Thom polynomials and Schur functions: the singularities $I_{2,2}(-)$

Piotr Pragacz*
Institute of Mathematics of Polish Academy of Sciences
Śniadeckich 8, 00-956 Warszawa, Poland
P.Pragacz@impan.gov.pl

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To the memory of Professor Stanisław Balcerzyk (1932-2005)

Abstract

We give the Thom polynomials for the singularities $I_{2,2}$ (in Mather's notation) associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ with parameter $k \geq 0$. Our computations combine the characterization of Thom polynomials via the "method of restriction equations" of Rimanyi et al. with the techniques of (super) Schur functions.

1 Introduction

The global behavior of singularities is governed by their *Thom polynomials* (cf. [31], [12], [1], [11], [28]). Knowing the Thom polynomial of a singularity η, denoted $\mathcal{T}^\eta$, one can compute the cohomology class represented by the η-points of a map. We do not attempt here to survey all activities related to computations of Thom polynomials which are difficult tasks in general.

In the present paper, following a series of papers by Rimanyi et al. [29], [28], [7], [2], we study the Thom polynomials for the singularities $I_{2,2}$ of the maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ with parameter $k \geq 0$.

The way of obtaining the thought Thom polynomial is through the solution of a system of linear equations, which is fine when we want to find one concrete Thom polynomial, say, for a fixed k. However, if we want to find the Thom polynomials for a series of singularities, associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ with k as a parameter, we have to solve *simultaneously*

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a countable family of systems of linear equations. As stated by Rimanyi in [28], p. 512 :

"However, another challenge is to find Thom polynomials containing k as a parameter."

We do it here for the restriction equations for the singularities $I_{2,2}$ and any k. In fact, the obtained functional equations in symmetric functions are of independent interest. The main novelty of the present paper over the previous articles on Thom polynomials, is an extensive use of *Schur functions*. Namely, instead of using *Chern monomial expansions* (as the authors of all previous papers constantly do), we use *Schur function expansions*. This puts a more transparent structure on computations of Thom polynomials. We hope that the expression for $\mathcal{T}^{I_{2,2}}$ given in Theorem 19), provides a strong support of this claim. In particular, we get in this way some recursive formulas (cf., e.g., Lemma 12) that are not so easy to find using other bases, for instance, the Chern monomial basis. In fact, different recursions play a prominent role in the present paper – apart from Lemma 12 see Eq. (57).

Another feature of using the Schur function expansions for Thom polynomials is that in all known to us cases, all the coefficients are nonnegative. In fact, we state the following:

Conjecture: *The coefficients of the Schur function expansion of a Thom polynomial are nonnegative.*[1]

To be more precise, we use here (the specializations of) *supersymmetric* Schur functions, also called *super-S-functions* or Schur functions *in difference of alphabets* together with their three basic properties: *vanishing*, *cancellation* and *factorization*, (cf. [3], [16], [22], [26], [17], [9], and [14]). These functions contain *resultants* among themselves. Their geometric significance was illuminated in the 80's in the author's study of *polynomials supported on degeneracy loci* (cf. [21]).

In the main body of the present paper, we give the Thom polynomials for the singularities $I_{2,2}$ (in Mather's notation) associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ with parameter $k \geq 0$. We do it via establishing the Schur function expansions for these Thom polynomials. We prove first in Lemma 11 that partitions indexing the Schur polynomials involved have not more than 3 parts. Then, in Lemma 12, we establish a recursive relation for Thom polynomials associated with successive values of the parameter k. This reduces the calculation to compute the (sub)sum indexed by partitions with precisely 2 parts. This is essentially done in Proposition 14 (see also Propositions 16, 17, 18).

[1]Note added in May 2006: this conjecture has been recently proved by A. Weber and the author in [27].

Our main result (Theorem 19 combined with Propositions 17, 18) gives an explicit presentation of the Thom polynomial for the singularities $I_{2,2}$ with parameter $k \geq 0$ as a $\mathbf{Z}$-combination of Schur functions. We give *closed algebraic* expressions for the coefficients of these expansions. "A bit" surprisingly, these coefficients are the same as the coefficients of the Schur function expansions of the Segre classes of the second symmetric power of a rank 2 vector bundle, computed in [30], [21], [13], and [23].

Our main result offers a generalization (to any $k \geq 0$) of the formulas obtained previously by Porteous [19] and Rimanyi [28] for $k = 0$ and $k = 1$, respectively.

In our calculations we use extensively the functorial λ-ring approach to symmetric functions developed mainly in Lascoux's book [14].

Main results of the present paper were announced in [24].

In the forthcoming article [25], the Schur function expansions of Thom polynomials for the singularities A_i are given. First, we give the Thom polynomials for the singularities A_i (any i) associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, with parameter $k \geq 0$. This is done under the additional assumption that $\Sigma^j = \emptyset$ for all $j \geq 2$. Second, we give the Thom polynomials – via their Schur function expansions – for the singularities A_3 (with parameter $k \geq 0$), this time with no additional assumptions on the degeneracy of the Σ^j's. Inspired by the present article, [24] and [25], Ozer Ozturk [18] computed the Thom polynomials for A_4 and $k = 2, 3$.

2 Recollections on Thom polynomials

Our main reference for this section is [28]. We start with recalling what we shall mean by a "singularity". Let $k \geq 0$ be a fixed integer. By *singularity* we shall mean an equivalence class of stable germs $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, where $\bullet \in \mathbf{N}$, under the equivalence generated by right-left equivalence (i.e. analytic reparametrizations of the source and target) and suspension (by suspension of a germ κ we mean its trivial unfolding: $(x, v) \mapsto (\kappa(x), v)$).

We recall[2] that the *Thom polynomial* $\mathcal{T}^\eta$ of a singularity η is a polynomial in the formal variables $c_1, c_2, \ldots$ that after the substitution

$$c_i = c_i(f^*TY - TX) = [c(f^*TY)/c(TX)]_i\,, \tag{1}$$

for a general map $f : X \to Y$ between complex analytic manifolds, evaluates the Poincaré dual of $[V^\eta(f)]$, where $V^\eta(f)$ is the cycle carried by the closure of the set

$$\{x \in X : \text{the singularity of } f \text{ at } x \text{ is } \eta\}\,. \tag{2}$$

[2]This statement is usually called the Thom-Damon theorem [31], [4].

By *codimension of a singularity* η, $\operatorname{codim}(\eta)$, we shall mean $\operatorname{codim}_X(V^\eta(f))$ for such an f. The concept of the polynomial $\mathcal{T}^\eta$ comes from Thom's fundamental paper [31]. For a detailed discussion of the *existence* of Thom polynomials, see, e.g., [1]. Thom polynomials associated with group actions were studied in [11].

According to Mather's classification, singularities are in one-to-one correspondence with finite dimensional $\mathbf{C}$-algebras. We shall use the following notation:

– A_i (of Thom-Boardman type Σ^{1_i}) will stand for the stable germs with local algebra $\mathbf{C}[[x]]/(x^{i+1})$, $i \geq 0$;

– $I_{a,b}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbf{C}[[x,y]]/(xy, x^a + y^b)$, $b \geq a \geq 2$;

– $III_{a,b}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbf{C}[[x,y]]/(xy, x^a, y^b)$, $b \geq a \geq 2$ (here $k \geq 1$).

Our computations of Thom polynomials for some of the above singularities, shall use the method which stems from a sequence of papers by Rimanyi et al. [29], [28], [7], [2]. We sketch briefly this method, refering the interested reader for more details to these papers, the main references being the last three mentioned items.

Let $k \geq 0$ be a fixed integer, and let $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ be a stable singularity with a prototype $\kappa : (\mathbf{C}^n, 0) \to (\mathbf{C}^{n+k}, 0)$. The *maximal compact subgroup of the right-left symmetry group*

$$\operatorname{Aut}\kappa = \{(\varphi, \psi) \in \operatorname{Diff}(\mathbf{C}^n, 0) \times \operatorname{Diff}(\mathbf{C}^{n+k}, 0) : \psi \circ \kappa \circ \varphi^{-1} = \kappa\} \tag{3}$$

of κ will be denoted by G_η. Even if $\operatorname{Aut}\kappa$ is much too large to be a finite dimensional Lie group, the concept of its maximal compact subgroup (up to conjugacy) can be defined in a sensible way (cf. [10] and [32]). In fact, G_η can be chosen so that images of its projections to the factors $\operatorname{Diff}(\mathbf{C}^n, 0)$ and $\operatorname{Diff}(\mathbf{C}^{n+k}, 0)$ are linear. Its representations via the projections on the source $\mathbf{C}^n$ and the target $\mathbf{C}^{n+k}$ will be denoted by $\lambda_1(\eta)$ and $\lambda_2(\eta)$. The vector bundles associated with the universal principal G_η-bundle $EG_\eta \to BG_\eta$ using the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$ will be called E'_η and E_η. The *total Chern class of the singularity* η is defined in $H^\bullet(BG_\eta; \mathbf{Z})$ by

$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)}\,. \tag{4}$$

The *Euler class* of η is defined in $H^{2\operatorname{codim}(\eta)}(BG_\eta; \mathbf{Z})$ by

$$e(\eta) := e(E'_\eta)\,. \tag{5}$$

In the following theorem we collect the information from [28], Theorem 2.4 and [7], Theorem 3.5, needed for the calculations in the present paper.

Theorem 1 *Suppose, for a singularity η, that the Euler classes of all singularities of smaller codimension than* $\operatorname{codim}(\eta)$*, are not zero-divisors* [3]*. Then we have*
(i) if $\xi \neq \eta$ *and* $\operatorname{codim}(\xi) \leq \operatorname{codim}(\eta)$*, then* $\mathcal{T}^\eta(c(\xi)) = 0$*;*
(ii) $\mathcal{T}^\eta(c(\eta)) = e(\eta)$*.*
This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^\eta$ in a unique way.

To use this method of determining the Thom polynomials for singularities, one needs their classification, see, e.g., [5].

In fact, the above is the "usual case" with singularities in the region where moduli (continuous families) of singularities do not occur. This will be the case of the singularities studied in the present paper. Indeed, the codimension of all these singularities does not exceed $6k + 8$, the lowest codimension when moduli of singularities start.

In Section 4, we shall use these equations to compute Thom polynomials. Sometimes it will be convenient not to work with the whole maximal compact subgroup G_η but with its suitable subgroup; this subgroup should be, however, as "close" to G_η as possible (cf. [28], p. 502). We shall denote this subgroup by the same symbol G_η.

Being challenged by [28], p. 512 and especially [2], we shall find Thom polynomials *containing k as a parameter* – this seems to be a (much) more difficult task than computing Thom polynomials for separate values of k, because one must solve *simultaneously* a countable family of systems of linear equations.

To effectively use Theorem 1 we need to study the maximal compact subgroups of singularities. We recall the following recipe from [28] pp. 505–507. Let η be a singularity whose prototype is $\kappa : (\mathbf{C}^n, 0) \to (\mathbf{C}^{n+k}, 0)$. The germ κ is the miniversal unfolding of another germ $\beta : (\mathbf{C}^m, 0) \to (\mathbf{C}^{m+k}, 0)$ with $d\beta = 0$. The group G_η is a subgroup of the maximal compact subgroup of the algebraic automorphism group of the local algebra Q_η of η times the unitary group $U(k-d)$, where d is the difference between the minimal number of relations and the number of generators of Q_η. With β well chosen, G_η acts as right-left symmetry group on β with representations μ_1 and μ_2. The representations λ_1 and λ_2 are

$$\lambda_1 = \mu_1 \oplus \mu_V \quad \text{and} \quad \lambda_2 = \mu_2 \oplus \mu_V \,, \tag{6}$$

where μ_V is the representation of G_η on the unfolding space $V = \mathbf{C}^{n-m}$ given, for $\alpha \in V$ and $(\varphi, \psi) \in G_\eta$, by

$$(\varphi, \psi)\ \alpha = \psi \circ \alpha \circ \varphi^{-1} \,. \tag{7}$$

[3] This is the so-called "Euler condition" (*loc.cit.*). The Euler condition holds true for the singularities $I_{2,2}$, for any $k \geq 0$.

For example, for the singularity of type A_i: $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, we have $G_{A_i} = U(1) \times U(k)$ with

$$\mu_1 = \rho_1, \quad \mu_2 = \rho_1^{i+1} \oplus \rho_k, \quad \mu_V = \oplus_{j=2}^{i}\, \rho_1^j \oplus \oplus_{j=1}^{i}(\rho_k \otimes \rho_1^{-1})\,, \tag{8}$$

where ρ_j denotes the standard representation of the unitary group $U(j)$. Hence we obtain assertion (i) of the following

Proposition 2 *(i) Let $\eta = A_i$; for any k, writing x and $y_1, \ldots, y_k$ for the Chern roots of the universal bundles on $BU(1)$ and $BU(k)$,*

$$c(A_i) = \frac{1 + (i+1)x}{1+x} \prod_{j=1}^{k} (1 + y_j)\,, \tag{9}$$

(ii) Let $\eta = I_{2,2}$. Denote by H the extension of $U(1) \times U(1)$ by $\mathbf{Z}/2\mathbf{Z}$ ("the group generated by multiplication on the coordinates and their exchange"). For $k \geq 0$, $G_\eta = H \times U(k)$. Hence, for the purpose of our computations we can use $G_\eta = U(1) \times U(1) \times U(k)$. Writing x_1, x_2 and $y_1, \ldots, y_k$ for the Chern roots of the universal bundles on two copies of $BU(1)$ and on $BU(k)$, we have

$$c(I_{2,2}) = \frac{(1+2x_1)(1+2x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k} (1 + y_j)\,, \tag{10}$$

$$e(I_{2,2}) = x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) \prod_{j=1}^{k} (x_1 - y_j)(x_2 - y_j)(x_1 + x_2 - y_j)\,. \tag{11}$$

(iii) Let $\eta = III_{2,2}$; for $k \geq 1$, $G_\eta = U(2) \times U(k-1)$, and writing x_1, x_2 and $y_1, \ldots, y_{k-1}$ for the Chern roots of the universal bundles on $BU(2)$ and $BU(k-1)$,

$$c(III_{2,2}) = \frac{(1+2x_1)(1+2x_2)(1+x_1+x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k-1} (1 + y_j)\,, \tag{12}$$

These assertions are obtained, in a standard way, following the instructions of [28], Sect. 4. See also [2].

Let η be a singularity. As it was illuminated in the author's paper [21], in the case of the singularities $\eta = \Sigma^i$, it is natural and useful to consider a certain (homogeneous) ideal in the polynomial ring $R = \mathbf{Z}[c_1, c_2, \ldots]$ whose component of minimal degree is generated by $\mathcal{T}^\eta$. Namely, we denote by $\mathcal{P}^\eta$ the *ideal in R of polynomials supported on $V^\eta(f)$*, where $f : X \to Y$ is a general map between complex analytic manifolds. (The notion of a "polynomial supported on a subscheme" can be found in [9], Appendix A.)

Keeping track of [21], we shall call $\mathcal{P}^\eta$ the *$\mathcal{P}$–ideal of the singularity* η. For example, the $\mathcal{P}$–ideal of the singularity

$$\Sigma^i : (\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$$

is

$$\mathcal{P}^{\Sigma^i} = \mathcal{P}_{m-i} \,,$$

where on the RHS we have the ideal studied extensively in [21] (cf. also [20], [22]). We shall use this ideal in the proof of Theorem 10.

In the present paper, it will be more handy to use, instead of k, a "shifted" parameter

$$r := k + 1 \,. \tag{13}$$

Sometimes, we shall write $\eta(r)$ for the singularity $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+r-1}, 0)$, and denote the Thom polynomial by $\mathcal{T}_r^\eta$ – to emphasize the dependence of both items on r.

Rather than the Chern classes

$$c_i(f^*TY - TX) = [f^*c(TY)/c(TX)]_i \,,$$

we shall use *Segre classes* S_i of the virtual bundle $TX^* - f^*(TY^*)$, i.e. complete symmetric functions $S_i(\mathbb{A}-\mathbb{B})$ for the alphabets of the *Chern roots* $\mathbb{A}, \mathbb{B}$ of TX^* and TY^*. The reader will find in the next section a summary of algebraic properties of the functions $S_i(\mathbb{A}-\mathbb{B})$, or, more generally, Schur functions $S_{(i_1,i_2,\ldots)}(\mathbb{A}-\mathbb{B})$, indexed by partitions, widely used in the present paper.

3 Recollections on Schur functions

In this section we collect needed notions related to symmetric functions. We adopt the functorial point of view of [14] for what concerns symmetric functions. Namely, given a commutative ring, we treat symmetric functions as operators acting on the ring. (Here, these commutative rings are mostly $\mathbf{Z}$-algebras generated by the Chern roots of the vector bundles from Proposition 2.)

Definition 3 *By an alphabet $\mathbb{A}$, we understand a (finite) multi-set of elements in a commutative ring.*

For $m \in \mathbf{N}$, by "an alphabet $\mathbb{A}_m$" we shall mean an alphabet $\mathbb{A} = (a_1, \ldots, a_m)$ (of cardinality m); ditto for $\mathbb{B}_n = (b_1, \ldots, b_n)$ and $\mathbb{X}_2 = (x_1, x_2)$.

Definition 4 *Given two alphabets $\mathbb{A}$, $\mathbb{B}$, the complete functions $S_i(\mathbb{A}-\mathbb{B})$ are defined by the generating series (with z an extra variable):*

$$\sum S_i(\mathbb{A}-\mathbb{B})z^i = \prod_{b\in\mathbb{B}}(1-bz)/\prod_{a\in\mathbb{A}}(1-az)\,. \tag{14}$$

So $S_i(\mathbb{A}-\mathbb{B})$ interpolates between $S_i(\mathbb{A})$ – the complete homogeneous symmetric function of degree i in $\mathbb{A}$ and $S_i(-\mathbb{B})$ – the ith elementary function in $\mathbb{B}$ times $(-1)^i$.

The notation $\mathbb{A}-\mathbb{B}$ is compatible with the multiplication of series:

$$\sum S_i(\mathbb{A}-\mathbb{B})z^i \cdot \sum S_j(\mathbb{A}'-\mathbb{B}')z^j = \sum S_i\big((\mathbb{A}+\mathbb{A}')-(\mathbb{B}+\mathbb{B}')\big)z^i\,, \tag{15}$$

the sum $\mathbb{A}+\mathbb{A}'$ denoting the union of two alphabets $\mathbb{A}$ and $\mathbb{A}'$.

Convention 5 *We shall often identify an alphabet $\mathbb{A}=\{a_1,\dots,a_m\}$ with the sum $a_1+\cdots+a_m$ and perform usual algebraic operations on such elements. For example, $\mathbb{A}b$ will denote the alphabet $(a_1b,\dots,a_mb)$. We will give priority to the algebraic notation over the set-theoretic one. In fact, in the following, we shall use mostly alphabets of variables.*

We have $(\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C})=\mathbb{A}-\mathbb{B}$, and this corresponds to simplification of the common factor for the rational series:

$$\sum S_i((\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C}))z^i = \sum S_i(\mathbb{A}-\mathbb{B})z^i\,. \tag{16}$$

Definition 6 *Given a partition $I=(0\le i_1\le i_2\le\ldots\le i_s)\in\mathbf{Z}^s$, and alphabets $\mathbb{A}$ and $\mathbb{B}$, the Schur function $S_I(\mathbb{A}-\mathbb{B})$ is*

$$S_I(\mathbb{A}-\mathbb{B}) := \Big|S_{i_p+p-q}(\mathbb{A}-\mathbb{B})\Big|_{1\le p,q\le s}\,. \tag{17}$$

These functions are often called *supersymmetric Schur functions* or *Schur functions in difference of alphabets*. Their properties were studied, among others, in [3], [16], [22], [26], [17], [9], and [14]; in the present paper, we shall use the notation and conventions from this last item).

For example,

$$S_{33344}(\mathbb{A}-\mathbb{B}) = \begin{vmatrix} S_3 & S_4 & S_5 & S_7 & S_8 \\ S_2 & S_3 & S_4 & S_6 & S_7 \\ S_1 & S_2 & S_3 & S_5 & S_6 \\ 1 & S_1 & S_2 & S_4 & S_5 \\ 0 & 1 & S_1 & S_3 & S_4 \end{vmatrix}\,,$$

where S_i means $S_i(\mathbb{A}-\mathbb{B})$.

By Eq. (16), we get the following *cancellation property*:

$$S_I((\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C})) = S_I(\mathbb{A}-\mathbb{B})\,. \tag{18}$$

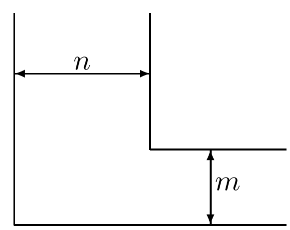
In the following, we shall identify partitions with their *Young diagrams*, as is customary (cf. [14]).

We record the following property (*loc.cit.*), justifying the notational remark from the end of Section 2; for a partition I,

$$S_I(\mathbb{A}{-}\mathbb{B}) = (-1)^{|I|} S_J(\mathbb{B}{-}\mathbb{A}) = S_J(\mathbb{B}^*{-}\mathbb{A}^*)\,, \tag{19}$$

where J is the conjugate partition of I (i.e. the consecutive rows of J are equal to the corresponding columns of I), and $\mathbb{A}^*$ denotes the alphabet $\{-a_1, -a_2, \ldots\}$.

Fix two positive integers m and n. We shall say that a partition $I = (0 < i_1 \le i_2 \le \cdots \le i_s)$ *is contained in* the (m,n)-hook if either $k \le m$, or $k > m$ and $i_{k-m} \le n$. Pictorially, this means that the Young diagram of I is contained in the "tickened" hook:



We record the following *vanishing property.* Given alphabets $\mathbb{A}$ and $\mathbb{B}$ of cardinalities m and n, if a partition I is not contained in the (m,n)-hook, then (*loc.cit.*):

$$S_I(\mathbb{A} - \mathbb{B}) = 0 \tag{20}$$

For example,

$$S_{3569}(\mathbb{A}_2 - \mathbb{B}_4) = S_{3569}(a_1{+}a_2{-}b_1{-}b_2{-}b_3{-}b_4) = 0$$

because 3569 is not contained in the $(2,4)$-hook.

In fact, we have the following result (*loc.cit.*).

Theorem 7 *If $\mathbb{A}_m$ and $\mathbb{B}_n$ are alphabets of variables, then the functions $S_I(\mathbb{A}_m - \mathbb{B}_n)$, for I runing over partitions contained in the (m,n)-hook, are* **Z***-linearly independent.*

(They form a **Z**-basis of the Abelian group of the so-called "supersymmetric functions" (*loc.cit.*).)

In the present paper, by a *symmetric function*, we shall mean a **Z**-linear combination of the operators $S_I(\bullet)$. In other words, speaking a bit informally, we treat Schur functions in a "functorial way" (cf. [14] for developments of the theory of symmetric functions in this spirit).

Definition 8 *Given two alphabets $\mathbb{A}, \mathbb{B}$, we define their resultant:*

$$R(\mathbb{A}, \mathbb{B}) := \prod_{a \in \mathbb{A},\, b \in \mathbb{B}} (a - b)\,. \tag{21}$$

This terminology is justified by the fact that $R(\mathbb{A}, \mathbb{B})$ is the classical resultant of the polynomials $R(x, \mathbb{A})$ and $R(x, \mathbb{B})$.

We have (*loc.cit.*)

$$R(\mathbb{A}_m, \mathbb{B}_n) = S_{(n^m)}(\mathbb{A} - \mathbb{B}) = \sum_I S_I(\mathbb{A}) S_{(n^m)/I}(-\mathbb{B})\,, \tag{22}$$

where the sum is over all partitions $I \subset (n^m)$.

When a partition is contained in the (m, n)-hook and at the same time it contains the rectangle (n^m), then we have the following *factorization property* (*loc.cit.*): for partitions $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_s)$,

$$S_{(j_1, \ldots, j_s, i_1 + n, \ldots, i_m + n)}(\mathbb{A}_m - \mathbb{B}_n) = S_I(\mathbb{A})\ R(\mathbb{A}, \mathbb{B})\ S_J(-\mathbb{B})\,. \tag{23}$$

The following convention stems from Lascoux's paper [15].

Convention 9 *We may need to specialize a letter to* 2*, but this must not be confused with taking two copies of* 1*. To allow one, nevertheless, specializing a letter to an (integer, or even complex) number r inside a symmetric function, without introducing intermediate variables, we write $\boxed{r}$ for this specialization. Boxes have to be treated as single variables. For example, $S_i(2) = \binom{i+1}{2}$ but $S_i(\boxed{2}) = 2^i$. A similar remark applies to* **Z***-linear combinations of variables. We have $S_2(\mathbb{X}_2) = x_1^2 + x_1x_2 + x_2^2$ but $S_2(\boxed{x_1 + x_2}) = x_1^2 + 2x_1x_2 + x_2^2$, $S_{11}(\mathbb{X}_2) = x_1x_2$ but $S_{11}(\boxed{x_1 + x_2}) = 0$, $S_2(3x) = 6x^2$ but $S_2(\boxed{3x}) = 9x^2$ etc.*

This convention will be used in the next section.

We end the present section with the following result which is a consequence of the author's study [21], [20], [22] of the $\mathcal{P}$-ideals of the singularities Σ^i.

Theorem 10 *Suppose that a singularity η is of Thom-Boardman type Σ^i. Then all summands in the Schur function expansion of $\mathcal{T}_r^\eta$ are indexed by partitions containing the rectangle partition $(r+i-1, \ldots, r+i-1)$ (i times).*

Proof. Since η is of Thom-Boardman type Σ^i, the Thom polynomial $\mathcal{T}_r^\eta$ belongs to the $\mathcal{P}$–ideal of the singularity Σ^i with parameter r. We also know by the Thom-Damon theorem (cf. [4]) that $\mathcal{T}_r^\eta$ is a **Z**-combination of Schur functions in $TX^* - f^*(TY^*)$. The assertion now follows by combining Theorem 3.4 from [21] with Lemma 2.5 from [20] (see also Claim in the proof of Theorem 5.3(i) in [22]). $\square$

4 Thom polynomial for $I_{2,2}(r)$

The codimension of $I_{2,2}(r)$, $r \geq 1$, is $3r+1$. The Thom polynomial for $I_{2,2}(1)$ is S_{22} (cf. [19]).

From now on, we shall assume that $r \geq 2$. The Thom polynomial for $I_{2,2}(2)$ is (cf. [28]):

$$S_{133} + 3S_{34}\,.$$

By virtue of Proposition 2, the equations from Theorem 1 characterizing the Thom polynomial for $I_{2,2}(r)$ are:

$$P(-\mathbb{B}_{r-1}) = P(x - \boxed{2x} - \mathbb{B}_{r-1}) = P(x - \boxed{3x} - \mathbb{B}_{r-1}) = 0\,, \tag{24}$$

and

$$P(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2} - \mathbb{B}_{r-1}) = x_1x_2(x_1 - 2x_2)(x_2 - 2x_1)\ R(\mathbb{X}_2 + \boxed{x_1 + x_2}, \mathbb{B}_{r-1})\,. \tag{25}$$

Here, without loss of generality, we assume that x, x_1, x_2, and $\mathbb{B}_{r-1}$ are variables. Moreover, $P(\bullet)$ denotes a symmetric function. For the remainder of this paper, we set

$$\mathbb{D} := \boxed{2x_1} + \boxed{2x_2} + \boxed{x_1 + x_2}\,. \tag{26}$$

Then, additionally, for variables x_1, x_2 and an alphabet $\mathbb{B}_{r-2}$, we have the vanishing imposed by $III_{2,2}$:

$$P(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0\,. \tag{27}$$

Indeed, the singularities $\neq I_{2,2}$ with codimension $\leq \mathrm{codim}(I_{2,2})$ are: A_0, A_1, A_2, $III_{2,2}$.

For $r \geq 1$, we set

$$\mathcal{T}_r(\bullet) := \mathcal{T}_r^{I_{2,2}}(\bullet)\,. \tag{28}$$

Lemma 11 *(i) A partition appearing nontrivially in the Schur function expansion of $\mathcal{T}_r$ contains the rectangular partition $(r+1, r+1)$.*

(ii) A partition appearing nontrivially in the Schur function expansion of $\mathcal{T}_r$ has at most three parts.

Proof. (i) Since the singularity $I_{2,2}$ is of Thom-Boardman type Σ^2, this is a particular case of Theorem 10.

(ii) We can assume that $r \geq 3$. In addition to information contained in (i), we shall use Eq. (27):

$$\mathcal{T}_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0\,.$$

By virtue of (i), we can use factorization property (23) to all summands of

$$\mathcal{T}_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = \sum_I \alpha_I S_I(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) \tag{29}$$

(we assume that $\alpha_I \neq 0$). We divide each summand of this last polynomial by the resultant

$$R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2}) .$$

Suppose that the resulting factor of S_I is:

$$S_{p,q}(\mathbb{X}_2)\ S_J(-\mathbb{D} - \mathbb{B}_{r-2}) , \tag{30}$$

cf. (23). Since $|I| = 3r + 1$, we have

$$|J| \leq r - 1 . \tag{31}$$

Now, let us assume that I has more than 3 parts, that is J has more than 2 parts. This assumption (together with the inequality (31)) implies that

$$S_J(-\mathbb{B}_{r-2}) \neq 0$$

($\mathbb{B}_{r-2}$ is an alphabet of variables). Expanding (30), we get among summands the following one of largest possible degree $|J|$ in $\mathbb{B}_{r-2}$:

$$S_{p,q}(\mathbb{X}_2)\ S_J(-\mathbb{B}_{r-2}) \neq 0 . \tag{32}$$

Take in the sum

$$\sum_I \alpha_I S_{p,q}(\mathbb{X}_2)\ S_J(-\mathbb{D} - \mathbb{B}_{r-2})$$

the (sub)sum of all the nonzero summands of the form (30) with the largest possible weight of J. Since Schur polynomials are independent this (sub)sum is nonzero and moreover it is **Z**-linearly independent of other summands both in the sum indexed by partitions with ≥ 3 parts, and as well as in that indexed by partitions with 2 parts (this last sum does not depend on $\mathbb{B}_{r-2}$). Hence, there is no **Z**-linear combination of S_I's which involve nontrivially I with more than three parts and possibly also those with 3 and 2 parts, that satisfies Eq. (27). Assertion (ii) has been proved. □

(For example, S_{1144} cannot appear in the Schur function expansion of $\mathcal{T}_3$ because $S_{1144}(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_1)$ after division by the resultant contains the summand $S_{11}(-\mathbb{B}_1) = S_2(\mathbb{B}_1)$, which does not occur in similar expressions for $S_{55}, S_{46}, S_{244}, S_{145}$.)

The following lemma gives a recursive description of $\mathcal{T}_r$. Denote by Φ the linear endomorphism on the **Z**-module spanned by Schur functions indexed by partitions of length ≤ 3, that sends a Schur function S_{i_1,i_2,i_3} to S_{i_1+1,i_2+1,i_3+1}. Let $\overline{\mathcal{T}_r}$ denote the sum of those terms in the Schur function expansion of $\mathcal{T}_r$ which are indexed by partitions of length ≤ 2. Note that $\overline{\mathcal{T}}_1 = S_{22}$.

Lemma 12 *With this notation, for $r \geq 2$, we have the following recursive equation:*

$$\mathcal{T}_r = \overline{\mathcal{T}}_r + \Phi(\mathcal{T}_{r-1}) . \tag{33}$$

Proof. Write

$$\mathcal{T}_r = \sum_I \alpha_I S_I = \sum_J \alpha_J S_J + \sum_K \alpha_K S_K \,, \tag{34}$$

where J have 2 parts and $K = (k_1, k_2, k_3)$ have 3 parts (we assume that $\alpha_I \neq 0$). We set

$$Q = \sum_K \alpha_K S_{k_1-1,k_2-1,k_3-1} \,, \tag{35}$$

and our goal is to show that $Q = \mathcal{T}_{r-1}$. Since a partition I appearing nontrivially in the Schur function expansion of $\mathcal{T}_r$ must contain the partition $(r{+}1, r{+}1)$, then any partition K above contains the partition (r, r). Since this last partition is not contained in the $(1, r-1)$-hook, Eqs. (24) with r replaced by $r-1$ are automatically fulfilled by virtue of the vanishing property (20). Note that Eq. (27) is a particular case of Eq. (25). Indeed, specializing b_{r-1} to $\boxed{x_1{+}x_2}$ in Eq. (25), we get Eq. (27). Therefore it suffices to show that

$$Q(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2}) = x_1x_2(x_1{-}2x_2)(x_2{-}2x_1)\ R(\mathbb{X}_2{+}\boxed{x_1{+}x_2}, \mathbb{B}_{r-2}) \,. \tag{36}$$

where $\mathbb{E} = \boxed{2x_1} + \boxed{2x_2}$. We apply to each summand

$$\alpha_K S_{k_1-1,k_2-1,k_3-1}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2})$$

of $Q(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2})$ the factorization property (23), and divide it by the resultant

$$R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-2}) \,.$$

Suppose that the resulting factor is:

$$\alpha_K S_{a,b}(\mathbb{X}_2)\ S_c(-\mathbb{E} - \mathbb{B}_{r-2}) \,, \tag{37}$$

where $(k_1-1, k_2-1, k_3-1) = (c, r+a, r+b)$.

Performing the same division of

$$x_1x_2(x_1{-}2x_2)(x_2{-}2x_1)\ R(\mathbb{X}_2{+}\boxed{x_1{+}x_2}, \mathbb{B}_{r-2})$$

we get $R(\boxed{x_1{+}x_2}, \mathbb{B}_{r-2})$. Thus the wanted equation $Q = \mathcal{T}_{r-1}$ is equivalent to

$$\sum_{a+b+c=r-2} \alpha_K S_{a,b}(\mathbb{X}_2)\ S_c(-\mathbb{E} - \mathbb{B}_{r-2}) = R(\boxed{x_1{+}x_2}, \mathbb{B}_{r-2}) \,. \tag{38}$$

To prove Eq. (38) we use Eqs. (25) and (34) for $\mathcal{T}_r$:

$$\sum_I \alpha_I S_I(\mathbb{X}_2{-}\mathbb{E}{-}\mathbb{B}_{r-1}) = x_1x_2(x_1{-}2x_2)(x_2{-}2x_1)\ R(\mathbb{X}_2{+}\boxed{x_1{+}x_2}, \mathbb{B}_{r-1}) \,.$$

Using again the factorization property (this time w.r.t. the larger rectangle $(r+1)^2$) and dividing both sides of the last equation by the resultant

$$R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-1})\,.$$

we get the identity

$$\sum_{p+q+j=r-1} \alpha_I S_{p,q}(\mathbb{X}_2)\ S_j(-\mathbb{E} - \mathbb{B}_{r-1}) = R(\boxed{x_1+x_2}, \mathbb{B}_{r-1}). \tag{39}$$

Since

$$S_j(-\mathbb{E} - \mathbb{B}_{r-1}) = S_j(-\mathbb{E} - \mathbb{B}_{r-2}) - b_{r-1} S_{j-1}(-\mathbb{E} - \mathbb{B}_{r-2})$$

and

$$R(\boxed{x_1+x_2}, \mathbb{B}_{r-1}) = (x_1 + x_2 - b_{r-1}) R(\boxed{x_1+x_2}, \mathbb{B}_{r-2})\,,$$

taking the coefficients of $(-b_{r-1})$ in both sides of Eq. (39), we get the wanted Eq. (38). The lemma has been proved. $\square$

(For example, writing $\mathcal{T}_3 = \alpha S_{46} + \beta S_{55} + \gamma S_{244} + \delta S_{145}$, we get that

$$\gamma S_1(-\mathbb{E} - B_1) + \delta S_1(\mathbb{X}_2) = R(\boxed{x_1+x_2}, \mathbb{B}_1)$$

by taking the coeficients of $(-b_2)$ in both sides of

$$\alpha S_2(\mathbb{X}_2) + \beta S_{11}(\mathbb{X}_2) + \gamma S_2(-\mathbb{E}-\mathbb{B}_2) + \delta S_1(-\mathbb{E}-\mathbb{B}_2) S_1(X_2) = R(\boxed{x_1+x_2}, \mathbb{B}_2)\,.)$$

Iterating Eq. (33) gives

Corollary 13 *With the above notation, we have*

$$\mathcal{T}_r = \overline{\mathcal{T}}_r + \Phi(\overline{\mathcal{T}}_{r-1}) + \Phi^2(\overline{\mathcal{T}}_{r-2}) + \cdots + \Phi^{r-1}(\overline{\mathcal{T}}_1)\,. \tag{40}$$

Of course, $\overline{\mathcal{T}}_r$ is uniquely determined by its value on $\mathbb{X}_2$. The following result gives this value.

Proposition 14 *For any $r \geq 1$, we have*

$$\overline{\mathcal{T}}_r(\mathbb{X}_2) = (x_1 x_2)^{r+1}\ S_{r-1}(\mathbb{D})\,. \tag{41}$$

Proof. We use induction on r. For $r = 1, 2$, the assertion holds true. Suppose that the assertion is true for $\overline{\mathcal{T}}_i$ where $i < r$. We consider the Schur function expansion of $\mathcal{T}_r$:

$$\mathcal{T}_r = \sum_I \alpha_I S_I\,. \tag{42}$$

Fix a partition $I = (j, r+1+p, r+1+q)$ appearing nontrivially in (42). Note that j varies from 0 to $r-1$ because $|I| = 3r+1$. We obtain by the factorization property (23):

$$S_I(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = R \cdot S_j(-\mathbb{D} - \mathbb{B}_{r-2}) \cdot S_{p,q}(\mathbb{X}_2)\,.$$

where $R = R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2})$. Hence, using Eq. (40), we see that

$$\mathcal{T}_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = R \cdot \Big(\sum_{j=0}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2})\ \frac{\overline{\mathcal{T}}_{r-j}(\mathbb{X}_2)}{(x_1x_2)^{r-j+1}}\Big)\,. \tag{43}$$

By the induction assumption, for positive $j \le r-1$,

$$\overline{\mathcal{T}}_{r-j}(\mathbb{X}_2) = (x_1x_2)^{r-j+1}\ S_{r-1-j}(\mathbb{D})\,.$$

Substituting this to (43), and using the vanishing (27), we obtain

$$\sum_{j=1}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2}) S_{r-1-j}(\mathbb{D}) + \frac{\overline{\mathcal{T}}_r(\mathbb{X}_2)}{(x_1x_2)^{r+1}} = 0\,. \tag{44}$$

But we also have, by a formula for addition of alphabets,

$$\sum_{j=1}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2}) S_{r-1-j}(\mathbb{D}) + S_{r-1}(\mathbb{D}) = S_{r-1}(-\mathbb{B}_{r-2}) = 0\,. \tag{45}$$

Combining Eqs. (44) and (45) gives

$$\overline{\mathcal{T}}_r(\mathbb{X}_2) = (x_1x_2)^{r+1}\ S_{r-1}(\mathbb{D})\,,$$

that is, the assertion of the induction. The proof of the proposition is now complete. □

Corollary 15 *If S_{i_1,i_2} appears nontrivially in the Schur function expansion of $\overline{\mathcal{T}}_r$, then $i_1 = r+1+p$ and $i_2 = 2r-p$, where $0 \le 2p \le r-1$.*

The Schur function expansion of $S_i(\mathbb{D})$ was described in [21], [13], and [23, App. A3] in the context of the *Segre classes* of the second symmetric power of a rank 2 vector bundle. Indeed, $\mathbb{D}$ is the alphabet of the Chern roots of the second symmetric power of a rank 2 bundle with the Chern roots x_1, x_2.

Denote by $\langle p, q\rangle$ the coefficient of $S_{p,q} := S_{p,q}(\mathbb{X}_2)$ in $S_{p+q}(\mathbb{D})$, where $0 \le p \le q$. A proof of the next proposition, due to Lascoux with the help of *divided differences*, can be found in [23], p. 163–166. We give here another proof without divided differences.

Proposition 16 *For $p > 0$, we have*

$$\langle p, q \rangle = \langle p-1, q \rangle + \langle p, q-1 \rangle \,. \tag{46}$$

Proof. We have

$$S_i(\mathbb{D}) = \sum_{h=0}^{i} S_h(\boxed{2x_1}+\boxed{2x_2})S_{i-h}(\boxed{x_1+x_2}) = \sum_{h=0}^{i} 2^h S_h \cdot (x_1+x_2)^{i-h} \,, \tag{47}$$

and (cf., e.g., [17] I.4, Ex.3)

$$(x_1 + x_2)^j = \sum_{a,b \geq 0} \binom{a+b}{a} \frac{b-a+1}{b+1} S_{a,b} \,, \tag{48}$$

where $a+b = j$ and $a \leq b$. Combining Eqs. (47), (48), with the Pieri formula (cf., e.g., [14], [17]), we get for $0 \leq p \leq q$,

$$\langle p, q \rangle = \sum_{h=0}^{p+q} 2^h \sum_{h_1, h_2 \geq 0} \binom{p+q-h}{p-h_1} \frac{(q-h_2)-(p-h_1)+1}{q-h_2+1} \,, \tag{49}$$

where $h_1 + h_2 = h$ and $h_1 \leq p \leq q-h_2$.

We also compute the Schur function expansion of $S_{1,i-1}(\mathbb{D})$. Denote by $[p, q]$ the coefficient of $S_{p,q}$ in $S_{1,p+q-1}(\mathbb{D})$, $0 \leq p \leq q$. We have the following expansion for $S_{1,i-1}(\mathbb{D})$:

$$\begin{aligned} &\sum_{h=1}^{i} S_{(1,i-1)/(i-h)}(\boxed{2x_1}+\boxed{2x_2})S_h(\boxed{x_1+x_2}) \\ &= \sum_{h=1}^{i} 2^h S_h \cdot (x_1+x_2)^{i-h} + \sum_{h=1}^{i} 2^h S_{1,h-1} \cdot (x_1+x_2)^{i-h} \,. \end{aligned}$$

We get from both sums in the last line that for $p > 0$ the coefficient $[p, q]$ is equal *twice* the RHS of Eq. (49), that is,

$$[p, q] = 2\langle p, q \rangle \,. \tag{50}$$

We have by the Pieri formula

$$S_{i-1}(\mathbb{D}) \cdot S_1(\mathbb{D}) = S_{i-1}(\mathbb{D}) \cdot 3S_1 = S_i(\mathbb{D}) + S_{1,i-1}(\mathbb{D}) \,. \tag{51}$$

This equation implies that $S_{p,q}$ appears in $S_i(\mathbb{D})+S_{1,i-1}(\mathbb{D})$ with multiplicity $3(\langle p-1, q \rangle + \langle p-1, q \rangle)$ (we use the Pieri formula once again). The desired Eq. (46) now follows by virtue of Eq. (50). $\square$

We now pass to some "closed" algebraic expressions for the $\langle p, q \rangle$'s [4]. We have

$$\langle 0, q \rangle = S_q(\boxed{1} + \boxed{2}) = 1 + 2 + \cdots + 2^q = 2^{q+1} - 1 \,. \tag{52}$$

The following result was obtained in [30], [21], and [13].

[4]Note that Eq. (49) is not quite algebraic.

Proposition 17 *For $0 \leq p \leq q$, we have*

$$\langle p,q\rangle = \binom{p+q+1}{p+1} + \binom{p+q+1}{p+2} + \cdots + \binom{p+q+1}{q+1}. \tag{53}$$

We propose now an alternative expression involving powers of 2, which is a natural generalization of the equation $\langle 0,q\rangle = 2^{q+1}{-}1$, and which stems directly from Eq. (46). Namely, with the convention that $\binom{a}{0} = 1$ for any $a \in \mathbf{Z}$, we have

Proposition 18 *For $0 \leq p \leq q$,*

$$\langle p,q\rangle = 2^{p+q+1} - \sum_{s=0}^{p} \Big[\binom{p{+}q{-}2s{-}1}{p{-}s} - \binom{2p{-}2s{-}1}{p{-}s{+}1} \Big] 2^{2s}. \tag{54}$$

Proof. The proof uses double induction on p and q. We use Eq. (46) several times:

$$\begin{aligned}
\langle p,q\rangle &= \langle p{-}1,q\rangle + \langle p,q{-}1\rangle \\
&= \langle p{-}1,q\rangle + \langle p{-}1,q{-}1\rangle + \langle p,q{-}2\rangle \\
&= \ldots \\
&= \langle p{-}1,q\rangle + \cdots + \langle p{-}1,1\rangle + \langle p,0\rangle\,.
\end{aligned}$$

We know the values of all summands in the last row by the induction assumption (the last summand being equal to $2^{p+1}{-}1$). Using several times Eq. (52) as well as a well-known equality:

$$1 + \binom{a{+}1}{a} + \binom{a{+}2}{a} + \cdots + \binom{2a{-}2}{a} = \binom{2a{-}1}{a{+}1}$$

we get the desired induction assertion (54) for $\langle p,q\rangle$. $\square$

Using Proposition 14, we shall now give the Schur function expansion of $\overline{\mathcal{T}}_r$. Denote by d_{rj} the coefficient of $S_{r+j,2r+1-j}$ in $\overline{\mathcal{T}}_r$ for $r \geq 1$ and $j \geq 1$. By virtue of Corollary 15, $d_{rj} \neq 0$ entails $j \leq [(r+1)/2]$ (for example, the only Schur functions that can appear with nonzero coefficients in $\overline{\mathcal{T}}_5$ are $S_{6,10}$, S_{79}, and S_{88}), so that we have

$$\overline{\mathcal{T}}_r = \sum_{j=1}^{[(r+1)/2]} d_{rj}\; S_{r+j,2r+1-j}\,. \tag{55}$$

We have the following link between the d_{rj}'s and $\langle p,q\rangle$'s: suppose that $d_{rj} \neq 0$, then we have

$$d_{rj} = \langle j{-}1, r{-}j\rangle\,. \tag{56}$$

We may display the d_{rj}'s with the help of the following "Pascal triangle"-type matrix.

$$
\begin{array}{cccccc}
d_{11} & 0 & 0 & 0 & 0 & \dots \\
d_{21} & 0 & 0 & 0 & 0 & \dots \\
d_{31} & d_{32} & 0 & 0 & 0 & \dots \\
d_{41} & d_{42} & 0 & 0 & 0 & \dots \\
d_{51} & d_{52} & d_{53} & 0 & 0 & \dots \\
d_{61} & d_{62} & d_{63} & 0 & 0 & \dots \\
d_{71} & d_{72} & d_{73} & d_{74} & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
\quad = \quad
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & \dots \\
3 & 0 & 0 & 0 & 0 & \dots \\
7 & 3 & 0 & 0 & 0 & \dots \\
15 & 10 & 0 & 0 & 0 & \dots \\
31 & 25 & 10 & 0 & 0 & \dots \\
63 & 56 & 35 & 0 & 0 & \dots \\
127 & 119 & 91 & 35 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

By Proposition 16, if $d_{rj} > 0$, then we have

$$d_{rj} = d_{r-1,j-1} + d_{r-1,j}\,. \tag{57}$$

We have the following values of $\overline{\mathcal{T}}_1, \overline{\mathcal{T}}_2, \dots, \overline{\mathcal{T}}_6$:

$$
\begin{aligned}
&S_{22} \\
&3S_{34} \\
&7S_{46} + 3S_{55} \\
&15S_{58} + 10S_{67} \\
&31S_{6,10} + 25S_{79} + 10S_{88} \\
&63S_{7,11} + 56S_{8,10} + 35S_{99}\,.
\end{aligned}
$$

Summing up all our considerations, we get the main result of the present paper. It gives the desired Thom polynomial in a parametric form (the parameter being r).

Theorem 19 *For $r \geq 1$, the Thom polynomial for $I_{2,2}(r)$ is equal to*

$$\sum_{i=0}^{r-1} \sum_{\{j\geq 1:\ i+2j\leq r+1\}} d_{r-i,j}\ S_{i,r+j,2r-i-j+1}\,, \tag{58}$$

where, invoking Eq. (56), the coefficients $d_{r-i,j}$ are given by Eqs. (52) and (53) (or (54)).

We have the following values of $\mathcal{T}_1, \mathcal{T}_2 = \Phi(\mathcal{T}_1) + \overline{\mathcal{T}}_2, \dots, \mathcal{T}_6 = \Phi(\mathcal{T}_5) + \overline{\mathcal{T}}_6$:

$$
\begin{aligned}
&S_{22} \\
&S_{133}+3S_{34} \\
&S_{244}+3S_{145}+7S_{46}+3S_{55} \\
&S_{355}+3S_{256}+7S_{157}+3S_{166}+15S_{58}+10S_{67} \\
&S_{466}+3S_{367}+7S_{268}+3S_{277}+15S_{169}+10S_{178}+31S_{6,10}+25S_{79}+10S_{88} \\
&S_{577}+3S_{489}+7S_{379}+3S_{388}+15S_{2,7,10}+10S_{289}+31S_{1,7,11}+25S_{1,8,10}+10S_{189}+ \\
&63S_{7,11}+56S_{8,10}+35S_{99}\,.
\end{aligned}
$$

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Note Schur function expansions of some (other) Thom polynomials were studied in [6] as I has been informed by Feher. After completion of the first version of this paper, I received the preprint [8] on some Chern monomial expansions of Thom polynomials and among them such an expression for the Thom series of $I_{2,2}$ with the following comment (see p.5): *"Strictly speaking we have not proved the Thom series of $I_{2,2}$, just obtained overwhelming computer evidence for it."* We stress that our expression is of different form (a $\mathbf{Z}$-linear combination of Schur functions), and for the moment we do not know how to pass from it to the one in [8].

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