ENVELOPES OF HOLOMORPHY AND HOLOMORPHIC DISCS

BURGLIND JÖRICKE

Abstract. The envelope of holomorphy of an arbitrary domain in a two-dimensional Stein manifold is identified with a connected component of the set of equivalence classes of analytic discs immersed into the Stein manifold with boundary in the domain. This implies, in particular, that for each of its points the envelope of holomorphy contains an embedded (non-singular) Riemann surface (and also an immersed analytic disc) passing through this point with boundary contained in the natural embedding of the original domain into its envelope of holomorphy. Moreover, it says, that analytic continuation to a neighbourhood of an arbitrary point of the envelope of holomorphy can be performed by applying the continuity principle once. Another corollary concerns representation of certain elements of the fundamental group of the domain by boundaries of analytic discs. A particular case is the following. Given a contact three-manifold with Stein filling, any element of the fundamental group of the contact manifold whose representatives are contractible in the filling can be represented by the boundary of an immersed analytic disc.

0. Introduction

The notion of the envelope of holomorphy of domains in $\mathbb{C}^n$ (or, more generally, in Stein manifolds) is as classical as the notion of pseudoconvex domains. Nevertheless, basic questions about envelopes of holomorphy are open. For instance, not much is known in general about the number of sheets of the envelope of holomorphy. It is not clear in general when the envelope of holomorphy is single-sheeted or at least (say smoothly) equivalent to a domain in the same Stein manifold (see e.g. [22]) .

One of the most interesting problems in this respect is to understand invariants of the envelope of holomorphy in terms of invariants of the original domain. It is known that the first Betti number of the envelope of holomorphy does not excceed that of the original domain [13]. Moreover, it is proved in [13] that the natural homomorphism between fundamental groups is surjective. In the same vein the natural map between first Cech cohomologies is injective [20], but in the general situation not too much is known beyond these results. Naive hopes are not justified (see e.g. the paper [2].)

The problem of understanding invariants of the envelope of holomorphy in terms of invariants of the domain is even interesting in the following particular case. The domain is a suitable one-sided neighbourhood of the boundary of a strictly pseudoconvex domain a Stein manifold (for instance, it equals the set $\{-\varepsilon < \rho < 0\}$ for a strictly plurisubharmonic defining function and a small positive constant ε) and the envelope of holomorphy is the domain itself. This case reduces to understanding the topology of the Stein fillings of a contact manifold in terms of the topology of the contact manifold and is well-known to symplectic geometers. Despite recent progress and breakthroughs many problems remain open. For instance, there are examples of contact three-manifolds that have a Stein filling with second Betti number strictly exceeding that of the three-manifold and an estimate of the second Betti number of Stein fillings of a given contact three-manifold is not known in general. For a contemporary account see [17].

2000 *Mathematics Subject Classification.* 32A40; 32E35;53D10.

Key words and phrases. envelopes of holomorphy, continuity principle, holomorphic discs, Riemann surfaces, Stein filling, planar trees.

The general problem motivates the search for a geometric description of the envelope of holomorphy. It is well-known that any domain in a Stein manifold has an envelope of holomorphy. Several constructions are known (see e.g. [10], [18], [14]). It is not obvious how to obtain from these constructions geometric information about the envelope of holomorphy.

We give here a new description of the envelope of holomorphy of a domain in a Stein manifold in terms of equivalence classes of analytic discs. This description, in particular, implies that analytic continuation to a neighbourhood of each point in the envelope of holomorphy can be performed by applying the continuity principle once along a family of immersed analytic discs (see below for details).

The approach has further geometric consequences which were not known before. To mention only one of them, for each of its points the envelope of holomorphy contains an embedded (non-singular) Riemann surface (and also an immersed analytic disc) passing through this point with boundary contained in the natural embedding of the original domain into its envelope of holomorphy. This is in contrast to what is known for polynomial hulls.

In the paper we focus on the case of Stein manifolds of dimension 2, which is in several aspects the most interesting case. We believe that the main results are true in higher dimensions. It seems they are true even in more general situations and we intend to work this out later.

1. Statement of results

Denote by X^2 a Stein surface, i.e. a two-dimensional Stein manifold. Let $G \subset X^2$ be a domain. For the description of the envelope of holomorphy we use analytic discs immersed into X^2 with boundary in G. More precisely, we need the following definition.

Definition 1. *Consider a holomorphic immersion from a neigbourhood of the closed unit disc $\overline{\mathbb{D}} \subset \mathbb{C}$ into X^2. The restriction $d : \overline{\mathbb{D}} \looparrowright X^2$ is an analytic disc.*

If the boundary $d(\partial \mathbb{D})$ of the disc is contained in G we will call the disc a G-disc. The set of G-discs is denoted by $\mathcal{G}$.

Fix a metric on X^2. For this we fix a proper holomorphic embedding $\mathfrak{F} : X^2 :\to \mathbb{C}^n$ into Euclidean space C^n of suitable dimension n and pull back the metric induced on $\mathfrak{F}X^2$ by $\mathbb{C}^n$. (By [3] one can always take $n = 4$.) Having in mind this metric on X^2 we will usually endow the set $\mathcal{G}$ of G-discs with the topology of C^1-convergence on the closed disc $\overline{\mathbb{D}}$.

When dealing with an individual G-disc we usually consider the generic case when its boundary is embedded. The following definition selects those G-discs which participate in the continuity principle.

Definition 2. *A G-disc d is G-homotopic to a constant, or for short d is a G_0-disc, if there is a continuous family of G-discs joining d to a constant disc. The set of G_0-discs is denoted by $\mathcal{G}_0$.*

More detailed, the existence of the G-homotopy means, that there is a continuous mapping $F(t, z), \quad t \in I = [0, 1]$, z in a neighbourhood of $\overline{\mathbb{D}}$, such that for each $t \in (0, 1]$ the mapping $z \to F(t, z)$ is a G-disc, moreover, $F(1, z) = d(z)$ and the mapping $z \to F(0, z)$ maps the disc to a point which is then automatically contained in G.

Notice that the existence of a G-homotopy to a constant is equivalent to the existence of a G-homotopy to an analytic disc which is embedded into G and whose image has small diameter. In other words, $\mathcal{G}_0$ is the connected component of $\mathcal{G}$ that contains small analytic discs embedded into G.

For convenience, in the sequel we will frequently use two ways of notation for a continuous map $\mathcal{A}$ defined on a subset of $\mathbb{R} \times \mathbb{C}$, namely $\mathcal{A}(t, z) = \mathcal{A}_t(z)$.

The reason to consider G_0-discs is the following lemma which can be considered as continuity principle applied to G.

Lemma 1. *Any G_0-disc $d : \overline{\mathbb{D}} \to X^2$ can be lifted to a (uniquely defined) immersion $\tilde{d}$, $\tilde{d} : \overline{\mathbb{D}} \looparrowright \tilde{G}$ into the envelope of holomorphy $\tilde{G}$ of G, such that $\mathcal{P} \circ \tilde{d} = d$ and $\tilde{d}(\partial \mathbb{D}) \subset i(G)$.*

Here $\mathcal{P} : \tilde{G} \to X^2$ is the natural projection and $i : G \to \tilde{G}$ is the natural embedding of G into the envelope of holomorphy $\tilde{G}$ with $\mathcal{P} \circ i = id$ on G.

Note that the lifted disc $\tilde{d}$ may have less self-intersections than the disc d. We do not know a description of those G_0-discs which lift to embedded discs in the envelope of holomorphy.

The proof of the lemma will be given below in section 3.

We are interested in the whole image $d(\mathbb{D})$, but it will be convenient to obtain each point in the image as center of another analytic disc obtained by precomposing with an automorphism of the unit disc. In detail, let d be a G-disc and $p = d(z)$, $z \in \mathbb{D}$. Denote by φ_z an automorphism of the unit disc $\mathbb{D}$ which maps 0 to z and consider $d \circ \varphi_z : \overline{\mathbb{D}} \to X^2$. The disc $d \circ \varphi_z$ is a G-disc with center $p = d \circ \varphi_z(0)$. Multiple points of an immersed disc $p = d(z_1) = d(z_2)$ correspond to centers of different discs $d \circ \varphi_{z_1}$ and $d \circ \varphi_{z_2}$.

Points in the envelope of holomorphy may occur as centers of many different lifted G-discs. Introduce an equivalence relation in the set $\mathcal{G}_0$ of G_0-discs. Notice that equivalent discs have the same center.

Definition 3. *The equivalence relation on $\mathcal{G}_0$ is the relation generated by the following two conditions.*

1. *$\mathcal{G}_0$-discs contained in G and having common center are equivalent.*
2. *Equivalence is preserved under homotopies of equally centered G-disc pairs.*

Equivalently, in condition (1) we may consider analytic discs with images of small diameters embedded into G instead of all $\mathcal{G}_0$-discs with image in G.

The second condition can be rephrased in more detail as follows. A homotopy of pairs of equally centered $\mathcal{G}_0$-discs is a continuous family of ordered pairs of G-discs, i.e. a continuous family of pairs of mappings $(F_1(t,z), F_2(t,z)), t \in I$, z in a neighbourhood of $\overline{\mathbb{D}}$, such that for each $t \in [0,1]$ both mappings $F_j(t,z)$, $z \in \overline{\mathbb{D}}$, $j = 1,2$, define G_0-discs and their centers $p(t) = F_1(t,0) = F_2(t,0)$ coincide (but may depend on the parameter t).

Condition (2) says the following: Suppose the initial pair of discs of the homotopy (i.e. the pair corresponding to the parameter $t = 0$) consists of equivalent discs, then so does the terminating pair (i.e. the pair corresponding to the parameter $t = 0$).

In section 2 below we describe a construction which leads to building all possible pairs of equivalent G_0-discs according to definition 3. The construction will be given in terms of trees. The motivation for considering the introduced equivalence relation is the following lemma which will be proved in section 3.

Lemma 2. *Centers of equivalent $\mathcal{G}_0$-discs lift to the same point in the envelope of holomorphy: If d_1 and d_2 are equivalent G_0-discs then $\tilde{d}_1(0) = \tilde{d}_2(0) \in \tilde{G}$.*

Our main theorem is the following.

Theorem 1. *Let G be a domain in a Stein surface X^2. Then the set of equivalence classes of G_0-discs can be equipped with the structure of a connected Riemann domain $\hat{G}$ over X^2. The natural projection $\hat{\mathcal{P}} : \hat{G} \to X^2$ assigns to each equivalence class of discs their common center. There is a natural embedding $\hat{i} : G \to \hat{G}$, $\hat{\mathcal{P}} \circ \hat{i} = id$, which assigns to a point in G the equivalence class represented by discs embedded into G (of small diameter) and centered at this point.*

The Riemann domain $\hat{G}$ coincides with the envelope of holomorphy $\tilde{G}$ of G.

The number of sheets of $\tilde{G}$ over a point $p \in X^2$ equals the number of equivalence classes of G_0-discs with center p.

It has been a classical fact that the whole envelope of holomorphy $\tilde{G}$ of a domain G in a Stein manifold X^2 can be covered by the following successive procedure.

Put $\mathcal{D}_0 = i(G) \subset \tilde{G}$. Consider analytic discs immersed into $\tilde{G}$ with boundary in $\mathcal{D}_0$ and call them $\tilde{\mathcal{D}}_0$-discs. See definition 1, but now G is replaced by $\mathcal{D}_0 = i(G)$ and X^2 is replaced by $\tilde{G}$. A continuous family of $\tilde{\mathcal{D}}_0$-discs which jons a given $\tilde{\mathcal{D}}_0$-disc d with a constant disc is called a continuity-principle-family. The points in the image of d are said to be reachable by applying the continuity principle once. See definition 2 with G replaced by $\mathcal{D}_0 = i(G)$ and X^2 replaced by $\tilde{G}$.

By the continuity principle (see e.g. [6]) any analytic function in $i(G)$ has analytic continuation to a neighbourhood of the image of d. This distinguishes the present situation from that of lemma 1. The discs of the family in Lemma 1 are immersed into X^2 rather than into $\tilde{G}$. In the situation of Lemma 1 near self-intersection points of the disc multi-valued analytic continuation may occur.

Let $D_{j+1},\ j = 0, 1, ...,$ be the open subset of $\tilde{G}$ obtained from D_j by adding all points of $\tilde{G}$ reached from D_j by applying the continuity principle once. The classical fact is that $\tilde{G}$ is equal to the union of all D_j.

The theorem states that, actually, all points of the envelope of holomorphy $\tilde{G}$ can be reached from $i(G)$ by applying the continuity principle only once. Moreover, another observation of Theorem 1 is the following. Information about the topology of the envelope of holomorphy is contained in the intersection behaviour of homotopies of $\mathcal{G}_0$-discs (which depends on the Stein manifold in which the domain is included).

Notice that there is no unique definition of Riemann domains in the literature. Here we use the terminology of Grauert (see [8]). In this terminology a Riemann domain over an n-dimensional Stein manifold X^n is a complex manifold of dimension n with no more than countably many connected components which admits a locally biholomorphic mapping (called projection) to X^n. Such Riemann domains are separable ([11]). We do not require (as done e.g. in [10]) that analytic functions on a Riemann domain separate points.

Together with the projection $\hat{\mathcal{P}} : \hat{G} \to X^2$ we will use the projection $\mathcal{P}_0 : \mathcal{G}_0 \to X^2$ which assigns to each individual G_0-disc its center, and the mapping $\hat{\mathcal{P}_0} : \mathcal{G}_0 \to \hat{G}$ which assigns to each G_0-disc the equivalence class it represents. Notice that $\mathcal{P}_0 = \hat{\mathcal{P}} \circ \hat{\mathcal{P}_0}$. Later we will use liftings of mappings with respect to different projections. For instance, let E be a topological space and $\psi : E \to X^2$ be a continuous mapping. A continuous mapping $\overset{\circ}{\psi} : E \to \mathcal{G}_0$ is a lift of ψ to $\mathcal{G}_0$ if $\mathcal{P}_0 \circ \overset{\circ}{\psi} = \psi$. Respectively, a continuous mapping $\hat{\psi} : E \to \hat{G}$ with $\hat{\mathcal{P}} \circ \hat{\psi} = \psi$ is a lift of ψ to $\hat{G}$. To specify which lift is meant we will either indicate the projection itself or the source and the target space of the projection.

As a corollary of the theorem we obtain the following result which was surprisingly not known before.

Corollary 1. *Let G be a domain in a Stein manifold X^2 and $\tilde{G}$ its envelope of holomorphy. Then for each of its point p the envelope of holomorphy $\tilde{G}$ contains a (non-singular) embedded Riemann surface (and also an immersed analytic disc) passing through p and having its boundary in $i(G)$.*

The proof of the corollary will be given below in section 11.

Corollary 1 should be contrasted to counterexamples known for polynomial hulls. Namely, there are compact subsets K of $\mathbb{C}^n, n \geq 2$, with the following property. There is a point in the polynomial hull $\hat{K}$ such that for any small enough neighbourhood U of K there is no Riemann surface with boundary in U passing through this point.

The following question seems natural.

Question 1. *For a point $p \in \tilde{G}$, what is the minimal genus of a (non-singular) Riemann surface in $\tilde{G}$ passing through p with boundary in $i(G)$?*

This genus may serve as a measure how "far" the point p is from $i(G)$.

The second corollary states that for each closed orientable surface in $\tilde{G}$ there is a homotopy that moves a big part of it to $i(G)$; what remains in $\tilde{G} \setminus i(G)$ is an immersed analytic disc in $\tilde{G}$ with boundary in $i(G)$. We may assume that the disc is either empty or belongs to $\mathcal{G} \setminus \mathcal{G}_0$.

Corollary 2. *Let G and $\tilde{G}$ be as in the preceding corollary. Let $f : S \hookrightarrow \tilde{G}$ be a connected closed orientable surface embedded into $\tilde{G}$. Then there exists a homotopy to a (singular) surface $F : S \to \tilde{G}$ (F a continuous mapping), such that either $F(S)$ is contained in $i(G)$ or there is a disc $\Delta \subset S$ such that $F(S \setminus \Delta)$ is contained in G and (for a suitable complex structure on Δ) $F : \overline{\Delta} \to \tilde{G}$ is an immersed analytic disc in the envelope of holomorphy $\tilde{G}$.*
In particular, $F : S \to \tilde{G}$ represents the same homology class in $H_2(\tilde{G})$ as the original surface.

The condition that f is an embedding can be skipped. It is sufficient that f is continuous.

The obstruction to move a surface $f : S \hookrightarrow \tilde{G}$ to the lift $i(G)$ of the original domain can be described in different terms.

Denote by $\mathfrak{L}^a$ the set of loops in G that bound analytic discs in X^2 (equipped with the topology of C^1 convergence). Let $\mathfrak{L}^a_0$ be the connected component of $\mathfrak{L}^a$ which contains constant loops. In the situation of Corollary 2 a non-trivial analytic disc $F : \overline{\Delta} \to \tilde{G}$ emerges from the existence of a non-contractible closed curve in the set $\mathfrak{L}^a_0$ (see below section 11).

There is a variant of Corollary 2 for surfaces with boundary in $i(G)$. We formulate only the following special case of it.

Denote by φ the natural homomorphism from $\pi_1(G)$ to $\pi_1(\tilde{G})$ which is induced by inclusion $i : G \to \tilde{G}$. It is known that φ is surjective ([13]). (Notice that this result of ([13]) can also be obtained as an immediate consequence of Theorem 1, see below section 11.)

Corollary 3. *Any element of the fundamental group of G which is in the kernel of φ can be represented by a loop in $i(G)$ which bounds an analytic disc that is immersed into $\tilde{G}$.*

A reformulation of the corollary is the following. Any loop in $i(G)$ which is contractible in $\tilde{G}$ is homotopic in $i(G)$ to a loop that bounds an immersed analytic disc in $\tilde{G}$.

The corollary can be slightly strengthened. Namely, given any point $p \in \tilde{G}$, the analytic disc of Corollary 3 may be taken to pass through p. An analoguous remark holds for Corollary 2.

We do not know which elements of the kernel φ can be represented by boundaries of *embedded* holomorphic discs.

We state separately the versions of Corollary 2 and 3 for Stein fillings. A relatively compact strictly pseudoconvex domain Ω in a Stein surface is a Stein filling of the contact three-manifold M^3 if M^3 is contactomorphic to $\partial\Omega$ with the contact structure induced by the complex tangencies.

Corollary 4. *Let Ω be a relatively compact strictly pseudoconvex domain in a Stein surface X^2 with boundary $\partial\Omega = M^3$. Let $f : S \hookrightarrow \overline{\Omega}$ be a connected closed orientable surface embedded into $\overline{\Omega}$. Then there exists a homotopy to a (singular) surface $F : S \to \overline{\Omega}$ (F a continuous mapping), such that either $F(S)$ is contained in $\partial\Omega = M^3$ or there is a disc $\Delta \subset S$ such that $F(S \setminus \Delta)$ is contained in M^3 and (with a suitable complex structure on Δ) $F : \overline{\Delta} \to \overline{\Omega}$ is an immersed analytic disc in $\overline{\Omega}$ with boundary in M^3.*

In particular, $F : S \to \overline{\Omega}$ represents the same homology class in $H_2(\overline{\Omega})$ as the original surface.

Corollary 5. *Let as before Ω be a relatively compact strictly pseudoconvex domain in a Stein surface X^2 with boundary $\partial\Omega = M^3$. Denote by φ the homomorphism from $\pi_1(M^3)$ to $\pi_1(\overline{\Omega})$ induced by inclusion $M^3 \hookrightarrow \overline{\Omega}$.*

Then any element in the kernel $ker\varphi$ can be represented by the boundary of an analytic disc immersed into $\overline{\Omega}$.

Again, for any point $p \in \Omega$ the disc can be chosen passing through p.

We do not know whether in the situation of Corollary 5 one can always find an *embedded* analytic disc (in other words whether a "holomorphic version" of the loop theorem holds) or whether the minimal number of self-intersections of analytic discs whose boundaries represent a given element of the fundamental group of M^3 determines a non-trivial invariant depending on the contact manifold M^3, the filling Ω and the element of the fundamental group.

Stepan Orevkov proposed to consider the following example where Ω is a tubular neighbourhood of a Lagrangian torus in $\mathbb{C}^2$. In this case all elements in the kernel of the homomorphism φ can be represented by boundaries of embedded analytic discs.

Example. Let T be the tube domain $\Delta \oplus i\mathbb{R}^2$ where Δ is the unit disc $\Delta \overset{def}{=} \{x_1^2 + x_2^2 < 1\}$ in $\mathbb{R}^2$. The map $\exp : (z_1, z_2) \to (\exp(z_1), \exp(z_2))$ is a covering from T onto a neighbourhood Ω of the standard torus $\partial\mathbb{D} \times \partial\mathbb{D}$. The image of $\partial\Delta \times \{0\}$ (with counterclockwise orientation) under the aforementioned mapping represents a generator of the kernel of the homomorphism $\varphi : \pi_1(\partial\Omega) \to \pi_1(\overline{\Omega})$. The analytic discs $f_\pm(z) = (z, \mp iz), z \in \overline{\mathbb{D}}$, are embedded into the closure of T. Their boundaries are homotopic to $\partial\Delta \times \{0\}$ with counterclockwise, respectively, clockwise orientation.

The images of the discs under the map exp are embedded analytic discs in $\overline{\Omega}$ whose boundaries represent a generator of the kernel φ, respectively, its inverse. Multiples of the generator can be represented by the boundary of the following embedded discs. Consider N discs $f_{+,j}(z) = (z, -iz + i\phi_j)$ in $\overline{T}, z \in \overline{\mathbb{D}}$, for N different points $\phi_j \in [0, \pi)$. The analytic discs $\exp \circ f_{+,j}$ are embedded and pairwise disjoint. Join the boundaries of two consecutive discs by a Legendrian arc. Suppose all Legendrian arcs are pairwise disjoint, without self-intersections and meet the union of the boundaries of the discs exactly at the endpoints . Approximate the union of all the analytic discs and all the arcs by a single analytic disc. (See below section 11 for details.) In the same way we proceed with multiples of the inverse of the generator. The boundaries of such discs represent all elements of the kernel.

Question 2. *Let p, q and r be pairwise relatively prime integers and $\varepsilon \neq 0$ a small complex number. Consider the Milnor-Brieskorn spheres $M(p,q,r) \stackrel{def}{=} \{z_1^p + z_2^q + z_3^r = \varepsilon\} \bigcap S^5 \subset \mathbb{C}^3$ and their natural filling. What is the minimal numbers of self-intersections of an analytic disc whose boundary represents a given element of the fundamental group of $M(p,q,r)$? What are these numbers for a collection of elements that generate the fundamental group in the sense of semigroups?*

We conclude with the following observation for the case $M^3 = \partial\Omega$ is a homology sphere. Consider any embedded loop $f : \partial\mathbb{D} \to M^3$ which bounds an analytic disc in the filling $\overline{\Omega}$. We may always assume that the loop passes through a given base point in M^3 (see below the sketch of Lemma 23). The loop determines a unique element s_f of the second homology $H_2(\overline{\Omega})$. Indeed, consider the analytic disc $f : \overline{\mathbb{D}} \to \Omega$ bounded by this loop and attach to it along the loop a compact surface with boundary, the surface contained in M^3. We obtain a closed surface in $\overline{\Omega}$. Since $H_2(M^3) = 0$ the homology class represented by the closed surface in $H_2(\overline{\Omega})$ does not depend on the choice of the surface contained in M^3 that was attached to the loop. Further, two loops f_1 and f_2, $f_j : \partial\mathbb{D} \to M^3$ for $j = 1, 2$, both bounding analytic discs in $\overline{\Omega}$ determine the same element in $H_2(\overline{\Omega})$ if they are homotopic in $\partial\Omega$ through loops bounding analytic discs. We do not have a satisfactory description of such homotopies. Notice that the set of homotopy classes of boundaries of analytic discs (passing through a given base point) has the structure of a semigroup.

The present work was done at the Max-Planck-Institut für Mathematik and at Toulouse University with a CNRS grant. The author gratefully acknowledges the unbureaucratic support and hospitality of these institutions. The author would like to thank N.Kruzhilin, S.Nemirovski and S.Orevkov for enlightening discussions and a group of visitors of a Mittag-Leffler semester, including N.Kruzhilin, L.Lempert, S.Nemirovski, S.Orevkov and A.Tumanov for their interest. The author is also grateful to F.Forstneric and L.Stout for useful information concerning references.

2. A constructive description of the equivalence condition

Call a pair of equally centered G_0-discs an *ec*-pair for short.

Lemma 3. *The set of all pairs of equivalent G_0-discs can be constructed by successively choosing and applying a finite number of times one of the following procedures.*

(i) *Take a pair of small equally centered embedded analytic discs contained in G.*
(ii) *Take a pair of G_0-discs that is homotopic through ec-pairs to a pair of equivalent discs.*
(iii) *Let $d_1, d_2, ..., d_N$ be G_0-discs such that consecutive discs $d_k, d_{k+1}, k = 1, 2, ..., N-1$, are equivalent. Take the pair (d_1, d_N).*

Proof. Procedures (i) and (ii) give pairs of equivalent discs by conditions (1) and (2) of definition 3, respectively. Since an equivalence relation is transitive (iii) gives pairs of equivalent discs.

It remains to see that all pairs of equivalent discs can be obtained in this way. Consider the property of a pair of discs to belong to the set constructed by the procedure described in Lemma 3. This is an equivalence relation since it is symmetric and transitive. Moreover, it satisfies conditions (1) and (2), and it is minimal with the latter property. Therefore it coincides with the previous equivalence relation.

□

Lemma 3 allows to characterize pairs of equivalent discs as those for which there exists an associated planar rooted tree. (Such a tree is not uniquely determined for a given pair of discs.) This goes as follows.

Recall that a rooted tree is a connected graph without simple closed paths with a vertex chosen as root. If the root of the tree is not a multiple vertex we call the rooted tree simple. Vertices that are different from the root and have only one adjacent edge are called leaves. For each pair of vertices there is a unique path joining them. This allows to orient the edges of the graph "towards the root". We call the two endpoints of an oriented edge its *minus*-end and its *plus*-end respectively. (Orientation is towards the *plus*-end.)

We will consider trees that are (embedded) subsets of the plane with edges being straight line segments. The following additional structure is given. Edges whose *plus*-end is a common vertex of the graph (incoming edges for this vertex) will be given a label and placed in the following way. When surrounding the common vertex counterclockwise starting from a point on the first labeled edge, we meet the edges in the order prescribed by labeling. There is at most one edge whose minus end is a given vertex (outgoing edge for this vertex). The outgoing edge is always placed between the last and the first labeled incoming edge (with respect to counterclockwise orientation).

Pairs of discs constructed by lemma 3 produce planar rooted trees in the following way.

Pairs of small equally centered embedded analytic discs contained in G correspond to leaves. A single leaf (see procedure (i)) can be considered as a tree without edges with its root coinciding with its leaf.

Providing procedure (ii) with a pair of discs corresponds to attaching an edge to the root of its tree. The attached edge corresponds to the homotopy of *ec*-pairs, in particular, each point on the edge corresponds to a single *ec*-pair. The *minus*-end of the attached edge is the root of the previous tree, it corresponds to the original pair of equivalent discs, the *plus*-end is the root of the new tree, it corresponds to the pair of discs obtained from the original one by applying procedure (ii).

Procedure (iii) obtains a pair of discs d_1, d_N from the pairs (d_1, d_2),...,(d_{N-1}, d_N) of equivalent discs. This procedure corresponds to gluing trees together along their common root. More, detailed, consider the rooted trees $T_1,\ T_2,\ ,...,\ T_{N-1}$ corresponding to the aforementioned pairs together with their label. Identify their roots. The obtained tree may be represented as subset of the plane, so that the previous trees are ordered counterclockwise around the common root. We obtain a new rooted tree, its root corresponds to the pair (d_1, d_N).

We proved the following lemma.

Lemma 4. *To each pair of equivalent G_0-discs corresponds a planar rooted tree such that the root of the tree corresponds to this pair. Leaves correspond to pairs of small equally centered analytic discs embedded into G. Edges correspond to ec-homotopies. For each multiple vertex those edges that have the vertex as plus-end are ordered. In this order their ends correspond to pairs (d_1, d_2),(d_2, d_3),..., (d_{N-1}, d_N). The respective multiple vertex of the tree corresponds to the pair (d_1, d_N).*

There is a continuous mapping $\hat{\Phi}_T : T \to X^2$. It assigns to each point of T the class represented by the equivalent discs corresponding to this point. The mapping $\Phi_T = \hat{\mathcal{P}} \circ \hat{\Phi}_T$ assigns to each point of the tree the center of the equivalent discs corresponding to this point.

Consider a planar tree T that has a non-trivial edge. Its complement $\hat{\mathbb{C}} \setminus T$ in the Riemann sphere is a simply connected domain. Consider a conformal mapping $\phi : \mathbb{D} \to \hat{\mathbb{C}} \setminus T$. The mapping ϕ extends continuously to the closed disc $\overline{\mathbb{D}}$. Consider the boundary curve $\phi : \partial\mathbb{D} \to \mathbb{C}$ of the conformal mapping and reverse its orientation. Note that this curve is the limit of the simple closed curves $\phi(|z| = r),\ r < 1,\ r \to 1$, oriented suitably. The image of the limit curve is contained in the tree T. We may think about the curve "surrounding the tree counterclockwise along its sides." We have in mind that we associate to each edge of the tree its left side and its right side (copies of the edge which are the limit of its shifts to the left, respectively to the right, when moving along the edge according to orientation; recall that trees are oriented "towards the root").

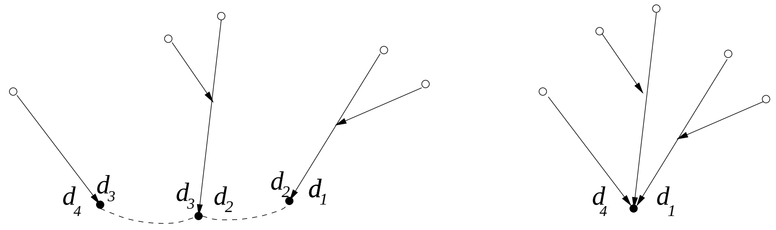

FIGURE 1. Planar rooted trees associated to pairs of equivalent discs (leaves indicated by white dots, roots by black dots)

Definition 4. *For a planar tree T the non-parametrized curve represented by the curve $\phi(\partial\mathbb{D})$ with reversed orientation is called the pellicle of the tree T.*

The punctured pellicle of the tree is obtained by removing from the pellicle the point over the root and adding instead two endpoints over the root.

This means that the initial point of the punctured pellicle is related to the tree in the following way. Consider all edges of the tree adjacent to the root and have them labeled as above, i.e. counterclockwise when traveling around the root. Take the point over the root on the left side of the first labeled edge. This is the initial point of the punctured pellicle of the tree.

Respectively, the terminating endpoint of the punctured pellicle is the point over the root on the right side of the last labeled edge.

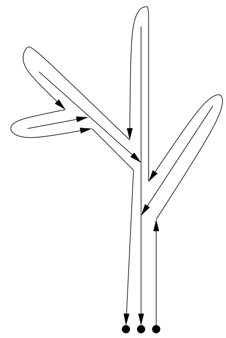

FIGURE 2. A planar rooted tree T and a curve approximating its punctured pellicle

We will parametrise the punctured pellicle by an interval (standardly it will be the unit interval $[0,1]$) with affine parametrization on the sides of the edges. We denote the punctured pellicle by $m_T : [0,1] \to \mathbb{C}$. The image of m_T covers the open edges of the tree T twice and covers the vertices with, maybe, higher multiplicity.

We need the following definitions.

Definition 5. *Let α be a curve in the plane and let $\Phi \circ \alpha$ be a curve in X^2. A curve $\overset{\circ}{\alpha}$ in $\mathcal{G}_0$ for which $\mathcal{P}_0 \circ \overset{\circ}{\alpha} = \Phi \circ \alpha$ is called a halo assigned to α and Φ.*

Notice that the halo is a continuously varying family of analytic discs around points in the image of the curve $\mathcal{P}_0 \circ \overset{\circ}{\alpha}$ in X^2. The latter curve is the curve of centers of the discs constituting the halo. The curve $\overset{\circ}{\alpha}$ can be considered as a mapping with values in X^2 of the trivial disc fibration over the curve α. The restriction of the mapping to the respective circle fibration has values in G.

Definition 6. *A planar rooted tree T with punctured pellicle m_T together with a continuous mapping $\Phi_T : T \to X^2$ is called a dendrite. The mapping $\Phi_T \circ m_T$ is called the punctured pellicle of the dendrite (opposed to the punctured pellicle m_T of the underlying tree). If the mapping $\Phi_T \circ m_T$ lifts to a mapping $\overset{\circ}{m}_T$ to $\mathcal{G}_0$ (i.e. $\mathcal{P}_0 \circ \overset{\circ}{m}_T = \Phi_T \circ m_T$) we call $\overset{\circ}{m}_T$ the punctured halo of the dendrite. The set $(T, m_T, \Phi_T, \overset{\circ}{m}_T)$ is called a dendrite with punctured halo and denoted by $\mathbf{T}$.*

Recall that for each point $\Phi_T \circ m_T(t)$ in the punctured pellicle of the dendrite the value of the halo at this point is an analytic disc centered at this point.

Note that we do not require here that the tree is associated to a pair of equivalent discs. In particular, we do not require that the values of Φ at the leaves are contained in G and the values of $\overset{\circ}{m}_T$ at the leaves are discs embedded into G.

The following lemma holds.

Lemma 5. *Let (d_1, d_2) be a pair of equivalent G_0-discs. Then there exists a dendrite $(T, m_T, \Phi_T, \overset{\circ}{m}_T)$ with punctured halo $\overset{\circ}{m}_T$ such that (for standard parametrization) $\overset{\circ}{m}_T(0) = d_1$ and $\overset{\circ}{m}_T(1) = d_2$.*

Moreover, at each of the leaves of the tree the value of $\overset{\circ}{m}_T$ is an analytic disc of small diameter embedded into G and its center, the value of $\Phi_T \circ m_T$, is a point in G.

Further, there is a lift $\hat{\Phi}_T : T \to \hat{G}$ of Φ_T, $\hat{\mathcal{P}} \circ \hat{\Phi}_T = \Phi_T$, such that $\hat{\mathcal{P}}_0 \circ \overset{\circ}{m}_T = \hat{\Phi}_T \circ m_T$.

A dendrite with the properties described in Lemma 5 is said to be associated to the pair (d_1, d_2) of equivalent discs.

Proof of Lemma 5. Let T be the planar rooted tree associated to the pair (d_1, d_2) by Lemma 4. Let Φ_T be the mapping from the tree into X^2 defined in that Lemma. We want to show that for the punctured pellicle m_T of the tree T the mapping $\Phi_T \circ m_T$ lifts to a continuous mapping $\overset{\circ}{m}_T$ with $\overset{\circ}{m}_T(0) = d_1$ and $\overset{\circ}{m}_T(1) = d_2$.

Recall that edges of the tree T correspond to homotopies of (ordered) *ec*-pairs. A homotopy of pairs of G_0-discs consists of two homotopies of G_0-discs, namely the homotopies defined by the first labeled, respectively second labeled, discs. Assign the first homotopy of G_0-discs to the left side (i.e. to the first side when surrounding the edge counterclockwise starting from the root), and the second homotopy to the right side of the edge.

The statement of the lemma can be proved by induction using the successive procedure of construction described in lemma 3.

First we consider trees consisting of an edge adjacent to a leaf. Change slightly those pairs of discs which correspond to points close to the leaf so that the pair associated to the leaf itself consists of two equal discs. Then the above described procedure gives a continuous mapping from the punctured pellicle of the edge into the set of G_0-discs with the desired values at the sides over the root. The value of the punctured halo at the leaf is a small disc embedded into G.

In the case corresponding to procedure (iii) there are several rooted trees $T_j,\ J = 1, ..., N-1$, and we assume that for each tree T_j there is a continuous lift $\overset{\circ}{m}_{T_j}$ of $\Phi_{T_j} \circ m_{T_j}$ to $\mathcal{G}_0$ which coincides at the left, respectively right sides over the roots with d_j, respectively d_{j+1}. The trees are glued together at their root and placed in the plane counterclockwise around the common root. The punctured pellicle of the new tree is obtained by gluing the right side over the root of T_j to the left side over the root of T_{j+1}. It is clear now that the values of the punctured halo of the trees T_j match so that for the new tree T we obtain a continuous lift of $\Phi_T \circ m_T$ into $\mathcal{G}_0$. At the leaves the halo takes values in the set of small analytic discs embedded into G.

The general case corresponding to (ii) is easier and left to the reader. $\square$

We will identify rooted trees realized as subsets of $\mathbb{C}$ if there is a piecewise affine homeomorphism of the plane mapping one tree to the other fixing the root and mappings edges (i.e. straight line segments joining vertices) to edges. We will identify the parametrised punctured pellicle and halo of such trees if they are obtained by precomposing with the mentioned homeomorphism.

We will not distinguish between different parametrizations of the pellicles and of the halo for a given embedding of a tree into $\mathbb{C}$ if the parametrization does not play a role.

3. Plan of Proof of the theorem

The proof of the theorem is divided into three steps according to the following propositions.

Proposition 1. *The set of equivalence classes of G_0-discs can be equipped with the structure of a connected Riemann domain $(\hat{G}, \hat{\mathcal{P}})$ over X^2. The projection $\hat{\mathcal{P}}$ associates to each equivalence class its center. There is a natural embedding $\hat{i} : G \to \hat{G}$ of G into $\hat{G}$, such that $\hat{\mathcal{P}} \circ \hat{i} = id$ on G.*

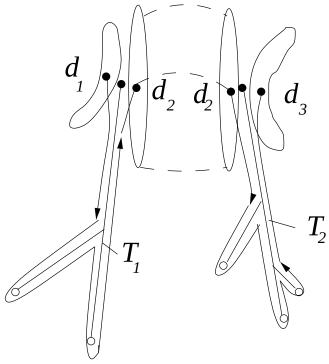

FIGURE 3. Matching the halo at common endpoints of punctured pellicles of two trees

Proposition 2. *For each analytic function on G its push-forward to $\hat{\imath}(G)$ extends to an analytic function on $\hat{G}$.*

The most subtle part of the proof of the theorem is the following proposition.

Proposition 3. *The Riemann domain $\hat{G}$ is pseudoconvex.*

The concept of pseudoconvexity of Riemann domains over $\mathbb{C}^n$ goes back to Oka ([15]). Oka showed that pseudoconvex Riemann domains over $\mathbb{C}^n$ are holomorphically convex (i.e. hulls of compacts with respect to analytic functions on the Riemann domain are compact.) In the paper [1] the notion of pseudoconvexity of an arbitrary complex manifold is introduced. Moreover, the authors present several equivalent characterizations of pseudoconvexity and extend Oka's result to Riemann domains over arbitrary Stein manifolds. Together with results of Grauert ([8]) this implies the following theorem.

Theorem DGO. *A pseudoconvex Riemann domain over a Stein manifold is a Stein manifold.*

This theorem shows, in particular, that holomorphic functions on pseudoconvex Riemann domains separate points (see [8] and [1]).

The three propositions imply the Theorem 1. Indeed, propositions 1 and 3 show that the set of equivalence classes of G_0-discs can be equipped with the structure of a Riemann domain $(\hat{G}, \hat{\mathcal{P}})$ over X^2, and moreover $\hat{G}$ is a Stein manifold. Proposition 2 shows that $\hat{G}$ is a holomorphic extension of G (see [10] , chapter 5..4). Therefore $\hat{G}$ coincides (up to a holomorphic isomorphism) with the envelope of holomorphy $\tilde{G}$ (see [10] , theorem 5.4.3). □

We will provide now proofs of the propositions.

Proof of Proposition 1. We start with the construction of a complex atlas on the set of equivalence classes of G_0-discs. Take an equivalence class $\hat{d}$ and choose a representative $d \in \hat{d}$. Denote the point $d(0) \in X^2$ by p. Associate to d a Riemann domain $\mathcal{R}_d = (V_d, F_d)$ over X^2 such that d lifts to it as an embedded disc and, moreover, $\mathcal{R}_d$ is foliated by analytic discs close to the lifted one. Such a Riemann domain may be constructed in a standard way. Take a small tubular neighbourhood $V_d = (1+\varepsilon)\mathbb{D} \times \delta\mathbb{D}$ of $\overline{\mathbb{D}} \times \{0\}$ in $\mathbb{C}^2$. Here $\varepsilon > 0$, $\delta > 0$ are small numbers. Put $F_d(z_1, 0) = d(z_1)$, $|z_1| < 1+\varepsilon$, and choose a holomorphic vector field $\mathcal{V} : V_d \to TX^2$ such that $\mathcal{V}|(1+\varepsilon)\mathbb{D} \times \{0\}$ is transversal to $F_d(z, 0)$, $z \in (1+\varepsilon)\mathbb{D}$. Denote by Φ its flow. Then, taking $F_d(z_1, z_2) = \Phi_{z_2}(F_d(z_1, 0))$ and shrinking the Riemann domain (V_d, F_d) if necessary, we arrive at a Riemann domain that has the required properties. For each z_2, $|z_2| < \delta$, the analytic disc $F_d \mid \overline{\mathbb{D}} \times \{z_2\}$ is a G_0-disc since the central disc d is a G_0-disc.

Consider now the set of equivalence classes of G_0-discs. Take an arbitrary element $\hat{d}$ of this set, choose a representative d and associate to it a Riemann domain $\mathcal{R}_d$. We want to define a Euclidean set in the set of equivalence classes that contains $\hat{d}$. For this purpose we use the discs of the foliation of $\mathcal{R}_d$ in the following way. Choose a neighbourhood N_d of zero in V_d so that F_d is biholomorphic from N_d onto a neighbourhood Q_d of p in X^2. Associate to each point $q \in Q_d$ the unique disc d^q of the foliation of $\mathcal{R}_d$ which passes through q, normalized so that q becomes its center. Take the equivalence class $\hat{d}^q$ which is represented by d^q. Define the set $\hat{N}^d = \{\hat{d}^q : q \in Q_d\}$ and the mapping $\hat{\mathcal{P}}_d : \hat{N}^d \to Q_d$, $\hat{\mathcal{P}}_d(\hat{d}^q) = q$. Call this set a standard neighbourhood of $\hat{d}$ associated to the representative $d \in \hat{d}$, the Riemann domain $\mathcal{R}_d$ and the set Q_d. Call $\hat{\mathcal{P}}_d$ the related standard projection.

The following lemma implies that standard neighbourhoods form a basis of a Hausdorff topology in the set of equivalence classes of G_0-discs.

Lemma 6. *Let $\hat{d}_1$, and $\hat{d}_2$ respectively, be equivalence classes of G_0-discs. Suppose $\hat{N}_1$ and $\hat{N}_2$ are standard neighbourhoods of $\hat{d}_1$ and $\hat{d}_2$, respectively, and $\hat{\mathcal{P}}_1 : \hat{N}_1 \to Q_1$ and $\hat{\mathcal{P}}_2 : \hat{N}_2 \to Q_2$ are the related standard projections onto the open subsets Q_1 and Q_2 of X^2. Suppose $\hat{N}_1$ and $\hat{N}_2$ intersect. Let $\hat{d}$ be a point in their intersection, hence $\hat{\mathcal{P}}_1(\hat{d}) = \hat{\mathcal{P}}_2(\hat{d})$. Denote the latter point by p. It is contained in $Q_1 \cap Q_2$.*

Then $\hat{N}_1$ and $\hat{N}_2$ intersect over the whole connected component Q^p of the intersection $Q_1 \cap Q_2$ which contains p. In other words, for $q \in Q^p$ the inclusion $\hat{\mathcal{P}}_1^{-1}(q) = \hat{\mathcal{P}}_2^{-1}(q) \subset \hat{N}_1 \cap \hat{N}_2$ holds.

It is clear from the lemma, that standard neighbourhoods form the basis of a topology. The lemma also implies that this topology is Hausdorff. Indeed, equivalence classes of G_0-discs with different center have obviously non-intersecting standard neighbourhoods. Let now $\hat{d}_1$, and $\hat{d}_2$ be distinct equivalence classes with equal center. Take standard neighbourhoods $\hat{\mathcal{P}}_j : \hat{N}_j \to Q_j$ of $\hat{d}_j$, $j = 1, 2$. Let Q^{p_0} be the connected component of $Q_1 \bigcap Q_2$ that contains the common center p_0 of $\hat{d}_1$ and $\hat{d}_2$. Then by the lemma $\hat{\mathcal{P}}_j^{-1}(Q^{p_0})$ are disjoint standard neighbourhoods of the $\hat{d}_j$.

Proof of lemma 6. Let q be any point in Q^p. Join p with q by a curve γ in Q^p, $\gamma : [0, 1] \to Q^p$, $\gamma(0) = p$, $\gamma(1) = q$. Let $\hat{\gamma}_j \overset{def}{=} \hat{\mathcal{P}}_j^{-1} \circ \gamma$, $j = 1, 2$. By construction the equivalence class $\hat{\gamma}_j(t)$, $j = 1, 2$, $t \in [0, 1]$, is represented by the unique disc of the foliation of $\mathcal{R}_{d_j}$ which passes through $\gamma(t)$ normalized so that its center becomes $\gamma(t)$. Denote the respective normalized disc by ${d_j}^{\gamma(t)}$. For $t = 0$ the discs ${d_j}^{\gamma(t)}$, $j = 1, 2$, coincide with the central discs d_j of the foliation.

By the conditions of the lemma the discs d_1 and d_2 are equivalent, hence for $t = 0$ the pair $({d_1}^{\gamma(t)}, {d_2}^{\gamma(t)})$ consists of equivalent discs. Therefore, by Definition 3 (see (ii)) for each $t \in [0, 1]$ the pair consists of equivalent discs. For $t = 1$ the pair coincides with $({d_1}^q, {d_2}^q)$. By construction the respective equivalence classes $\hat{d}_1{}^q = \hat{d}_2{}^q$ coincide with the respective points of $\hat{N}_j$ over q. The lemma is proved. □

The standard neighbourhoods equip the set of equivalence classes of G_0-discs with the structure of a Riemann domain over X^2 which we denote by $(\hat{G}, \hat{\mathcal{P}})$. The projection $\hat{\mathcal{P}}$ assigns to each equivalence class of G_0-discs its center.

Prove that there is a natural holomorphic embedding of G into $\hat{G}$. Indeed, take any point $p \in G$. All analytic discs with center p and sufficiently small diameter are entirely contained in G and equivalent to each other (see Definition 3, (i)). Associate to $p \in G$ this equivalence class of discs which is a point $\hat{p} \in \hat{G}$. The mapping $\hat{\imath}$, which maps p to $\hat{p}$ is locally biholomorphic according to the way an atlas is introduced on $\hat{G}$. The mapping is globally injective and $\hat{\mathcal{P}} \circ \hat{\imath}$ is the identity mapping on G. Hence $\hat{\imath}$ is biholomorphic onto its image.

It remains to show that $\hat{G}$ is connected. This is an easy consequence of the following two lemmas which will also be needed further.

Lemma 7. *Let $d : \overline{\mathbb{D}} \to X^2$ be a G_0-disc. Let U be the connected component of $\{\zeta \in \overline{\mathbb{D}} : d(\zeta) \in G\}$ which contains $\partial\mathbb{D}$. Then for any $z \in U \cap \mathbb{D}$ the disc $d \circ \varphi_z$ is equivalent to (small) discs centered at $d(z) = d \circ \varphi_z(0)$ and contained entirely in G.*

Lemma 8. *Consider the set of analytic discs $d : \overline{\mathbb{D}} \to X^2$ such that d extends to an analytic mapping in a neighbourhood of $\overline{\mathbb{D}}$. Endow the set with the topology of C^1-convergence on the closed disc $\overline{\mathbb{D}}$. Then the set of G_0-discs is open in this space and the mapping which assigns to each G_0-disc its equivalence class in $\hat{G}$ is continuous.*

Postpone the proof of the lemmas for a moment and finish the proof of proposition 1.

End of proof of proposition 1.

We show that any point in $\hat{G}$ can be connected with a point in $\hat{i}(G)$ by a path. Let $\hat{d} \in \hat{G}$ and let d be a representative of $\hat{d}$. Take a segment $[0,r] \subset \mathbb{D}$ in the unit disc with $d(r) \in G$. Then $d \circ \varphi_t,\ t \in [0,r]$, is a (continuous) curve of G_0-discs. By lemma 7 the disc $d \circ \varphi_r$ is equivalent to small discs through $d(r) \in G$ that are entirely contained in G. Taking equivalence classes $\hat{d}_t = \widehat{d \circ \varphi_t},\ t \in [0,r]$, and applying lemma 8 we obtain a curve in $\hat{G}$ with $\hat{d}_0 = \hat{d}$ and $\hat{d}_r \in \hat{i}(G)$. The proposition 1 is proved. □

Proof of lemma 7. Since d is a G_0-disc there is a homotopy of G_0-discs $d_s,\ s \in [0,1]$, which joins $d_1 = d$ with a small disc d_0 embedded into G. Consider a continuous path z_s in $\mathbb{D}$, $s \in [0,1]$, such that for each s the point z_s is in the connected component U_s of $\{\zeta \in \overline{\mathbb{D}} : d_s(\zeta) \in G\}$ which contains $\partial\mathbb{D}$. The normalized discs $d_s \circ \varphi_{z_s}$ are centered at $d_s(z_s) \in G$.

Consider a second continuous family of G_0-discs $D_s,\ s \in [0,1]$, consisting of small analytic discs embedded into G and centered at $d_s(z_s)$. Then the two discs $d_0 \circ \varphi_{z_0}$ and D_0 are equivalent, hence so are the discs $d_1 \circ \varphi_{z_1}$ and D_1 (see conditions (1) and (2) defining the equivalence relation). □

Proof of lemma 8. Let d be a G_0-disc and $\hat{d} \in \hat{G}$ its equivalence class. Choose a Riemann domain $\mathcal{R}_d = (V_d, F_d)$ foliated by G_0-discs with d being the central leaf. Let $N_d \subset V_d$ be a neighbourhood of zero and let $Q_d \subset X^2$ be a neighbourhood of $d(0)$ in X^2 such that $F_d : N_d \to Q_d$ is biholomorphic. Let $D : \mathbb{D} \to X^2$ be an analytic disc that is close to d in the topology of C^1-convergence on $\overline{\mathbb{D}}$ such that D extends analytically to a neighbourhood of $\overline{\mathbb{D}}$. Then D is an immersion of a neighbourhood of $\overline{\mathbb{D}}$ with $D(\partial\mathbb{D}) \subset G$ and $D(0)$ is close to $d(0)$. After possibly decreasing the neighbourhood of $\overline{\mathbb{D}}$ on which D is given there is a unique lift of D to the Riemann domain $\mathcal{R}_d$ that passes through the point $F_d^{-1}(D(0))$. The lifted disc is equivalent to the disc of the foliation of $\mathbb{R}_d$ that passes through this point. Continuity of the mapping and openess of the set of G_0-discs are now clear. □

The following two lemmas concern genericity of one-parameter families of analytic discs and will be used in the sequel. Denote the unit interval by $I = [0,1]$.

Lemma 9. *Let $\varepsilon > 0$ be a small number. Any continuous mapping $F : I \times (1+\varepsilon)\mathbb{D} \to X^2$ that is fiberwise holomorphic can be approximated uniformly on $I \times (1+\frac{\varepsilon}{2})\mathbb{D}$ by a continuous mapping that is fiberwise a holomorphic immersion.*

The approximation may be done keeping the centers of the discs fixed.

Lemma 10. *Let ε be a small positive number. A continuous mapping $F : I \times (1+\varepsilon)\mathbb{D} \to X^2$ that is fiberwise a holomorphic immersion can be approximated uniformly on $I \times (1+\frac{\varepsilon}{2})\mathbb{D}$ by a holomorphic mapping $\mathcal{F}$ in a neighbourhood of $I \times (1+\frac{\varepsilon}{2})\mathbb{D}$ that is fiberwise a holomorphic immersion. Moreover, the approximation can be made in such a way that $\mathcal{F}$ coincides with F on $\{1\} \times (1+\frac{\varepsilon}{2})\mathbb{D}$ and is locally biholomorphic in a neighbourhood of $\{1\} \times (1+\frac{1}{2}\varepsilon)\mathbb{D}$.*

Proof of Lemma 10. Assume first that X^2 equals $\mathbb{C}^2$. Decreasing $\epsilon > 0$ we may replace F by a C^1-mapping which coincides with the previous one on $\{1\} \times (1+\varepsilon)\mathbb{D}$ and has injective differential on $[1-\delta, 1] \times (1+\varepsilon)\mathbb{D}$ for some small positive number δ. This can be done so that the new mapping is uniformly close to the old one and is fiberwise a holomorphic immersion. Denote the new mapping as before by F.

The mapping F can be expressed by Taylor series in the z-variable that converge uniformly for $t \in I$ and $z \in (1+\frac{3}{4}\varepsilon)\mathbb{D}$:

$$F(t,z) = \sum_{k=0}^{\infty} a_k(t) z^k.$$

We obtain a uniform estimate for the coefficients

$$|a_k(t)| \le M(1+\frac{3}{4}\varepsilon)^{-k}, \quad k = 1,2,..., \quad t \in I,$$

for a constant M not depending on k and t. A similar estimate holds for the t-derivatives $a_k'(t)$ of the coefficients. The functions

$$F_N(t,z) = \sum_{k=0}^{\infty} a_k(1) z^k + \sum_{k=0}^{N} (a_k(t) - a_k(1)) z^k$$

converge to F uniformly on $I \times (1+\frac{\varepsilon}{2})\mathbb{D}$ and $\frac{\partial}{\partial t} F_N(t,z)$ converge uniformly to $\frac{\partial}{\partial t} F(t,z)$ on this set. It remains to approximate finitely many of the a_k in $C^1([0,1])$ by analytic functions in a neighbourhood of $[0,1]$ so that their value at 1 is fixed and the derivative at 1 converges to $a_k'(1)$.

For general Stein surfaces X^2 we consider a holomorphic embedding $\mathfrak{F} : X^2 \to \mathbb{C}^4$ and proceed as above with the coordinate functions of the mapping $\mathfrak{F} \circ F$. The image of the approximating mappings is contained in a small tubular neighbourhood of $\mathfrak{F}X^2$. It remains to compose with a holomorphic projection of the tubular neighbourhood onto $\mathfrak{F}X^2$. □

Proof of Lemma 9. The lemma follows from the holomorphic transversality theorem ([12], see also [4]) by standard dimension counting. For convenience of the reader we give the short argument.

After uniform approximation on $I \times (1+\varepsilon)\mathbb{D}$ we may assume that the mapping F is holomorphic on $Y^2 \overset{def}{=} U \times (1+\frac{3}{4}\varepsilon)\mathbb{D}$ for a neighbourhood U of I in $\mathbb{C}$, in other words F is a holomorphic mapping from the Stein surface Y^2 into the complex manifold X^2. We may assume that the restriction $F|[0,1] \times \{0\}$ is the same as before and the mapping is a fiberwise immersion near the set $U \times \{0\}$.

Denote by A the set of all elements in the space of 1-jets $J^1_{hol}(Y^2, X^2)$ of holomorphic mappings from Y^2 to X^2 which have vanishing derivatives in the z-direction. A is an analytic submanifold of $J^1_{hol}(Y^2, X^2)$. A mapping $\mathcal{F}$ from a subset of Y^2 to X^2 is fiberwise (for fixed t-variable) an immersion if its 1-jet extension $j^1\mathcal{F}$ avoids A.

Since the 1-jet extension of F restricted to $|U \times \{0\}$ avoids A, by the holomorphic transversality theorem ([12], see also [4]) the mapping F can be uniformly approximated on relatively compact open subsets $\overset{\circ}{Y}$ of Y^2 by holomorphic mappings $\mathcal{F}$ with 1-jet extension transversal to A, fixing its 1-jet on $U \times \{0\}$. Take for $\overset{\circ}{Y}$ a set of the form $\overset{\circ}{U} \times (1+\frac{\varepsilon}{2})\mathbb{D}$ for a relatively compact open subset $\overset{\circ}{U}$ of U containing I.

Note that A has real codimension 4 in $J^1_{hol}(Y^2, X^2)$ and $j^1\mathcal{F}$ maps the real 4-dimensional manifold $\overset{\circ}{Y}$ into $J^1_{hol}(Y^2, X^2)$. Hence for a curve $J \subset \overset{\circ}{U}$ which is a small perturbation of I the restriction of $\mathcal{F}$ to $J \times (1+\frac{\varepsilon}{2})\mathbb{D}$ has the desired property: its 1-jet extension avoids A. □

Proof of lemma 1. Consider the subsets $\mathfrak{c} \overset{def}{=} ([0,1) \times \overline{\mathbb{D}}) \bigcup ([0,1] \times \partial\mathbb{D})$ and $\mathfrak{c}_0 = (\{0\} \times \overline{\mathbb{D}}) \cup ([0,1] \times \partial\mathbb{D})$ of $\mathbb{R} \times \mathbb{C}$ and their convex hull $\mathfrak{C} \overset{def}{=} [0,1] \times \overline{\mathbb{D}}$.

Recall that the most elementary version of the continuity principle states that any holomorphic function in a neighbourhood of the set $\mathfrak{c}$ (more generally in a neighbourhood of $\mathfrak{c}_0$) in $\mathbb{C}^2$ extends to a holomorphic function in a neighbourhood of $\mathfrak{C}$ in $\mathbb{C}^2$.

The proof is completely elementary: The Cauchy type integral over the circles $\{t\} \times (1+\varepsilon)\partial\mathbb{D}$ ($\varepsilon > 0$ small and $t \in [0,1]$) defines an analytic function in a neighbourhood of $\mathfrak{C}$ which coincides with the original function in a neighbourhood of the bottom disc $\{0\} \times \overline{\mathbb{D}}$.

Let d be a G_0-disc. Let $\mathcal{F}$ be the mapping of lemma 10. For any analytic function g in G the function $g \circ \mathcal{F}$ is analytic in a neighbourhood U of $\mathfrak{c}_0 = (\{0\} \times \overline{\mathbb{D}}) \cup ([0,1] \times \partial\mathbb{D})$. By the continuity principle $g \circ \mathcal{F}$ extends analytically to a neighbourhood of $\mathfrak{C} = [0,1] \times \overline{\mathbb{D}}$, in particular it extends analytically to a neighbourhood V of $\{1\} \times \overline{\mathbb{D}}$.

The neighbourhood V together with the mapping $\mathcal{F}$ define a Riemann domain over X^2. Use the mapping $\mathcal{F}$ to glue the Riemann domain to the domain G along a suitable connected neighbourhood of $\{1\} \times \partial\mathbb{D}$. Any analytic function g on G extends analytically to the union of G with the Riemann domain.

Identify points in the union which are not separated by extensions of holomorphic functions on G. This factorization gives a Hausdorff space (see [10] for the case of $\mathbb{C}^2$ and [18] for the general case), and hence a Riemann domain which is an extension domain of G the points of which are separated by analytic functions. It is biholomorphically equivalent to a subset of the envelope of holomorphy (the biholomorphic mapping being compatible with projection), see e.g. [10]. The described procedure gives an immersion $\tilde{d}$ of the G_0-disc into $\tilde{G}$ such that $d = \tilde{\mathcal{P}} \circ \tilde{d}$ and $\tilde{d}(\partial\mathbb{D})$ is contained in $\tilde{i}(G)$. The lemma is proved. $\square$

Proof of lemma 2. The lemma is true for two discs of small diameter embedded into G. Indeed, the mapping $\hat{i}$ maps the center of both of them to the same point in $\hat{i}(G)$. The statement of the lemma is preserved under homotopies of pairs of equally centered G-discs. Indeed, let $(F_1(t,\cdot), F_2(t,\cdot)), t \in I$, be such a homotopy. Suppose for $d_j = F_j(0,\cdot)$ the desired equality $\tilde{d}_1(0) = \tilde{d}_2(0)$ holds.

Apply lemma 1 to each disc $F_j(t,\cdot)$ with $t \in I$, $j = 1, 2$. We obtain a unique lift $\tilde{F}_j(t,\cdot)$ of each of the discs to $\tilde{G}$. As in the proof of lemma 8 for fixed j the lifts of the discs depend continuously on the parameter t. For $j = 1, 2$ the curve $\tilde{F}_j(t, 0)$ is a lift to $\tilde{G}$ of the same curve in X^2, namely, of the curve of the common centers $F_1(t, 0) = F_2(t, 0)$ of the pairs. Since by assumption the lifts of the centers coincide for $t = 0$, by uniqueness the lifts of the whole curve coincide. The lemma is proved. $\square$

Proof of proposition 2. Take for each equivalence class of G-discs a representative and consider the lift of its center to the envelope of holomorphy $\tilde{G}$, (see Lemma 1). By Lemma 2 this point does not depend on the choice of the representative but only on the equivalence class. This defines a continuous mapping $\rho : \hat{G} \to \tilde{G}$ which respects projections: $\tilde{\mathcal{P}} \circ \rho = \hat{\mathcal{P}}$. Hence ρ is locally biholomorphic.

This map maps the set $\hat{i}(G)$ to $\tilde{i}(G)$ so that $\tilde{\mathcal{P}} \circ \rho = \hat{\mathcal{P}}$ on $\hat{i}(G)$. The analytic continuation of functions from $\tilde{i}(G)$ to the envelope of holomorphy $\tilde{G}$ determines analytic continuation of functions from $\hat{i}(G)$ to $\hat{G}$. The statement of the proposition follows. $\square$

4. Pseudoconvexity of the Riemann domain $\hat{G}$

We come to the most subtle part of the proof of the theorem, namely the proof of Proposition 3. In this section we reduce Proposition 3 to a lemma with which it is more convenient to work.

Our goal is to prove that the Riemann domain $\hat{G}$ is p_7^*-convex in the sense of Docquier and Grauert (see [1], p. 105/ 106). Docquier and Grauert proved that this convexity notion is the weakest of the equivalent conditions for pseudoconvexity of a Riemann domain over a Stein manifold.

Recall the notion of p_7^*-convexity for convenience of the reader. Denote by $\mathcal{C}\mathbb{D}^2$ the set $\mathbb{D}^2 \cup (\overline{\mathbb{D}} \times \partial\mathbb{D})$. This subset of the closed bidisc is obtained by removing from $\overline{\mathbb{D}}^2$ its "open face" $\partial\mathbb{D} \times \mathbb{D}$. Following Grauert we denote by $\tilde{\partial}\hat{G}$ the "boundary of $\hat{G}$ in the sense of ends" defined by filters ([1], p. 104, [6], p.100). The notion of p_7^*-convexity uses the definition of an R-mapping. An R-mapping into the Riemann domain $\hat{G}$ is a continuous mapping ϕ from the closed unit bidisc $\overline{\mathbb{D}}^2$ into the closure $\hat{G} \cup \tilde{\partial}\hat{G}$ of the Riemann domain $\hat{G}$ that has the following properties.

- (I) $\phi(\overline{\mathbb{D}}^2) \not\subset \hat{G}$,
- (II) $\phi(\mathcal{C}\mathbb{D}^2) \subset \hat{G}$
- (III) The mapping $\hat{\mathcal{P}} \circ \phi$ extends to a biholomorphic mapping of a neighbourhood of the closed bidisc $\overline{\mathbb{D}}^2$ into X^2.

According to the definition of Docquier and Grauert $\hat{G}$ is p_7^*-convex, equivalently pseudoconvex, if each end $p \in \tilde{\partial}\hat{G}$ of $\hat{G}$ has a neighbourhood $U(p)$ in $\hat{G} \cup \tilde{\partial}\hat{G}$ such that no R-mapping with image in $U(p)$ exists. We will prove that any mapping satisfying (II) and (III) will violate (I). More precisely, denoting the extension of the mapping $\hat{\mathcal{P}} \circ \phi$ to a neighbourhood of the closed bidisc (see (III)) by Ψ and the mapping ϕ extended to a neighbourhood of $\mathcal{C}\mathbb{D}^2$ in $\mathbb{C}^2$ by $\hat{\Psi}$, proposition 3 reduces to the following statement.

Proposition 3'. *Let Ψ be a biholomorphic mapping from a neighbourhood $\mathcal{N}(\overline{\mathbb{D}}^2) \subset \mathbb{C}^2$ of the closed bidisc onto a subset of X^2. Suppose the restriction of Ψ to a neighbourhood $\mathcal{N}(\mathcal{C}\mathbb{D}^2)$ of $\mathcal{C}\mathbb{D}^2$ lifts to a biholomorphic mapping $\hat{\Psi}$ onto a subset of $\hat{G}$ such that $\hat{\mathcal{P}} \circ \hat{\Psi} = \Psi$ on $\mathcal{N}(\mathcal{C}\mathbb{D}^2)$. Then* *the mapping Ψ lifts to a biholomorphic mapping, again denoted by $\hat{\Psi}$, from a neighbourhood of the closed bidisc onto a subset of $\hat{G}$, such that $\hat{\mathcal{P}} \circ \hat{\Psi} = \Psi$ on this neighbourhood.*

To prove proposition 3' we have to show that for any point p in the face $\partial\mathbb{D} \times \mathbb{D}$ $(= \overline{\mathbb{D}}^2 \setminus \mathcal{C}\mathbb{D}^2)$ of the bidisc there is a neighbourhood U of p and a lift of the mapping $\Psi \mid U$ to $\hat{G}$ which coincides with $\hat{\Psi}$ on $U \cap \mathbb{D}^2$. After rotation in the first variable we may assume that $p \in \{1\} \times \mathbb{D}$.

Consider the intersections of the closed bidisc, respectively of the set $\mathcal{C}\mathbb{D}^2$, with the set $[0,1] \times \overline{\mathbb{D}}$. The first intersection is equal to $\mathfrak{C} = [0,1] \times \overline{\mathbb{D}}$, the second equals $\mathfrak{c} = ([0,1) \times \overline{\mathbb{D}}) \bigcup ([0,1] \times \partial\mathbb{D})$.

It will be enough to prove proposition 3' for $\mathcal{N}(\overline{\mathbb{D}}^2)$ replaced by a neighbourhood of $\mathfrak{C}$ and $\mathcal{N}(\mathcal{C}\mathbb{D}^2)$ replaced by a neighbourhood of $\mathfrak{c}$. Moreover, since lifting is an open property it is enough to prove the following proposition.

Proposition 3". *Suppose $\Psi : \mathfrak{C} \to X^2$ is a continuous mapping which is fiberwise a holomorphic immersion (of a neighbourhood of the closed disc $\overline{\mathbb{D}}$ in $\mathbb{C}$ into X^2). Suppose $\Psi|\mathfrak{c}$ lifts to a continuous mapping $\hat{\Psi} : \mathfrak{c} \to \hat{G}$ with $\hat{P} \circ \hat{\Psi} = \Psi$. Then the mapping Ψ on the whole set $\mathfrak{C}$ admits a lift to $\hat{G}$.*

Recall the following reformulation of the property to admit a lift to $\hat{G}$.

A mapping Ψ from a set $E \subset \mathfrak{C}$ into X^2 lifts to a mapping $\hat{\Psi} : E \to \hat{G}$ iff for each point $(t,z) \in E$ there exists a G_0-disc $d_{(t,z)}$ with center at $\Psi(t,z)$ which represents the equivalence class $\hat{\Psi}(t,z) = \hat{d}_{(t,z)}$ and, moreover, the equivalence classes $\hat{d}_{(t,z)}$ depend continuously on (t,z).

Let $\Psi : \mathfrak{C} \to X^2$ be a mapping for which the restriction to $\mathfrak{c}$ lifts to a continuous mapping into $\hat{G}$. Write $\Psi_t(\cdot) \overset{def}{=} \Psi(t,\cdot)$ and let $\hat{\Psi}_t(\cdot)$ be the lifted mapping where it is defined.

The following simple lemma allows to modify the family Ψ_t to obtain a family with a stronger property of the initial disc: Namely, one can assume that the initial disc has small diameter and is embedded into G instead of assuming that through each of its points there is a G_0-disc.

Lemma 11. *Under the conditions of Proposition 3" there is a continuous family of analytic discs $\Phi_t = \Phi(t,\cdot)$, $\Phi : \mathfrak{C} = [0,1] \times \overline{\mathbb{D}} \to X^2$, which coincides for t close to 1 with the family of the previous discs, i.e. $\Phi(1,z) = \Psi(1,z)$ for $z \in \overline{\mathbb{D}}$ and t close to 1, and has the following properties:*

1. *$\Phi \mid \mathfrak{c}$ lifts to a mapping $\hat{\Phi} : \mathfrak{c} \to \hat{G}$.*
2. *The lift $\hat{\Phi}_0 : \overline{\mathbb{D}} \to \hat{G}$ of the disc Φ_0 is embedded into $\hat{i}(G)$. Its projection $\Phi_0(\overline{\mathbb{D}}) = \hat{P} \circ \hat{\Phi}_0(\overline{\mathbb{D}})$ is an analytic disc of small diameter embedded into G.*

Proof. We will extend the family $\Psi(t,z)$ for negative values of t and reparametrize in the parameter t to obtain property (2).

The extension is constructed as follows. According to the conditions the disc $\Psi_0 = \Psi(0,\cdot)$ lifts to a mapping $\hat{\Psi}_0 : \overline{\mathbb{D}} \to \hat{G}$.

For $t \in [-1,0]$ we define a mapping $\hat{\Psi}_t$ as a contraction of $\hat{\Psi}_0$ along the radius. More precisely, choose a small enough positive number σ and define $\hat{\Psi}_t(z) \overset{def}{=} \hat{\Psi}_0(\rho(t)z)$, $z \in \overline{\mathbb{D}}$, for an orientation preserving diffeomorphism $\rho : [-1,0] \to [\sigma,1]$.

Connect the center $\hat{\Psi}_0(0)$ of the lifted disc $\hat{\Psi}_0$ with a point on $\hat{i}(G)$ by a curve $\hat{h} : [-2,-1] \to \hat{G}$. Associate to the curve a continuous family of analytic discs $\hat{\Psi}_t : \overline{\mathbb{D}} \to \hat{G}$, $t \in [-2,-1]$, such that the curve of centers $\hat{\Psi}_t(0)$ coincides with $\hat{h}(t)$, $t \in [-2,-1]$ and the analytic disc $\hat{\Psi}_{-1}$ coincides with the previous analytic disc $z \to \hat{\Psi}_0(\sigma z)$. If $\sigma > 0$ is small enough such a family can be found. Indeed, one can take small analytic discs embedded into $\hat{G}$ with center $\hat{h}$. Moreover, this family can be chosen so that $\hat{\Psi}_{-2}$ is an embedding into $\hat{i}(G)$. Projecting to X^2 gives a family $\Psi_t = \hat{\mathcal{P}} \circ \hat{\Psi}_t$, $t \in [-2,-1]$, which is a continuous extension of the family Ψ_t, $t \in [0,1]$.

The mapping Φ is obtained by changing the parameter t by an orientation preserving diffeomorphism of the interval $[-2,1,]$ onto $[0,1]$ which is the identity near 1. $\square$

Lemma 13 below will be the key for proving proposition 3". We will state the Lemma after formulating the weaker lemma 12 which considers a single analytic disc instead of a family of discs. Lemma 12 is easier to state than Lemma 13. Later we will formulate a more elaborate version of lemma 12 which will be used in the proof of the corollaries (see Lemmas 17 and 18 below).

Lemma 12. *Let $\Phi : \overline{\mathbb{D}} \to X^2$ be an analytic disc such that its boundary lifts to $\hat{G}$. Then through each point $\Phi(z), z \in \mathbb{D}$, passes a G-disc (but maybe, not a G_0-disc).*

Lemma 13. *Let $\Phi : \mathfrak{C} \to X^2$ be a continuous family of analytic discs that satisfy conditions (1) and (2) of Lemma 11. Then the mapping Φ lifts to a mapping $\hat{\Phi} : \mathfrak{C} \to \hat{G}$.*

Lemmas 11 and 13 imply proposition 3". In the following sections we will prove Lemmas 12 and 13.

5. Neurons

This section is based on the key observation stated in Lemma 14 below. Start with the following definition.

Definition 7. *Let α be a piecewise smooth curve in the plane. (It may be a mapping of a closed interval or of the circle). We call a piecewise smooth curve α^* in the plane an excrescence of α if α^* is obtained by cutting α at finitely many points and pasting each time on the "right" of α (according to its orientation) the punctured pellicle of a planar rooted tree. We require that the trees are pairwise disjoint and meet α exactly at their roots.*

Let σ be a continuous mapping of the image of α into X^2 which has a continuous lift $\hat{\sigma}$ to $\hat{G}$, $\hat{\mathcal{P}} \circ \hat{\sigma} = \sigma$.

Suppose there is an excrescence α^* and extensions σ^* and $\hat{\sigma}^*$ of σ and $\hat{\sigma}$ defined on the image of α^*, $\hat{\mathcal{P}} \circ \hat{\sigma}^* = \sigma^*$, with the following property. There is a halo $\overset{\circ}{\alpha}{}^*$ for which $\hat{\mathcal{P}}_0 \circ \overset{\circ}{\alpha}{}^* = \hat{\sigma}^* \circ \alpha^*$.

Then we say that α has an excrescence α^* with halo $\overset{\circ}{\alpha}{}^*$ associated to $\hat{\sigma}$.

Lemma 14. *Let α be a piecewise smooth curve in the plane such that small shifts to the right of the smooth parts do not meet the curve. Let σ be a continuous mapping from its image into X^2 which admits a lift $\hat{\sigma}$ to $\hat{G}$. Then there exists an excrescence α^* with halo $\overset{\circ}{\alpha}{}^*$ associated to $\hat{\sigma}$.*

Proof. Let α be a mapping of the unit circle into X^2. (For mappings of an interval the proof is the same.) Cover the circle by a finite number of closed arcs with pairwise disjoint interior so that on each arc one can choose a continuous family of $\mathcal{G}_0$-discs representing $\hat{\sigma} \circ \alpha$. At each common endpoint of two of the closed arcs we obtain two equivalent G_0-discs d_j^- and d_j^+ (limits from the left, respectively from the right of the point). Consider for each of the discontinuity points t_j a tree T_j rooted at $\alpha(t_j)$ and corresponding to the respective pairs of equivalent G_0-discs by Lemma 4. Realize the trees as pairwise disjoint subsets of the plane, each attached to the curve on its "right" side and meeting the curve exactly at the root. Associate to each tree T_j the structure of a dendrite with halo $\overset{\circ}{m}_{T_j}$ such that $\overset{\circ}{m}_{T_j}$ takes the value d_j^- at the initial point and the value d_j^+ at the terminating point of the punctured pellicle of the tree T_j. Cut the curve at each discontinuity point and paste the punctured pellicle of the respective tree. Denote the obtained curve by α^*. Extend σ and $\hat{\sigma}$ by the mappings Φ_{T_j} and $\hat{\Phi}_{T_j}$ (see Lemma 5) to each of the trees and hence to each punctured pellicle and denote the extended mappings by σ^* and $\hat{\sigma}^*$. By the choice of the dendrites the mapping $\sigma^* \circ \alpha^*$ lifts to $\mathcal{G}_0$. The lift is the required halo $\overset{\circ}{\alpha}{}^*$.

□

Lemma 14 will be applied, in particular, to boundaries of analytic discs. We need the following terminology. It will be convenient to consider analytic discs up to reparametrization by conformal mappings of simply connected planar domains to the unit disc.

Definition 8. *1)(Generalized disc) Let D be a relatively compact simply connected domain in the complex plane with smooth boundary. Let T_j be a finite collection of pairwise disjoint planar trees. Suppose the trees have pairwise different root on ∂D and meet the closure $\overline{D}$ of the domain exactly at the root. Denote by T the union $\bigcup T_j$ of the trees. The set $\nu = \overline{D} \bigcup T$ is called a generalized disc, the set $\nu \setminus D$ is called the boundary of the generalized disc ν and the excrescence of ∂D (traveled counterclockwise) determined by the union of the trees is called the pellicle of the generalized disc ν and is denoted by m.*

2) (Preneurons) Suppose, moreover, that there is a continuous mapping $\Phi : \nu \to X^2$ that is analytic on D. Then the triple (ν, m, Φ) is called a preneuron. We will call $\Phi \circ m$ the pellicle of the preneuron.
Points on the circle which are not roots of attached trees are called regular points.

3) (Halo of a preneuron) If the pellicle $\Phi \circ m$ of the preneuron admits a continuous lift $\overset{\circ}{m}$ to $\mathcal{G}_0$ then the preneuron together with the mapping $\overset{\circ}{m}$ is called a preneuron with a halo.

4) (Main body) The restriction of the mapping Φ to the closure of the domain, $\Phi : \overline{D} \to X^2$, is called the main body of the preneuron.

5) (Axon and neuron) A non-empty dendrite whose tree consists of a single edge with leaf mapped into G (or consists of a single leaf mapped into G) is called an axon. A preneuron with an axon attached is called a neuron. A halo of a neuron is a lift $\overset{\circ}{m}$ of the mapping $\Phi \circ m$ to $\mathcal{G}_0$ with the additional property that the value of $\overset{\circ}{m}$ at the leaf of the axon is a small disc embedded into G.

6) (Continuity) We will say that a family ν_t of generalized discs depends continuously on the real parameter t if suitable parametrizations m_t of their pellicles are continuous functions in all parameters. A family of (pre)neurons (ν_t, m_t, Φ_t) is continuous if in addition the mapping $\Phi_t \circ m_t$ is continuous in all parameters. For continuity of a family of neurons with halo we have to add the condition that the mappings $\overset{\circ}{m}_t$ are continuous in all parameters.

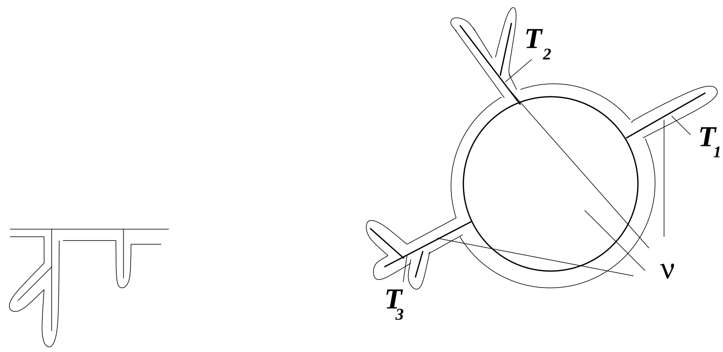

FIGURE 4. a) An excrescence of an interval and b) a generalized disc and a surrounding curve that approximates the pellicle

With this terminology, any analytic disc in X^2 is a preneuron, but it admits the structure of a neuron only if some part of its boundary is contained in G. In the latter case any boundary point contained in G can be chosen to serve a one-vertex (or degenerate) axon. There are many ways to extend the unit disc to a generalized disc and to give it the structure of a preneuron whose main body is the original disc. If the generalized disc has non-empty trees attached and Φ maps at least one leaf of certain tree into G the preneuron can be given the structure of a neuron. This is always the case if a non-empty tree of the generalized disc together with the mapping Φ form a dendrite related to a pair of equivalent discs according to lemma 5. Any edge of its tree that is adjacent to a leaf may serve as the tree of an axon. Notice that the notion of the halo of a neuron is stronger than that of the halo of a preneuron.

The main reason for constructing neurons out of analytic discs is the following fact: If an analytic disc is performed into the main body of a neuron with halo then the neuron structure may be used for obtaining G-discs which approximate the original disc uniformly along compacts (see below the proof of Lemma 12; for a refinement of this assertion see the proof of Lemma 13).

The following lemma extends Lemma 14 to preneurons.

Lemma 15. *Suppose the pellicle of a preneuron $n = (\nu, m, \Phi)$, $\Phi \circ m \to X^2$, has a lift $\hat{m}$ to $\hat{G}$. Then there is a neuron with halo $n^* = (\nu^*, m^*, \Phi, \overset{\circ}{m}{}^*)$ whose generalized disc ν^* contains ν with the following properties. The pellicle m^* of ν^* is an excrescence of the pellicle m of ν such that the halo $\overset{\circ}{m}{}^*$ of m^* is associated to $\hat{m}$. The values of $\overset{\circ}{m}{}^*$ over each leaf of a tree contained in $\nu^* \setminus \nu$ (not only over the leaf of the axon) is a small disc embedded into G.*

The lemma can be rephrased as follows. If the boundary of a preneuron lifts to $\hat{G}$ then after further attachment of dendrites a neuron is obtained with the following property. There is a closed curve $\gamma : \partial D \to \mathcal{L}_0^a$ meeting the set of small discs contained in G and such that the curve described by the centers of the discs $\gamma(\zeta)$, $\zeta \in \partial D$, coincides with the pellicle of the neuron.

Proof. Apply lemma 14 to the pellicle m of the generalized disc ν. We obtain an excrescence m^* which is the pellicle of a generalized disc ν^*, which is obtained from ν by attaching further trees (either with root at the circle or with root at a tree of ν). Moreover, m^* is chosen so that the mappings Φ and $\hat{\Phi}$ extend to the image of m^* in such a way that $\Phi \circ m^*$ lifts to a halo $\overset{\circ}{m}{}^*$ with $\hat{\mathcal{P}}_0 \circ \overset{\circ}{m}{}^* = \hat{\Phi} \circ m^*$. We may assume that ν^* differs from ν by at least one non-trivial tree corresponding to a pair of equivalent discs. We obtained a neuron $n^* = (\nu^*, m^*, \Phi, \overset{\circ}{m}{}^*)$ with halo. The second assertion of the lemma is clear. $\square$

Let $n = (\nu, m, \Phi, \overset{\circ}{m})$ be a neuron. Parametrize the pellicle m of ν by the unit circle $\partial\mathbb{D}$. Consider the evaluation mapping of the halo $\overset{\circ}{m}$: $\overset{\circ}{m}(\zeta, z) \overset{def}{=} \overset{\circ}{m}(\zeta)(z)$, $\zeta \in \partial\mathbb{D}$, $z \in \overline{\mathbb{D}}$. This evaluation mapping is a continuous mapping from the the set $\partial\mathbb{D} \times \overline{\mathbb{D}}$ into X^2 which is holomorphic on the disc fibers. (Recall that the mapping $\overset{\circ}{m}_D$ is a continuous mapping of ∂D into the space $A^1(\mathbb{D})$ of holomorphic mappings from the unit disc into X^2 that have C^1 extension to the closed unit disc.) Let $m(\zeta_0)$ be the tip of the axon tree of the neuron. Consider the (image of the) disc fiber $\overset{\circ}{m}(\zeta_0)(\overline{\mathbb{D}})$ and the union of all (images of) circle fibers $\bigcup_{\zeta \in \partial\mathbb{D}} \overset{\circ}{m}(\zeta)(\partial\mathbb{D})$. The union of the two sets, $\kappa_n \overset{def}{=} \bigcup_{\zeta \in \partial\mathbb{D}} \overset{\circ}{m}(\zeta)(\partial\mathbb{D}) \bigcup \overset{\circ}{m}(\zeta_0)(\overline{\mathbb{D}})$ is a compact subset of G associated to the neuron n.

The **idea of the proof of Lemma 12** in case $X^2 = \mathbb{C}^2$ is the following (see below section 7 for details).

Let $\Phi : \overline{\mathbb{D}} \to \mathbb{C}^2$ be an analytic disc with boundary lifting to $\hat{G}$. Lemma 15 produces a neuron n with halo whose main body coincides with the analytic disc Φ. A neuron can be considered as a degenerate analytic disc. Mergelyan's theorem allows uniform approximation of the neuron by a true analytic disc ("fattening of dendrites", see below section 6).

The domain of definition of the disc is a simply connected smoothly bounded domain D, whose closure contains the generalized disc of the neuron and approximates it.

If the original neuron had a halo the approximating disc-neuron may be given a halo. Denote the new disc-neuron with halo by $(D, m_D, \Phi_D, \overset{\circ}{m}_D)$. Here m_D just denotes the boundary curve of the domain D. In other words, the disc-neuron is an analytic disc $\Phi_D : \overline{D} \to X^2$ with a halo $\overset{\circ}{m}_D : \partial D \to \mathcal{G}_0$. The halo defines the following (image of a) torus $\bigcup_{\zeta \in \partial D} \overset{\circ}{m}_D(\zeta)(\partial\mathbb{D})$ consisting of the union of the boundaries of G_0-discs. Call them meridians of the torus. The torus is a compact subset of G contained in a small neighbourhood of κ_n. Approximate solutions of the Riemann-Hilbert boundary value problem for this torus produce holomorphic discs $f_D : \overline{D} \to X^2$ with boundary in a small neighbourhood of the torus. Such discs are G-discs. Approximate solutions of Riemann-Hilbert boundary value problems are constructed in [5]. There is a closed arc $\Gamma \subset \partial D$ (an arc that is close to the tip of the axon tree of n) such that for $\zeta \in \Gamma$ the meridian $\overset{\circ}{m}_D(\zeta)(\partial\mathbb{D})$ bounds an analytic disc *of small diameter contained in* G. This implies the following additional property of approximate solutions of the Riemann-Hilbert boundary value problem. Given any compact subset $K \subset D \bigcup \Gamma$, after possibly squeezing some meridians along the analytic discs bounded by them, the value $\max_K |f_D - \Phi_D|$ is small compared to the distance of κ_n to the boundary of G. Hence, for each point in $\Phi_D(K)$ a small translation of the disc $f_D : \overline{D} \to X^2$ produces a G-disc through this point. For more detail see below section 7.

We give an argument different from that in [5] to construct approximate solutions of the Riemann-Hilbert boundary value problem. For a curve $\zeta \to (\zeta, g(\zeta))$, $\zeta \in \partial D$, with $g(\zeta) \in \overset{\circ}{m}_D(\zeta)(\partial\mathbb{D})$ for each $\zeta \in \partial D$, we consider the winding number around the meridians. For the approximate solutions of the Riemann-Hilbert boundary value problem given in [5] the winding number of the boundary curve grows uncontrolled with the rate of approximation. This fact and the hope to handle more general situations are the reasons to choose here an argument that differs from that in [5]. Namely, instead of squeezing some meridians of the original torus, we take an open arc $\overset{\circ}{\Gamma}$ whose closure is contained in the interior $Int\,\Gamma$ such that $K \cap \partial\mathbb{D} \subset \overset{\circ}{\Gamma}$, and approximate the mapping $\overset{\circ}{m}_D : \partial D \to \mathcal{G}_0$ on $\partial D \setminus \overset{\circ}{\Gamma}$ by a continuous mapping $\overset{\circ}{M}$ from $\overline{D}$ into $\mathcal{G}$ that is holomorphic on D. Moreover, $\overset{\circ}{M}$ is chosen so that the evaluation mapping of $\overset{\circ}{M}(\zeta)$ at the point $0 \in \mathbb{D}$ equals $\Phi_D(\zeta)$ for each point $\zeta \in \overline{D}$. Here we call a mapping from D into $\mathcal{G}$ holomorphic if it is locally the sum of a power series with coefficients being $\mathcal{G}$-discs. (The metric in $\mathcal{G}$ is the C^1-norm of the mappings on $\overline{\mathbb{D}}$. We use the notion of holomorphic mappings $\overset{\circ}{M}$ into $\mathcal{G}$ only here for the purpose of explaining the concept. Later we will only use the evaluation mapping $\overset{\circ}{M}(\zeta)(z)$, $\zeta \in \overline{D}$, $z \in \overline{\mathbb{D}}$, of such a holomorphic mapping which is a continuous mapping from $\overline{D} \times \overline{\mathbb{D}}$ that is holomorphic on the interior of this set.) The approximating mapping $\overset{\circ}{M}$ defines a new torus over the boundary of the domain D. The part of the new torus over $\partial D \setminus \overset{\circ}{\Gamma}$ is close to the respective part of the old torus. Squeeze the meridians corresponding to points in Γ as much as needed along the analytic discs bounded by them. The thus obtained tori are still contained in a small neighbourhood of κ_n . There are exact solutions of the corresponding Riemann-Hilbert boundary value problem with winding number of the boundary curve not depending on the rate of approximation and of squeezing of meridians. For details see below section 7.

The **proof of lemma 13** is more subtle. Under the conditions of Lemma 13 there is a homotopy of the disc Φ_1 to an analytic disc Φ_0 where Φ_0 is embedded into G and lifts to $\hat{\imath}(G)$. The homotopy consists of analytic discs Φ_t whose boundaries lift to $\hat{G}$. We have to take a G-disc related to Φ_1 as constructed by Lemma 12 and find a G-disc homotopy to an analytic disc embedded into G.

The key point is to obtain a continuous family ϕ_t of neurons with continuously changing halo and continuously changing axons such that for t in neighbourhoods of 0 and of 1 the main bodies of the ϕ_t coincide with the analytic discs Φ_t.

Indeed, the scheme of proof of lemma 12 applies not only for an individual neuron with halo but also for continuous families of such neurons. This observation allows to obtain from the aforementioned continuous family of neurons a homotopy of G-discs. The homotopy of G-discs joins the given G-disc obtained in lemma 12 to a disc embedded into G. The conclusion is that each point in $\Phi_1(\mathbb{D})$ is contained in the projection of $\hat{G}$. The existence of a continuous lift of Φ_1 to $\hat{G}$ follows from lemma 7 (see below section 8 for details).

The first step towards the construction of the continuous family of neurons ϕ_t (see below Lemma 19) is to convert the continuously family of analytic discs $\Phi_t : \overline{\mathbb{D}} \to X^2$ into a *piecewise* continuous family of preneurons with the following property. To each of the preneurons an axon can be attached and the axons can be chosen continuously depending on the parameter t.

The tips of the axons form a curve that is mapped into G. Fatten the axons continuously depending on t (see section 6 below). We obtain a piecewise continuous family Ψ_t of neurons and a fixed arc Γ of the circle mapped into G by all Ψ_t. More precisely, the mapping $(t, z) \to \Psi_t$ is a continuous mapping from $[0, 1] \times \Gamma$ into G. We may assume that $1 \in \Gamma$.

The mapping Ψ, $\Psi(t, \zeta) \overset{def}{=} \Psi_t(\zeta)$, restricted the set $[0, 1] \times \Gamma$ lifts to $\mathcal{G}_0$. Indeed, any continuous mapping $\overset{\circ}{\Psi}$ into the set of small discs embedded into G such that the center of $\overset{\circ}{\Psi}(t, \zeta)$ equals $\Psi(t, \zeta)$ may serve.

Attaching further dendrites we associate with each of the thus obtained neurons a new neuron n_t which has already a halo. We do it in such a way that the halo on $[0, 1] \times \Gamma$ equals the above chosen one and the family n_t is piecewise continuous.

From the piecewise continuous family we get a continuous family of neurons in the following way. Let t_0 be a discontinuity point of the family n_t. Let $n_{t_0}^-$, and $n_{t_0}^+$ respectively, be the limit neurons at t_0 from the left and, from the right respectively. We show that we can attach a dendrite $\mathfrak{T}_{t_0}$ to $n_{t_0}^+$ at a point of Γ in such a way that $n_{t_0}^+ \cup \mathfrak{T}_{\mathbf{t_0}}$ has a halo and there is a homotopy of neurons with halo joining $n_{t_0}^-$ with $n_{t_0}^+ \cup \mathfrak{T}_{\mathbf{t_0}}$. A continuously changing copy of the dendrite $\mathfrak{T}_{t_0}$ will be attached to all neurons n_t with $t > t_0$. We proceed in this way with each discontinuity point of the family n_t.

The most subtle part of the aforementioned proof is the construction of the homotopy joining $n_{t_0}^-$ with $n_{t_0}^+ \cup \mathfrak{T}_{\mathbf{t_0}}$ (see below Lemma 20). This construction will be a procedure which preserves the main body (which is common for $n_{t_0}^-$ and $n_{t_0}^+$) and can be considered as continuously "peeling off the halo of the left neuron $n_{t_0}^-$" starting at a point in Γ and letting "grow the halo of the right neuron $n_{t_0}^+$ on the peeled places and symmetrically on the inside of the removed peel".

6. Partial fattening of dendrites.

Here we describe in detail the procedure of "fattening dendrites" which is used in the proof of Lemmas 12 and 13. In the proof of lemma 12 the procedure is applied to a single neuron. In the proof of lemma 13 it is applied to a family of neurons. We will describe the version for families.

Consider a single generalized disc $\nu = \overline{\mathbb{D}} \cup \bigcup T_j$. For each tree T_j we consider a connected open (in the topology induced on T_j by $\mathbb{C}$) subset $S_j \subset T_j$ which contains the root of T_j. The closure $\overline{S}_j$ of S_j is again a tree with root coinciding with that of T_j. Each set $\overline{S}_j$ contains together with each point the path on T_j connecting it with the root of T_j. A rooted tree $\overline{S}_j$ obtained in this way is called a subtree of T_j.

Any connected component of $T_j \setminus S_j$ is also a tree (if the set is not empty). A vertex of such a component may belong to $\overline{S}_j$. Since T_j is a tree there is exactly one such point in each connected component. (This point may be a multiple vertex.) With this point chosen as root the connected component becomes a rooted tree. Note that a connected component of $T_j \setminus S_j$ may consist of several trees adjacent to this root.

Provide a "cutting of trees" : replace each tree T_j by $\overline{S}_j$. Denote by S the union of trees $\bigcup \overline{S}_j$ and consider the generalized disc $\nu_S = \overline{\mathbb{D}} \cup S$. For a positive number τ_0 we associate to ν_S a family $E_S^\tau,\ \tau \in (0, \tau_0,]$, of bounded smoothly bounded simply connected domains with the following properties.

(1) The sets $E_S^\tau \setminus \mathbb{D},\ \tau \in (0, \tau_0]$, are contained in a small neighbourhood of S (i.e. $E_S^\tau \setminus \mathbb{D}$ are fattenings of S).
(2) For each $\tau \in (0, \tau_0]$ the set E_S^τ contains $\mathbb{D} \bigcup \bigcup S_j$. Moreover for each τ and each j all leaves of $\overline{S}_j$ are on the boundary of E_S^τ and E_S^τ does not intersect $\bigcup(T_j \setminus S_j)$.
(3) The family decreases, i.e. $E_S^{\tau_1} \subset E_S^{\tau_2}$ for $0 < \tau_1 < \tau_2 \le \tau_0$. Moreover, the family is continuous and converges to ν_S for $\tau \to 0$. We put $E_S^0 \overset{def}{=} \nu_S (= \lim_{\tau \to 0} E_S^\tau)$.

Consider the set $\nu^\tau \overset{def}{=} \overline{E_S^\tau} \cup \bigcup T_j$ for $\tau \in [0, \tau_0]$. Note that $\nu^0 = \nu$. The ν^τ are generalized discs. The trees of ν^τ correspond to the connected components of $T_j \setminus S_j$.

The described procedure is a "partial fattening of trees". The sets $E_S^\tau \setminus \overline{\mathbb{D}}$ are the fattenings of S. We always assume that the connected components $E_{S,j}^\tau$ of $E_S^\tau \setminus \overline{\mathbb{D}}$ are in a one-to-one correspondence with the trees S_j.

Note that for a continuous family $\nu_t,\ t \in [0, \tau_0]$, of generalized discs and continuous families of unions of subtrees $\bigcup(\overline{S}_j)_t$ of $\bigcup(T_j)_t$ the "partial fattening of trees" can be arranged continuously depending on the parameter t. In other words, it can be made so that it leads to a family ν_t^τ which is continuous in both parameters t and τ.

In the following lemma we consider neurons. The lemma extends the procedure of partial fattening of trees to a "partial fattening of dendrites". For each t the generalized disc is the union of the closed unit disc with attached trees.

Lemma 16. *Suppose $n_t = (\nu_t, m_t, \Phi_t), t \in [0, 1]$, is a continuous family of neurons. Let $S_t = \bigcup \overline{(S_j)_t}$ be a continuous family of unions of subtrees of the trees of their generalized discs $T_t = \bigcup(T_j)_t$. Let $\nu_t^\tau = \overline{E_t^\tau} \cup \bigcup(T_j)_t,\ t \in [0, 1],\ \tau \in [0, \tau_0]$, be a continuous family of generalized discs*

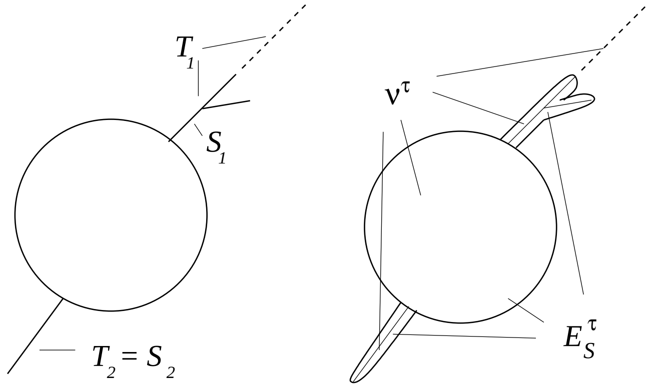

FIGURE 5. Partial fattenings of trees of a generalized disc

obtained from the ν_t *by fattening the trees constituting* S_t. *Then there is a continuous family of mappings* $\Phi_t^\tau : \nu_t^\tau \to X^2$, $t \in [0,1]$, $\tau \in [0,\tau_0]$, *that are holomorphic on the interior* E_t^τ *of* ν_t^τ *such that* $\Phi_t^0 = \Phi_t$. *If the restriction of* Φ *to* $\bigcup_{t\in[0,1]}\{t\} \times (\nu_t \setminus \mathbb{D})$ *has a lift* $\hat{\Phi}$ *to* $\hat{G}$ *then the restrictions of* Φ^τ *to* $\bigcup_{t\in[0,1]}\{t\} \times \partial E_t^\tau$, $\tau \in [0,\tau_0]$, *have lifts* $\hat{\Phi}^\tau$ *depending continuously on* τ.

Let $m_{j,t}$ *be the punctured pellicle of* $(T_j)_t$ *and* $m_{j,t}^\tau$ *the arc of the pellicle of* ν_t^τ *whose image is contained in* $\partial E_{S,j}^\tau \bigcup (T_j \setminus S_j)$. *If for some* j *all dendrites* $(\mathbf{T}_j)_t = ((T_j)_t, m_{t,j}, \Phi_t|(T_j)_t)$, $t \in [0,1]$, *have punctured halo* $\mathring{m}_{j,t}$ *associated to* $\hat{\Phi}$ *that depends continuously on t then (possibly after decreasing* τ_0*) also the curves* $m_{j,t}^\tau$ *have a halo* $\mathring{m}_{j,t}^{\tau}$ *associated to* $\hat{\Phi}^\tau$ *that depends continuously on* t *and* τ *and converges to* $\mathring{m}_{j,t}$ *for* $\tau \to 0$.

Proof. **In case** $X^2 = \mathbb{C}^2$ the first assertion of the lemma is a standard approximation lemma for the coordinate functions of the mappings Φ_t. Let E_t^τ be the generalized discs obtained by fattening the trees constituting S. The idea of proof of this approximation lemma is to extend for each t the function Φ_t to a continuous function in the whole plane $\mathbb{C}$ and to smoothen the extension (in dependence on τ) in such a way that the $\overline{\partial}$-derivative is small near points of $(\nu_S)_t$ and vanishes on a big compact subset of $\mathbb{D}$. For details we refer to the book [19] (see the proof of theorem 20.5). The construction can be made continuously depending on t and τ. The approximating function Φ_t^τ is obtained by correcting the extended and smoothened function by the solution of a $\overline{\partial}$-equation related to the interior of ν_t^τ.

Prove the second assertion for the case $X^2 = \mathbb{C}^2$. For suitable parametrizations of $m_{t,j}$ and $m_{t,j}^\tau$ by $s \in [0,1]$ we have uniform convergence $m_{t,j}^\tau \to m_{t,j}$ for $\tau \to 0$, hence the arc $\Phi_{t,j}^\tau \circ m_{t,j}^\tau$ in X^2 converges to the arc $\Phi_{t,j} \circ m_{t,j}$ for $\tau \to 0$. It remains to make for $s \in [0,1]$ and small τ the following choice for $\mathring{m}_{j,t}^{\tau}$. Take the parallel translation in $\mathbb{C}^2$ of the G_0-disc $\mathring{m}_{t,j}(s)$ for which the center equals $\Phi_{t,j}^\tau \circ m_{t,j}^\tau(s)$.

For general Stein surfaces X^2 we consider a holomorphic embedding $\mathfrak{F} : X^2 \to \mathbb{C}^4$. The approximation of $\mathfrak{F} \circ \Phi_t$ works as in the proof of the first assertion for $\mathbb{C}^2$. Given the halo $\mathfrak{F} \circ \mathring{m}_{t,j}$ on $\mathfrak{F} \circ m_{t,j}$, the halo on the approximating arcs in $\mathbb{C}^4$ can be chosen by using small translations. It remains to compose all constructed mappings (they all have image in a small tubular neighbourhood of $\mathfrak{F}(X^2)$) with a holomorphic projection from the tubular neighbourhood onto $\mathfrak{F}(X^2)$. The assertions of the Lemma are proved in the case of general Stein surfaces.

□

7. PROOF OF LEMMA 12

The proof of Lemma 12 is based on the following approximation lemmas which will be needed also in section 11 below. Let D be a bounded, smoothly bounded simply connected domain in the complex plane and let $\Gamma \subset \partial D$ be an arc. Put $\mathcal{S}_{\partial D} \overset{def}{=} (\overline{D} \times \{0\}) \bigcup ((\partial D) \times \overline{\mathbb{D}})$. Notice that suitable neighbourhoods of $\mathcal{S}_{\partial D}$ are usually called Hartogs figures. In other words, $\mathcal{S}_{\partial D}$ is the core of Hartogs figures. Denote the compact subset $(\overline{\partial D} \times \partial \mathbb{D}) \bigcup (\Gamma \times \overline{\mathbb{D}})$ of $\mathcal{S}_{\partial D}$ by $\mathcal{Q}_\Gamma$.

Recall that for defining a metric on X^2 we fixed a holomorphic embedding of X^2 into $\mathbb{C}^4$ and pulled back the Euclidean metric. ε-approximation of mappings into X^2 refers to this metric. Note that the second part of Lemma 17 below concerns continuous families of mappings and is needed in the proof of Lemma 13.

Denote by $A_{X^2}(D \times \mathbb{D})$ the space of continuous mappings from $\overline{D} \times \overline{\mathbb{D}}$ into X^2 that are holomorphic on the interior $D \times \mathbb{D}$.

Lemma 17. *Let $J_D : \mathcal{S}_{\partial D} \to X^2$ be a continuous mapping that is analytic on $D \times \{0\}$ and fiberwise analytic on $\partial D \times \overline{\mathbb{D}}$. Let $\Gamma \subset \partial D$ be a closed arc.*

Then for each positive number ε and each neighbourhood V of $J_D(\mathcal{S}_{\partial D})$ in X^2 there exists a mapping $\mathcal{H} \in A_{X^2}(D \times \mathbb{D})$, such that

1. *$\mathcal{H}|\overline{D} \times \{0\} = J_D|\overline{D} \times \{0\}$,*
2. *$\mathcal{H}(\partial D \times \partial \mathbb{D})$ is contained in an ε-neighbourhood of $J_D(\mathcal{Q}_\Gamma)$.*
3. *the image of $\mathcal{H}$ is contained in V, moreover, for each compact subset K of $D \bigcup \Gamma$ the mapping $\mathcal{H}$ can be chosen so that for each $\zeta \in K$ the whole fiber $\mathcal{H}(\{\zeta\} \times \overline{\mathbb{D}})$ is contained in an ε-neighbourhood of $\Phi_D(\zeta)$.*

Suppose D_t, $t \in [0,1]$, is a continuous family of simply connected bounded and smoothly bounded planar domains. Let $\mathfrak{A}_t$ be continuously changing closed arcs, $\mathfrak{A}_t \subset \partial D_t$. Let further K_t, $t \in [0,1]$, be a family of compact subsets of $D_t \cup \mathfrak{A}_t$ depending continuously on the parameter t (hence $\bigcup_{t\in[0,1]} \{t\} \times K_t$ is a compact subset of $\mathbb{R} \times \mathbb{C}$). Consider the continuously changing family of sets $\mathcal{S}_{\partial D_t}$ and $\mathcal{Q}^t \overset{def}{=} (\overline{\partial D_t} \times \partial \mathbb{D}) \bigcup (\mathfrak{A}_t \times \overline{\mathbb{D}})$.

Suppose $J^t_{D_t} : \mathcal{S}_{\partial D_t} \to X^2$, $t \in [0,1]$, is a continuous family of mappings, each of it being analytic on all analytic discs contained in $\mathcal{S}_{\partial D_t}$.

Then for any number $\varepsilon > 0$ there exists a continuous family of mappings $\mathcal{H}_t \in A_{X^2}(D_t \times \mathbb{D})$, such that each $\mathcal{H}_t$, $t \in [0,1]$, satisfies conditions (1),(2) and (3) above with respect to the objects specified for the number t.

Fix K. Let $\overset{\circ}{\Gamma}$ be as in section 5 an open arc, $\overset{\circ}{\Gamma} \Subset Int\,\Gamma$, $K \subset \mathbb{D} \bigcup \overset{\circ}{\Gamma}$. Denote the set $(\overline{D} \times \{0\}) \bigcup (\partial D \setminus \overset{\circ}{\Gamma} \times \overline{\mathbb{D}})$ by $S_{(\partial D \setminus \overset{\circ}{\Gamma})}$. The proof of Lemma 17 is based on the following variant of the Weierstraß approximation theorem for the arc $\partial D \setminus \overset{\circ}{\Gamma}$.

Lemma 18. *For any positive number ε and any neighbourhood V of $J_D(\mathcal{S}_{\partial D})$ there exists a neighbourhood U of $S_{(\partial D \setminus \overset{\circ}{\Gamma})}$ in $\overline{D} \times \overline{\mathbb{D}}$ and a continuous mapping $\mathfrak{H} : U \to V \subset X^2$ that is holomorphic on the interior $Int\,U$ of U such that $\mathfrak{H}|\overline{D} \times \{0\} = J_D|\overline{D} \times \{0\}$ for $\zeta \in \overline{D}$ and $\mathfrak{H}$ is uniformly ε-close to J_D on $(\partial D \setminus \overset{\circ}{\Gamma}) \times \overline{\mathbb{D}}$.*

Proof. In case X^2 is different from $\mathbb{C}^2$ we compose the mapping J_D with the holomorphic embedding $\mathfrak{F}$ of X^2 into $\mathbb{C}^4$. Denote the composition by $\mathbf{J}_D$. The target space for this mappings is $\mathbb{C}^4$. In case $X^2 = \mathbb{C}^2$ the target space was $\mathbb{C}^2$ from the beginning. For unifying notation we use the fat letter $\mathbf{J}_D$ for the mapping J_D in this case as well. So in any case $\mathbf{J}_D$ is a mapping into some $\mathbb{C}^n$ (either $n = 2$ or $n = 4$).

Notice that for $r \in (0,1)$, $r \to 1$, the mappings $\mathbf{J}_{D,r}$, $\mathbf{J}_{D,r}(\zeta, z) \overset{def}{=} \mathbf{J}_D(\zeta, rz)$, $\zeta \in \partial D \setminus \overset{\circ}{\Gamma}, z \in \overline{\mathbb{D}}$, converge uniformly to $\mathbf{J}_D(\zeta, z)$, $\zeta \in \partial D \setminus \overset{\circ}{\Gamma}, z \in \overline{\mathbb{D}}$.

Write the mapping $\mathbf{J}_D|(\partial D \setminus \overset{\circ}{\Gamma}) \times \overline{\mathbb{D}}$ in form of power series:

$$\sum_{k=0}^{\infty} a_k(\zeta) z^k, \quad \zeta \in \partial D \setminus \overset{\circ}{\Gamma}, \quad z \in \overline{\mathbb{D}}.$$

Choose a number $r < 1$ sufficiently close to 1 and a big enough number N so that the mapping

$$\mathbf{J}_{D,r,N}(\zeta,z)\overset{def}{=}\sum_{k=0}^{N}a_k(\zeta)r^k z^k,\quad \zeta\in\partial D\setminus\overset{\circ}{\Gamma},\quad z\in\overline{\mathbb{D}},$$

approximates the mapping $\mathbf{J}_D$ sufficiently well on $(\partial D\setminus\overset{\circ}{\Gamma})\times\overline{\mathbb{D}}$. Note that both mappings, $\mathbf{J}_D$ and $\mathbf{J}_{D,r,N}$ coincide on $\partial D\setminus\overset{\circ}{\Gamma}\times\{0\}$ with $\mathbf{J}_D$. Approximate each of the coefficients $a_k(\zeta),\ k=1,...,N$, uniformly for $\zeta\in\partial D\setminus\overset{\circ}{\Gamma}$ by holomorphic mappings from a neighbourhood of $\overline{D}$ to $\mathbb{C}^4$. We obtain a continuous mapping $\mathcal{I}$ from $\overline{D}\times\overline{\mathbb{D}}$ into $\mathbb{C}^n$ which is holomorphic on $D\times\mathbb{D}$, approximates the mapping $\mathbf{J}_D$ uniformly on $(\partial D\setminus\overset{\circ}{\Gamma})\times\ \overline{\mathbb{D}}$ and coincides with $\mathbf{J}_D$ on $\overline{D}\times\{0\}$.

Being uniformly close to $\mathbf{J}_D$ on $S_{(\partial D\setminus\overset{\circ}{\Gamma})}$ the mapping $\mathcal{I}$ maps a neighbourhood U of this set (in $\overline{D}\times\overline{\mathbb{D}}$) into a small tubular neighbourhood of $\mathfrak{F}(X^2)$. (Recall that $\mathbf{J}_D(S_{(\partial D\setminus\overset{\circ}{\Gamma})})\subset\mathfrak{F}(X^2)$.) Consider the composition $\mathfrak{Pr}\circ\mathcal{I}$ of the mapping $\mathcal{I}$ with a holomorphic projection $\mathfrak{Pr}$ of a tubular neighbourhood of $\mathfrak{F}(X^2)$ onto $\mathfrak{F}(X^2)$ and apply to it the inverse of $\mathfrak{F}$ we obtain a holomorphic mapping $\mathfrak{H}$ from U into X^2 that approximates J_D on $(\partial D\setminus\overset{\circ}{\Gamma})\times\ \overline{\mathbb{D}}$. If U is chosen small enough depending on V the image of $\mathfrak{H}$ is contained in V.

□

Proof of Lemma 17. Notice that for each $\zeta\in\partial D\setminus\overset{\circ}{\Gamma}$ the set U of Lemma 18 contains the fiber $\{\zeta\}\times\overline{\mathbb{D}}$. For $\zeta\in\overset{\circ}{\Gamma}$ the set U may not contain the respective fibers but it contains a small neighbourhood of $\overset{\circ}{\Gamma}\times\{0\}$. We want to shrink the fibers over points in Γ suitably. Take a smooth positive function ρ on ∂D that equals 1 outside Γ, does not exceed 1 everywhere on ∂D and is as small as needed in a neighbourhood of the closure of $\overset{\circ}{\Gamma}$.

Consider an analytic function g on D with boundary values having absolute value ρ. The function g is smooth up to the boundary if ρ is smooth. (Recall that D has smooth boundary.) Moreover, on the compact subset K of $D\bigcup\overset{\circ}{\Gamma}$ the absolute value $|g|$ of the function does not exceed a small constant depending on the compact set K and the function ρ and tending to 0 if the maximum of the function ρ on $\overset{\circ}{\Gamma}$ tends to 0 . This is a consequence of an estimate of the harmonic measure of $\overset{\circ}{\Gamma}$ on K.

Define the mapping $\Upsilon^g,\ \Upsilon^g(\zeta,z)\overset{def}{=}(\zeta,g(\zeta)z)$ of the closed bidisc $\overline{D}\times\overline{\mathbb{D}}$ onto U^g, $U_g\overset{def}{=}\{(\zeta,z)\in\overline{D}\times\overline{\mathbb{D}}:|z|\le|g(\zeta)|\}$. With a suitable choice of ρ for each fixed $z\in\overline{\mathbb{D}}$ the distance $|\Upsilon^g(\zeta,z)-(\zeta,0)|$ is as close as needed uniformly for $\zeta\in K$.

Increasing the compact subset K of $D\bigcup\overset{\circ}{\Gamma}$ we may assume that each point ζ outside the compact K is as close as needed to $\partial D\setminus\overset{\circ}{\Gamma}$. Therefore the choice of the function ρ can be made in such a way that the set U_g is contained in the small neighbourhood U of $\mathcal{S}_{\partial D\setminus\overset{\circ}{\Gamma}}$ in $\overline{D}\times\overline{\mathbb{D}}$.

Let $\mathcal{H}$ be the composition of the mapping $\mathfrak{H}$ with the mapping $\Upsilon^g,\ \Upsilon^g(\zeta,z)\overset{def}{=}(\zeta,g(\zeta)z)$ of the closed bidisc $\overline{D}\times\overline{\mathbb{D}}$ onto U^g, $\mathcal{H}=\mathfrak{H}\circ\Upsilon^g$. The mapping $\mathcal{H}$ has the required properties.

Indeed, since ρ has absolute value 1 on $\partial D\setminus\Gamma$ and absolute value not exceeding 1 on $\Gamma\setminus\overset{\circ}{\Gamma}$ the set $\mathcal{H}(\partial D\setminus\overset{\circ}{\Gamma}\times\partial\mathbb{D})$ is contained in a small neighbourhood of $J_D(\mathcal{Q}_\Gamma)$. (See Lemma 18 for the properties of $\mathfrak{H}$ and use the fact that $\mathcal{Q}_\Gamma\supset(\partial D\setminus\overset{\circ}{\Gamma}\times\partial\mathbb{D})\bigcup(\Gamma\setminus\overset{\circ}{\Gamma}\times\overline{\mathbb{D}})$.) If ρ is small enough on $\overset{\circ}{\Gamma}$ then also $\mathcal{H}(\overset{\circ}{\Gamma}\times\partial\mathbb{D})$ is contained in a small neighbourhood of $J_D(\mathcal{Q}_\Gamma)$.

Property (3) is a consequence of the properties of Υ^g.

The proof of the respective assertion for continuous families of mappings $J^t_{D_t}$ is straightforward.

Lemma 17 is proved. □

Proof of Lemma 12. Let $\Phi:\overline{\mathbb{D}}\to X^2$ be an analytic disc whose boundary lifts to a mapping $\hat{\Phi}:\partial\mathbb{D}\to\hat{G}$. Lemma 15 produces a neuron $n=(\nu,m,\Phi)$ which has halo $\overset{\circ}{m}$ associated to $\hat{\Phi}$ and has the disc as main body. Apply Lemma 16 ("fattening of dendrites") for the single neuron

n, its halo and the set of all trees of its generalized disc ν, so that we obtain a true analytic disc with halo $(D, m_D, \Phi_D, \mathring{m}_D)$. We assume that Φ_D is an ε-approximation of Φ and $\mathring{m}_D$ is an ε-approximation of $\mathring{m}$. The evaluation mapping of the halo $\mathring{m}_D$ defines a continuous mapping from the set $\partial D \times \overline{\mathbb{D}}$ into X^2 which is fiberwise holomorphic. Moreover, $\mathring{m}_D(\zeta)(0) = \Phi_D(\zeta)$ for all $\zeta \in \partial D$. Thus, $\Phi_D : \overline{D} \to X^2$ and $\mathring{m}_D : \partial D \times \overline{\mathbb{D}} \to X^2$ define a continuous mapping J_D from the set $\mathcal{S}_{\partial D} = (\overline{D} \times \{0\}) \bigcup ((\partial D) \times \overline{\mathbb{D}})$ into X^2.

Let $\Phi_D : \Gamma \to X^2$, $\Gamma \subset \partial D$, be a closed arc of the pellicle of D that is close enough to the tip of the axon tree of the original neuron. Then for the subset $\mathcal{Q}_\Gamma = (\overline{\partial D} \times \partial \mathbb{D}) \bigcup (\Gamma \times \overline{\mathbb{D}})$ of $\mathcal{S}_{\partial D}$ the set $J_D(\mathcal{Q}_\Gamma)$ is contained in a 2ε-neighbourhood of $\kappa_n \subset G$ (see the definition of κ_n after the proof of Lemma 15). An application of Lemma 17 with the same number ε and with a compact subset K of $\mathbb{D} \bigcup \mathring{\Gamma}$ provides a mapping $\mathcal{H} \in A_{X^2}(D \times \mathbb{D})$, such that $\mathcal{H}(\partial D \times \partial \mathbb{D})$ is contained in an ε-neighbourhood of $J_D(Q_\Gamma)$ and for each fixed $\zeta \in K$ the fiber $\mathcal{H}(\{\zeta\} \times \overline{\mathbb{D}})$ is ε-close to $\Phi_D(\zeta)$ on K.

For each $z \in \partial \mathbb{D}$ the disc $f^z(\zeta) = \mathcal{H}(\zeta, z)$, $\zeta \in \overline{D}$, has its boundary in a 3ε-neighbourhood of $\kappa_n \subset G$. The family $f^{rz}, r \in [0,1]$, provides a homotopy joining $\Phi_D(\cdot) = J_D(\cdot, 0)$ and f^z. If $\mathcal{H}$ is chosen to satisfy (3) for given $K \subset D \cup \Gamma$ then $\max_K |\Phi_D - f^{rz}| < \varepsilon$ for each $r \in [0,1]$. Choose the point $z \in \partial \mathbb{D}$. An ε-approximation of f^z provides an immersed analytic disc, hence a G-disc provided ε is small.

In case $X^2 = \mathbb{C}^2$ a suitable translation of the disc passes through $\Phi(p)$ and has boundary contained in a 5ε-neighbourhood of κ_n.

In the case of general Stein manifolds X^2 translations can be replaced by diffeomorphisms close to the identity from a suitable relatively compact subset of X^2 onto another subset of X^2. Such diffeomophisms are defined as compositions of the holomorphic embedding $\mathfrak{F}$ of X^2 into $\mathbb{C}^4$, a small translation in $\mathbb{C}^4$, a holomorphic projection of a tubular neighbourhood of $\mathfrak{F}(X^2)$ to $\mathfrak{F}(X^2)$ and the inverse of the mapping $\mathfrak{F}$.

We proved that through each point of $\Phi_D(K)$ passes a G-disc. Given $\zeta \in \mathbb{D}$ the compact set K can be chosen to contain ζ. Lemma 12 is proved. □

8. A piecewise continuous family of neurons with continuously changing axon

This paragraph is a preparation for the proof of Lemma 13.

Let $\Phi_t : \overline{\mathbb{D}} \to X^2$, $t \in [0,1]$, be a continuous family of analytic discs enjoying properties (1) and (2) of Lemma 11. The following lemma allows a further improvement of the properties of the family of analytic discs without changing the discs Φ_0 and Φ_1.

Lemma 19. *There is a continuous family of analytic discs $\Psi_t : \overline{\mathbb{D}} \to X^2$, $t \in [0,1]$, coinciding with the previous family Φ_t for t close to 0 and close to 1 such that condition (1) and (2) of Lemma 11 hold and the following additional condition is satisfied.*

The curve $\alpha(t) = (t, 1)$, $t \in [0,1]$, in $[0,1] \times \partial \mathbb{D} \subset \mathfrak{c}$ has the following property: the mapping $\Psi_t(\alpha(t))$, $t \in [0,1]$, admits a lift $\mathring{\alpha}$ to $\mathcal{G}_0$ such that $\hat{\mathcal{P}}_0 \circ \mathring{\alpha} = \hat{\psi}_t(\alpha(t))$.

Proof. Consider the mapping $\Phi(t, z) \stackrel{def}{=} \Phi_t(z)$, $t \in [0,1]$, $z \in \overline{\mathbb{D}}$ with values in X^2. By the condition (1) of Lemma 11 the restriction of this mapping to $[0,1] \times \partial \mathbb{D}$ lifts to $\hat{G}$, hence the mapping $\Phi \circ \alpha(t)$, $t \in [0,1]$, lifts to $\hat{G}$. The curve α is contained in the cylinder $[0,1] \times \partial \mathbb{D}$. It can therefore be considered as a planar curve and Lemma 14 applies. It will be convenient to realize the excrescence of α in a slightly different way. Namely, consider a tree and its punctured pellicle which participate in the construction of the excrescence of α in the cylinder. Let the root of the considered tree be the point $(t_i, 1)$ of the cylinder. We may assume that all points t_i are contained in the open interval $(0,1)$. We take another realization of the tree and its pellicle, namely, we consider a tree T_i in the complex plane with root at the point 1 that meets the closed disc $\overline{\mathbb{D}}$ exactly at the root and which is a homeomorphic copy of the tree in the cylinder. Call the product of the one-point set $\{t_i\}$ with the punctured pellicle of the tree $T_i \subset \mathbb{C}$ the punctured pellicle of $\{t_i\} \times T_i$. Cut α at the point $(t_i, 1)$ and paste the punctured pellicle of the tree $\{t_i\} \times T_i$. Doing this with all trees we obtain the realization of the excrescence α^* we will work with.

The trees T_i define a piecewise continuous family of generalized discs $\nu_t, t \in [0,1]$, given by the relation $\nu_t \stackrel{def}{=} \overline{\mathbb{D}}$, if t is not equal to one of the t_j, and $\nu_t \stackrel{def}{=} \overline{\mathbb{D}} \bigcup T_j$, if $t = t_j$. The new curve α^* has values in $\bigcup_{t\in[0,1]}\{t\} \times \nu_t$. By Lemma 14 there are continuous extensions of the mappings Φ and $\hat{\Phi}$ to the image of α^* such that the curve $\Phi \circ \alpha^*$ has a lift to $\mathcal{G}_0$ that is associated to $\hat{\Phi}$. Take a C^0-small deformation of the curve α^* which fixes the punctured pellicles of the trees and provides small changes of the original part α of the curve α^* so that the image of the deformation of the part α of α^* is the union of finitely many vertical segments of the form $I_z \times \{z\}$ for an interval $I_z \subset [0,1]$ and a point $z \in \partial\mathbb{D}$, and finitely many horizontal arcs of the form $\{t_j\} \times \beta_j$ for one of the aforementioned points $t_j \in [0,1]$ and an arc β_j in the unit circle. We may assume that the perturbed curve coincides with the previous one near the points $(0,1)$ and $(1,1)$. Denote the approximating curve again by α^*. Still, $\Phi \circ \alpha^*$ has a lift to $\mathcal{G}_0$ that is associated to $\hat{\Phi}$.

Consider the piecewise continuous family of generalized discs $\nu_t,\ t \in [0,1]$, that was defined above. Notice that the image of α^* is contained in $\bigcup_{t\in[0,1]}\{t\} \times (\nu_t \setminus \mathbb{D})$ and the mappings Φ and $\hat{\Phi}$ extend continuously to the union $\{t\} \times (\nu_t \setminus \mathbb{D})$ of the boundaries of the generalized discs.

Replace the family of generalized discs ν_t by a *continuous* family of generalized discs ν_t^* in the following way. Choose small disjoint intervals $I_j \subset (0,1)$ around t_j and define a continuous family of trees $T(t),\ t \in [0,1]$, with root 1 such that $T(t_j) = T_j$ and $T(t)$ is equal to a one point (degenerate) tree for t close to the endpoints of the I_j and outside the I_j. This is possible since each rooted tree is contractible to its root. Put $\nu_t^* \stackrel{def}{=} \nu_t \cup T(t)$.

The intervals and the contractions of the trees can be chosen in such a way that the mappings Φ and $\hat{\Phi}$ extend continuously to $\bigcup_{t\in[0,1]}\{t\} \times (\nu_t^* \setminus \mathbb{D})$. Denote the extended mappings again by Φ and $\hat{\Phi}$.

Lemma 16 provides fattenings of the dendrites $\mathbf{T}_t$ depending continuously on the parameter t. This yields a continuous family of simply connected domains $D_t,\ t \in [0,1]$, and a continuous mapping $\psi : \bigcup_{t\in[0,1]}\{t\} \times \overline{D_t} \to X^2$ which is holomorphic on each $\{t\} \times D_t$, approximates Φ uniformly on $\bigcup\{t\} \times \nu_t$ and coincides with Φ for values of t close to 0 and close to 1. Moreover, the restriction of the mapping ψ to the set $\bigcup_{t\in[0,1)}\{t\} \times \overline{D}_t \bigcup \bigcup_{t\in[0,1]}\{t\} \times \partial D_t)$ lifts to a mapping $\hat{\psi}$ into $\hat{G}$ which coincides with $\hat{\Phi}$ for t close to 0 and close to 1.

Deform the arcs of α^* contained in the set $t = t_j$ into arcs that are C^0-close to the previous ones and run along the boundary $\{t_j\} \times \partial D_{t_j}$. Denote the deformed curve by α^0.

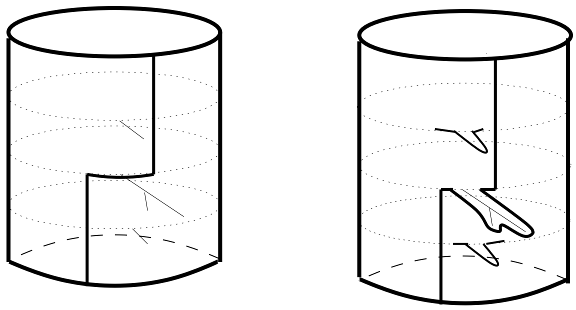

FIGURE 6. Fattening of trees of a family and deformation of the excrescence

Provide a further deformation of the curve so that its t-coordinate is strictly increasing. Parametrize the thus obtained curve by the t-coordinate of its image and denote it again by α^0. The mapping $\psi \circ \alpha^0$ admits a lift to $\mathcal{G}_0$ which is associated to $\hat{\psi}$.

Choose a continuous family of conformal mappings $\varphi_t : \mathbb{D} \to D_t$ (which extend to a continuous family of homeomorphisms between the closed unit disc and the closures of the domains) that map the point $1 \in \partial\mathbb{D}$ to the point $\alpha^0(t) \in \partial D_t$. The mappings $\Psi_t \stackrel{def}{=} \psi_t \circ \varphi_t$ (with $\psi_t(z) = \psi(t,z)$ for $t \in [0,1]$ and $z \in \overline{\mathbb{D}}$) have the desired property.

□

Choose an arc Γ of the unit circle containing the point 1 so that the mapping $(t,\zeta) \to \Psi(t,\zeta) = \Psi_t(\zeta),\ (t,\zeta) \in [0,1]\times\Gamma$, lifts to a continuous mapping $\overset{\circ}{m} : [0,1]\times\Gamma \to \mathcal{G}_0$ for which $\hat{\mathcal{P}}_0 \circ \overset{\circ}{m} = \hat{\Psi}$. In other words, the analytic discs Ψ_t have continuously changing halo on Γ that is associated to $\hat{\Psi}_t$.

According to Lemma 15 by attaching dendrites each disc $\Psi_t : \overline{\mathbb{D}} \to X^2$ can be performed into a neuron with halo associated to the lift $\hat{\Psi}_t|\partial\mathbb{D}$. This can be done so that the halo of the neuron on Γ coincides with $\overset{\circ}{m}(t,\cdot),\ t\in[0,1]$. In particular, for each t the arc Γ consists of regular points for the neuron. Further, the attaching of dendrites may be done in such a way that the neurons depend piecewise continuously on the parameter t.

Consider the constructed neurons as preneurons and attach for each $t\in[0,1]$ an axon $\mathbf{T}_t^{ax}$ to the respective (pre)neuron such that the root of its tree is the regular point 1. The trees T_t^{ax} of the axons $\mathbf{T}_t^{ax}$ are chosen to depend continuously on t, for t close to 1 being equal to the edge $T_t^{ax} = [1,2]$ which is orthogonal to the unit circle, and degenerated to a point for t close to 0. In particular, the tips of the axon trees, $\mathfrak{a}_t$ depend continuously on t. Since the restrictions of the halo of the (pre)neurons to Γ depend continuously on the parameter, the halo of the axon $\mathbf{T}_t^{ax}$ may be chosen to depend continuously on t. We define it in the following way. Let $m_{T_t}(\tau),\ \tau\in[0,1]$, parametrize the punctured pellicle of T_t^{ax}. The parametrization is chosen symmetric with respect to the sides of the edge T_t^{ax}, i.e. $m_{T_t}(\tau) = m_{T_t}(1-\tau),\ \tau\in[0,1]$. For the halo on the first side, $\overset{\circ}{m}_{T_t}(\tau),\ \tau\in[0,1/2]$, we choose a G_0-homotopy of the disc $\overset{\circ}{m}_t(0)$ to a disc embedded into G which is the value of the halo over the tip of the axon. The halo on the second side is chosen symmetrically.

We obtain a piecewise continuous family of neurons with halo, which we denote by $n_t = (\nu_t, m_t, \Psi_t, \overset{\circ}{m}_t),\ t\in[0,1]$. The neurons have a continuously changing axon attached whose halo at the tip is a small analytic disc embedded into G. For t close to 0 the neuron coincides with the original analytic disc which is embedded into G. For t close to 1 the main body of the neuron coincides with the original disc.

In the next section we obtain from this family a continuous family of neurons with halo with a continuously changing axon attached.

9. A continuous family of neurons. "Peeling"

This section is the key of the proof of Lemma 13.

Let t_0 be the first discontinuity point of the constructed family n_t of neurons with halo. Denote by $n_{t_0}^{\pm} = (\nu_{t_0}^{\pm}, m_{t_0}^{\pm}, \Psi_{t_0}^{\pm}, \overset{\circ}{m}{}_{t_0}^{\pm})$ the respective limits from the left and from the right. Note that the main bodies $\Psi_{t_0}^{\pm} : \overline{\mathbb{D}} \to X^2$ of the neurons $n_{t_0}^{\pm}$ coincide. Moreover, the values of $\overset{\circ}{m}{}_{t_0}^{\pm}$ coincide on Γ.

There may be no homotopy joining the neurons $n_{t_0}^{-}$ and $n_{t_0}^{+}$. The following lemma shows that there is such a homotopy after attaching a special dendrite to $n_{t_0}^{+}$.

Lemma 20. *There is a dendrite $\mathfrak{T}_{t_0}$ with punctured halo and a neuron with halo $n_{t_0}^0 = n_{t_0}^+ \cup \mathfrak{T}_{t_0}$ obtained in the following way. The tree of $\mathfrak{T}_{t_0}$ is attached to the generalized disc $\nu_{t_0}^+$ of $n_{t_0}^+$ at a point $\zeta^*\in\Gamma$. The pellicle (respectively, the halo) of the neuron $n_{t_0}^+$ punctured at ζ^* and the punctured pellicle (respectively, the punctured halo) of $\mathfrak{T}_{t_0}$ match and define the pellicle (respectively, the halo) of the neuron $n_{t_0}^+\cup\mathfrak{T}_{t_0}$. Moreover, there is a homotopy of neurons with halo joining the neuron $n_{t_0}^-$ with the neuron $n_{t_0}^0 = n_{t_0}^+\cup\mathfrak{T}_{t_0}$.*

Proof. In the proof we will skip everywhere the index t_0.

To ease reading we will first work out the proof in simple but typical situations before giving the formal proof in the general situation.

Step 1 of the proof. Peeling for n^+-regular points ζ. Let $\zeta_0 = \exp(i\theta_0),\ \theta_0 > 0$, be a point in Γ (counterclockwise from 1). We let a one-edge dendrite grow out of n^- at the point ζ_0 and let its root run counterclockwise along the circle. More precisely, let $\zeta = \exp(it_\zeta),\ t_\zeta > \theta_0$, be a point on the unit circle situated counterclockwise from ζ_0. Let γ_ζ be the arc between ζ_0 and ζ,

$\gamma_\zeta = \{\gamma_\zeta(t) = \exp(it) : \theta_0 \le t \le t_\zeta\}$. Assume that all points of γ_ζ are regular for the neuron n^+. Let e_ζ be a closed straight line segment attached to $\partial\mathbb{D}$ at the endpoint ζ of γ_ζ which is transversal to $\partial\mathbb{D}$ and meets $\overline{\mathbb{D}}$ exactly at ζ.

Consider the generalized disc $\nu_\zeta \stackrel{def}{=} \nu^- \cup e_\zeta$. Give it the structure of a neuron $n_\zeta = (\nu_\zeta, \Psi_\zeta, m_\zeta, \mathring{m}_\zeta)$ with halo in the following way.

Remove the point ζ_0 from the unit circle and close up the arc by adding two points over ζ_0. We refer to this set as the punctured circle (punctured at ζ_0). In the same way we define the pellicle of ν^- punctured at ζ_0. Denote by O_ζ the union of the closed arc $\overline{\partial\mathbb{D} \setminus \gamma_\zeta}$ of the circle with the "outer" side of e_ζ (i.e. the "right" side of the edge e_ζ with orientation towards the root, in other words, the second side when surrounding the edge counterclockwise starting from the root). This side is pasted to $\overline{\partial\mathbb{D} \setminus \gamma_\zeta}$ at the point ζ.

Consider the excrescence $\mathcal{E}^-$ of the punctured circle which is equal to the pellicle of n^- punctured at ζ_0. Let A_ζ be a homeomorphism of $\mathcal{E}^-$ onto an excrescence O^*_ζ of O_ζ. Suppose A_ζ is the identity on $\partial\mathbb{D} \setminus \gamma_\zeta$, maps γ_ζ onto e_ζ and fixes ζ. Moreover, assume that A_ζ is affine on each segment of $\mathcal{E}^-$ that is contained in an edge of an n^--tree.

Assign a halo to O^*_ζ in the following way. Let for some interval I the mapping $m_{\mathcal{E}^-}(t)$, $t \in I$, be a parametrization of $\mathcal{E}^-$. Then $m_{O^*_\zeta}(t) \stackrel{def}{=} A_\zeta \circ m_{\mathcal{E}^-}(t)$, $t \in I$, parametrizes O^*_ζ and we put $\mathring{m}_{O^*_\zeta}(t) \stackrel{def}{=} \mathring{m}_{\mathcal{E}^-}(t)$, $t \in I$.

Assign to the arc $t \to \gamma_\zeta(t)$, $t \in [\theta_0, t_\zeta]$, the n^+-halo: choose the parametrization $m^+(t) = \gamma_\zeta(t), t \in [\theta_0, t_\zeta]$, for the arc of the pellicle of n^+ and put $\mathring{m}_{\gamma_\zeta}(t) = \mathring{m}^+(t)$, $t \in [\theta_0, t_\zeta]$.

Finally, parametrize the "inner" (i.e. "left") side e^l_ζ of e_ζ by $e^l_\zeta(t) = A_\zeta \circ m^+(t), t \in [\theta_0, t_\zeta]$, and define the halo on e^l_ζ by $\mathring{m}_{e^l_\zeta}(t) = \mathring{m}^+(t)$, $t \in [\theta_0, t_\zeta]$. Note that the halo on e^l_ζ is "A_ζ-symmetric" (i.e symmetric with respect to the homeomorphism A_ζ) to the halo on γ_ζ.

The three arcs O^*_ζ, γ_ζ and e^l_ζ cover the pellicle of the generalized disc ν_ζ. The values of the halo match at the common endpoints of the arcs. Indeed, they match at the tip of e_ζ because this point is the image of $\zeta_0 \in \Gamma$ under the map A_ζ and for points in Γ the n^+-halo takes the same value as the n^--halo. They also match at the point ζ because A_ζ fixes this point.

We obtained a neuron n_ζ with halo. It has a distinguished attached dendrite $\mathbf{e}_\zeta$.

The construction proceeds as long as no n^+- dendrite is attached to the interior of the arc γ_ζ. It is arranged so that it provides a family of neurons n_ζ that depend continuously on the parameter ζ so that the values of the halo of each of it is contained in the union of the set of values of the halo of $n^+_{t_0}$ and $n^-_{t_0}$. Notice that the parametrization of the pellicle of n^+ can be chosen so that the arc $t \to m^+(t)$, $t \in [\theta_0, t_\zeta]$, of the pellicle of the generalized disc ν^+ is identical to the arc $t \to \gamma_\zeta(t)$ of the circle.

Step 2. Reaching edge-like dendrites of n^+. Suppose the construction of step 1 has been made up to a point $\zeta \in \partial\mathbb{D}$. We obtained a continuous family of neurons joining n^- with a neuron $n_\zeta = (\nu_\zeta, \Psi_\zeta, m_\zeta, \mathring{m}_\zeta)$. Recall that no n^+-neuron is attached to the interior $Int(\gamma_\zeta)$ so that $\gamma_\zeta = m^+([\theta_0, t_\zeta]) \subset \partial\mathbb{D}$ for a parameter t_ζ.

Suppose that ζ is the root of a tree T_ζ of the neuron n^+. Hence t_ζ parametrizes the initial point of the pellicle of the tree T_ζ. Let t'_ζ parametrize the terminating point of the pellicle of T_ζ.

Denote by $\mathcal{B}_\zeta$ the (closed) ray that bisects the angle between γ_ζ and the edge e_ζ obtained at step 1 (more precisely, the angle between the tangent ray to γ_ζ at ζ and e_ζ; we mean the angle which is covered moving in counterclockwise direction around the point ζ.) Choose a closed convex cone U_ζ with vertex ζ and non-empty interior which is symmetric with respect to reflection in the symmetry ray $\mathcal{B}_\zeta$, (hence, it contains $\mathcal{B}_\zeta$) and is contained in the sector between γ_ζ and e_ζ.

Suppose the tree T_ζ of the n^+-dendrite $\mathbf{T}_\zeta$ attached at ζ consists of a single edge.

Our goal is to construct a continuous family of neurons which differ only by a dendrite whose tree is attached at ζ and situated inside the cone U_ζ. The family is constructed so that it joins the neuron n_ζ with a neuron n'_ζ so that n'_ζ has the following property. The pellicle of its generalized disc ν'_ζ contains an arc that coincides with $t \to m^+(t)$, $t \in [\theta_0, t'_\zeta]$ (i.e. the arc is constituted by γ_ζ together with the punctured pellicle of T_ζ).

For defining the family of neurons it is enough to describe the family of dendrites.

Realize T_ζ as a straight line segment in the plane in the direction of $\mathcal{B}_\zeta$ meeting the generalized disc ν_ζ exactly at ζ. Reparametrize the punctured pellicle m_{T_ζ} of T_ζ by the interval $[0,1]$ and symmetrically with respect to its sides. More precisely, denote by $m_{T_\zeta} : [0,1] \to T_\zeta$ the (reparametrized) punctured pellicle of T_ζ. We require that this mapping has the following symmetry property: for each $t \in [0,1]$ the points $m_{T_\zeta}(t)$ and $m_{T_\zeta}(1-t)$ are at different sides of the pellicle over the same point. The halo $\overset{\circ}{m}_{T_\zeta}$ is reparametrized accordingly by the interval $[0,1]$.

Construct a continuous family of dendrites $\mathbf{T}_\zeta^s$, $s \in [0,1]$, with punctured halo, the tree T_ζ^s of which has root ζ and such that

- for each s the tree T_ζ^s is contained in U_ζ and meets the boundary of U_ζ exactly at ζ;
- for each s the values of the punctured halo of $\mathbf{T}_\zeta^s$ at the initial and terminating point coincide and are equal to $\overset{\circ}{m}_{T_\zeta}(0)$; the dendrites are mirror symmetric with respect to reflection in the symmetry ray $\mathcal{B}_\zeta$;
- $\mathbf{T}_\zeta^0$ is a one-point dendrite;
- $\mathbf{T}_\zeta^1$ consists of the union of two dendrites attached at ζ ("dendrite twins"). The first of the two dendrites (i.e. its underlying tree, its punctured pellicle and punctured halo) is a homeomorphic copy of $\mathbf{T}_\zeta$ and is (by a slight abuse) denoted again by $\mathbf{T}_\zeta$. Its tree is placed in the closed part U_ζ^- of the cone U_ζ which is clockwise from $\mathcal{B}_\zeta$, and meets the boundary of U_ζ^- exactly at ζ;
  The second dendrite is mirror symmetric to the first one with respect to reflection in the symmetry ray $\mathcal{B}_\zeta$ and is denoted by $\mathbf{T}_\zeta^*$.
  The value of the punctured halo of the dendrite $\mathbf{T}_\zeta^1$ at the point that lies over ζ between the dendrite twins coincides with the value at the terminating point of the pellicle of $\mathbf{T}_\zeta$.

We call this procedure "growing of dendrite twins" (see below Lemma 21 for the general case).

The construction is the following. For $s=0$ we obtain a one-point dendrite $\mathbf{T}_\zeta^0$. The procedure of attaching this dendrite $\mathbf{T}_\zeta^0$ does not change n_ζ.

For $s \in (0,1/2]$ the tree T_ζ^s of the dendrite $\mathbf{T}_\zeta^s$ is an edge and consists of the points $m_{T_\zeta}([0,s])$.

Parametrize the pellicle of the tree T_ζ^s by the interval $[0,2s]$ and symmetrically with respect to the sides of the tree: take $m_{T_\zeta^s}|[0,s] \overset{def}{=} m_{T_\zeta}|[0,s]$ (parametrization of the first side of the tree), and symmetrically, $m_{T_\zeta^s}(\tau) \overset{def}{=} m_{T_\zeta}(2s-\tau)$ for $\tau \in [s,2s]$ (parametrization of the second side of the tree).

Respectively, the halo of the dendrite $\mathbf{T}_\zeta^s$ is defined by the relations $\overset{\circ}{m}_{T_\zeta^s}|[0,s] \overset{def}{=} \overset{\circ}{m}_{T_\zeta}|[0,s]$ on the first side, and symmetrically, $\overset{\circ}{m}_{T_\zeta^s}(\tau) \overset{def}{=} \overset{\circ}{m}_{T_\zeta^s}(s-\tau)$, $\tau \in [s,2s]$, on the second side of the dendrite.

For $s \in (1/2,1]$ the tree T_ζ^s of the dendrite becomes a letter "Y" which is symmetric with respect to the symmetry ray.

Describe the tree T_ζ^s. Denote by a_ζ^s the segment $m_{T_\zeta^s}([0,1-s])$ of T_ζ which is adjacent to ζ (note that the number $1-s$ is less than $\frac{1}{2}$). Denote the remaining segment $b_\zeta^s \overset{def}{=} \overline{T_\zeta \setminus a_\zeta^s}$. The segment $a_\zeta^s \subset \mathcal{B}_\zeta$ is the "trunk" of the letter Y. The "first branch of the letter Y" is the image $\mathcal{R}_\zeta^s(b_\zeta^s)$ of b_ζ^s under a rotation $\mathcal{R}_\zeta^s$ around the common endpoint $m_{T_\zeta^s}(1-s)$ of a_ζ^s and b_ζ^s. The rotation is chosen so that the rotated segment $\mathcal{R}_\zeta^s(b_\zeta^s)$ is placed in U_ζ^- and meets the boundary of U_ζ^- exactly at $m_{T_\zeta^s}(1-s)$. The rotations $\mathcal{R}_\zeta^s$ are chosen continuously depending on s.

The second branch of the letter Y is chosen symmetric to the first one with respect to mirror reflection in the symmetry ray $\mathcal{B}_\zeta$.

Describe the punctured pellicle m_ζ^s of the tree T_ζ^s and the halo of the dendrite $\mathbf{T}_\zeta^s$. The part of the pellicle of T_ζ^s corresponding to the first side of a_ζ^s coincides with the corresponding part of the pellicle of T_ζ: $m_\zeta^s(\tau) = m_\zeta(\tau)$ for $\tau \in [0,1-s]$. Respectively, for the halo the relation $\overset{\circ}{m}{}^s{}_\zeta(\tau) \overset{def}{=} \overset{\circ}{m}_\zeta(\tau)$ for $\tau \in [0,1-s]$ holds.

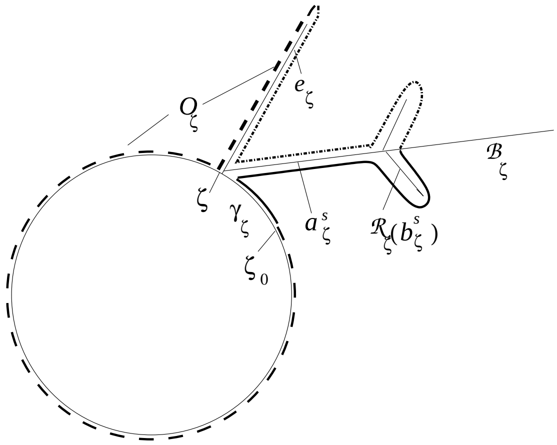

FIGURE 7. "Peeling" in case of a single n^+-edge at ζ

For τ in the interval $[1-s, s]$ the relation is $m^s_\zeta(\tau) = \mathcal{R}^s_\zeta \circ m_\zeta(\tau)$. This part of the pellicle m^s_ζ surrounds $\mathcal{R}^s_\zeta(b^s_\zeta)$. The halo of $\mathbf{T}^s_\zeta$ for those parameters τ is defined by the halo of $\mathbf{T}_\zeta$: we put $\overset{\circ}{m}{}^s_\zeta(\tau) = \overset{\circ}{m}_\zeta(\tau)$ for $\tau \in [1-s, s]$.

The remaining part of the punctured pellicle and punctured halo of the dendrite $\mathbf{T}^s_\zeta$ is mirror symmetric to the just described part.

For $s = 1$ we arrive at a mirror symmetric pair of dendrites with the properties described above. The construction for this case is completed.

Step 3. The general case. Let ζ_0 be as above a point in Γ situated counterclockwise from the root 1 of the axon. Suppose $\zeta \in \partial\mathbb{D} \setminus \Gamma$ is reached by moving counterclockwise from ζ_0 and γ_ζ is the closed arc of the circle between ζ_0 and ζ (counterclockwise traveling). Let m^+ parametrize the pellicle of n^+ punctured at ζ_0, $m^+(\theta_0) = \zeta_0$.

If ζ is a regular point for n^+ then there is a unique parameter t_ζ in the pellicle of n^+ for which the equality $m^+(t_\zeta) = \zeta$ holds. If ζ is not regular for n^+ then there is a finite collection of increasing parameters $t^1_\zeta, ..., t^l_\zeta$ for which $m^+(t^j_\zeta) = \zeta$. Here t^1_ζ parametrises the initial point of the n^+-tree attached at ζ and t^l_ζ parametrises its terminating point. The points t^j_ζ and t^{j+1}_ζ parametrise the initial, respectively the terminating, points of the simple trees constituting the tree at ζ.

The plan is the following. Let $\zeta \in \partial\mathbb{D} \setminus \Gamma$ be any point counterclockwise from ζ_0 and let t_ζ denote one of the parameters for which $m^+(t_\zeta) = \zeta$. Assume a neuron n_{t_ζ} is constructed such that the pellicle of its tree contains the arc $\tau \to m^+(\tau), \tau \in [\theta_0, t_\zeta]$. We will construct a neuron such that an arc of its pellicle coincides with $\tau \to m^+(\tau)$, $\tau \in [\theta_0, t]$, for some parameters $t > t_\zeta$.

Here is the precise description of the induction hypothesis.

Suppose a neuron n_{t_ζ} is constructed with the following properties. Its generalized tree ν_ζ has an edge e_ζ attached at ζ. Let as in step 2 $\mathcal{B}_\zeta$ be the (closed) ray that bisects the angle between γ_ζ and the edge e_ζ. The main property of n_{t_ζ} is the following. The pellicle of ν_ζ (considered as a curve parametrized by the unit circle $\partial\mathbb{D}$) has a partition into three parts each reparametrized by an interval.

(1) The first part is the excrescence O^*_ζ of O_ζ. Its halo is defined by $\mathcal{E}^-$ as in step 1.
(2) The second part is the excrescence $\gamma^*_\zeta(t) \overset{def}{=} m^+(t),\ t \in [\theta_0, t_\zeta]$, of γ_ζ. We assume the excrescence is chosen so that its image $m^+([\theta_0, t_\zeta])$ is situated clockwise from $\mathcal{B}_\zeta$ and meets $\mathcal{B}_\zeta$ exactly at the points $m^+(t^j_\zeta) = \zeta,\ j = 1, ..., i$, where $t^i_\zeta = t_\zeta$. The halo on the second part is defined by $\overset{\circ}{m}{}^+(t),\ t \in [\theta_0, t_\zeta]$.
(3) To define the third part we consider an extension of the homeomorphism A_ζ from $\mathcal{E}^-$ to the image $m^+([\theta_0, t_\zeta])$ such that A_ζ is affine on each straight line segment contained

in $m^+([\theta_0, t_\zeta])$. Moreover, the image $A_\zeta \circ m^+([\theta_0, t_\zeta])$ is contained in the closed angle between $\mathcal{B}_\zeta$ and e_ζ (i.e. counterclockwise from $\mathcal{B}_\zeta$) and meets $\mathcal{B}_\zeta$ exactly at the points $m^+(t_\zeta^j) = \zeta,\ j = 1, ..., i,$.

The third part is the excrescence $(e_\zeta^l)^*(t) = A_\zeta \circ m^+(t),\ t \in [\theta_0, t_\zeta]$. The halo is defined by $\overset{\circ}{m}_{e(^l_\zeta)^*}(t) = \overset{\circ}{m}{}^+(t),\ t \in [\theta_0, t_\zeta]$.

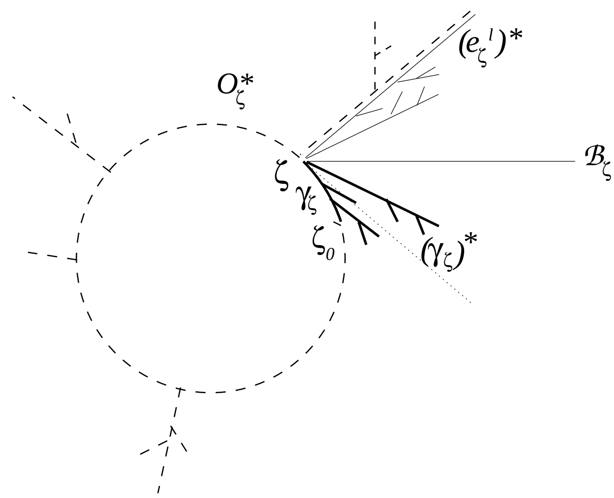

FIGURE 8. "Peeling": The generalized disc ν_{t_ζ}

Two possibilities may arise.

(a) Points in the pellicle of n^+ parametrized by $t > t_\zeta$ and close to t_ζ are regular points contained in $\partial\mathbb{D}$.

(b) t_ζ is the initial point of one of the simple trees that constitute the n^+-tree attached at ζ. We denote this tree for short by T_ζ and the respective dendrite by $\mathbf{T}_\zeta$. (Notice that the structure of the whole n^+-dendrite that is attached at ζ does not play a role in the proof.) Let t'_ζ parametrize the terminating point of the pellicle of T_ζ. So $m^+(t'_\zeta) = \zeta$ and the arc $t \to m^+(t),\ t \in [t_\zeta, t'_\zeta]$, of the pellicle of ν^+ is the punctured pellicle of the tree T_ζ.

Here are the constructions in cases (a) and (b). In the first case (a) we proceed like in step 1 of the proof. We change the root of the main edge e_ζ in counterclockwise direction along the circle and let the edge grow. More precisely, let $\zeta' \in \partial\mathbb{D}$ be counterclockwise of ζ and let the arc between ζ and ζ' consist of regular points. At ζ' we attach an edge $e_{\zeta'}$ and equip it with the following structure. For a segment of $e_{\zeta'}$ adjacent to the leaf we take an excrescence on each of the sides of the edge (and the respective halo on it) that is homeomorphic to the respective one for e_ζ. For the remaining segment of $e_{\zeta'}$ that is adjacent to the root ζ' we proceed as in step 1.

In the second case (b) we will construct a continuous family of neurons that connects n_ζ with a neuron n'_ζ so that the final neuron n'_ζ has the following properties. As for the original neuron the pellicle of n'_ζ has a decomposition into three parts satisfying properties (1), (2) and (3) with t_ζ replaced by $t_{\zeta'}$. Thus, the pellicle of n'_ζ contains the arc $t \to m^+(t),\ t \in [\theta_0, t'_\zeta]$. (Recall that $t \to m^+(t),\ t \in [t_\zeta, t'_\zeta]$, is the punctured pellicle of T_ζ).

To construct the family of neurons it is enough to construct the respective family of dendrites attached to n_ζ at the point ζ. The following lemma provides this construction.

Lemma 21. *(On growing of dendrite twins) Let γ_ζ, e_ζ and $\mathcal{B}_\zeta$ be as above. Let U_ζ be a closed convex cone with vertex ζ which is symmetric with respect to reflection in $\mathcal{B}_\zeta$ and contained in the sector between γ_ζ and e_ζ. Denote by U_ζ^- the closed part of U_ζ which is situated clockwise from $\mathcal{B}_\zeta$. Let $\mathbf{T} = (T, m_T, \Phi_T, \overset{\circ}{m}_T)$ be a dendrite with halo. Suppose the pellicle of the underlying tree T is parametrized by $[t', t'']$.*

Consider a point $\xi \in \mathcal{B}_\zeta$ and a closed convex cone $U_\xi \subset U_\zeta$ with vertex ξ which is symmetric with respect to reflection in $\mathcal{B}_\zeta$.

Then there exists a continuous family of dendrites $\mathbf{T}_\xi^s$, $s \in [0,1]$, with root ξ with the following properties:

1. *For all $s \in [0,1]$ the tree of $\mathbf{T}_\xi^s$ is contained in the cone U_ξ and meets the boundary of the cone exactly at ξ.*
2. *For all s the values of the halo $\overset{\circ}{m}{}^s_{T_\xi}$ at the initial point and at the terminating point coincide and equal $\overset{\circ}{m}{}^+_T(t')$. The dendrites are mirror symmetric for reflection in the ray $\mathcal{B}_\zeta$.*
3. *The dendrite $\mathbf{T}_\xi^0$ is a one-point dendrite.*
4. *The dendrite $\mathbf{T}_\xi^1$ is a dendrite twin attached at ξ. The tree of the first labeled twin dendrite is contained in $U_\xi^- \overset{def}{=} U_\zeta^- \cap U_\xi$ and meets the boundary of U_ξ^- exactly at ξ. The first labeled dendrite (i.e. its underlying tree, its punctured pellicle and punctured halo) is a homeomorphic copy of $\mathbf{T}$. The second dendrite is mirror symmetric to the first one with respect to reflection in the symmetry ray. The value of the halo of the dendrite twin at the point between the twins equals $\overset{\circ}{m}_T(t'')$.*

Proof. If the tree of the dendrite $\mathbf{T}$ consists of a single edge the proof was given in step 2. Prove the lemma by induction on the number of edges of the tree $\mathbf{T}$.

Suppose first that the (planar) tree T is not simple, i.e. it has more than one (non-empty) edges *adjacent* to the root. Then the tree is the union of two (planar) trees T' and T'' with the same root labeled so that T'' is counterclockwise of T'. Each of the trees T' and T'' has less edges than T. By induction hypothesis the required family $\mathbf{T'}_\xi^s$, $s \in [0,1]$, of dendrites rooted at ξ exists for the first dendrite $\mathbf{T}'$. The final dendrite $\mathbf{T'}_\xi^1$ is the union of mirror symmetric twins. The first of the twins is denoted by $\mathbf{T'}_\xi$ (situated clockwise from $\mathcal{B}_\zeta$) and the second twin is denoted by $(\mathbf{T'}_\xi)^*$ (situated counterclockwise from $\mathcal{B}_\zeta$)).

Consider a smaller closed convex cone $\overset{\circ}{U}_\xi \subset U_\xi$ with vertex ξ which is symmetric with respect to $\mathcal{B}_\zeta$ and meets the trees T'_ξ and $(T'_\xi)^*$ exactly at the point ξ.

An application of the induction hypothesis to the point ξ, the cone $\overset{\circ}{U}_\xi$ and the second dendrite $\mathbf{T}''$ finishes the proof in this case.

Consider the remaining case when the tree T is simple, i.e. it has a single edge E adjacent to its root. Realize E as a segment E_ξ with initial point ξ on the symmetry ray $\mathcal{B}_\zeta$ (traveled in positive direction of $\mathcal{B}_\zeta$). Associate to the tree E_ξ the following dendrite $\mathbf{E}_\xi^-$ with halo. The tree of $\mathbf{E}_\xi^-$ is chosen equal to E_ξ. The halo on the first side of E_ξ is taken to coincide with the halo of $\mathbf{T}$ along the first side of the edge E of its tree. The halo on the second side of E_ξ is chosen symmetrically. There is a continuous family of dendrites which join the one-point dendrite with root ξ with the dendrite $\mathbf{E}_\xi^-$.

Denote by $\mathbf{T}^E$ the dendrite obtained by removing $\mathbf{E}$ from $\mathbf{T}$. In other words, the tree of $\mathbf{T}^E$ equals $T^E \overset{def}{=} T \setminus E$. The halo of the dendrite $\mathbf{T}^E$ is the restriction of the halo of $\mathbf{T}$.

The tree T^E has an edge less than T. The induction hypothesis applied to T^E, the endpoint η of the tree E_ξ and a closed convex cone $U_\eta \subset U_\xi$ symmetric with respect to $\mathcal{B}_\zeta$ gives a continuous family of dendrites that join the one-point dendrite at η with a dendrite twin $\mathbf{T}_\eta^E \bigcup (\mathbf{T}_\eta^E)^*$ rooted at η. Here $T_\eta^E \subset U_\eta$ is situated clockwise from $\mathcal{B}_\zeta$ and $(T_\eta^E)^* \subset U_\eta$ is counterclockwise from $\mathcal{B}_\zeta$. Cut the punctured pellicle of E_ξ^- at the tip η and paste the punctured pellicle of $T_\eta^E \bigcup (T_\eta^E)^*$.

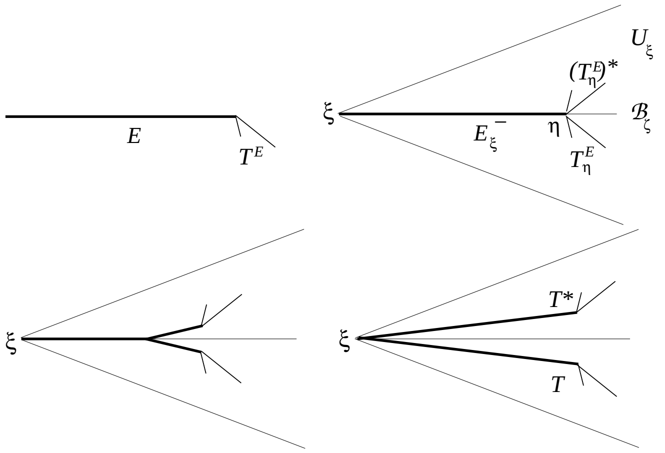

FIGURE 9. Growing of dendrite twins

The punctured halo of the twin dendrite $\mathbf{T}^E_\eta \bigcup (\mathbf{T}^E_\eta)^*$ matches with that of $\mathbf{E}^-_\xi$ at the point η. The result of pasting is a dendrite with halo which can be joined with the one-point dendrite at ξ by a continuous family.

The rest of the construction is based, as in step 2, on splitting the segment E_ξ into a letter Y but with copies of T^E_ξ (respectively $(T^E_\xi)^*$) attached at the tip of the first branch (respectively, of the second branch) of the letter Y.

It remains to define the halo on the Y. The pellicle of the Y punctured at the bottom point has a partition into three arcs: the part, seen from the right (the union of the first side of the steam and the first side of the first branch), the part seen from above (the union of the second side of the first branch and the first side of the second branch) and the part seen from the left (the union of the second side of the second branch and the second side of the steam). The halo on the part seen from the right (respectively seen from the left) is the halo on the first side (respectively on the second side) of $\mathbf{E}^-_\xi$ after a change of variables. The halo on the part seen from above is defined as in step 2.

We defined a continuous family of dendrites with halo. The final dendrite of the family is the required twin dendrite. The proof of lemma 21 is finished.

□

To finish step 3 of the proof of Lemma 20 we apply Lemma 21 to the dendrite $\mathbf{T}_\zeta$, the point ζ and a closed convex cone U_ζ contained in the sector between γ_ζ and e_ζ which meets the trees of n_ζ at most at ζ. The desired continuous family of neurons is obtained by pasting the constructed family of dendrites obtained in Lemma 21.

The general "peeling"-procedure described in step 3 can be continued until a point $\zeta^* \in \Gamma \subset \partial\mathbb{D}$ situated clockwise (within Γ) from the point 1 is reached.

By assumption $\Gamma \setminus \{1\}$ consists of regular points for both, n^+ and n^-, and the n^+-halo coincides with the n^--halo on $\Gamma \setminus \{1\}$. Hence, the obtained neuron n_{ζ^*} has the required property: it differs from n^+ by a dendrite attached at ζ^*. Lemma 20 is proved. □

Lemma 20 yields a continuous family of neurons with halo that joins the neuron n_0 with the neuron $n^+_{t_0} \cup \mathfrak{T}_{t_0}$. By a change of the t-variables we may assume that the parameter set is again the interval $[0, t_0]$. For t close to 0 the new neurons coincide with the previous ones and for $t = t_0$ the new neuron coincides with $n^+_{t_0} \cup \mathfrak{T}_{t_0}$.

For all $t > t_0$ we attach to the neuron n_t a dendrite $\mathfrak{T}_t$ with halo and root ζ^* of the underlying tree. The family of dendrites with halo $\mathfrak{T}_t$ is chosen continuously depending on t and converging to $\mathfrak{T}_{t_0}$ for $t \to t_0$. A continuous choice of the dendrites can be made since the halo of the neurons on the arc Γ changes continuously.

We obtain a piecewise continuous family of neurons with halo. Moreover, the family of neurons has one discontinuity point less than the previous family. Shrink the arc Γ (keeping the same notation) so that the arc still contains the point 1 and $\Gamma \setminus \{1\}$ is free from roots of attached trees for all $t \in [0,1]$.

Consider all (finitely many) discontinuity points t_j (in increasing order) of the family n_t. Apply Lemma 20 successively to each n_{t_i} and attach to the $n_t,\ t \geq t_i$, dendrites that depend continuously on t. At each step the arc Γ is shrinken suitably.

We arrive at a continuous family of neurons with halo. Denote the neurons by
$N_t = (\nu_t', \phi_t, M_t, \overset{\circ}{M}_t)$. All generalized discs ν_t' coincide with the closed unit disc with a number of trees attached. In particular, each generalized disc ν_t' contains the tree T_t^{ax} of an axon attached at the point 1. For each t there is a number t' such that the restricted mapping $\phi_t|\overline{\mathbb{D}}$ coincides with the original mapping $\Phi_{t'}$ from Lemma 13. Moreover, for t close to 1 the restrictions coincide with the mappings from Lemma 13: $\phi_t|\overline{\mathbb{D}} = \Phi_t$ for t close to 1. For t close to 0 the generalized discs ν_t' coincide with the unit disc $\overline{\mathbb{D}}$ and the neurons coincide with the original analytic discs of Lemma 13. They are small discs embedded into G and the values of their halo are small discs embedded into G. For all t the halo $\overset{\circ}{M}_t$ is associated to the lift of $\Phi_t|\partial\mathbb{D}$ to $\hat{G}$. In other words, the restriction of the mapping $\hat{M}_t = \hat{\mathcal{P}}_0 \circ \overset{\circ}{M}_t$ to $\partial\mathbb{D}$ coincides with $\hat{\Phi}_t|\partial\mathbb{D}$.

In the sequel we need also the following property of the neurons. Choose parametrizations $M_t(\xi),\ t \in [0,1],\ \xi \in \partial\mathbb{D}$, of the pellicle of ν_t which depend continuously on t. The property is the following. There exists a compact subset κ of G such that $\bigcup_{t\in[0,1],\xi\in\partial\mathbb{D}} \overset{\circ}{M}_t(\xi)(\partial\mathbb{D}) \subset \kappa$. Moreover, let for each t the point $M_t(\xi_0)$ be the tip of the axon tree of ν_t. Then $\bigcup_{t\in[0,1]} \overset{\circ}{M}_t(\xi_0)(\overline{\mathbb{D}}) \subset \kappa$ and, hence, in particular, $\bigcup_{t\in[0,1]} \phi_t \circ M_t(\xi_0) \subset \kappa$ and $\hat{\phi}_t \circ M_t(\xi_0) \subset \hat{i}(G)$.

10. Proof of lemma 13

Using the continuous family of neurons N_t with halo obtained in the previous section the proof of Lemma 13 can be completed essentially along the same lines as the proof of lemma 12. Here are the details.

Fix an $\varepsilon > 0$ which is small compared to the distance of κ to the boundary of G. Apply the procedure of continuous fattening of dendrites (Lemma 16) to all neurons N_t and all attached dendrites. We obtain a continuous family of analytic discs with continuously varying halo, denoted by $(D_t, m_t, \psi_t, \overset{\circ}{m}_t),\ t \in [0,1]$, for which $\max_{\nu_t'} |\psi_t - \phi_t| < \varepsilon$ and $\overset{\circ}{m}_t$ is ε-close to the halo $\overset{\circ}{M}_t$ of the respective original neuron. (We abuse notation for the pellicle and the halo using the same letter as for the objects related to the original family Φ_t). The sets $\overline{D}_t$ are closures of continuously changing bounded simply connected and smoothly bounded domains in the complex plane. The sets $\overline{D}_t$ are obtained from the closed unit disc by attaching "closed thickened trees". The "closed thickened axons" play a special role. These are thin closed neighbourhoods of the interiors of the axon trees T_t^{ax} that depend continuously on t and are pasted to the closed unit disc along an arc of the circle. For each t the tip $\mathfrak{a}_t$ of the axon is the only point of the axon that is located on the boundary of the respective domain D_t.

Since for each t the inclusion $\hat{\phi}_t(\mathfrak{a}_t) \in \hat{i}(G)$ holds, there are closed arcs $\mathfrak{A}_t$ contained in ∂D_t, $\mathfrak{a}_t \in \mathfrak{A}_t$, for which $\hat{\psi}_t(\mathfrak{A}_t) \subset \hat{i}(G)$, provided ε is small enough. Choose continuously changing open arcs $\mathfrak{A}_t^0$ which are relatively compact in $Int\,\mathfrak{A}_t$ with $\mathfrak{a}_t \in \mathfrak{A}_t^0$.

Use the same notation as in the proof of lemma 12: $\mathcal{S}_{\partial D_t} \overset{def}{=} (\overline{D}_t \times \{0\}) \bigcup (\partial D_t \times \overline{\mathbb{D}})$, $\mathcal{S}_{\partial D_t \setminus \mathfrak{A}_t^0} \overset{def}{=} (\overline{D}_t \times \{0\}) \bigcup (\partial D_t \setminus \mathfrak{A}_t^0 \times \overline{\mathbb{D}})$ and $\mathcal{Q}^t \overset{def}{=} (\partial D_t \times \partial\mathbb{D}) \bigcup (\mathfrak{A}_t \times \overline{\mathbb{D}})$.

Define, as in the proof of Lemma 12, for each t a mapping $\mathcal{J}_t$ on $\mathcal{S}_{\partial D_t}$ which equals ψ_t on the central disc $\overline{D}_t \times \{0\}$ and is equal to the evaluation map for $\overset{\circ}{m}_t$ on the disc fibers over ∂D_t. The mappings $\mathcal{J}_t$ depend continuously on t.

For each neighbourhood $\mathcal{V}$ of κ the number ε and the arcs $\mathfrak{A}_t$ may be chosen so that
$\bigcup_{t\in[0,1]} \mathcal{J}_t(\mathcal{Q}_t) \subset \mathcal{V}$.

Let K_t denote the following compact subset of $\mathbb{D} \bigcup T_t^{ax}$: $K_t \stackrel{def}{=} r\overline{\mathbb{D}} \bigcup [r,1] \bigcup T_t^{ax}$.(Recall that for each t we denote by T_t^{ax} the tree of the axon of the neuron n_t.) Note that K_t is a compact subset of $D_t \bigcup \mathfrak{A}_t^0$. For t close to 1 $K_t = r\overline{\mathbb{D}} \bigcup [0,2]$.

By Lemma 17 there is a continuous family of mappings $\mathcal{H}_t \in A_{X^2}(D_t \times \mathbb{D})$ such that for an arbitrary point $z \in \partial\mathbb{D}$ the mappings $f_t^z,\ t \in [0,1],\ f_t^z(\zeta) \stackrel{def}{=} \mathcal{H}_t(\zeta, z),\ \zeta \in \overline{D}_t$, define a continuous family of analytic discs with boundary in $\mathcal{V}$ satisfying the inequality $\max_{K_t} |\psi_t - f_t^z| < \varepsilon$ for all $t \in [0,1]$. Moreover, by the special choice of $\psi_0 = \Phi_0$ Lemma 17 implies that the disc $f_0^z(\overline{\mathbb{D}})$ is entirely contained in G.

Fix a point $z \in \partial\mathbb{D}$. An application of lemma 9 to the family f_t^z produces a family of *immersed* discs f_t with all above listed properties preserved. In particular the boundaries of the discs $f_t(\partial\mathbb{D})$ are contained in $\mathcal{V}$.

Take an arbitrary point $p \in \Phi_1(\mathbb{D})$. Choosing r close enough to 1 we may assume that $p \in \Phi_1(K_1)$. Further, we may assume that the family f_t is chosen so that $p \in f_1(K_1)$. (This can be achieved considering, in case $X^2 = \mathbb{C}^2$, small translations of the discs of the family and in the general case by applying compactly defined holomorphic mappings close to the identity on X^2.) We proved that p is contained in the projection $\hat{\mathcal{P}}(\hat{G})$.

To choose a standard lift of a neighbourhood of p to $\hat{G}$ we reparametrize f_1. More precisely, consider the composition $f_1 \circ \varphi_1$ with a conformal mapping φ_1 from the unit disc onto D_1 such that $f_1 \circ \varphi_1(0) = p$. For a number $\mathsf{r} < 1$ and close to 1 we consider the function $\zeta \to f_1 \circ \varphi_1(\mathsf{r}\zeta)$ and denote it by d_p. Let $\hat{d}_p$ be the equivalence class represented by d_p.

Consider a standard neighbourhood $\hat{\mathcal{P}} : \hat{V} \to Q_p$ of $\hat{d}_p$ associated to the representative d_p (see section 3). Here Q_p is a neighbourhood of p in X^2, $\hat{\mathcal{P}}$ is biholomorphic and for $q \in Q_p$ the classes $\hat{d}_q = (\hat{\mathcal{P}}|\hat{V})^{-1}(q)$ are represented by a continuous family of analytic discs d_q. For $q = p$ the disc coincides with the one defined before.

It remains to see that this standard lift of Q_p to $\hat{G}$ is compatible with the lift $\hat{\Phi}$ of Φ. More precisely, let $(t, z') \in [0,1) \times \mathbb{D}$ be close to $(1, z)$, so that $q \stackrel{def}{=} \Phi(t, z')$ is contained in Q_p. We have to prove the following Lemma 22.

Lemma 22. *The equivalence classes $\hat{d}_q$ and $\hat{\Phi}(t, z')$ coincide.*

Proof of Lemma 22. Recall that for t close to 1 $\hat{\phi}_t|\partial\mathbb{D} = \hat{\Phi}_t|\partial\mathbb{D}$. For $t < 1$ close to 1 we extend $\hat{\phi}_t$ to $\overline{\mathbb{D}}$ by $\hat{\phi}_t|\overline{\mathbb{D}} \stackrel{def}{=} \hat{\Phi}_t$. It is enough to find two curves $\hat{\gamma}_d$ and $\hat{\gamma}_\Phi$ in $\hat{G}$ with equal projections $\hat{\mathcal{P}} \circ \hat{\gamma}_d = \hat{\mathcal{P}} \circ \hat{\gamma}_\Phi$ such that for the initial points of the curves $\hat{\gamma}_d(0) = \hat{d}_q$ and $\hat{\gamma}_\Phi(0) = \hat{\Phi}(t, z')(= \hat{\phi}(t, z'))$ and the terminating points of the curves $\hat{\gamma}_d$ and $\hat{\gamma}_\Phi$ coincide.

Each curve will be the sum of two curves. To define the first part of $\hat{\gamma}_\Phi$ we choose a number $\mathfrak{a} \in [0,2]$ close to 2 and let $\beta : [0, \mathfrak{a}] \to \{t\} \times K_t$ be a curve that joins the point (t, z') with the point $(t, \mathfrak{a})$. Recall that for t close to 1 the set K_t has the form $r\overline{\mathbb{D}} \cup [0,2]$. Define the first part of $\hat{\gamma}_\Phi$ by $\hat{\gamma}_\Phi(\tau) = \hat{\phi}(\beta(\tau)),\ \tau \in [0, \mathfrak{a}]$. Hence, as required $\hat{\gamma}_\Phi(0) = \hat{\phi}(t, z')$. For the projected curve we have $\hat{\mathcal{P}} \circ \hat{\gamma}_\Phi(\tau) = \phi(\beta(\tau)), \tau \in [0, \mathfrak{a}]$.

Since for t close to 1 the point 2 is the tip of the axon tree T_t^{ax} the inclusions $\phi_1(2) = \phi(1,2) \in \kappa$, $\hat{\phi}(1,2) = \hat{i} \circ \phi(1,2) \in \hat{i}(G)$ hold (see the end of section 9). Hence we may assume that $\phi(\beta)(\mathfrak{a})$ is contained in the neighbourhood $\mathcal{V}$ of κ and $\hat{\phi}(\beta(\mathfrak{a})) = \hat{i} \circ \phi(\beta(\mathfrak{a})$ is in $\hat{i}(G)$.

To define the first part of $\hat{\gamma}_d$ we find a continuous family of G-discs d^τ that are all close to d_q and have center $d^\tau(0) = \phi(\beta(\tau))$ so that $d^0 = d_q$. For this we recall that f_1 is 2ε-close to ϕ_1 on K_1 (since it is ε-close to ψ_1 on K_1 and ψ_1 is ε-close to ϕ_1 on ν_1) and $\phi_1 = \Phi_1$ on $\overline{\mathbb{D}}$. Also, d_p differs from f_1 by a reparametrization. Further, if (t, z') is close to $(1, z)$ then d_q is ε-close to d_p on $\overline{\mathbb{D}}$. Moreover, for t close to 1 $\max_{K_1} |\phi(t, z) - \phi(1, z)| < \varepsilon$. Hence, in case $X^2 = \mathbb{C}^2$, there are points $z_\tau \in \mathbb{D}$ depending continuously on $\tau \in [0, \mathfrak{a}]$ and a continuous family of translations d_q^τ of d_q such that the relation $d_q^\tau(z_\tau) = \phi(\beta(\tau)),\ \tau \in [0, \mathfrak{a}]$, holds and $d_q^0 = d_q$. For general X^2 instead of translations one can use a continuous family of compactly defined holomorphic maps close to the identity on X^2. Renormalize the discs d_q^τ so that the centers become $\phi(\beta(\tau)))$ and let $\hat{\gamma}_d(\tau)$ be the equivalence class represented by the renormalized disc $d^\tau \stackrel{def}{=} d_q^\tau \circ \varphi_{z_\tau}$.

For defining the second part of the curves we consider an arc $\gamma : [\mathfrak{a}, 3] \to \mathcal{V} \subset G \subset X^2$ which joins the point $\phi(\beta(\mathfrak{a})))$ with a point q_1 in the image $d_p(\mathbb{D}) = f_1 \circ \varphi_1(\mathrm{r}\mathbb{D})$ which is close to $\phi_1(2) = \phi(1,2) \in \kappa$. Define $\hat{\gamma}_\Phi$ on $[\mathfrak{a}, 3]$ to coincide with the lift $\hat{i} \circ \gamma$ of γ.

To define $\hat{\gamma}_d$ on $[\mathfrak{a}, 3]$ we consider again a continuous family of small perturbations of d_q such that for each τ the respective disc passes through $\gamma(\tau)$, for $\tau = \mathfrak{a}$ the disc coincides with the disc $d_q^{\mathfrak{a}}$ defined before and for $\tau = 3$ the disc equals d_p.

Reparametrize the discs so that the centers become $\gamma(\tau)$, and consider the equivalence classes represented by the reparametized discs. We obtain a curve $\hat{\gamma}_d | [\mathfrak{a}, 3]$ which is the second part of $\hat{\gamma}_d$. Note that $\hat{\gamma}_d(3)$ is represented by a reparametrization of d_p for which the center is the point q_1. With a suitable choice of $q_1 \in d_p(\mathbb{D})$ we may assume that the conditions of Lemma 7 are satisfied and, hence, $\hat{\gamma}_d(3)$ coincides with the class represented by small discs in G centered at $\gamma_d(3)$. Since the same is true for $\hat{\phi}(\gamma(3))$ the proof of Lemma 22 is completed. □

Lemma 13 and, hence, the theorem are proved. □

11. Proof of the corollaries

Proof of Corollary 1. By Theorem 1 and Lemma 1 for each point p in the envelope of holomorphy $\tilde{G}$ there exists an immersed analytic disc $\tilde{d} : \overline{\mathbb{D}} \to \tilde{G}$ such that $\tilde{d}(0) = p$ and $\tilde{d}(\partial\mathbb{D}) \subset i(G) \subset \tilde{G}$.

We may assume that $\tilde{d}$ extends to an analytic immersion of $(1+\varepsilon)\mathbb{D}$ for some positive number ε. The mapping can be uniformly approximated on $(1 + 1/2\varepsilon)\mathbb{D}$ by an immersion of the disc with only double self-intersection points and transversal self-intersection. This is a standard Morse-Sard type argument. The obtained disc can be considered as a nodal curve with boundary, i.e. as a singular Riemann surface with boundary all singularities of which are nodal singularities. By results of Ivashkovich and Shevchishin on the moduli space of Riemann surfaces (see [9], theorem 3.4 and lemma 3.8) the nodal curve is uniformly close to a smooth Riemann surface embedded into $\tilde{G}$. □

Remark. Theorem 1 implies the result of [13] that the natural homomorphism $\varphi : \pi_1(G) \to \pi_1(\tilde{G})$ induced by inclusion is surjective. Indeed, by the following argument any closed curve γ in $\tilde{G}$ is homotopic in $\tilde{G}$ to a curve in $i(G)$. Take an excrescence γ^* of γ, $\gamma^* : \partial\mathbb{D} \to \tilde{G}$, which lifts to a mapping $\overset{\circ}{\gamma}{}^* : \partial\mathbb{D} \to \mathcal{G}_0$. Note that γ^* is homotopic to γ. Let $\overset{\circ}{\gamma}{}^*(\zeta)(z)$, $\zeta \in \partial\mathbb{D}$, $z \in \overline{\mathbb{D}}$, be the evaluation mapping. The curve $\zeta \to \overset{\circ}{\gamma}{}^*(\zeta)(1)$ is homotopic in $\tilde{G}$ to $\gamma^*$and contained in $i(G)$.

Proof of Corollaries 2 and 3. Consider the following slightly more general situation which includes the case of each of the two corollaries. Let S be an orientable compact connected surface with or without boundary. Let $f : S \to \tilde{G}$ be a continuous mapping. If the boundary ∂S is not empty we will assume that $f(\partial S) \subset i(G)$. In case of a closed surface S we think about $f : S \to \tilde{G}$ representing a homology class in $H_2(\tilde{G})$. The case when $S = b^2$ is a disc corresponds to the homotopy of the loop representing an element in the kernel of the homomorphism φ in Corollary 3. We may always deform the surface so that $f(S)$ contains the point p. Say $p = f(\zeta^*)$.

Since $\tilde{G} = \hat{G}$ and locally each mapping into $\hat{G}$ lifts to a mapping into $\mathcal{G}_0$ we may consider a simplicial decomposition of S which is fine enough so that the following properties hold:

(1) On each 2-simplex σ_j of the decomposition there is a continuous lift $\overset{\circ}{f}_j : \sigma_j \to \mathcal{G}_0$ of $f_j \overset{def}{=} f \mid \sigma_j$ to $\mathcal{G}_0$.
(2) Consider an arbitrary edge e_k of the simplicial complex. Let σ_i and σ_j be the adjacent 2-simplices. For $\zeta \in e_k$ we denote by $(\overset{\circ}{f}_i(\zeta), \overset{\circ}{f}_j(\zeta))$ the equivalent discs corresponding to the two simplices by property (1). We require that there is a family of dendrites $\mathbf{T}_{i,j}(\zeta)$ with punctured halo associated to the family $(\overset{\circ}{f}_i(\zeta), \overset{\circ}{f}_j(\zeta))$ of pairs of discs by lemma 5, depending continuously on the point ζ and such that the underlying trees of the dendrites are homeomorphic.
(3) The point ζ^* is a vertex of the simplicial decomposition.

We will use now properties (1) and (2) to obtain a homotopy of the mapping f to a new mapping $f^1 : S \to \tilde{G}$ with the following property. There is a tree $\mathfrak{T} \subset S$ such that $f^1|S \setminus \mathfrak{T}$ lifts to $\mathcal{G}_0$. Moreover, the lifted mapping extends continuously to the pellicle of $\mathfrak{T}$ (the latter defined in the above sense assuming a simply connected neighbourhood of $\mathfrak{T}$ in S being extended to a sphere).

To find a suitable tree $\mathfrak{T}$ we will color each 1-simplex either white or black in such a way that the union of black simplices constitutes a (connected) tree wich contains each of the vertices of the triangulation. The coloring is done as follows. Since S is connected the union of all 1-simplices (edges) of the triangulation is connected. If the boundary ∂S is not empty then all edges contained in it are colored white. Since for each 2-simplex no more than one adjacent edge is contained in ∂S the union of uncolored edges is connected and contains all vertices of the triangulation. If the union of uncolored edges contains a closed loop we give white color to one of the edges constituting the loop. The union of uncolored edges still constitutes a connected set and contains all vertices. After finitely many steps the union of uncolored edges is a connected set without closed loops containing all vertices. Color the so far uncolored edges black. We obtained a coloring with the desired properties. Denote the tree constituted by the union of all black edges by $\mathfrak{T}'$.

Consider the barycentric subdivision of the simplicial complex. Associate to each edge e_k of the original complex the union $\tilde{\sigma}_k$ of those four 2-simplices of the subdivision that contain a "half" of e_k. The $\tilde{\sigma}_k$ have pairwise disjoint interior and cover S.

Let e_k be a white edge. We describe now a homotopy of the restriction $f|\tilde{\sigma}_k$ to a mapping $f^1|\tilde{\sigma}_k$ which fixes the values at the boundary of $\tilde{\sigma}_k$. Let σ_i and σ_j be the 2-simplices of the original simplicial complex that are adjacent to e_k and let $\mathbf{T}_{i,j}(\zeta)$, $\zeta \in e_k$, be the dendrites associated to e_k according to property 2. Let further $m_{i,j}(t,\zeta)$, $t \in [0,1], \zeta \in e_k$, be a parametrization of the pellicles of the trees $T_{i,j}(\zeta)$ depending continuously on ζ.

Cut $\tilde{\sigma}_k$ along e_k and glue back the union $\bigcup_{t\in[0,1],\zeta\in e_k} m_{i,j}(t,\zeta)$ with the natural gluing homeomorphism on the two sides of e_k (the point $m_{i,j}(0,\zeta)$ (respectively, the point $m_{i,j}(1,\zeta)$) is identified with the point on the side of σ_i (respectively, σ_j) over $\zeta \in e_k$). We obtain a (singular) closed square σ_k^*. The mapping $f|\tilde{\sigma}_k$ extends to a continuous mapping on $\tilde{\sigma}_k \cup \bigcup_{t\in[0,1],\zeta\in e_k} T_{i,j}(\zeta)$, moreover, it extends to a continuous mapping f_k^* on σ_k^* which lifts to $\mathcal{G}_0$. Moreover, reparametrize σ_k^* in the following way. Consider disjoint trees $T_0 = T_0^k$ and $T_1 = T_1^k$, both homeomorphic to the underlying tree of the dendrites $\mathbf{T}_{i,j}(\zeta)$, having their root respectively at the endpoints ζ_0 and ζ_1 of the edge e_k, being contained in $\tilde{\sigma}_k$ and each meeting the boundary of $\tilde{\sigma}_k$ exactly at its root.

Let φ be a homeomorphism of the set $\tilde{\sigma}_k \setminus (T_0 \cup T_1)$ onto $\sigma_k^* \setminus (T_{i,j}(\zeta_0) \cup T_{i,j}(\zeta_1))$ which is the identity on the boundary $\partial\tilde{\sigma}_k$. Require, moreover, that φ extends continuously to the pellicle of T_0 (T_1, respectively) and maps it homeomorphically onto the pellicle of $T_{i,j}(\zeta_0)$ ($T_{i,j}(\zeta_1)$, respectively). Put $f^1|\tilde{\sigma}_k \setminus (T_0 \cup T_1) \overset{def}{=} f_k^* \circ \varphi|\tilde{\sigma}_k \setminus (T_0 \cup T_1)$. This mapping extends to a continuous mapping on $\tilde{\sigma}_k$, also denoted by f^1. Since each rooted tree is contractible to its root and the construction an be made for subtrees and so that it depends countinuously on the choice of subtrees, the mappings $f|\tilde{\sigma}_k$ and $f^1|\tilde{\sigma}_k$ are homotopic.

As required, the restriction $f^1|\tilde{\sigma}_k \setminus (T_0 \cup T_1)$ lifts to $\mathcal{G}_0$. The lift extends continuously to the punctured pellicle of T_0 and T_1. Attach the trees $T_0 = T_0^k$ and $T_1 = T_1^k$ to $\mathfrak{T}'$.

Proceed in the same way with each of the white edges. We obtain a new tree $\mathfrak{T} \subset S$ and a homotopy of f on the whole of S to a mapping f^1. The restriction $f^1|S \setminus \mathfrak{T}$ of the final mapping f^1 admits a lift $\overset{\circ}{f^1}$ to $\mathcal{G}_0$ which extends continuously to the pellicle of the tree $\mathfrak{T}$.

Approximate the mapping $f^1 : \mathfrak{T} \to \tilde{G}$ of the tree by a true analytic disc $f^2 : \overline{\Delta} \to \tilde{G}$. Here Δ denotes a small simply connected neighbourhood of $\mathfrak{T}$ on S which we endow with complex structure. Extend the mapping to a continuous mapping $f^2 : S \to \tilde{G}$ which equals f^1 outside a small neighbourhood of the closure $\overline{\Delta}$. If f^2 is close to f^1 on S then the two mappings are homotopic and $f^2|S \setminus \Delta$ lifts to a mapping $\overset{\circ}{f^2}|S \setminus \Delta \to \mathcal{G}_0$.

Note that the (images of the) circle fibers $\bigcup_{\zeta\in S\setminus\Delta} \overset{\circ}{f^2}(\zeta)(\partial\mathbb{D})$ are contained in G. Moreover, there is an open subset U_0 of S such that for $\zeta \in U_0$ the (image of the) whole disc fiber $\overset{\circ}{f^2}(\zeta)(\overline{\mathbb{D}})$ is contained in G. For each k the points in $\tilde{\sigma}_k$ which are close to a leaf of T_0^k or T_1^k belong to

U_0. If Δ is a sufficiently small neighbourhood of $\mathfrak{T}$ its boundary $\partial\Delta$ intersects U_0 since $\mathfrak{T}$ contains the trees T_0^k and T_1^k for each white edge e_k . Hence, the mapping $f^2|\Delta$ is a disc neuron and the restriction $\overset{\circ}{f}{}^2|\partial\Delta$ is its halo.

We may consider the lift $\overset{\circ}{f}{}^2$ of f^2 up to changing it on the set U_0. More precisely, consider lifts $\overset{\circ}{F}{}^2$ of f^2 on $S \setminus \Delta$ such that $\overset{\circ}{F}{}^2 = \overset{\circ}{f}{}^2$ outside U_0 and for all ζ in U_0 the property $\overset{\circ}{F}{}^2(\zeta)(\overline{\mathbb{D}}) \subset G$ holds. We call such lifts $\overset{\circ}{F}{}^2$ admissible changes of $\overset{\circ}{f}{}^2$.

Lemma 17 and 18 apply to $f^2|\overline{\Delta}$ and its halo (and the Stein manifold $\tilde{G}$). Lemma 18 provides an approximation (take, for instance, the mapping $\mathfrak{H}(\zeta,\cdot)$ in the notation of lemma 17) of $\overset{\circ}{f}{}^2(\zeta)$, $\zeta \in \partial\Delta \setminus U_0$, and (the proof of) Lemma 17 states that after an admissible change on U_0 we obtain a new lift $\overset{\circ}{f}{}^3$ on $\partial\Delta$ of the same mapping $f^2|\partial\Delta$ such that the Riemann-Hilbert boundary value problem is solvable: There exists a section $\partial\Delta \ni \zeta \to \overset{\circ}{f}{}^3(\zeta)(g(\zeta)) \in \bigcup_{\zeta\in\partial\Delta} \overset{\circ}{f}{}^3(\zeta)(\partial\mathbb{D})$ which coincides with the boundary values of an analytic disc in $\tilde{G}$. This disc is a $\mathcal{G}$-disc. Denote it by $F(\zeta)$, $\zeta \in \Delta$. The mappings $\overline{\Delta} \ni \zeta \to \overset{\circ}{f}{}^3(\zeta)(rg(\zeta)) \in \bigcup_{\zeta\in\partial\Delta} \overset{\circ}{f}{}^3(\zeta)(\overline{\mathbb{D}})$, $r \in [0,1]$, provide a homotopy of mappings into $\tilde{G}$ joining $f^2|\overline{\Delta}$ with $F|\overline{\Delta}$.

Extend $\overset{\circ}{f}{}^3$ to the whole set $S\setminus\Delta$ as a continuous lift of f^2 such that the extended mapping equals $\overset{\circ}{f}{}^2$ outside a neighbourhood of $\partial\Delta$. Denote the mapping again by $\overset{\circ}{f}{}^3$. After admissible changes of the mapping $\overset{\circ}{f}{}^3$ on U_0 it remains to find a section $S \setminus \Delta \ni \zeta \to \overset{\circ}{f}{}^3(g(\zeta)) \in \bigcup_{\zeta\in S\setminus\Delta} \overset{\circ}{f}{}^3(\zeta)(\partial\mathbb{D})$ extending the section found before on $\partial\Delta$. Since U_0 intersects $\tilde{\sigma}_k$ for each white edge e_k this is always possible. The new mapping F is now defined on $S\setminus\Delta$ by this section: $F(\zeta) = \overset{\circ}{f}{}^3(g(\zeta)), \zeta \in S \setminus \Delta$, and the homotopy is given by $\overset{\circ}{f}{}^3(rg(\zeta))$, $r \in [0,1]$.

Note that the disc Δ contains the point ζ^*. The construction can be made in such a way that F is close to f in a neighbourhood of ζ^*. A small perturbation of the surface $F : S \to \tilde{G}$ will pass through p.

Corollaries 2 and 3 are proved. □

Proof of Corollaries 4 and 5. The proof uses Corollaries 2 and 3. Let Ω be a strictly pseudoconvex domain in a Stein surface X^2, $\Omega = \{\rho < 0\}$ for a strictly plurisubharmonic function ρ defined in a neighbourhood of the closure $\overline{\Omega}$ of Ω. Let $G = \{0 < \rho < \varepsilon\}$ for a small positive number ε so that ρ does not have critical points in G. Then $\tilde{G} = \Omega_\varepsilon \overset{def}{=} \{\rho < \varepsilon\}$. Denote by $\mathfrak{I}$ a retraction of Ω_ε onto $\overline{\Omega}$.

Let $f : S \to \overline{\Omega}$ be a continuous mapping of an orientable connected compact surface. If the boundary ∂S is not empty we require that $f(\partial S) \subset \overline{\Omega}$. Consider f as a mapping into $\tilde{G} = \Omega_\varepsilon$. If ∂S is not empty we perturb the mapping slightly so that $f(\partial S) \subset G$. By the proof of the Corollaries 2 and 3 there is a homotopy of f (in Ω_ε) to a mapping $F_1 : S \to \Omega_\varepsilon$ and a disc $\Delta \subset S$ such that $F_1|\overline{\Delta}$ is an analytic disc and $F_1(S \setminus \Delta)$ is contained in G. We may assume that Δ is not empty. After a small perturbation of F_1 the analytic disc $F_1(\Delta)$ has no self-intersection points on $\partial\Omega$ and intersects $\partial\Omega$ transversally. Let Δ_1 be the subset of Δ that is mapped into Ω: $\Delta_1 \overset{def}{=} \{\zeta \in \Delta : F_1(\zeta) \in \Omega\}$. By the maximum principle for the function ρ the set Δ_1 is the union of simply connected planar domains. If Δ_1 is connected then $\mathfrak{I} \circ F_1$ is the desired mapping.

If Δ_1 is not connected, let $\delta_1, ..., \delta_N$ be its connected components. There are pairwise disjoint arcs $\gamma_1, ...\gamma_{N-1}$ on Δ without self-intersections such that γ_i joins a point in $\partial\delta_i$ with a point in $\partial\delta_{i+1}$ and does not meet the union of the $\overline{\delta}_i$ otherwise. After a further (small) homotopy of the mapping $F_1|\Delta \setminus \bigcup \overline{\delta}_i$ inside $\Omega_\varepsilon \setminus G$ which fixes the mapping on the union of the boundaries $\bigcup \partial\delta_i$ we may assume that the arcs $F_1(\gamma_i)$ are contained in $\partial\Omega$, are pairwise disjoint without self-intersection points and meet the union of the $F_1(\partial\delta_i)$ exactly at the endpoints of the arcs. After approximating the arcs and the mapping F_1 we may assume that the arcs are Legendrian arcs in $\partial\Omega$. (It is well-known in contact geometry that arbitrary curves in contact manifolds may be C^0 approximated by Legendrian curves, for an elementary proof see, e.g. [7]). We arrived at the union of analytic discs with Legendrian arcs $F_1 : \cup\overline{\delta}_i \bigcup \cup\gamma_i \to \overline{\Omega}$.

Lemma 23. *Let $E \subset \mathbb{C}$ be a connected compact simply connected set consisting of the union of pairwise disjoint closed discs and pairwise disjoint arcs meeting the discs at most at their endpoints. Let Ω be a relatively compact strictly pseudoconvex domain in a Stein surface X^2 and let $f : E \to \overline{\Omega}$ be a continuous mapping for which the restriction to each closed disc in E is an analytic disc with boundary in $\partial\Omega$ and each of the arcs is a Legendrian arc in $\partial\Omega$.*

Then the mapping can be approximated by a true analytic disc $F : \Delta \to \overline{\Omega}$ with boundary in $\partial\Omega$. Here Δ is a simply connected planar domain with $E \subset \overline{\Delta}$ and Δ is contained in a small neighbourhood of E. Moreover, if z is the tip of an arc in E (not contained in the boundary of any of the closed discs in E) then Δ can be chosen so that $z \in \partial\Delta$ and $F(z) = f(z)$.

The Lemma seems to be folklore but we have no direct reference. After the proof of the Corollaries we will sketch the proof.

The lemma allows to find a homotopy of F_1 to a mapping $F_2 : S \to \Omega_\varepsilon$ such that for a simply connected domain $\Delta_2 \subset \Delta$ the restriction $F_2|\Delta_2$ is an analytic disc with boundary in $\partial\Omega$ and the set $F(S \setminus \Delta_2)$ is contained in $\Omega_\varepsilon \setminus \Omega$. Composing F_2 with the retraction $\mathfrak{I}$ finishes the proof. $\square$

It remains to sketch the proof of Lemma 23. Notice that the Lemma was used in the example in section 1 and also implies the following fact. The boundary of the disc of Corollary 5 which represents an element of the fundamental group of $\partial\Omega$ can be chosen to pass through a given base point $p \in \partial\Omega$.

Sketch of the proof of Lemma 23. Notice that after approximating we may assume that for each analytic disc $f(\overline{\delta}_j)$ contained in $f(E)$ the mapping f extends to an analytic immersion of a larger disc $\delta' \supset \overline{\delta}$ to a neighbourhood of $\overline{\Omega}$ in X^2 (keeping the condition $f(\partial\delta) \subset \partial\Omega$). Consider a small connected neighbourhood V of $f(E \setminus Int\, E)$. (The set $f(E \setminus Int\, E)$ is the union of the boundaries of the analytic discs contained in $f(E)$ and the Legendrian arcs. Notice that $f(E \setminus Int\, E) \subset \partial\Omega$.) With each of the analytic discs $f_i : \delta_i' \to X^2$ we associate (as in section 3) a Riemann domain $\mathcal{R}_i$ over X^2 (biholomorphic to $\delta_i' \times \varepsilon_i\mathbb{D}$ for some $\varepsilon_i > 0$) to which the disc lifts as an embedded disc. Consider the disjoint union of the Riemann domains $\mathcal{R}_i$ and glue each $\mathcal{R}_i$ in a natural way to V along a neighbourhood of the respective circle $f(\partial\delta_i)$. Shrinking the Riemann domains and the domain V suitably we obtain a (strictly) pseudoconvex Riemann domain $\mathcal{R}$ over X^2 which is diffeomorphic to a ball (see [21] where the method of gluing tubular neighbourhoods of arcs to strictly pseudoconvex domains to obtain strictly pseudoconvex domains appeared first).

Denote by M the lift of $V \cap \partial\Omega$ to $\mathcal{R}$. M is a relatively closed hypersurface in $\mathcal{R}$ which is strictly pseudoconvex from one side. The lifts to $\mathcal{R}$ of the analytic discs contained in $f(E)$ extend to embedded relatively closed analytic discs in $\mathcal{R}$, denoted by $F_i(\overline{\mathbb{D}})$. Denote the lifts of the arcs in $f(E)$ by γ_i. The γ_i are Legendrian arcs in M. To each γ_i we associate a chain of small analytic discs $g_k : \overline{\mathbb{D}} \to \mathcal{R},\ k = 1, ..., N$, so that $g_k(\partial\mathbb{D}) \subset M$, $g_1(-1)$ is an endpoint of γ_i, $g_k(1) = g_{k+1}(-1),\ k = 1, ..., N-1$, and $g_N(1)$ is the other endpoint of γ_i. The discs may be taken to be intersections with the pseudoconvex side of M of complex lines in suitable coordinates. By further shrinking the Riemann domain we assume that these discs extend to relatively closed embedded analytic discs in $\mathcal{R}$ which meet transversally and do not meet the $F_i(\overline{\mathbb{D}})$ except at $g_1(-1)$ and possibly $g_N(1)$. We may assume that the latter intersections are also transversal. We obtained a finite collection of relatively closed discs in $\mathcal{R}$. Since $\mathcal{R}$ is diffeomorphic to a ball, each disc is the zero set $\{\mathcal{F}_i = 0\}$ of an analytic function $\mathcal{F}_i$ on $\mathcal{R}$. For a generic choice of a small number η the set $X_\eta \overset{def}{=} \{\prod \mathcal{F}_i = \eta\} \cap \overline{\Omega}$ is an analytic disc (see, e.g. [16], Lemma 3.7). If γ_i is an arc with the second endpoint not contained in the boundary of any of the analytic discs $F_i(\partial\mathbb{D})$ we may adjust the choice of the last small disc g_N and the number η so that the boundary of the disc X_η passes through the endpoint of γ_i.

The lemma is proved. $\square$

References

[1] F. Docquier, H. Grauert, *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Math.Ann. **140** (1960), 94-123.

[2] J.-E. Fornaess, W.R. Zame, *Riemann domains and envelopes of holomorphy*, Duke Math.J. **50** (1983), 273-283.

[3] Y. Eliashberg, M. Gromov, *Embeddings of Stein manifolds of dimension n into the affine space of dimension* $3n/2+1$, Ann. of Math. textbf136 (1992), 123-135.

[4] F. Fortstneric, *Holomorphic flexibility properties of complex manifolds*, Amer. J. Math. **128** (2006), 239-270.

[5] F. Forstneric, J. Globevnik, *Discs in pseudoconvex domains*, Comment.Math.Helvetici **67** (1992), 129-145.

[6] K. Fritsche, H. Grauert, *From holomorphic functions to complex manifolds*, Graduate Texts in Mathematics,**213**, Springer, New York, Berlin, Heidelberg, 2002.

[7] H. Geiges, *Contact Geometry*, Handbook of differential geometry. Vol. **II**, 315-382, Elsevier/North-Holland, Amsterdam, 2006.

[8] H. Grauert, *Charakterisierung der holomorph vollständigen komplexen Räume*, Math.Ann. **129** (1955), 233-259.

[9] S.M. Ivashkovich, V.V. Shevchishin, *Deformations of non-compact complex curves and envelopes of meromorphy of spheres*, Sbornik:Mathematics **189:9** (1998), 1295-1333.

[10] L. Hörmander, *An Introduction to Complec Analysis in Several Variables*, Third Edition (revised), North Holland Math. Library, Amsterdam, New York, Oxford, Tokio, (1990).

[11] M. Jurchescu, *On a theorem of Stoilow*, Math. Ann., **138** (1959), 332-334.

[12] Sh. Kaliman, M. Zaidenberg *A transversality theorem for holomorphic mappings and stability of Eisenman-Kobayashi measures*, Trans.Amer.Math.Soc. **348**, no. 2,(1996), 1-12.

[13] H. Kerner, *Überlagerungen und Holomorphiehüllen*, Math.Ann. **144** (1961), 126-134.

[14] B. Malgrange, *Lectures on the theory of functions of several complex variables*, Tata Institute of fundamenbtal research, Bombay (1958) (Reissued 1965).

[15] K. Oka, *Sur les fonctions analytiques de plusieurs variables. IX. - Domains finis sans point critique interieur*, Japanese J. Math. **23** (1953), 87-155.

[16] S. Orevkov, *An algebraic curve in the unit ball in* $\mathbf{C}^2$ *that passes through the origin and all of whose boundary components are arbitrarily short* (Russian), Tr. Mat. Inst. Steklova **253** (2006), 135-157.

[17] B. Ozbagci, A. Stipsicz, *Surgery on contact 3-manifolds and Stein surfaces*, Bolyai Society Mathematical Society and Springer (2004).

[18] H. Rossi, *On Envelopes of Holomorphy*, Comm. Pur Appl. Math. **XVI** (1963), 9 -17.

[19] W. Rudin, *Real and complex analysis*, Third Edition, McGraw-Hill Book Company, New York etc (1987).

[20] H.L. Royden, *One-dimensional cohomology in domains of holomorphy*, Ann. Math. **78** (1963), 197-200.

[21] N. Shcherbina, *Decomposition of a common boundary of two domains of holomorphy into analytic curves* (Russian), Izv. Akad. Nauk SSSR Ser. Mat.,**46** (1982), 1106-1123, 1136.

[22] E.L. Stout, *A domain whose envelope of holomorphy is not a domain*, Ann. Polon. Math. **89** (2006), 197-201.

Max-Planck-Institut für Mathematik, P.O.Box: 7280, 53072 Bonn, Germany

E-mail address: `joericke@mpim-bonn.mpg.de`