forall\, f, g \in \mathbb{K}[gl(m)^*]. \tag{7.6}$$

Here $\circ : \mathbb{K}[gl(m)^*]^{\otimes 2} \to \mathbb{K}[gl(m)^*]$ stands for the commutative pointwise product of functions on $gl(m)^*$ and superscripts of $r_{\pm}$ denote the following actions

$$r_-^{\mathrm{ad,ad}}(f \otimes g) := \sum_{i<j}^{m} \mathrm{ad}\, e_i^j(f) \wedge \mathrm{ad}\, e_j^i(g), \quad r_+^{\mathrm{l,r}}(f \otimes g) := \sum_{i,j} (e_i^j \triangleright f) \otimes (g \triangleleft e_j^i)\,. \tag{7.7}$$

Note that the brackets

$$\{f,\, g\}_- = \circ\, r_-^{\mathrm{ad,ad}}(f \otimes g) \quad \text{and} \quad \{f, g\}_+ = \circ\, r_+^{\mathrm{l,r}}(f \otimes g) - \circ\, r_+^{\mathrm{r,l}}(f \otimes g)$$

are *not* Poisson, since they do not obey the Jacobi identity.

The bracket $\{\,\}_+$ is $gl(m)$-covariant, that is

$$\mathrm{ad}\, X(\{f,g\}_+) = \{\mathrm{ad}\, X(f), g\}_+ + \{f, \mathrm{ad}\, X(g)\}_+, \qquad \forall f,g \in \mathbb{K}[gl(m)^*], \quad X \in gl(m).$$

The bracket (7.6) restricts onto any $gl(m)$-orbit $\mathcal{O} \subset gl(m)^*$, since for any $f \in I_{\mathcal{O}}$, where $I_{\mathcal{O}}$ is an ideal of functions vanishing on this orbit, and for any $g \in \mathbb{K}[gl(m)^*]$ we have

$$\{f,g\}_r \in I_{\mathcal{O}}\,.$$

This property is evident for the component $\{\,\}_-$, since it is defined via the adjoint vector fields. The proof for the component $\{\,\}_+$ is given in [D]. In particular, the bracket (7.6) can be restricted on the variety $c_1 := \sum l_i^i = c$ where $c \in \mathbb{K}$ is a constant. On setting $c = 0$, we get a Poisson bracket on the algebra $\mathbb{K}[sl(m)^*]$.

Remark 33 Being restricted on $\mathbb{K}[\mathfrak{g}^*]$ where $\mathfrak{g} = sl(m)$, the bracket $\{\,\}_+$ admits the following interpretation [G4, D]. Consider the space $\mathfrak{g}^{\otimes 2}$ as an adjoint $\mathfrak{g}$-module. It decomposes into the direct sum of submodules

$$\mathfrak{g}^{\otimes 2} = \mathfrak{g}_s \oplus \mathfrak{g}_a,$$

where $\mathfrak{g}_s(\mathfrak{g}_a)$ is the symmetric (skew-symmetric) subspace of $\mathfrak{g}^{\otimes 2}$. For $n > 2$ there exist subspaces $\mathfrak{g}_+ \subset \mathfrak{g}_s$ and $\mathfrak{g}_- \subset \mathfrak{g}_a$ which are isomorphic to $\mathfrak{g}$ itself as adjoint $\mathfrak{g}$-modules. Therefore there is a unique (up to a factor) nontrivial $\mathfrak{g}$-covariant morphism $\beta : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 2}$ sending $\mathfrak{g}_-$ to $\mathfrak{g}_+$.

Since the space of linear functions on $\mathfrak{g}^*$ with the Poisson-Lie bracket is isomorphic to $\mathfrak{g}$ as a Lie algebra, the morphism β can also be defined on the whole algebra $\mathbb{K}[\mathfrak{g}^*]$ (via the Leibnitz rule) and it gives a bracket which coincides (up to a factor) with $\{\,\}_+$. Note that for $n = 2$ the component $\mathfrak{g}_+$ vanishes and therefore the bracket $\{\,\}_+$ vanishes as well.

Proposition 34 *The bracket* (7.6) *is compatible with the Poisson-Lie one (their Schouten bracket vanishes) and, therefore, any bracket of the pencil* (1.4) *is Poisson.*

This claim is an immediate corollary of the fact that the family of algebras $\mathcal{L}(R_q, \hbar)$ is a two parametric deformation of the commutative algebra $\mathbb{K}[gl(m)^*]$, but it can be also verified by direct calculations [D].

Now, let us consider the case $n = 2$ in more detail. As we noticed above, in this case the component $\{\,\}_+$ of the Poisson bracket $\{\,\}_r$ vanishes and we have $\{\,\}_r = \{\,\}_-$. Let $\{H, E, F\}$ be the Cartan-Chevalley generators of $sl(2)$. Then $r_- = E \wedge F$ (see (7.4)) and the Poisson bracket (7.6) reads

$$\{a,b\}_r = -\mathrm{ad}\, E(a)\, \mathrm{ad}\, F(b) + \mathrm{ad}\, F(a)\, \mathrm{ad}\, E(b).$$

Take the generators $\{e, f, h\}$ of $\mathbb{K}[sl(2)^*]$ which corresponds to the Cartan-Chevalley generators under the isomorphism of $sl(2)$ and the Lie algebra of linear functions on $sl(2)^*$. A simple calculation on the base of (7.5) gives

$$\{h,e\}_r = -2eh\,, \quad \{h,f\}_r = 2fh\,, \quad \{e,f\}_r = -h^2\,.$$

Note, that this differs from the Poisson-Lie bracket only by the factor $-h$ and, therefore, each leaf of the bracket $\{\,\}_r$ lies in a leaf of the Poisson-Lie bracket. Moreover, the element $c_2 = h^2/2 + 2ef$ is central with respect to the both brackets and hence the corresponding Poisson pencil can be restricted onto the quotient $\mathbb{K}[sl(2)^*]/\langle c_2 - c\rangle$ for any $c \in \mathbb{K}$.

Let $\mathbb{K} = \mathbb{C}$ and elements

$$x = \frac{1}{2}\,(e-f), \quad y = \frac{i}{2}\,(e+f), \quad z = \frac{i}{2}\,h$$

be the generators of $su(2)$. In these generators we get the Poisson pencil $\{\,\}_{PL,r}$ where

$$\{x,y\}_{PL}=z,\qquad \{y,z\}_{PL}=x,\qquad \{z,x\}_{PL}=y,$$
$$\{x,y\}_r=z^2,\qquad \{y,z\}_r=xz,\qquad \{z,x\}_r=yz.$$

Here we have renormalized the bracket $\{\,\}_r$ since this does not affect the Poisson pencil. The quadratic central element takes the form $c_2=x^2+y^2+z^2$.

A particular bracket of this Poisson pencil (namely, $\{\,\}_{PL}-\{\,\}_r$) appeared in [Sh] (see Appendix by J.-H.Lu and A.Weinstein) in studding a semiclassical counterpart of the quantum sphere. In cited paper, the quantum sphere was represented as an operator algebra. This approach is based on the work [P], where the quantum sphere was treated to be a C^*-algebra and its irreducible representations (as C^*-algebra) were classified. Finally, in [Sh] the quantum sphere was presented in terms of functional analysis.

Our method of constructing the quantum sphere (or quantum hyperboloid what is the same over the field $\mathbb{K}=\mathbb{C}$) is completely different. First of all, we quantize the KKS bracket[8] on the sphere and realize the resulting quantum algebra as the quotient

$$U(su(2)_\hbar)/\langle x^2+y^2+z^2-c\rangle\,,\quad c\in\mathbb{K}\,,\ c\neq 0. \tag{7.8}$$

We are interested in finite dimensional representations of this algebra. There exists a set of negative values $c^{(k)}=-\hbar^2k(k+2)/4$, $k\in\mathbb{N}$, of the parameter c such that the quotient algebra (7.8) admits a finite dimensional representation if $c=c^{(k)}$ for some $k\in\mathbb{N}$.

Returning to the generators of the algebra $sl(2)$, we obtain a one-parameter family of algebras

$$\mathcal{SL}^c(\hbar)=U(sl(2)_\hbar)/\langle\frac{1}{2}\,H^2+EF+FE-c\rangle\,.$$

Any algebra $\mathcal{SL}^c(\hbar)$ of this family as well as $\mathcal{SL}^c=\mathbb{K}[sl(2)^*]/\langle\frac{h^2}{2}+ef+fe-c\rangle$, being equipped with the $sl(2)$-action, can be expanded into a multiplicity free direct sum of $sl(2)$-modules

$$\mathcal{SL}^c\cong\bigoplus_{k\geq 0}V_k\,.$$

Let $\alpha:\mathcal{SL}^c\to\mathcal{SL}^c(\hbar)$ be an $sl(2)$-invariant map sending the highest weight elements $e^{\otimes k}\in\mathcal{SL}^c$ to $E^{\otimes k}\in\mathcal{SL}^c(\hbar)$. This requirement defines the map α completely. Now, the commutative algebra $\mathcal{SL}^c$ can be equipped with a new noncommutative product $\star$ coming from the algebra $\mathcal{SL}^c(\hbar)$

$$f\star_\hbar g=\alpha^{-1}(\alpha(f)\circ\alpha(g)),\qquad f,g\in\mathcal{SL}^c \tag{7.9}$$

where $\circ$ is the product in the algebra $\mathcal{SL}^c(\hbar)$. Thus, we have the quantized KKS bracket on the hyperboloid $c_2=c$ in the spirit of deformation quantization scheme — introducing a new product in the initial space of commutative functions. Note, that according to [R], our algebraic quantization cannot be extended on the function space $C^\infty[S^2]$.

Now, deform the algebra $\mathbb{K}[sl(2)^*]$ in q and $\hbar$ "directions" simultaneously. We get the mREA (5.8) with R-matrix given in (1.3) where we should set $m=2$. Extracting the R-traceless elements from the set of four mREA generators, we come to a unital associative algebra, generated by three linearly independent elements $\{\hat H,\hat E,\hat F\}$ subject to the system of commutation relations

$$\begin{aligned}
&q^2\hat H\hat E-\hat E\hat H=2_q\hbar\,\hat E\,,\\
&\hat H\hat F-q^2\hat F\hat H=-2_q\hbar\,\hat F\,,\\
&q(\hat E\hat F-\hat F\hat E)=\hat H\Big(\hbar-\frac{(q^2-1)}{2_q}\,\hat H\Big)\,.
\end{aligned}$$

[8]Recall, that KKS bracket is a restriction of the Poisson-Lie bracket on a coadjoint orbit of the Lie group in the space dual to its Lie algebra.

Denote this algebra by $\mathcal{SL}(q,\hbar)$. The element $C_q = \frac{\hat{H}^2}{2_q} + q^{-1}\hat{E}\hat{F} + q\hat{F}\hat{E}$ is central and is called *the braided Casimir*. Let us put

$$\mathcal{SL}^c(q,\hbar) = \mathcal{SL}(q,\hbar)/\langle C_q - c\rangle.$$

We call this algebra the quantum hyperboloid or (considering it over the field $\mathbb{K} = \mathbb{C}$) the quantum sphere. It is a two-parameter deformation of the initial commutative algebra $\mathcal{SL}^c$. For a generic value of q it is possible to define a map $\alpha_q : \mathcal{SL}^c \to \mathcal{SL}^c(q,\hbar)$ similar to the map α (but without the equivariance property) and represent the product in $\mathcal{SL}^c$ in the spirit of relation (7.9).

As in the case of algebra (7.8), there exists a series of values $c = c_k$, such that the corresponding quotient algebra $\mathcal{SL}^{c_k}(q,\hbar)$ has a finite dimentional equivariant representation. Its construction is described in section 6. When $q \to 1$, we get the representation of the algebra $\mathcal{SL}^{c_k}(\hbar)$. By contrast, the representation theory of the quantum sphere suggested in [P] has nothing in common with the theory of finite dimensional representations of $sl(2)$ (or $su(2)$).

In general, by quantizing the KKS bracket on a semisimple orbit we represent the quantum algebra as an appropriate quotient of the enveloping algebra $U(\mathfrak{g}_\hbar)$ with $\mathfrak{g} = gl(m)$ or $sl(m)$. (Note, that if such an orbit is not generic the problem of looking for defining relations of the corresponding "quantum orbit" is somewhat subtle, cf. [DM]). Finally, we additionally deform this quotient in "q-direction" and get some quotient of the algebra $\mathcal{L}(q,\hbar)$.

Observe, that on a generic orbit in $\mathfrak{g}^*$ where $\mathfrak{g}$ is a simple Lie algebra there exists a family of nonequivalent Poisson brackets, giving rise to $U_q(\mathfrak{g})$-covariant algebras. One of them is the reduced Sklyanin bracket. It is often described in terms of the Bruhat decomposition (cf. [LW]). The classification of all these brackets and their deformational quantization are given in [DGS] and [D]. The reduced Sklyanin bracket can be also quantized in terms of the so-called Hopf-Galois extension (cf. [DGH]). But only the bracket (7.6), restricted on a semisimple orbit, is compatible with the KKS bracket and the quantization of the corresponding Poisson pencil can be realized in the spirit of affine algebraic geometry, i.e. via generators and relations between them.

Note, that on the sphere (hyperboloid) the reduced Sklyanin bracket coincides with one of the bracket from the Poisson pencil $\{\,,\}_{KKS,r}$ (it is also true for any symmetric orbit). So, it can be quantized via different approaches. However, for $m > 2$ and for higher dimensional orbits the notion of quantum orbits should be concretized. It essentially depends on the bracket to be quantized.

As for the other classical simple algebras $\mathfrak{g}$ of the B, C or D series, there is no two-parameter deformation of the algebra $\mathbb{K}[\mathfrak{g}^*]$ (cf. [D]). Though a quadratic-linear algebra (similar to $\mathcal{L}(R_q,\hbar)$) can be constructed in this case (cf. [DGG] for detail), we note that neither this algebra nor the associated quadratic algebra is a deformation of its classical counterpart.

Let us complete this section by considering a semiclassical analog of the quantum trace in the spirit of [G2]. In that paper there were considered Poisson pencils similar to the above ones but generated by triangular classical r-matrices (they give rise to involutive braidings). The main difference is that the result of the "double quantization" of Poisson pencil from [G2] was treated as the enveloping algebra of a generalized Lie algebra and its finite dimensional representations formed a tensor (not a quasitensor) category.

As is known, on any symplectic variety there is a Liouville (or invariant, or symplectic) measure $d\mu$ with basic property $\int\{f,g\}d\mu = 0$. In the framework of the deformation quantization this measure gives rise to the trace with usual properties (cf. [GR]). It is just the case of the KKS bracket on a semisimple orbit. For non-symplectic Poisson brackets one usually tries to describe its symplectic leaves and to quantize them separately, i.e., to associate an operator algebra to each of leaves. In the framework of our approach we are not dealing with quantizing leaves of the bracket $\{\,,\}_r$ or any bracket from the Poisson pencil $\{\,,\}_{KKS,r}$ but we quantize this Poisson pencil as a whole. In other words, we simultaneously q-deform all algebras arising from "$\hbar$-quantization" and arrive to operator algebras with deformed traces.

Consider the Poisson pencil $\{\,,\}_{KKS,r}$ on a semisimple orbit $\mathcal{O} \subset su(m)^*$. The bracket $\{\,,\}_r$ is not symplectic, therefore the pencil involved has no Liouville measure on the whole orbit (a similar case was considered in [G2]). Nevertheless, the following proposition holds true independently on the concrete form of the matrix r.

Proposition 35 *Let $\{\,,\}_{KKS,r}$ be the Poisson pencil on a semisimple orbit $\mathcal{O}\subset\mathfrak{g}^*$, where $\mathfrak{g}=su(m)$ (or its complexification) and $d\mu$ is the Liouville measure for the bracket $\{\,,\}_{KKS}$. Then the quantity*

$$\langle a,b\rangle=\int_{\mathcal{O}}\{a,b\}_r\,d\mu \tag{7.10}$$

is a cocycle with respect to the bracket $\{\,,\}_{KKS}$, i.e.

$$\langle a,\{b,c\}_{KKS}\rangle+\langle b,\{c,a\}_{KKS}\rangle+\langle c,\{a,b\}_{KKS}\rangle=0.$$

This statement is a simple consequence of the fact that the brackets $\{\,,\}_{KKS}$ and $\{\,,\}_r$ are compatible. The cocycle (7.10) is treated as an infinitesimal term of the deformation of the pairing $a\otimes b\mapsto\int_{\mathcal{O}}ab\,d\mu$ [G2].

In a similar way we consider an infinitesimal term of the deformation of the pairing $A\otimes B\mapsto\mathrm{Tr}(A\circ B)$. For this end, we use the relation

$$\mathrm{Tr}_R\circ(\mathcal{R}_{12}L_1\mathcal{R}_{21}L_2-L_2\mathcal{R}_{12}L_1\mathcal{R}_{21})=0$$

where matrix elements of the matrices L_1 and L_2 belong to $\mathrm{End}(V)$, the symbol $\circ$ stands for the product (4.12) in this algebra and the operation Tr_R is applied to each matrix element. The above relation holds true due to $\mathrm{Tr}_R\,l_i^j=\delta_i^j$.

Then, developing the R-matrix and the R-trace into a series in ν:

$$\mathcal{R}=I+\nu\,\mathsf{r}+O(\nu^2)\,,\qquad \mathrm{Tr}_R=\mathrm{Tr}+\nu\,b+O(\nu^2)\,,$$

(r is given by (7.2)) we get the explicit form of the operation $b\circ$ on the skew-symmetric subspace $\wedge^2(\mathrm{End}(V))$

$$b\circ(L_1\otimes L_2-L_2\otimes L_1)=-\mathrm{Tr}\circ(\mathsf{r}_{12}L_1L_2+L_1\mathsf{r}_{21}L_2-L_2\mathsf{r}_{12}L_1-L_2L_1\mathsf{r}_{21})\,.$$

Having thus defined the operation $b\circ$ on the basis elements, we directly get the general expression (see (7.4–(7.7) for notations)

$$b\circ(A\otimes B-B\otimes A)=\mathrm{Tr}\circ(-r_-^{\mathrm{ad,ad}}(A\otimes B)-r_+^{\mathrm{r,l}}(A\otimes B)+r_+^{\mathrm{l,r}}(A\otimes B)),\quad A,B\in\mathrm{End}(V)\,. \tag{7.11}$$

Thus, we have got the skew-symmetrized linear term of deformation of the pairing

$$A\otimes B\mapsto\mathrm{Tr}(A\circ B)\,.$$

Proposition 36 *The quantity $\langle A,B\rangle=-b\circ(A\otimes B-B\otimes A)$ is a cocycle on the Lie algebra $gl(m)$, i.e.*

$$\langle A,[B,C]\rangle+\langle B,[C,A]\rangle+\langle C,[A,B]\rangle=0.$$

It reduces to the Lie algebra $sl(m)$.

It is not difficult to write the explicit form of the cocycle $\langle A,B\rangle$. Indeed, one can show that the second and third terms in the right hand side of (7.11) give no contribution to this cocycle and by using cyclic property of the usual trace we get

$$\langle A,B\rangle=\mathrm{Tr}\,([A,B]\circ\sum_{\alpha>0}[X_\alpha,X_{-\alpha}])=\mathrm{Tr}\,([A,B]\circ\sum_{\alpha>0}H_\alpha),$$

where the sum is going over the set of all positive roots.

Appendix

In this section we collect some facts and definitions on the theory of the A_{k-1} series Hecke algebras $H_k(q)$, used in the main text of the paper. For a detailed review of the subject the reader is referred to [OP1]. Throughout this section we shall use the definitions and notations of that paper. At the end of the section we give the proof of Proposition 7, formulated in section 2.

By definition, the Hecke algebra of A_{k-1} series is a unital associative algebra $H_k(q)$ over a field $\mathbb{K}$ generated by the elements σ_i, $1 \le i \le k-1$, subject to the following commutation relations

$$\begin{aligned} &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} && 1 \le i \le k-2 \\ &\sigma_i\sigma_j = \sigma_j\sigma_i && |i-j| \ge 2 \\ &\sigma_i^2 = 1_H - (q-q^{-1})\,\sigma_i && 1 \le i \le k-1. \end{aligned}$$

Here 1_H is the unit of the algebra, $q \in \mathbb{K}$ is a nonzero element of the ground field. Below we shall suppose $\mathbb{K}$ to be a field of complex numbers $\mathbb{C}$ or the field of rational functions $\mathbb{C}(q)$ of the formal variable q.

At a generic value of q the Hecke algebra $H_p(q)$ is semisimple and isomorphic to the group algebra of the k-th order symmetric group $\mathbb{K}[\mathfrak{S}_k]$ [We]. Therefore, being considered as the regular two-sided $H_k(q)$-module, the Hecke algebra $H_k(q)$ can be presented as the direct sum of simple ideals (the Wedderburn-Artin theorem)

$$H_k(q) = \bigoplus_{\lambda\vdash k} M^\lambda$$

labelled by partitions λ of the integer k. Under the left (right) action of the Hecke algebra the submodules M^λ are reducible and can be further decomposed into the direct sum of the corresponding equivalent one-sided (left or right) submodules

$$M^\lambda = \bigoplus_{a=1}^{d_\lambda} M^{(\lambda,a)}_{l(r)}$$

where d_λ is the number of the standard Young tableaux (λ, a) corresponding to the partition λ [Mac]. The index a enumerates standard tableaux in accordance with some ordering (say, lexicographical).

In each ideal M^λ one can fix a linear basis of "matrix units" e^λ_{ab} with the multiplication table

$$e^\lambda_{ab}\, e^\mu_{cd} = \delta^{\lambda\mu}\delta_{bc}\, e^\mu_{ad}\,.$$

A subset e^λ_{ab}, $1 \le b \le d_\lambda$, (with a fixed value of the first index) forms the basis of the right module $M_r^{(\lambda,a)}$ while fixing the second index gives the basis of the left module $M_l^{(\lambda,b)}$.

The diagonal elements e^λ_{aa} denoted shortly as e^λ_a form the set of primitive idempotents of the Hecke algebra $H_k(q)$. The idempotents e^λ_a are explicitly constructed as some polynomials in the Jucys-Murphy elements $\mathcal{J}_p$, $1 \le p \le k$, (see [OP1] for details) which are defined by the iterative rule

$$\mathcal{J}_1 = 1_H, \quad \mathcal{J}_{p+1} = \sigma_p\, \mathcal{J}_p\, \sigma_p.$$

The set of Jucys-Murphy elements form a basis of the maximal commutative subalgebra of $H_k(q)$. An important property of these elements reads

$$\mathcal{J}_p\, e^\lambda_a = e^\lambda_a\, \mathcal{J}_p = j_p(\lambda,a) e^\lambda_a, \quad j_p(\lambda,a) = q^{2(c_p-r_p)} \in \mathbb{K}. \tag{A.1}$$

Here the positive integers c_p and r_p are the numbers of the column and the row of the Young tableau (λ, a) which contain the box with integer p. Given below is a simple example for a Young tableau of the partition $\lambda = (3, 2, 1)$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \Rightarrow \quad \begin{array}{ll} j_1 = 1 & j_4 = q^4 \\ j_2 = q^{-2} & j_5 = q^{-4} \\ j_3 = q^2 & j_6 = 1 \end{array}$$

Any two idempotents e_a^λ and e_b^λ corresponding to the different tableaux (λ, a) and (λ, b) of a partition $\lambda \vdash k$ can be transformed into each other by the two sided action of some invertible elements of the Hecke algebra $H_k(q)$ [OP1]

$$e_a^\lambda = x_{ab}^\lambda \, e_b^\lambda \, y_{ab}^\lambda \,, \qquad x_{ab}^\lambda, \, y_{ab}^\lambda \in H_k(q). \tag{A.2}$$

Consider now a Hecke symmetry $R : V^{\otimes 2} \to V^{\otimes 2}$, and define a special representation ρ_R of a Hecke algebra $H_k(q)$ in the tensor product $V^{\otimes p}$, $p \geq k$, by the rule

$$\begin{aligned} &\rho_R(1_H) = \mathrm{id}_V^{\otimes p}, \\ &\rho_R(\sigma_i) = \mathrm{id}_V^{\otimes (i-1)} \otimes R_i \otimes \mathrm{id}_V^{\otimes (p-i-1)}, \quad 1 \leq i \leq k-1, \\ &\rho_R(xy) = \rho_R(x)\rho_R(y), \quad \forall \, x, y \in H_k(q), \end{aligned} \tag{A.3}$$

recall, that $R_i := R_{ii+1}$. The fact that ρ_R is a representation follows immediately from (1.1) and (1.2).

Let the bi-rank of R be $(m|n)$, that is its HP series $P_-(t)$ is of the form (3.3). Consider the partitions

$$\begin{aligned} \lambda_{m,n} &:= ((n+1)^{m+1}) \qquad & \lambda_{m,n} &\vdash (m+1)(n+1) \\ \lambda^-_{m,n} &:= ((n+1)^m, n) \qquad & \lambda^-_{m,n} &\vdash mn + m + n \,. \end{aligned} \tag{A.4}$$

In graphic form the partition $\lambda_{m,n}$ is represented by a rectangular diagram with $m+1$ rows of the length $n+1$, while the diagram of $\lambda^-_{m,n}$ is obtained from the former one by removing one box in the right lower corner of the rectangle. Note, that $\lambda^-_{m,n} \in \mathsf{H}(m, n)$, while the partition $\lambda_{m,n}$ is the minimal one, not belonging to the hook $\mathsf{H}(m, n)$ (see Definition 5).

Listed below are the properties of representations ρ_R which follow immediately from Proposition 6.

i) The images $E_a^\lambda = \rho_R(e_a^\lambda) \neq 0$, $e_a^\lambda \in H_k(q)$, for all $2 \leq k < (m+1)(n+1)$;

ii) the representation ρ_R of $H_{(m+1)(n+1)}(q)$ possesses a kernel generated by

$$\rho_R(e_a^{\lambda_{m,n}}) = 0, \quad 1 \leq a \leq d_{\lambda_{m,n}} \,,$$

while $\rho_R(e_a^\mu) \neq 0$ for all $\mu \vdash (m+1)(n+1)$, $\mu \neq \lambda_{m,n}$;

iii) for any integer $p \geq (m+1)(n+1)$ and for any partition $\nu \vdash p$ one has

$$\rho_R(e_a^\nu) = 0 \quad \Leftrightarrow \quad \lambda_{m,n} \subset \nu,$$

where inclusion $\mu = (\mu_1, \mu_2, \ldots) \subset \nu = (\nu_1, \nu_2, \ldots)$ means that $\mu_i \leq \nu_i \; \forall \, i$.

Proof of Proposition 7.

Let us denote $p := (m+1)(n+1)$ for more compact writing of the formulae below. In the Hecke algebra $H_p(q)$ we extract the Hecke subalgebra $H_{p-1}(q) \subset H_p(q)$, generated by $\sigma_i \in H_p(q)$, $1 \leq i \leq p-2$. Fix a standard Young tableau $(\lambda_{m,n}, a)$ (see definition (A.4) above) and consider the idempotents $e_{a^-}^{\lambda^-_{m,n}} \in H_{p-1}(q)$ and $e_a^{\lambda_{m,n}} \in H_p(q)$. Here the notation $(\lambda^-_{m,n}, a^-)$ refers to a special choice of the corresponding Young tableau: it is properly included into the Young tableau $(\lambda_{m,n}, a)$. In other words, the integers from 1 to $p-1$ occupy the same positions in the tableau $(\lambda^-_{m,n}, a^-)$ as they do in the tableau $(\lambda_{m,n}, a)$. Note, that since we consider the standard Young tableaux, the only possible position for the number p is the box in the right lower corner of the rectangular tableaux $(\lambda_{m,n}, a)$.

Let us now apply the map $\rho_R : H_p(q) \to \mathrm{End}(V^{\otimes n})$ to the relation (cf. [OP1])

$$e_a^{\lambda_{m,n}} = e_{a^-}^{\lambda^-_{m,n}} \frac{(\mathcal{J}_p - q^{2(n+1)} 1_H)}{(q^{2(n-m)} - q^{2(n+1)})} \frac{(\mathcal{J}_p - q^{-2(m+1)} 1_H)}{(q^{2(n-m)} - q^{-2(m+1)})}.$$

Denoting $\rho_R(\mathcal{J}_k) := J_k$ and taking into account the item ii) of the above properties of ρ_R we get the identity

$$0 = E_{a^-}^{\lambda^-_{m,n}} \frac{(J_p - q^{2(n+1)} I)}{(q^{2(n-m)} - q^{2(n+1)})} \frac{(J_p - q^{-2(m+1)} I)}{(q^{2(n-m)} - q^{-2(m+1)})}$$

where $E_{a^-}^{\lambda^-_{m,n}} \neq 0$ due to i), the letter I stands for the identity operator on the space $V^{\otimes p}$.

We shall calculate the trace tr of the above identity in the last (p-th) component of the tensor product $V^{\otimes p}$, where tr coincides up to a factor with the categorical R-trace (4.15)

$$\mathsf{tr}(X) := Tr\,(C \cdot X).$$

It is clear, that $\mathsf{tr}(I) = Tr\,C$ is the object we are interested in.

Since the matrix $E_{a^-}^{\lambda^-_{m,n}}$ is a polynomial in J_k with $k < p$, it can be drawn out of the trace in the p-th space and we come to

$$0 = E_{a^-}^{\lambda^-_{m,n}} \mathsf{tr}_{(p)} \Big(J_p^2 - (q^{2(n+1)} + q^{-2(m+1)}) J_p + q^{2(n-m)} I \Big). \tag{A.5}$$

Consider separately the traces of the terms of the above identity. Introducing auxiliary shorthand notation $\omega := q - q^{-1}$ we find

$$\mathsf{tr}_{(p)}(J_p) = \mathsf{tr}_{(p)} \Big(R_{p-1} J_{p-1} (R_{p-1}^{-1} + \omega I) \Big) = \omega J_{p-1} + I_{p-1}\, \mathsf{tr}_{(p-1)}(J_{p-1}).$$

In the above line of transformations we have used the iterative definition of the Jucys-Murphy element, the Hecke condition for R and properties (2.4) and (2.8) of tr listed in section 2. Since the trace in (A.5) is multiplied by the idempotent, we can replace the Jucys-Murhy element J_{p-1} by the corresponding "eigenvalue" j_{p-1} defined in (A.1)

$$E_{a^-}^{\lambda^-_{m,n}} \mathsf{tr}_{(p)}(J_p) = E_{a^-}^{\lambda^-_{m,n}} \Big(\omega j_{p-1} + \mathsf{tr}_{(p-1)}(J_{p-1}) \Big). \tag{A.6}$$

To simplify the formulae, we shall omit the symbol of idempotent and shall perform all calculations, bearing in mind the possibility to replace each free of trace Jucys-Murphy element J_k by the corresponding number j_k.

So, the calculation of $\mathsf{tr}_{(p)}(J_p)$ is completed by the straightforward induction on the base of relation (A.6)

$$\mathsf{tr}_{(p)}(J_p) = \omega \sum_{k=1}^{p-1} j_k + \mathsf{tr}(I),$$

where we have taken into account that $J_1 = I$ by definition.

Transform now the term with the second power of J_p:

$$
\begin{aligned}
\mathsf{tr}_{(p)}(J_p^2) &= \mathsf{tr}_{(p)}\Big(R_{p-1}J_{p-1}R_{p-1}(R_{p-1}^{-1}+\omega I)J_{p-1}R_{p-1}\Big)\\
&= \mathsf{tr}_{(p)}\Big(R_{p-1}J_{p-1}^2(R_{p-1}^{-1}+\omega I)\Big)+\omega\, J_{p-1}\mathsf{tr}_{(p)}\Big(J_p(R_{p-1}^{-1}+\omega I)\Big)\\
&= 2\omega\, J_{p-1}^2+\omega^2\, J_{p-1}\mathsf{tr}_{(p)}(J_p)+\mathsf{tr}_{(p-1)}(J_{p-1}^2)\\
&= 2\omega\, j_{p-1}^2+\omega^2\, j_{p-1}\mathsf{tr}_{(p)}(J_p)+\mathsf{tr}_{(p-1)}(J_{p-1}^2).
\end{aligned}
$$

Substituting the value of $\mathsf{tr}_{(p)}(J_p)$ we get the base for the inductive calculation

$$\mathsf{tr}_{(p)}(J_p^2) = 2\omega\, j_{p-1}^2+\omega^2 j_{p-1}\,\mathsf{tr}(I)+\omega^3 j_{p-1}\sum_{k=1}^{p-1} j_k+\mathsf{tr}_{(p-1)}(J_{p-1}^2).$$

This immediately leads to the following expression

$$\mathsf{tr}_{(p)}(J_p^2) = 2\omega\sum_{k=1}^{p-1} j_k^2+\omega^3\sum_{k=1}^{p-1} j_k\sum_{s=1}^{k} j_s+\Big(1+\omega^2\sum_{k=1}^{p-1} j_k\Big)\,\mathsf{tr}(I).$$

Substituting all the calculated components into identity (A.5) and taking into account that $E_{a^-}^{\lambda^-_{m,n}}\neq 0$, we find the following linear equation for $\mathsf{tr}(I)$

$$\alpha\,\mathsf{tr}(I)+\beta = 0$$

with

$$
\begin{aligned}
\alpha &= 1+q^{2(n-m)}-q^{2(n+1)}-q^{-2(m+1)}+\omega^2\sum_{k=1}^{p-1} j_k\\
\beta &= \omega\left(2+\frac{\omega^2}{2}\right)\sum_{k=1}^{p-1} j_k^2+\frac{\omega^3}{2}\Big(\sum_{k=1}^{p-1} j_k\Big)^2-\omega\Big(q^{2(n+1)}+q^{-2(m+1)}\Big)\sum_{k=1}^{p-1} j_k\,,
\end{aligned}
$$

where in finding the coefficient β we used the identity

$$\sum_{k=1}^{p-1} j_k\sum_{s=1}^{k} j_s = \frac{1}{2}\sum_{k=1}^{p-1} j_k^2+\frac{1}{2}\Big(\sum_{k=1}^{p-1} j_k\Big)^2.$$

Taking into account the definition of j_k and the form of the diagram $\lambda^-_{m,n} = ((n+1)^m, n)$, we can easily calculate the sum of eigenvalues j_k

$$
\begin{aligned}
\sum_{k=1}^{p-1} j_k = \sum_{k=1}^{p-1} q^{2(c_k-r_k)} &= (1+q^2+\ldots+q^{2n})(1+q^{-2}+\ldots+q^{-2m})-q^{2(n-m)}\\
&= q^{n-m}(n+1)_q(m+1)_q-q^{2(n-m)}\,,
\end{aligned}
$$

and therefore

$$\sum_{k=1}^{p-1} j_k^2 = \sum_{k=1}^{p-1}(q^2)^{2(c_k-r_k)} = q^{2(n-m)}(n+1)_{q^2}(m+1)_{q^2}-q^{4(n-m)}\,.$$

Now by a short calculation we simplify the coefficient α to the form

$$\alpha = -\omega^2\, q^{2(n-m)}.$$

The transformation of β is more involved though straightforward too. In this way, one should use the following identity

$$k_{q^2} = \frac{q^{2k} - q^{-2k}}{q^2 - q^{-2}} = \frac{(q^k - q^{-k})}{(q - q^{-1})} \frac{(q^k + q^{-k})}{(q + q^{-1})} = k_q \, \frac{q^k + q^{-k}}{2_q}.$$

Omitting routine calculations we present the final result

$$\beta = \omega^2 \, q^{3(n-m)} (m-n)_q.$$

So, we finally get

$$\mathsf{tr}(I) = Tr\, C = -\frac{\beta}{\alpha} = q^{n-m}(m-n)_q.$$

This completes the proof. ∎

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