Configuration spaces of $\mathbb{C}$ and $\mathbb{CP}^1$: some analytic properties

Vladimir Lin

Department of Mathematics
Technion-Israel Institute of
Technology
32000 Haifa

Israel

Max-Planck-Institut
für Mathematik
Vivatsgasse 7
53111 Bonn

Germany

MPI2003-98

This work was started in 2000 and completed in 2003, both times during my stay at the Max-Planck-Institut für Mathematik in Bonn. I am deeply grateful to MPIM for hospitality and especially for giving me a fortunate opportunity to meet in 2000 Marat Gizatullin, discuss with him some examples and learn from him about the Eisenstein automorphism of the space of non-degenerate binary cubic forms. This renovated my old interest to configuration spaces and stimulated the present work.

Abstract.

I mean... You know...

We study certain analytic properties of the ordered and unordered configuration spaces $\mathcal{C}_o^n(X) = \{(q_1, ..., q_n) \in X^n \mid q_i \neq q_j \; \forall\, i \neq j\}$ and $\mathcal{C}^n(X) = \{Q \subset X \mid \#Q = n\}$ of simply connected algebraic curves $X = \mathbb{C}$ and $X = \mathbb{CP}^1$.

For $n \geq 3$ the braid group $B_n(X) = \pi_1(\mathcal{C}^n(X))$ of the curve X is non-abelian. A map $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ is called *cyclic* if the subgroup $F_*(\pi_1(\mathcal{C}^n(X))) \subset \pi_1(\mathcal{C}^k(X))$ is cyclic; otherwise F is called *non-cyclic*.

The diagonal $\operatorname{Aut} X$ action in X^n induces $\operatorname{Aut} X$ actions in $\mathcal{C}_o^n(X)$ and $\mathcal{C}^n(X)$. A morphism (holomorphic or regular) $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ is called *orbit-like* if $F(\mathcal{C}^n(X)) \subset (\operatorname{Aut} X)Q^*$ for some $Q^* \in \mathcal{C}^k(X)$. An endomorphism F of $\mathcal{C}^n(X)$ is called *tame* if there is a morphism $T\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ such that $F(Q) = T(Q)Q$ for all $Q \in \mathcal{C}^n(X)$. Tame endomorphisms preserve each $\operatorname{Aut} X$ orbit in $\mathcal{C}^n(X)$. We prove that *for $n > 4$ and $k \geq t(X) := \dim_{\mathbb{C}} \operatorname{Aut} X$ an endomorphism F of $\mathcal{C}^n(X)$ is tame if and only if it is non-cyclic, and a morphism $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ is orbit-like if and only if it is cyclic* ("Tame Map Theorem" and "Cyclic Map Theorem").

To study non-cyclic maps we establish first certain algebraic properties of the braid groups $B_n(X)$, including the stability of the pure braid group $PB_n(X) \subset B_n(X)$ under all non-cyclic endomorphisms of $B_n(X)$. This implies that any non-cyclic endomorphism of $\mathcal{C}^n(X)$ lifts to an $\mathbf{S}(n)$ equivariant endomorphism of $\mathcal{C}_o^n(X)$. To study the latter one, we construct a new invariant of affine varieties Z. We endow the set $L(Z)$ of all non-constant holomorphic functions $Z \to \mathbb{C} \setminus \{0, 1\}$ with a certain natural structure of a (finite) simplicial complex. In many interesting cases the correspondence $Z \mapsto L(Z)$ is a contravariant functor (this is surely the case when restricting to dominant morphisms). We describe the simplices of the complex $L(\mathcal{C}_o^n(X))$ explicitly, compute $\dim_{\mathbb{R}} L(\mathcal{C}_o^n(X))$ and show that any $\mathbf{S}(n)$ equivariant endomorphism of $\mathcal{C}_o^n(X)$ produces the corresponding simplicial self-map of $L(\mathcal{C}_o^n(X))$. We determine the $\mathbf{S}(n)$ orbits of simplices $\Delta \subset L(\mathcal{C}_o^n(X))$ and find their normal forms with respect to this action. Eventually, this provides a complete description of $\mathbf{S}(n)$ equivariant endomorphisms of $\mathcal{C}_o^n(X)$ and proves Tame Map Theorem. This theorem shows that $\mathcal{C}^n(X))/\operatorname{Aut} \mathcal{C}^n(X)) = \mathcal{C}^n(X))/\operatorname{Aut} X$ *and the holomorphic homotopy classes of non-cyclic endomorphisms of $\mathcal{C}^n(X)$ are in a natural $1-1$ correspondence with the homotopy classes of all continuous maps $\mathcal{C}^n(X) \to \mathcal{K}(X)$, where $\mathcal{K}(X)$ is a maximal compact subgroup of* $\operatorname{Aut} X$. It also implies that $\dim_{\mathbb{C}} F(\mathcal{C}^n(X)) \geq n - t(X) + 1$ *for any $n > 4$ and any non-cyclic endomorphism F of $\mathcal{C}^n(X)$.*

The study of cyclic maps involves the Liouville property of $\mathcal{C}^n(X)$ and the holomorphic direct decomposition $\mathcal{C}_o^k(X) \cong \operatorname{Aut} X \times \mathcal{C}_o^m(\mathbb{C} \setminus \{0, 1\})$, where $m = k - t(X)$ and $\mathcal{C}_o^m(\mathbb{C} \setminus \{0, 1\})$ is the ordered configuration space of $\mathbb{C} \setminus \{0, 1\}$.

We prove that *for $n > t(X) + 1$ and any non-cyclic morphism $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ there is a point $Q \in \mathcal{C}^n(X)$ such that $Q \cap F(Q) \neq \varnothing$* ("Linked Map Theorem"). The proof involves an "explicit" description of the holomorphic universal covering map $\Pi\colon \widetilde{\operatorname{Aut} X} \times \mathbf{T}(0, m) \to \mathcal{C}^n(X)$, where $m = n + 3 - t(X)$, $\widetilde{\operatorname{Aut} X}$ is the universal cover of $\operatorname{Aut} X$ and $\mathbf{T}(0, m)$ is the Teichmüller space of type $(0, m)$. The construction of Π involves the universal Teichmüller family $\mathbf{V}'(0, m) \to \mathbf{T}(0, m)$. To prove Linked Map Theorem, we use the Hubbard-Earl-Kra theorem, which says that the universal Teichmüller family of type (g, m) has no holomorphic sections if $\dim_{\mathbb{C}} \mathbf{T}(g, m) > 1$.

The above results apply to a version of the 13th Hilbert problem. We prove that for $n > \max\{4, m + 2\}$ the algebraic function $[x : y] = U_n(z)$ on $\mathbb{CP}^n$ defined by the "universal" equation $z_0 x^n + z_1 x^{n-1} y + ... + z_n y^n = 0$ cannot be represented as a composition of algebraic functions of m variables in such a way that a representing function-composition $F \supseteq U_n$ and U_n itself have the same branch loci.

Contents

1. Introduction

1.1. Configuration spaces. The (unordered) n^{th} *configuration space* $\mathcal{C}^n(X)$ of a space X consists of all n point subsets $Q \subseteq X$. It may be also viewed as the *regular orbit space* of the natural $\mathbf{S}(n)$ action in X^n. The standard left action of the symmetric group $\mathbf{S}(n)$ on X^n defined by

$$(\sigma, q) \mapsto \sigma q = (q_{\sigma^{-1}(1)}, ..., q_{\sigma^{-1}(n)}) \quad \forall\, \sigma \in \mathbf{S}(n) \text{ and } \forall\, q = (q_1, ..., q_n) \in X^n \tag{1.1}$$

is free on the *ordered configuration space*

$$\mathcal{C}_o^n(X) = \{q = (q_1, ..., q_n) \in X^n \mid q_i \neq q_j \ \forall\, i \neq j\}\,; \tag{1.2}$$

this provides the unbranched $\mathbf{S}(n)$ Galois covering

$$\begin{aligned} &p\colon \mathcal{C}_o^n(X) \to \mathcal{C}^n(X) = \mathcal{C}_o^n(X)/\mathbf{S}(n)\,, \\ &p(q) = p(q_1, ..., q_n) = \{q_1, ..., q_n\} = Q \in \mathcal{C}^n(X) \quad \forall\, q = (q_1, ..., q_n) \in \mathcal{C}_o^n(X)\,. \end{aligned} \tag{1.3}$$

If X is a complex or complex algebraic manifold then both $\mathcal{C}_o^n(X)$ and $\mathcal{C}^n(X)$ are so.

This paper is devoted to the configuration spaces $\mathcal{C}^n(\mathbb{C})$ and $\mathcal{C}^n(\mathbb{CP}^1)$, which are irreducible complex affine algebraic manifolds of dimension n; $\mathcal{C}^n(\mathbb{C})$ may be viewed as a Zariski open subset of $\mathcal{C}^n(\mathbb{CP}^1)$. We are especially interested in the morphisms of these spaces in the analytic or algebraic category.

Throughout the paper, unless otherwise specified, we assume that $n \geq 3$ and X is one of the curves $\mathbb{C}$, $\mathbb{CP}^1$.

1.2. The fundamental groups $\pi_1(\mathcal{C}^n(X))$ and $\pi_1(\mathcal{C}_o^n(X))$. The fundamental group $\pi_1(\mathcal{C}^n(X))$ and its normal subgroup $\pi_1(\mathcal{C}_o^n(X))$ are quite the classical objects known as the *braid group* $B_n(X)$ and the *pure braid group* $PB_n(X)$ respectively. They fit into the exact sequence

$$1 \to PB_n(X) \xrightarrow{p_*} B_n(X) \xrightarrow{\mu} \mathbf{S}(n) \to 1 \tag{1.4}$$

related to the Galois covering (1.3); the epimorphism $\mu\colon B_n(X) \to \mathbf{S}(n)$ is called *standard*. In more details, $\pi_1(\mathcal{C}^n(\mathbb{C})) = B_n$ and $\pi_1(\mathcal{C}_o^n(\mathbb{C})) = PB_n$ are the classical Artin braid group and the pure Artin braid group respectively. Similarly, $\pi_1(\mathcal{C}^n(\mathbb{PC}^1)) = B_n(S^2)$ and $\pi_1(\mathcal{C}_o^n(\mathbb{PC}^1)) = PB_n(S^2)$ are the braid group and the pure braid group of the two dimensional sphere S^2. Notice that for $n \geq 3$ the braid groups B_n and $B_n(S^2)$ are non-abelian. Their abelianizations are cyclic: $B_n/B_n' \cong \mathbb{Z}$ and $B_n(S^2)/B_n'(S^2) \cong \mathbb{Z}/(2n-2)\mathbb{Z}$.

1.3. Tame, degenerate tame and orbit-like morphisms. Let $\operatorname{Aut} X$ be the automorphism group of X; thus, $\operatorname{Aut} X$ is either the group $\mathbf{Aff}\,(\mathbb{C})$ of all affine transformations of $\mathbb{C}$ or the group $\mathbf{PSL}(2, \mathbb{C})$ of all Möbius transformations of the Riemann sphere. The diagonal $\operatorname{Aut} X$ action in $\mathcal{C}^n(X)$

$$\mathcal{C}^n(X) \ni Q = \{q_1, ..., q_n\} \mapsto \{Aq_1, ..., Aq_n\} \stackrel{\text{def}}{=\!=} AQ \in \mathcal{C}^n(X) \quad \forall\, A \in \operatorname{Aut} X \tag{1.5}$$

gives rise to the simplest examples of automorphisms of $\mathcal{C}^n(X)$: every $A \in \operatorname{Aut} X$ induces the automorphism $\mathcal{C}^n(X) \ni Q \mapsto AQ \in \mathcal{C}^n(X)$. These examples may be modified by letting $A = A(Q)$ depend on a point $Q \in \mathcal{C}^n(X)$ holomorphically and applying $A(Q)$ either to Q itself or to a chosen and marked point $Q^* \in \mathcal{C}^n(X)$. To be more precise, we give the following definition.

***Definition* 1.1.** For any morphism $T\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ we define the corresponding *tame endomorphism* $F_T\colon \mathcal{C}^n(X) \to \mathcal{C}^n(X)$ by

$$F_T(Q) = T(Q)Q = \{T(Q)q_1, ..., T(Q)q_n\} \ \ \forall\, Q = \{q_1, ..., q_n\} \in \mathcal{C}^n(X)\,. \tag{1.6}$$

Choosing a point $Q^* = \{q_1^*, ..., q_k^*\} \in \mathcal{C}^k(X)$, we define also the *degenerate tame* morphism $F_{T,Q^*}\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$,

$$F_{T,Q^*}(Q) = T(Q)Q^* = \{T(Q)q_1^*, ..., T(Q)q_k^*\} \ \ \forall\, Q = \{q_1, ..., q_n\} \in \mathcal{C}^n(X)\,; \tag{1.7}$$

if $k = n$, we obtain a degenerate tame endomorphism of $\mathcal{C}^n(X)$. Furthermore, a morphism $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ is said to be *orbit-like* if its image $F(\mathcal{C}^n(X))$ is contained in an orbit of $\operatorname{Aut} X$ action in $\mathcal{C}^k(X)$. ◯

***Remark* 1.2. *a*)** Of course, any degenerate tame morphism is orbit-like. Since the stabilizer $St_{Q^*} = \{A \in \operatorname{Aut} X \mid AQ^* = Q^*\}$ of a generic point $Q^* \in \mathcal{C}^k(X)$ is trivial, an orbit-like morphism whose image is contained in the orbit of such a point Q^* is degenerate tame.

***b*)** We may extend the above definition to continuous maps by saying that a continuous map $T\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ provides the tame continuous map $F_T\colon \mathcal{C}^n(X) \to \mathcal{C}^n(X)$ and the degenerate tame continuous map $F_{T,Q^*}\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ defined by (1.6) and (1.7) respectively. Any tame map F keeps each orbit of the $\operatorname{Aut} X$ action in $\mathcal{C}^n(X)$ stable, that is, $F((\operatorname{Aut} X)Q) \subseteq (\operatorname{Aut} X)Q$ for all $Q \in \mathcal{C}^n(X)$. Any degenerate tame map is orbit-like, that is, it carries the whole space $\mathcal{C}^n(X)$ into a single $\operatorname{Aut} X$ orbit in $\mathcal{C}^k(X)$. These facts are of a little moment since tame and degenerate tame maps are the rarities in the class of all continuous maps of configuration spaces. However, due to Tame Map Theorem formulated below, a "generic" endomorphism is tame and hence it keeps each $\operatorname{Aut} X$ orbit stable. Furthermore, by a quite elementary reason, any morphism $\mathcal{C}^n(X) \to \mathcal{C}^k(X)$ must carry $\operatorname{Aut} X$ orbits to $\operatorname{Aut} X$ orbits; see Section 8.2, Proposition 8.5. ◯

1.4. Cyclic homomorphisms and cyclic maps. It will be convenient to introduce the following notions.

***Definition* 1.3.** A group homomorphism $\varphi\colon\ G \to H$ is said to be *cyclic*, *abelian* or *solvable* if its image $\varphi(G)$ is a cyclic, abelian or solvable subgroup of the group H respectively. A homomorphism $\varphi\colon\ G \to H$ is abelian if and only if its restriction to the commutator subgroup $G' = [G, G]$ of the group G is trivial. If the abelianization G/G' of G is a cyclic group then any abelian homomorphism of G is cyclic.

A continuous map $F\colon \mathcal{X} \to \mathcal{Y}$ of arcwise connected spaces is called *cyclic*, *abelian* or *solvable* if the induced homomorphism $F_*\colon \pi_1(\mathcal{X}) \to \pi_1(\mathcal{Y})$ of the fundamental groups is cyclic, abelian or solvable respectively. ◯

1.5. Main results. Any degenerate tame morphism is cyclic since $\pi_1(\operatorname{Aut}\mathbb{C}) \cong \mathbb{Z}$ and $\pi_1(\operatorname{Aut}\mathbb{CP}^1) \cong \mathbb{Z}/2\mathbb{Z}$. In fact we will show in Section 8 that for $n > 4$ a morphism $\mathcal{C}^n(X) \to \mathcal{C}^k(X)$ is cyclic if and only if it is orbit-like.

In contrast to orbit-like morphisms, for $n \geq 3$ any tame endomorphism of $\mathcal{C}^n(X)$ is non-cyclic (see Remark 2.9(***b***)). Our first main result tells that the for $n > 4$ the converse is also true.

Theorem 1.4 (Tame Map Theorem)**.** *For $n > 4$ every non-cyclic endomorphism F of $\mathcal{C}^n(X)$ is tame. That is, there exists a unique morphism $T\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ such that*

$$F(Q) = F_T(Q) \stackrel{\text{def}}{=\!=} T(Q)Q \quad \text{for all } Q \in \mathcal{C}^n(X)\,.$$

In particular, for $n > 4$ every automorphism of $\mathcal{C}^n(X)$ is tame. ○

In view of the above notes about cyclic morphisms, this theorem shows that for endomorphisms of $\mathcal{C}^n(X)$ the properties to be tame or orbit-like form a dichotomy, which actually coincides with the dichotomy *"non-cyclic – cyclic"*:

Corollary 1.5. *For $n > 4$ the classes of non-cyclic and cyclic endomorphisms of $\mathcal{C}^n(X)$ coincide with the classes of tame and orbit-like endomorphisms respectively.* ○

Tame Map Theorem has also the following two immediate consequences.

Corollary 1.6. *For $n > 4$ the holomorphic homotopy classes of non-cyclic holomorphic maps*[1] *$\mathcal{C}^n(X) \to \mathcal{C}^n(X)$ are in the natural one-to-one correspondence with the holomorphic homotopy classes of holomorphic maps $\mathcal{C}^n(X) \to \operatorname{Aut} X$. By the famous Grauert theorem, the latter classes are in the natural one-to-one correspondence with the usual homotopy classes of all continuous maps $\mathcal{C}^n(X) \to \operatorname{Aut} X$, or, which is the same, with the homotopy classes of all continuous maps $\mathcal{C}^n(X) \to \mathcal{K}(X)$, where $\mathcal{K}(X)$ is a maximal compact subgroup of the group $\operatorname{Aut} X$.* ○

Corollary 1.7. *Let $n > 4$ and $G = \operatorname{Aut} \mathcal{C}^n(X)$. Then the orbits of the natural G action in $\mathcal{C}^n(X)$ coincide with the orbits of the diagonal $\operatorname{Aut} X$ action in $\mathcal{C}^n(X)$.*

In particular, the orbit space $\mathcal{C}^n(\mathbb{CP}^1)/\operatorname{Aut} \mathcal{C}^n(\mathbb{CP}^1)$ may be identified to the orbit space $\mathcal{C}^n(\mathbb{CP}^1)/\mathbf{PSL}(2,\mathbb{C})$, which, in turn, may be viewed as the moduli space $M(0,n)$ of the Riemann sphere with n unordered punctures. ○

***Remark* 1.8.** For $X = \mathbb{C}$ one may take $\mathcal{K}(\mathbb{C})$ to be the subgroup $\mathbb{T} \subset \mathbf{Aff}\,\mathbb{C}$ consisting of all rotations $z \mapsto e^{it}z$, $t \in \mathbb{R}$. Thereby, the holomorphic homotopy classes of non-cyclic holomorphic maps $\mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^n(\mathbb{C})$ are in the one-to-one correspondence with the homotopy classes of all continuous maps $\mathcal{C}^n(\mathbb{C}) \to \mathbb{T}$, which, in turn, may be identified with the elements of the group $H^1(\mathcal{C}^n(\mathbb{C}), \mathbb{Z}) \cong \operatorname{Hom}(B_n, \mathbb{Z}) \cong \mathbb{Z}$.

For $X = \mathbb{CP}^1$ one may take $\mathcal{K}(\mathbb{CP}^1) \subset \mathbf{PSL}(2,\mathbb{C})$ to be the subgroup isomorphic to $\mathbf{SO}(3)$ and consisting of all Möbius transformations of the form

$$z \mapsto \frac{az - \bar{c}}{cz + \bar{a}}\,, \quad |a|^2 + |c|^2 = 1\,.$$

Thus, the holomorphic homotopy classes of non-cyclic holomorphic endomorphisms of $\mathcal{C}^n(\mathbb{CP}^1)$ are in the one-to-one correspondence with the homotopy classes of all continuous maps $\mathcal{C}^n(\mathbb{CP}^1) \to \mathbf{SO}(3)$. ○

[1] That is, the classes of holomorphic maps homotopic to each other within the space of all holomorphic maps.

***Remark* 1.9.** Consider the space of all non-zero complex binary forms of degree n

$$\phi(x,y;z) = z_0x^n + z_1x^{n-1}y + \cdots + z_{n-1}xy^{n-1} + z_ny^n\,, \tag{1.8}$$

where z is regarded either as a non-zero point $(z_0,\cdots,z_n) \in \mathbb{C}^{n+1}$ or as a point $[z_0:\cdots:z_n] \in \mathbb{CP}^n$. In the latter case the form $\phi(x,y;z)$ is called *projective*, meaning that proportional forms are identified. The discriminant $D_n(z)$ of a form $\phi(x,y;z)$ is a homogeneous polynomial in $z_0,...,z_n$ of degree $2(n-1)$:

$$D_n(z) = \begin{vmatrix} 1 & z_1 & z_2 & ... & z_{n-2} & z_{n-1} & z_n & 0 & ... & 0 \\ 0 & z_0 & z_1 & ... & ... & ... & z_{n-1} & z_n & .... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & z_0 & z_1 & z_2 & .... & z_{n-1} & z_n \\ n & (n-1)z_1 & (n-2)z_2 & ... & 2z_{n-2} & z_{n-1} & 0 & 0 & ... & 0 \\ 0 & nz_0 & (n-1)z_1 & ... & ... & 2z_{n-2} & z_{n-1} & 0 & .... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 & nz_0 & (n-1)z_1 & ... & 2z_{n-2} & z_{n-1} \end{vmatrix};$$

$\phi(x,y;z)$ is said to be *non-degenerate* if $D_n(z) \neq 0$. The set

$$\mathcal{F}^n = \{\phi = \phi(x,y;z) \mid z = [z_0 : ... : z_n] \in \mathbb{CP}^n\,, \ \ D_n(z) \neq 0\} \tag{1.9}$$

is called *the space of all non-degenerate complex projective binary forms of degree n.* For every $Q = \{q_1,...,q_n\} \in \mathcal{C}^n(\mathbb{CP}^1)$ there is a unique form $\phi = \phi(x,y;z) \in \mathcal{F}^n$ whose zero set $Z_\phi = \{[x:y] \in \mathbb{CP}^1 \mid \phi(x,y;z) = 0\}$ coincides with Q. This provides the natural identification $\mathcal{C}^n(\mathbb{CP}^1) = \mathcal{F}^n$.

Under this identification, the subspace $\mathcal{C}^n(\mathbb{C}) \subset \mathcal{C}^n(\mathbb{CP}^1)$ consists of all $\phi([x:y],z) \in \mathcal{F}^n$ with $z_0 \neq 0$; it may be viewed as the space $\mathbf{G}_n$ of all complex polynomials $p_n(t;w) = t^n + w_1t^{n-1} + ... + w_n$ with simple roots. We also regard $\mathbf{G}_n$ as a domain in $\mathbb{C}^n$ defined by

$$\mathbf{G}_n = \{w = (w_1,...,w_n) \in \mathbb{C}^n \mid d_n(w) \neq 0\}\,, \tag{1.10}$$

where $d_n(w) = d_n(w_1,...,w_n) = D_n(1,w_1,...,w_n)$ is the discriminant of $p_n(t;w)$. The discriminant map

$$d_n\colon \mathcal{C}^n(\mathbb{C}) = \mathbf{G}_n \ni w \mapsto d_n(w) \in \mathbb{C}^* \tag{1.11}$$

is a holomorphic locally trivial fiber bundle; the induced homomorphism of the fundamental groups d_{n*} coincides with the abelianization homomorphism $\chi\colon B_n \to \mathbb{Z}$.

In terms of binary forms, a tame endomorphism of $\mathcal{C}^n(\mathbb{CP}^1) = \mathcal{F}^n$ is just a projective change of variables $[x:y] \mapsto T(\phi)[x:y]$, which depends holomorphically on a form ϕ itself. A tame endomorphism of the space $\mathcal{C}^n(\mathbb{C}) = \mathbf{G}_n$ is the transformation of polynomials $p_n(t;w)$ corresponding to an affine change of the variables $T\colon t \mapsto t' = a(w)t + b(w)$ holomorphically depending on $w \in \mathbf{G}_n$. ○

Our second main result concerns morphisms of configuration spaces which do not necessarily coincide to each other. It will be convenient to make the following definition and notation.

***Definition* 1.10.** Let $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ be a map of configuration spaces. Then either $Q \cap F(Q) = \varnothing$ for all $Q \in \mathcal{C}^n(X)$ or there is a point $Q_0 \in \mathcal{C}^n(X)$ such that

$Q_0 \cap F(Q_0) \neq \varnothing$. In the first case F is called *disjoint*; in the second case we say that F is *linked.*

***Notation* 1.11.** We set $t(X) = \dim_{\mathbb{C}} \operatorname{Aut} X$, i. e., $t(X) = 2$ for $X = \mathbb{C}$ and $t(X) = 3$ for $X = \mathbb{CP}^1$. ◯

In all dimensions there are smooth maps $\mathcal{C}^n(X) \to \mathcal{C}^k(X)$ of both kinds. For holomorphic maps the situation changes drastically.

Theorem 1.12 (Linked Map Theorem). *For $n > t(X) + 1$ and $k \geq 1$ every holomorphic map $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ is linked.* ◯

This theorem is actually a reflection of the fact that the universal Teichmüller family of type $(0, n+3)$ does not possess holomorphic sections. In Section 3 we will relate configuration spaces to Teichmüller spaces. The proof of Linked Map Theorem and some its applications to algebraic functions will be postponed till Section 7. Examples of disjoint holomorphic maps in small dimension will be presented in Section 14. In Section 11 we will discuss some applications of Linked Map Theorem to the 13th Hilbert problem for algebraic functions.

1.6. Some historical remarks. My interest to holomorphic mappings $\mathbf{G}_n \to \mathbf{G}_k$ of polynomials with simple roots was motivated by their relations to a version of the 13th Hilbert problem for algebraic functions. Endomorphisms of $\mathcal{C}^n(\mathbb{C}) = \mathbf{G}_n$ were completely described in the 1970's (see my papers [**Lin72b, Lin72c, Lin79**], where some results about morphsms $\mathbf{G}_n \to \mathbf{G}_k$ were obtained as well). However, the complete proofs were never published, since they contained some very long and unpleasant combinatorial computations.

In the 1990's I found that certain morphisms $\mathbf{G}_n \to \mathbf{G}_k$ are related to the universal Teichmüller family of type $(0, n+1)$, which led to the proof of Linked Map Theorem for $X = \mathbb{C}$ in the form presented in Remark 7.7(***a***) (see [**Lin96b**]).

On the other hand, in [**Lin96b**] (see also Section 14.2 of the present paper) a disjoint polynomial map $F\colon \mathcal{C}^3(\mathcal{C}) \to \mathcal{C}^6(\mathcal{C})$ has been constructed. In view of the result mentioned above, the latter map is so special that its novelty must look dubious. During my stay at Max-Planck-Institut für Mathematik in Bonn in 2000, I met Marat Gizatullin (whose knowledge of and love to the classical mathematical literature are exceptional) and asked him whether he have seen such a map somewhere. After a few days he came back and told that he did not meet such a monster but instead he found out that in 1844 Gotthold Eisenstein has discovered a very exciting automorphism of non-degenerate binary cubic forms (see [**Eis1844**] and Section 14.3 of the present paper).

This Eisenstein's example renovated my interest to endomorphisms of configuration spaces. In the present work my old approach is improved and simplified and now it applies to both $\mathcal{C}^n(\mathbb{C})$ and $\mathcal{C}^n(\mathbb{CP}^1)$.

It should be also mentioned that endomorphisms of the configuration spaces $\mathcal{C}^n(\mathbb{C}^*)$ have been described by V. Zinde [**Zin77**].

Recently my student Y. Feler proved that for $n > 4$ all automorphisms of the configuration space $\mathcal{C}^n(\mathbb{T}^2)$ of a torus $\mathbb{T}^2 = \mathbb{C}/(a\mathbb{Z} + b\mathbb{Z})$ are tame. Together with my old results quoted above, results of V. Zinde, [**Zin77**], and the ones of the present paper, this gives a realistic hope to complete the description of endomorphisms of configuration spaces for all non-hyperbolic Riemann surfaces. Furthermore, Feler studied certain configuration spaces of different kind, say the spaces $\mathcal{C}^n_{aff}(\mathbb{C}^2)$ and $\mathcal{C}^n_{prj}(\mathbb{CP}^2)$ consisting of subsets $Q = \{q_1, ..., q_n\}$ in $\mathbb{C}^2$ or in $\mathbb{CP}^2$ whose points $q_1, ..., q_n$ are in general affine and general projective position respectively. (My attention to such spaces was attracted by A. M. Vershik.) We have in mind to publish these and some other results about configuration spaces in a joint paper.

2. Strategy of the proof of Tame Map Theorem

The question about the nature of endomorphisms of configuration spaces belongs to analytic or algebraic geometry and seemingly is a rather complicated one. We will reduce it, step by step, to a series of algebraic or combinatorial tasks, which look and as a matter of fact are simpler.

In this section we try first to find out what the inevitable attributes of tame endomorphisms and those of certain accompanying objects ought to be. Then we formulate the statements which should be proved in order to ensure that non-cyclic endomorphisms and the corresponding objects of their "suite" actually possess such properties. The accomplishment of this approach in the consequent sections eventually will lead to the desired result.

Some proofs named ***"Explanation"*** also are placed here but only when they are completely trivial. We start with an obvious observation.

Lifting and invariance properties. Tame and degenerate tame endomorphisms F of $\mathcal{C}^n(X)$ have in common the following

Lifting property: *there exists an endomorphism $f\colon \mathcal{C}^n_o(X) \to \mathcal{C}^n_o(X)$ which fits into the commutative diagram*

$$\begin{array}{ccc} \mathcal{C}^n_o(X) & \xrightarrow{f} & \mathcal{C}^n_o(X) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{C}^n(X) & \xrightarrow[F]{} & \mathcal{C}^n(X) \end{array} \tag{2.1}$$

Such an f may be defined as follows:

$$f(q) = f_T(q) \stackrel{\text{def}}{=\!=} T(Q)(q) = (T(Q)q_1, ..., T(Q)q_n) \ \text{ whenever } F = F_T\,, \tag{2.2}$$

and

$$f(q) = f_{T,Q^*}(q) \stackrel{\text{def}}{=\!=} T(Q)(q^*) = (T(Q)q^*_1, ..., T(Q)q^*_n) \ \text{ whenever } F = F_{T,Q^*}\,, \tag{2.3}$$

where $q = (q_1, ..., q_n) \in \mathcal{C}^n_o(X)$, $Q = \{q_1, ..., q_n\} = p(q) \in \mathcal{C}^n(X)$. ○

By the covering map theorem, a continuous map $F\colon \mathcal{C}^n(X) \to \mathcal{C}^n(X)$ possesses a continuous covering map f that fits into diagram (2.1) if and only if the following invariance condition is fulfilled:

Invariance property: *the induced endomorphism* $F_*\colon \pi_1(\mathcal{C}^n(X)) \to \pi_1(\mathcal{C}^n(X))$ *of the fundamental group respects the normal subgroup* $\pi_1(\mathcal{C}_o^n(X)) \lhd \pi_1(\mathcal{C}^n(X))$, *that is,*

$$F_*(\pi_1(\mathcal{C}_o^n(X))) \subseteq \pi_1(\mathcal{C}_o^n(X))\,. \tag{2.4}$$

2.1. Step 1: Lifting of non-cyclic continuous maps. Taking into account that tame endomorphisms of $\mathcal{C}^n(X)$ ***must*** lift to the covering $\mathcal{C}_o^n(X)$, we are going to prove the following

Theorem 2.1 (Lifting Theorem)**.** *For* $n > 4$ *every non-cyclic continuous map* $F\colon \mathcal{C}^n(X) \to \mathcal{C}^n(X)$ *lifts to a continuous map* $f\colon \mathcal{C}_o^n(X) \to \mathcal{C}_o^n(X)$ *so that* F *and* f *fit into commutative diagram* (2.1)*. Clearly* f *is an endomorphism if* F *is so.* ○

In view of the covering mapping theorem, this topological result is equivalent to the following algebraic

Theorem 2.2 (Invariance Theorem)**.** *Let* $n > 4$*. Then any non-cyclic endomorphism* φ *of the group* $B_n(X)$ *respects the normal subgroup* $PB_n(X)$*, i. e.,* $\varphi(PB_n(X)) \subseteq PB_n(X)$*. In fact* $PB_n(X) = \varphi^{-1}(PB_n(X))$*, which certainly implies the previous inclusion.* ○

Invariance Theorem is, in turn, an immediate consequence of the following two results:

Theorem 2.3 (Composition Theorem)**.** *Let* $n > 4$ *and let* $\varphi\colon B_n(X) \to B_k(X)$ *be a non-cyclic homomorphism. Then the composition*

$$\psi = \mu \circ \varphi\colon B_n(X) \xrightarrow{\varphi} B_k(X) \xrightarrow{\mu} \mathbf{S}(k) \tag{2.5}$$

of φ *and the standard epimorphism* $\mu\colon B_n(X) \to \mathbf{S}(n)$ *is non-cyclic as well.* ○

Theorem 2.4 (Kernel Theorem)**.** *For* $n > 4$ *the kernel* $\operatorname{Ker}\psi$ *of any non-cyclic homomorphism* $\psi\colon B_n(X) \to \mathbf{S}(n)$ *coincides with the pure braid group* $PB_n(X)$*.* ○

Explanation. For a non-cyclic $\varphi\colon B_n(X) \to B_n(X)$ Composition Theorem shows that $\psi = \mu \circ \varphi\colon B_n(X) \to \mathbf{S}(n)$ is non-cyclic and, by Kernel Theorem, $PB_n(X) = \operatorname{Ker}\psi = \operatorname{Ker}(\mu \circ \varphi) = \varphi^{-1}(\operatorname{Ker}\mu) = \varphi^{-1}(PB_n(X))$, which is exactly the statement of Invariance Theorem. □

2.2. Step 2: Commutator subgroup, homomorphisms $B_n(X) \to \mathbf{S}(n)$. To prove Composition and Kernel Theorems, we establish first some other important properties of braid groups $B_n(X)$.

Recall that a group G is said to be *perfect* if it coincides with its commutator subgroup $G' = [G, G]$, or, which is the same, G does not possess non-trivial homomorphisms to abelian groups.

We extensively make use of the following theorem proved in Section 4.2:

Theorem 2.5 (Commutator Theorem)**.** *The commutator subgroup* $B_n'(X)$ *of the braid group* $B_n(X)$ *is a perfect group whenever* $n > 4$*.* ○

In the next section we show (Theorem 3.3) that for any $n \geq 2$ perfect groups do not admit non-trivial homomorphisms into the pure braid group $PB_k(X)$. Together with Commutator Theorem, this implies:

Theorem 2.6 (HOMOMORPHISMS $B'_n(X) \to PB_k(X)$)**.** *For $n > 4$ the commutator subgroup $B'_n(X)$ of the braid group $B_n(X)$ does not possess non-trivial homomorphisms to the pure braid group $PB_k(X)$.* ○

Furthermore, in Section 4.3 we prove the following theorem:

Theorem 2.7 (SURJECTIVITY THEOREM)**.** *For $n \neq 4$ every non-cyclic homomorphism $\psi\colon B_n(X) \to \mathbf{S}(n)$ is surjective.* ○

These results, together with the classical Artin theorem about transitive representations $B_n \to \mathbf{S}(n)$ (see [**Art47b, Lin96b**]), lead to the proof of Composition and Kernel Theorems and thereby Invariance and Lifting Theorems as well (see Section 4 for the proofs).

2.3. Step 3: Strictly equivariant maps. With Lifting Theorem at our disposal, we proceed as follows. Since (1.3) is a Galois $\mathbf{S}(n)$ covering, any continuous lifting

$$f = (f_1, ..., f_n)\colon \mathcal{C}^n_o(X) \to \mathcal{C}^n_o(X) \tag{2.6}$$

of a continuous map

$$F\colon \mathcal{C}^n(X) \to \mathcal{C}^n(X) \tag{2.7}$$

is *equivariant* in the following sense:

Equivariance condition: *there exists an endomorphism $\alpha\colon \mathbf{S}(n) \to \mathbf{S}(n)$ such that*

$$f(\sigma q) = \alpha(\sigma) f(q) \ \text{ for all } q \in \mathcal{C}^n_o(X) \text{ and } \sigma \in \mathbf{S}(n)\,, \tag{2.8}$$

or, in terms of the components of f,

$$f_m(\sigma q) = f_{[\alpha(\sigma)]^{-1}(m)}(q) \ \text{ for all } m = 1, ..., n\,,\ q \in \mathcal{C}^n_o(X)\,,\ \sigma \in \mathbf{S}(n)\,.$$

Up to conjugation, the endomorphism α is uniquely defined by the original map F. That is, if endomorphisms α', α'' correspond to two liftings f', f'' of F then there is a permutation $s \in \mathbf{S}(n)$ such that $\alpha''(\sigma) = s^{-1}\alpha'(\sigma)s$ for all $\sigma \in \mathbf{S}(n)$.

Explanation. Given F which lifts to $\mathcal{C}^n_o(X)$, we may construct α as follows. Taking a continuous lifting f and choosing basepoints $q^\circ \in \mathcal{C}^n_o(X)$ and $Q^\circ = p(q^\circ) \in \mathcal{C}^n(X)$ we obtain the commutative diagram

$$\begin{tikzcd}
\pi_1(\mathcal{C}^n_o(X), q^\circ) \arrow[r, "f_*"] \arrow[d, "p_*"'] & \pi_1(\mathcal{C}^n_o(X), f(q^\circ)) \arrow[d, "p_*"] \\
\pi_1(\mathcal{C}^n(X), Q^\circ) \arrow[r, "F_*"'] \arrow[d, "\mu'"'] & \pi_1(\mathcal{C}^n(X), F(Q^\circ)) \arrow[d, "\mu''"] \\
\pi_1(\mathcal{C}^n(X), Q^\circ)/\pi_1(\mathcal{C}^n_o(X), q^\circ) \arrow[r, "\alpha"] \arrow[d] & \pi_1(\mathcal{C}^n(X), F(Q^\circ))/\pi_1(\mathcal{C}^n_o(X), f(q^\circ)), \arrow[d] \\
1 & 1
\end{tikzcd} \tag{2.9}$$

where μ' and μ'' are the natural epimorphisms onto the quotient groups and α is a unique homomorphism which makes the lower rectangle commutative. The latter quotient groups are identified with the symmetric groups of the ordered sets $q^\circ =$

$(q_1^\circ, ..., q_n^\circ)$ and $f(q^\circ) = (f_1(q^\circ), ..., f_n(q^\circ))$, that is, with $\mathbf{S}(n)$; thus, α becomes an endomorphism of $\mathbf{S}(n)$. Another choice of f and/or a basepoint $q^\circ \in \mathcal{C}_o^n(X)$ may lead to another endomorphism which must be conjugate to α. Furthermore, connecting $f(q^\circ)$ to q° and $F(Q^\circ)$ to Q° with paths $\gamma\colon [0,1] \to \mathcal{C}_o^n(X)$ and $p \circ \gamma\colon [0,1] \to \mathcal{C}^n(X)$ we come to identifications of pairs

$$\{\pi_1(\mathcal{C}_o^n(X), q^\circ) \lhd \pi_1(\mathcal{C}^n(X), Q^\circ)\} \cong \{PB_n(X) \lhd B_n(X)\} \text{ and}$$
$$\{\pi_1(\mathcal{C}_o^n(X), f(q^\circ)) \lhd \pi_1(\mathcal{C}^n(X), F(Q^\circ))\} \cong \{PB_n(X) \lhd B_n(X)\}$$

that conform to each other. Thereby (2.9) leads to the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & PB_n(X) \arrow[r, "p_*"] \arrow[d, "f_*"'] & B_n(X) \arrow[r, "\mu"] \arrow[d, "F_*"'] & \mathbf{S}(n) \arrow[r] \arrow[d, "\alpha"] & 1 \\ 1 \arrow[r] & PB_n(X) \arrow[r, "p_*"] & B_n(X) \arrow[r, "\mu"] & \mathbf{S}(n) \arrow[r] & 1 \end{tikzcd} \tag{2.10}$$

whose horizontal lines are exact sequences (1.4) of the covering $p\colon \mathcal{C}_o^n(X) \to \mathcal{C}^n(X)$ corresponding to the two pairs of basepoints $\{Q^\circ, q^\circ\}$ and $\{F(Q^\circ), f(q^\circ)\}$. $\square$

Definition* 2.8.** A continuous map $f\colon \mathcal{C}_o^n(X) \to \mathcal{C}_o^n(X)$ is said to be *strictly equivariant* if there exists an ***automorphism α of the group $\mathbf{S}(n)$ such that the above mentioned condition

$$f(\sigma q) = \alpha(\sigma) f(q) \text{ for all } q \in \mathcal{C}_o^n(X) \text{ and } \sigma \in \mathbf{S}(n) \tag{2.9}$$

is fulfilled. $\bigcirc$

Remark* 2.9.** ***a) For $n \neq 6$ any automorphism of $\mathbf{S}(n)$ is inner, and therefore (2.8) takes the form

$$\exists\, s \in \mathbf{S}(n) \text{ such that } f(\sigma q) = s\sigma s^{-1} f(q) \ \ \forall\, q \in \mathcal{C}_o^n(X) \text{ and } \forall\, \sigma \in \mathbf{S}(n)\,. \tag{2.11}$$

In fact any strictly equivariant *endomorphism* of $\mathcal{C}_o^n(X)$ satisfies (2.11) for all $n \geq 3$ (see ***Explanation*** in Section 2.4).

b) It is easily seen from diagram (2.10) that a tame endomorphism $F = F_T$ of $\mathcal{C}^n(X)$ must be non-cyclic. Indeed, let $f = f_T$ be the lifting of F defined according to (2.2), that is, $f(q) = T(p(q))q = (T(p(q))q_1, ..., T(p(q))q_n)$ for $q = (q_1, ..., q_n)$. As we know, f is equivariant; let α be the corresponding endomorphism of $\mathbf{S}(n)$. Then for any $\sigma \in \mathbf{S}(n)$ and all $q \in \mathcal{C}_o^n(X)$ we have $\alpha(\sigma)f(q) = f(\sigma q) = T(p(\sigma q))\sigma q = \sigma T(p(q))q = \sigma f(q)$; thereby $\alpha(\sigma) = \sigma$ and $\alpha = \mathrm{id}_{\mathbf{S}(n)}$. In particular, α is non-cyclic, and (2.10) implies that F_* is non-cyclic. $\bigcirc$

The following result is an easy consequence of Composition Theorem.

Theorem 2.10 (Equivariance Theorem)**.** *Let $n > 4$. Then any continuous lifting $f\colon \mathcal{C}_o^n(X) \to \mathcal{C}_o^n(X)$ of a non-cyclic continuous map $F\colon \mathcal{C}^n(X) \to \mathcal{C}^n(X)$ is strictly equivariant.* $\bigcirc$

Explanation. Since F is non-cyclic, the induced homomorphism F_* in (2.10) is non-cyclic and, by Composition Theorem, the composition $\alpha \circ \mu = \mu \circ F_*$ is non-cyclic as well. Thus, α is non-cyclic, that is, its image $\alpha(\mathbf{S}(n)) \cong \mathbf{S}(n)/\operatorname{Ker}\alpha$ is a non-cyclic group. For $n > 4$ the group $\mathbf{S}(n)$ possesses a single proper normal subgroup, the alternating group $\mathbf{A}(n)$, and the quotient group $\mathbf{S}(n)/\mathbf{A}(n) \cong \mathbb{Z}/2\mathbb{Z}$ is cyclic. Therefore $\operatorname{Ker}\alpha = \{1\}$ and α is an automorphism. $\square$

***Definition* 2.11.** A morphism $\tau\colon \mathcal{C}_o^n(X) \to \operatorname{Aut} X$ is called $\mathbf{S}(n)$ *invariant* if it is constant on each $\mathbf{S}(n)$ orbit in $\mathcal{C}_o^n(X)$, that is,

$$\tau(\sigma q) = \tau(q) \ \text{ for all } q \in \mathcal{C}_o^n(X) \text{ and } \sigma \in \mathbf{S}(n)\,.$$

An endomorphism $f\colon \mathcal{C}_o^n(X) \to \mathcal{C}_o^n(X)$ is said to be *tame* if there exists an $\mathbf{S}(n)$ invariant morphism $\tau\colon \mathcal{C}_o^n(X) \to \operatorname{Aut} X$ and a permutation $s \in \mathbf{S}(n)$ such that

$$f(q) = f_\tau^s(q) \stackrel{\text{def}}{=\!=} s\tau(q)q = (\tau(q)q_{s^{-1}(1)}, ..., \tau(q)q_{s^{-1}(n)}) \ \ \forall\, q = (q_1, ..., q_n) \in \mathcal{C}_o^n(X)\,.$$

Similarly, we say that f is *degenerate tame* if there are $q^\circ \in \mathcal{C}_o^n(X)$, $s \in \mathbf{S}(n)$ and an $\mathbf{S}(n)$ invariant morphism $\tau\colon \mathcal{C}_o^n(X) \to \operatorname{Aut} X$ such that

$$f(q) = f_{\tau,q^\circ}^s(q) \stackrel{\text{def}}{=\!=} s\tau(q)q^\circ \ \ \forall\, q \in \mathcal{C}_o^n(X)\,. \qquad \bigcirc$$

Any tame endomorphism $f = f_\tau^s$ of $\mathcal{C}_o^n(X)$ may be pushed down to a unique tame endomorphism $F = F_T$ of $\mathcal{C}^n(X)$ such that f and F fit into (2.1); T and F are defined by

$$T(Q) = \tau(q) \ \text{ and } \ F(Q) = F_T(Q) = T(Q)Q = pf_\tau^s(q)\,, \tag{2.12}$$

where $Q \in \mathcal{C}^n(X)$ and q is any point of the fiber $p^{-1}(Q) \subset \mathcal{C}_o^n(X)$.

Lifting and Equivariance Theorems show that in order to prove Tame Map Theorem it suffices to prove the following

Theorem 2.12 (EQUIVARIANT MAP THEOREM)**.** *For $n > 6 - t(X)$ every strictly equivariant endomorphism of $\mathcal{C}_o^n(X)$ is tame.* $\bigcirc$

2.4. Step 4: Coherence Theorem. The group $\operatorname{Aut} X$ is exactly $t(X)$ times transitive in X (see Notation 1.11). Let $n > t(X)$. Two ordered n point subsets $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \subset X$ may be carried to each other by an automorphism $A \in \operatorname{Aut} X$ if and only if they satisfy

$$\frac{y_1 - y_m}{y_1 - y_2} = \frac{x_1 - x_m}{x_1 - x_2} \quad \text{for all } m = 3, ..., n \quad \text{when} \quad X = \mathbb{C}$$

and

$$\frac{y_m - y_1}{y_m - y_2} : \frac{y_1 - y_3}{y_2 - y_3} = \frac{x_m - x_1}{x_m - x_2} : \frac{x_1 - x_3}{x_2 - x_3} \quad \text{for all } m = 4, ..., n \quad \text{when} \quad X = \mathbb{CP}^1\,.$$

When the corresponding condition is fulfilled a required $A \in \operatorname{Aut} X$ is uniquely defined and regular in $x, y \in \mathcal{C}_o^n(X)$.

Let $n > t(X)$ and let $f = (f_1, ..., f_n)$ be an endomorphism of $\mathcal{C}_o^n(X)$. If we are looking for a representation $f = f_\tau$ with a morphism $\tau\colon \mathcal{C}_o^n(X) \to \operatorname{Aut} X$, we must for every $q = (q_1, ..., q_n) \in \mathcal{C}_o^n(X)$ find $\tau(q) \in \operatorname{Aut} X$, which carries $(q_1, ..., q_n)$ to $(f_1(q), ..., f_n(q))$ and nicely depends on q. This is possible if and only if the ordered sets $x = (q_1, ..., q_n) \subset X$ and $y = (f_1(q), ..., f_n(q)) \subset X$ satisfy the necessary and sufficient condition mentioned above. Taking into account that permutations are admitted, we can conclude that Equivariant Map Theorem is an immediate consequence of the following result.

Theorem 2.13 (Coherence Theorem). *Let $n > 6 - t(X)$ and let $f = (f_1, ..., f_n)$ be a strictly equivariant endomorphism of $\mathcal{C}_o^n(X)$. Then there exists a permutation $\sigma \in \mathbf{S}(n)$ such that for all $q \in \mathcal{C}_o^n(X)$ and all $r = t(X) + 1, ..., n$*

$$\text{(SR)}\quad \frac{f_1(\sigma q) - f_r(\sigma q)}{f_1(\sigma q) - f_2(\sigma q)} = \frac{q_1 - q_r}{q_1 - q_2} \quad \text{if } X = \mathbb{C}\,,$$

$$\text{(CR)}\quad \frac{f_r(\sigma q) - f_1(\sigma q)}{f_r(\sigma q) - f_2(\sigma q)} : \frac{f_1(\sigma q) - f_3(\sigma q)}{f_2(\sigma q) - f_3(\sigma q)} = \frac{q_r - q_1}{q_r - q_2} : \frac{q_1 - q_3}{q_2 - q_3} \quad \text{if } X = \mathbb{CP}^1\,. \qquad \bigcirc$$

Explanation. For the sake of neatness, let us explain how to deduce Equivariant Map Theorem from Coherence Theorem. We start with the following simple remark, which will be useful in what follows:

Remark* 2.14.** For $n > t(X)$ there exist non-empty Zariski open subsets $U \subset \mathcal{C}_o^n(X)$ and $V \subset \mathcal{C}^n(X)$ such that ***a) if $Aq = sq$ for some $q \in U$, $A \in \operatorname{Aut} X$ and $s \in \mathbf{S}(n)$ then $A = \mathrm{id}$ and $s = 1$; ***b***) if $AQ = Q$ for some $Q \in V$ and $A \in \operatorname{Aut} X$ then $A = \mathrm{id}$. Indeed, the set of all points $q = (q_1, ..., q_n) \in \mathcal{C}_o^n(X)$ such that some its non-trivial permutation $sq = (q_{s^{-1}(1)}, ..., q_{s^{-1}(n)})$ may be also written as $Aq = (Aq_1, ..., Aq_n)$ with some $A \in \operatorname{Aut} X$ is a proper Zariski closed subset of $\mathcal{C}_o^n(X)$; this implies both (***a***) and (***b***). $\bigcirc$

For each $q = (q_1, ..., q_n) \in \mathcal{C}_o^n(X)$ we define the transformation $\tau(q) \in \operatorname{Aut} X$ by one of the following equations:

$$\frac{f_1(\sigma q) - \tau(q)x}{f_1(\sigma q) - f_2(\sigma q)} = \frac{q_1 - x}{q_1 - q_2} \quad \text{if } x \in X = \mathbb{C}\,, \tag{2.13}$$

$$\frac{\tau(q)x - f_1(\sigma q)}{\tau(q)x - f_2(\sigma q)} : \frac{f_1(\sigma q) - f_3(\sigma q)}{f_2(\sigma q) - f_3(\sigma q)} = \frac{x - q_1}{x - q_2} : \frac{q_1 - q_3}{q_2 - q_3} \quad \text{if } x \in X = \mathbb{CP}^1. \tag{2.14}$$

Clearly $\tau \colon \mathcal{C}_o^n(X) \to \operatorname{Aut} X$ is a morphism. Conditions (SR) and (CR) respectively imply that $\tau(q)q_m = f_m(\sigma q)$ for each $m = 1, ..., n$ and all $q = (q_1, ..., q_n) \in \mathcal{C}_o^n(X)$; thereby $\tau(q)q = f(\sigma q) = \alpha(\sigma)f(q)$, or, which is the same, $f(q) = \alpha(\sigma^{-1})\tau(q)q$ for all $q \in \mathcal{C}_o^n(X)$. To complete the proof, we must check that the morphism τ is $\mathbf{S}(n)$ invariant. For every $s \in \mathbf{S}(n)$ and all $q \in \mathcal{C}_o^n(X)$ we have

$$\begin{aligned} s\tau(sq)q &= \tau(sq)sq = f(\sigma sq) = \alpha(\sigma s)f(q) \\ &= \alpha(\sigma s)f(\sigma\sigma^{-1}q) = \alpha(\sigma s)\tau(\sigma^{-1}q)\sigma^{-1}q = \alpha(\sigma s)\sigma^{-1}\tau(\sigma^{-1}q)q\,, \end{aligned}$$

which can be rewritten as

$$[(\tau(sq))^{-1} \cdot \tau(\sigma^{-1}q)]q = \sigma\alpha(s^{-1}\sigma^{-1})sq\,, \tag{2.15}$$

where $(\tau(sq))^{-1} \cdot \tau(\sigma^{-1}q) \in \operatorname{Aut} X$ is the product in the group $\operatorname{Aut} X$. In view of Remark 2.14, this implies that $\sigma\alpha(s^{-1}\sigma^{-1})s = 1$ and $\tau(sq) = \tau(\sigma^{-1}q)$ for all $s \in \mathbf{S}(n)$ and all q in a non-empty Zariski open subset $U \subset \mathcal{C}_o^n(X)$. Since τ is continuous, the latter relation holds true for all $q \in \mathcal{C}_o^n(X)$ and all $s \in \mathbf{S}(n)$, which shows that the morphism $\tau \colon \mathcal{C}_o^n(X) \to \operatorname{Aut} X$ is $\mathbf{S}(n)$ invariant.

Moreover, it is easily seen that the second identity implies $\alpha(s) = \sigma^{-1}s\sigma$ for all $s \in \mathbf{S}(n)$. Thus, Coherence Theorem implies that *for all $n > 4$ the automorphism α related to a strictly equivariant endomorphism f of $\mathcal{C}_o^n(X)$ must be inner*; in view of Remark 2.9, this α must be inner also for $n = 3$ or 4. $\square$

2.5. Step 5: Simplicial complex of functions omitting two values. Let $f = (f_1, ..., f_n) \colon \mathcal{C}_o^n(X) \to \mathcal{C}_o^n(X)$ be a morphism and let i, j, k, l be distinct. Then

$$sr_{ijk}[f] = (f_k - f_i)/(f_k - f_j) \ \text{ and } \ cr_{ijkl}[f] = [(f_l - f_i)/(f_l - f_j)] : [(f_i - f_k)/(f_j - f_k)]$$

are meromorphic function on $\mathcal{C}_o^n(X)$ and do not take the values 0 and 1. In fact, if $X = \mathbb{C}$ then $sr_{ijk}[f]$ and $cr_{ijkl}[f]$ are both holomorphic; if $X = \mathbb{CP}^1$ this is certainly the case for $cr_{ijkl}[f]$.

In Section 6 (Lemma 6.1) we show that if f is a strictly equivariant endomorphism of $\mathcal{C}_o^n(X)$ then for $X = \mathbb{C}$ all $sr_{ijk}[f]$ and $cr_{ijkl}[f]$ are non-constant and for $X = \mathbb{CP}^1$ this is certainly the case for all $cr_{ijkl}[f]$.

Notice that the functions standing in the left hand sides of relations (SR) and (CR) up to a permutation of the variables $q_1, ..., q_n$ coincide with $sr_{r,2,1}[f]$ and $cr_{1,2,3,r}[f]$ respectively. Thus, what is written above applies to these functions.

It is well known that the set $L(Z)$ of all non-constant holomorphic functions $\lambda \colon Z \to \mathbb{C} \setminus \{0; 1\}$ on an irreducible quasi-projective variety Z is finite. In particular, the set $L(\mathcal{C}_o^n(X))$ is finite and all functions $sr_{ijk}[f]$ and $cr_{ijkl}[f]$ in the case $X = \mathbb{C}$ (or all $cr_{ijkl}[f]$ in the case $X = \mathbb{CP}^1$) must be its elements.

This observation suggests the following way to prove Coherence Theorem. First, compute $L(\mathcal{C}_o^n(X))$, that is, determine explicitly all non-constant holomorphic functions on $\mathcal{C}_o^n(X)$ omitting the values 0 and 1. Second, try to identify (perhaps after a suitable permutation of $q_1, ..., q_n$) the functions $sr_{r,2,1}[f]$ $(r = 3, ..., n)$ and/or $cr_{1,2,3,r}[f]$ $(r = 4, ..., n)$ with certain elements of $L(\mathcal{C}_o^n(X))$, and then... if we are lucky... who knows... To say the least, this looks as a finite combinatorial problem.

To execute the first part of this program, we prove the following theorem.

Theorem 2.15 (NON-CONSTANT HOLOMORPHIC FUNCTIONS $\mathcal{C}_o^n(X) \to \mathbb{C} \setminus \{0;\, 1\}$). ***a***) *Each non-constant holomorphic function $\mu \colon \mathcal{C}_o^n(\mathbb{C}) \to \mathbb{C} \setminus \{0;\, 1\}$ is either*

a simple ratio $sr_{ijk} = (q_k - q_i)/(q_k - q_j)$ or

a cross ratio a cross ratio $cr_{ijkl} = [(q_l - q_i)/(q_l - q_j)] : [(q_i - q_k)/(q_j - q_k)]$

(*for $n = 3$ only simple ratios survive*).

b) *Each non-constant holomorphic function $\mu \colon \mathcal{C}_o^n(\mathbb{CP}^1) \to \mathbb{C} \setminus \{0;\, 1\}$ is a cross ratio $cr_{ijkl} = [(q_l - q_i)/(q_l - q_j)] : [(q_i - q_k)/(q_j - q_k)]$ (for $n = 3$ no such function exists*). $\bigcirc$

To avoid unpleasant combinatorics in the course of the realization of the second part of our plan, we must try to find the *"genuine"* reason forcing the sequence of the functions in the left hand side of (SR) or (CR) to be of the form prescribed by the right hand side (perhaps after a suitable rearrangement).

To this end, we notice that the set $L(Z)$ of all non-constant holomorphic functions $\lambda\colon Z \to \mathbb{C}\setminus\{0;1\}$ on an irreducible quasi-projective variety Z carries a natural structure of a finite simplicial complex $L_{\vartriangle}(Z)$, which turns out to be amazingly useful for our purposes.[2]

A subset $\Delta^m = \{\mu_0, ..., \mu_m\}$ consisting of $m+1$ distinct elements of $L(Z)$ is declared to be an m *simplex* if the quotients $\mu_i : \mu_j \in L(z)$ for all $i \neq j$. In other words,

$$\Delta^m\colon Z \ni z \mapsto (\mu_0(z), ..., \mu_m(z)) \in \mathbb{C}^{m+1} \tag{2.16}$$

must be a holomorphic map such that all its components μ_i and all their pairwise quotients $\mu_i : \mu_j$ $(i \neq j)$ are non-constant and its image $\Delta^m(Z)$ is contained in the domain

$$\mathcal{M}^{m+1} = \{w = (w_0, ..., w_m) \in \mathbb{C}^{m+1} \mid w_i \neq 0, 1\ \forall\, i \text{ and } w_i \neq w_j\ \forall\, i \neq j\}\,. \tag{2.17}$$

For instance, *the sets of functions*

$$\{(q_1 - q_r)/(q_1 - q_2) \mid r = 3, ..., n\} \quad \text{and}$$
$$\{[(q_r - q_1)/(q_r - q_2)] : [(q_1 - q_3)/(q_2 - q_3)] \mid r = 4, ..., n\}$$

standing in the right hand sides of (SR) *and* (CR) *are simplices* of dimension $n-3$ and $n-4$ respectively.

For a holomorphic map $f\colon Y \to Z$ of irreducible quasi-projective varieties the correspondence

$$\lambda \mapsto f^*(\lambda) \stackrel{\text{def}}{=\!=} \lambda \circ f\colon Y \xrightarrow{f} Z \xrightarrow{\lambda} \mathbb{C}\setminus\{0;1\}$$

turns out to be a *simplicial map* $f^*\colon L_{\vartriangle}(Z) \to L_{\vartriangle}(Y)$, provided that $\lambda\circ f \neq \text{const}$ for each $\lambda \in L(Z)$ (this is certainly the case if f is dominant but the latter condition is not a necessary one). Moreover, the map f^* preserves dimension of simplices.

***Remark* 2.16.** The simplicial complex $L_{\vartriangle}(Z)$ itself and all its standard topological invariants are invariants of the underlying complex manifold Z with respect to biholomorphic isomorphisms. In Section 5.3 we shaw that $\dim L_{\vartriangle}(\mathcal{C}_o^n(\mathbb{C})) = n-3$ for $n \geq 3$ and $\dim L_{\vartriangle}(\mathcal{C}_o^n(\mathbb{CP}^1)) = n-4$ for $n \geq 4$, which plays important part in the proof of Coherence Theorem. It is not difficult to show that for $n \geq 4$ the Euler characteristic $\chi(L_{\vartriangle}(\mathcal{C}_o^n(\mathbb{CP}^1))) = n(n-1)(n-2)(13-3n)/4$; for $n > 4$ the complex $L_{\vartriangle}(\mathcal{C}_o^n(\mathbb{CP}^1))$ is connected and its highest homology group $H_{n-4}(L_{\vartriangle}(\mathcal{C}_o^n(\mathbb{CP}^1), \mathbb{Z}))$ is trivial. Moreover, $\pi_1(L_{\vartriangle}(\mathcal{C}_o^5(\mathbb{CP}^1))) = \mathbb{F}_{31}$ and $\operatorname{rank} H_1(L_{\vartriangle}(\mathcal{C}_o^6(\mathbb{CP}^1), \mathbb{Z})) = 151$. Does it mean anything?

The complex $L_{\vartriangle}(Z)$ always has the following rather special property: if $\{\mu_0, \mu_1\}$, $\{\mu_1, \mu_2\}$ and $\{\mu_0, \mu_2\}$ are simplices then the ordered set $\Delta^2 = [\mu_0, \mu_1, \mu_2]$ is an ordered 2 simplex and therefore the chain $c = [\mu_1, \mu_2] - [\mu_0, \mu_2] + [\mu_0, \mu_1] = \partial\Delta^2$ is a 1 cycle homologic to 0. This does not mean, however, that $H_1(L_{\vartriangle}(Z), \mathbb{Z}) = 0$, as some "longer" 1 cycle may be not a boundary. ◯

[2] I came to this almost obvious structure just very recently when the first version of this paper was already written. In this renewed version the proof does not become much shorter but its main ideas are now utterly clear. Strangely enough I paid no attention to this structure 30 years ago. But seemingly nobody did so...

We show (see Section 6, Lemma 6.1) that the above construction applies to every strictly equivariant endomorphism f of $\mathcal{C}_o^n(X)$. It follows that *the sets of functions standing in the left hand sides of* (SR) *and* (CR) *also are simplices* (of dimension $n-3$ and $n-4$ respectively).

The $\mathbf{S}(n)$ action in $\mathcal{C}_o^n(X)$ induces an $\mathbf{S}(n)$ action on the set of all simplices $\Delta^m \subseteq L(\mathcal{C}_o^n(X))$ of any fixed dimension m. In view of what was explained above, Coherence Theorem just tells us that *the simplices of functions standing in the left hand sides of* (SR) *and* (CR) *belong to the* $\mathbf{S}(n)$ *orbits of the simplices of functions standing in the right hand sides of* (SR) *and* (CR) *respectively.*

Using the explicit form of functions $\lambda \in L(\mathcal{C}_o^n(X))$ provided by Theorem 2.15, we will determine *all* the orbits of the latter $\mathbf{S}(n)$ action on the set of m simplices. In fact this will prove Coherence Theorem in the form presented in the previous paragraph and complete the proof of Tame Map Theorem.

3. Holomorphic universal covers and universal Teichmüller families

In this section we exhibit holomorphic universal coverings of certain configuration spaces, which will play the key part in the proof of Linked Map Theorem in Section 7. To my best knowledge, the holomorphic universal coverings of $\mathcal{C}^n(\mathbb{C})$ and some related spaces were for the first time described by S. Kaliman [**Kal75, Kal93**]. We do this in a different manner, which will be more convenient for our purposes.

At the end of the section we describe the standard normal series with free factors in the pure braid group PB_n and construct a similar series in $PB_n(S^2)$. In particular, this leads to the proof of Theorem 2.6 formulated in Section 2.2. We also present some information about higher homotopy groups of $\mathcal{C}^n(X)$.

Some of the results discussed below are certainly well-known, especially those dealing with homotopy groups; see, for instance, [**FoxNeu62, FadNeu62, FadHus01**]. Unlike the authors of the quoted sources, we systematically take into account that configuration spaces under consideration are complex manifolds. This was extensively used by V. Arnold but in fact goes back to A. Hurwitz and O. Zariski. There are certain advantages in doing so even when dealing with "purely topological" objects related to our configuration spaces. Say the direct decomposition (3.11) and projections (3.14) are "invisible" from the real point of view. (Notice that E. M. Feichtner and M. Ziegler [**FeiZie99**] pointed out the direct decomposition (3.11) in the case $X = \mathbb{CP}^1$ but seemingly they were not interested in its complex-analytic nature. What they denoted by $M_{0,n}$ and referred to as (ordered!) "moduli spaces of n-punctured complex projective lines" in the present paper is denoted by $\mathcal{D}^{n-3}(\mathbb{CP}^1)$ or $\mathcal{C}_o^{n-3}(\mathbb{C}^{②})$; see the next section.)

3.1. Holomorphic universal coverings of certain configuration spaces.

We need now to deal with configuration spaces of the line $\mathbb{C}$ punctured at several distinct points, say such as $\mathcal{C}_o^n(\mathbb{C} \setminus \{0, 1, ..., k-1\})$. To shorten notation, we set

$$\mathbb{C}^{ⓚ} = \mathbb{C} \setminus \{0, 1, ..., k-1\} \quad (k \geq 1)\,. \tag{3.1}$$

For instance, for $m > 1$

$$\mathcal{C}_o^m(\mathbb{C}^{②}) = \{z = (z_1, ..., z_m) \in \mathbb{C}^m \mid z_i \neq 0, 1\ \forall\, i \text{ and } z_i \neq z_j\ \forall\, i \neq j\} \tag{3.2}$$

(compare to (2.17)). To avoid the doubling in formulations related to two cases $X = \mathbb{C}$ and $X = \mathbb{CP}^1$ under consideration, we will assume that $n \geq t(X)$ (see Notation 1.11) and use the notation

$$\mathcal{D}^{n-t(X)}(X) \stackrel{\text{def}}{=\!=} \begin{cases} \{q = (q_1, ..., q_n) \in \mathcal{C}_o^n(\mathbb{C}) \mid q_{n-1} = 0,\ q_n = 1\} \qquad \text{if } X = \mathbb{C}\,, \\ \{q = (q_1, ..., q_n) \in \mathcal{C}_o^n(\mathbb{CP}^1) \mid q_{n-2} = 0,\ q_{n-1} = 1,\ q_n = \infty\} \\ \hfill \text{if } X = \mathbb{CP}^1\,. \end{cases}$$

For $n > t(X)$ we identify $\mathcal{D}^{n-t(X)}(X)$ and $\mathcal{C}_o^{n-t(X)}(\mathbb{C}^{②})$ via the correspondence

$$\begin{aligned} &\mathcal{D}^{n-2}(\mathbb{C}) \ni (q_1, ..., q_{n-2}, 0, 1) \stackrel{\cong}{\longleftrightarrow} (q_1, ..., q_{n-2}) \in \mathcal{C}_o^{n-2}(\mathbb{C}^{②})\,, \\ &\mathcal{D}^{n-3}(\mathbb{CP}^1) \ni (q_1, ..., q_{n-3}, 0, 1, \infty) \stackrel{\cong}{\longleftrightarrow} (q_1, ..., q_{n-3}) \in \mathcal{C}_o^{n-3}(\mathbb{C}^{②})\,. \end{aligned} \tag{3.3}$$

It is well known (see, for instance, [**Kal75, Kal93**]) that the holomorphic universal cover $\widetilde{\mathcal{C}_o^m(\mathbb{C}^{②})}$ of $\mathcal{C}_o^m(\mathbb{C}^{②})$ may be viewed as the Teichmüller space $\mathbf{T}(0, m+3)$ of the Riemann sphere with $m+3$ punctures.[3] The construction presented in the quoted papers makes use of the classical techniques related to quasi-conformal mappings, which is, in a sense, inevitable when dealing with Teichmüller spaces. We present here a more direct construction of holomorphic covering mappings[4]

$$\tau\colon \mathbf{T}(0, m+3) \to \mathcal{C}_o^m(\mathbb{C}^{②})\,, \tag{3.4}$$

$$\begin{aligned} &\pi\colon \widetilde{\operatorname{Aut} X} \times \mathbf{T}(0, n+3-t(X)) \to \mathcal{C}_o^n(X) \quad \text{and} \\ &\Pi\colon \widetilde{\operatorname{Aut} X} \times \mathbf{T}(0, n+3-t(X)) \to \mathcal{C}^n(X) \end{aligned} \tag{3.5}$$

based on the concept of the universal Teichmüller family, or, which is the same, the "universal Teichmüller curve"

$$\mathcal{T}(0, l) := \quad \rho\colon \mathbf{V}'(0, l) \to \mathbf{T}(0, l)\,, \tag{3.6}$$

which may be roughly described as follows.

Any point $\mathbf{t}$ of the Teichmüller space $\mathbf{T}(0, l)$ "is" a curve $\Gamma_{\mathbf{t}}$ of type $(0, l)$ (i. e., the Riemann sphere $\mathbb{CP}^1 \cong \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ punctured at certain l points and endowed with a marked free basis of its fundamental group, which we prefer to forget about). The natural projection

$$\rho\colon \mathbb{CP}^1 \times \mathbf{T}(0, l) \to \mathbf{T}(0, l) \tag{3.7}$$

possesses l holomorphic sections $\mathbf{s}_i\colon \mathbf{T}(0, l) \ni \mathbf{t} \longmapsto (s_i(\mathbf{t}), \mathbf{t}) \in \mathbb{CP}^1 \times \mathbf{T}(0, l)$ with the following properties: all the maps $s_1, ..., s_l\colon \mathbf{T}(0, l) \to \mathbb{CP}^1$ are holomorphic and take

[3] Notice that $\mathbf{T}(0, m+3)$ is homeomorphic to $\mathbb{R}^{2m}$; moreover, it is biholomorphic to a holomorphically convex Bergmann domain in $\mathbb{C}^m$.

[4] $\widetilde{\operatorname{Aut} X}$ is the holomorphic universal covering of the complex Lie group $\operatorname{Aut} X$; see (3.13).

distinct values at each point $\mathbf{t} \in \mathbf{T}(0,l)$,

$$s_{l-2}(\mathbf{t}) \equiv 0\,, \quad s_{l-1}(\mathbf{t}) \equiv 1\,, \quad s_l(\mathbf{t}) \equiv \infty^{\,5} \tag{3.8}$$

and, moreover, for every $\mathbf{t} \in \mathbf{T}(0,l)$ the curve $\Gamma(\mathbf{t}) \stackrel{\mathrm{def}}{=\!=} \mathbb{CP}^1 \setminus \{s_1(\mathbf{t}), ..., s_l(\mathbf{t})\}$ is "the same" curve $\Gamma_{\mathbf{t}}$ whose "shape" in $\mathbf{T}(0,l)$ is the point $\mathbf{t}$. We set

$$\mathbf{V}'(0,l) = [\mathbb{CP}^1 \times \mathbf{T}(0,l)] \setminus [\mathbf{s}_1(\mathbf{T}(0,l)) \cup ... \cup \mathbf{s}_l(\mathbf{T}(0,l))]\,; \tag{3.9}$$

restricting (3.7) to $\mathbf{V}'(0,l)$ we obtain (3.6). Because of normalization (3.8), the mappings $s_1, ..., s_{l-3}$ are just holomorphic functions on $s_i\colon \mathbf{T}(0,l) \to \mathbb{C} \setminus \{0,1\}$ distinct at each point $\mathbf{t} \in \mathbf{T}(0,l)$.

The family $\mathcal{T}(0,l) := \ \rho\colon \mathbf{V}'(0,l) \to \mathbf{T}(0,l)$ is *universal* in the following sense:

Universality: *Let $\mathcal{F} := \ \pi\colon X \to Y$ be a holomorphic family of curves of type $(0,l)$ over a simply connected base Y, i. e., π is a holomorphic map of complex manifolds, Y is simply connected and for each $y \in Y$ the fiber $\pi^{-1}(y)$ is a curve of type $(0,l)$. Then there exists a holomorphic map $\phi\colon Y \to \mathbf{T}(0,l)$ such that the induced family $\phi^*(\mathcal{T}(0,l))$ is equivalent to the family $\mathcal{F}$.*

With the above construction at our disposal, we take $l = m+3$ and define the required holomorphic covering map (3.4) as follows:

$$\tau\colon \mathbf{T}(0,m+3) \ni \mathbf{t} \mapsto (s_1(\mathbf{t}), ..., s_m(\mathbf{t})) \in \mathcal{C}_o^m(\mathbb{C}^{②})\,, \tag{3.10}$$

where $\mathbf{s}_i = (s_i, \cdot)\colon \mathbf{T}(0,m+3) \to \mathbb{CP}^1 \qquad (i = 1, ..., m+3)$ are the corresponding normalized sections of the projection $\mathbf{V}'(0,m+3) \to \mathbf{T}(0,m+3)$.

Furthermore, let $\Theta\colon \mathcal{C}_o^n(X) \to \operatorname{Aut} X$ denote the morphism whose value at any point $q = (q_1, ..., q_n) \in \mathcal{C}_o^n(X)$ is a unique element $\Theta(q) \in \operatorname{Aut} X$ that carries $(0,1)$ to (q_{n-1}, q_n) when $X = \mathbb{C}$ and $(0,1,\infty)$ to (q_{n-2}, q_{n-1}, q_n) when $X = \mathbb{CP}^1$. The diagonal $\operatorname{Aut} X$ action in $\mathcal{C}_o^n(X)$ is free and biregular, and each its orbit $(\operatorname{Aut} X)q$ intersects the submanifold $\mathcal{D}^{n-t(X)}(X)$ in a single point that is exactly $(\Theta(q))^{-1}q$. In view of (3.3), this provides us with the biregular isomorphisms

$$j_{X,n}\colon \mathcal{C}_o^n(X) \xrightarrow{\cong} \operatorname{Aut} X \times \mathcal{D}^{n-t(X)}(X) = \operatorname{Aut} X \times \mathcal{C}_o^{n-t(X)}(\mathbb{C}^{②})\,, \tag{3.11}$$

where $j_{X,n}$ and its inverse $j_{X,n}^{-1}$ are defined by

$$\begin{aligned} j_{X,n}(q) = (\Theta(q),\ (\Theta(q))^{-1}q)\,, \qquad & j_{X,n}^{-1}(A,z) = Az \\ & (q \in \mathcal{C}_o^n(X)\,, \quad A \in \operatorname{Aut} X\,, \quad z \in \mathcal{D}^{n-t(X)}(X))\,. \end{aligned} \tag{3.12}$$

Let $\mathbb{C} \rightthreetimes \mathbb{C}$ be the semi-direct product with the group multiplication $(b,\zeta) \cdot (b',\zeta') = (b + e^{\zeta}b', \zeta + \zeta')$ and the complex structure of $\mathbb{C} \times \mathbb{C}$. For $X = \mathbb{C}$ or $X = \mathbb{CP}^1$ the holomorphic universal covering $\pi_X\colon \widetilde{\operatorname{Aut} X} \to \operatorname{Aut} X$ is given by

$$\begin{aligned} &\pi_{\mathbb{C}}\colon \quad \mathbb{C} \rightthreetimes \mathbb{C} \ni (b,\zeta) \mapsto \pi_{\mathbb{C}}(b,\zeta) = A \in \mathbf{Aff}\ \mathbb{C}\,, \quad Az = e^{\zeta}z + b \ \ \forall\, z \in \mathbb{C}\,, \\ &\pi_{\mathbb{CP}^1}\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{SL}(2,\mathbb{C})/\{\pm I\} = \mathbf{PSL}(2,\mathbb{C}) = \operatorname{Aut} \mathbb{CP}^1\,. \end{aligned} \tag{3.13}$$

[5] This normalization is not a necessary but a very convenient one.

Finally, we define the required holomorphic covering maps (3.5) as follows:[6]

$$\begin{aligned}\pi\colon \widetilde{\operatorname{Aut} X}\times \mathbf{T}(0,n+3-t(X)) &\ni (A,\mathbf{t})\mapsto \pi(A,\mathbf{t})\\ &= (\pi_X(A)s_1(\mathbf{t}),...,\pi_X(A)s_n(\mathbf{t}))\in \mathcal{C}_o^n(X)\,,\\ \Pi\colon \widetilde{\operatorname{Aut} X}\times \mathbf{T}(0,n+3-t(X)) &\ni (A,\mathbf{t})\mapsto \Pi(A,\mathbf{t})\\ &= \{\pi_X(A)s_1(\mathbf{t}),...,\pi_X(A)s_n(\mathbf{t})\}\in \mathcal{C}^n(X)\,,\end{aligned}\tag{3.14}$$

where $\pi_X\colon \widetilde{\operatorname{Aut} X}\to \operatorname{Aut} X$ is defined by (3.13).

3.2. Homotopy groups and the standard normal series for $PB_n(X)$. According to (3.11), $\mathcal{C}_o^n(X)\cong \operatorname{Aut} X\times \mathcal{C}_o^{n-t(X)}(\mathbb{C}^{②})$. Hence for $n\ge 3$

$$\begin{aligned}\pi_1(\mathcal{C}_o^n(X)) &= \pi_1(\operatorname{Aut} X)\times \pi_1(\mathcal{C}_o^{n-t(X)}(\mathbb{C}^{②}))\\ &= \begin{cases}\mathbb{Z}\times \pi_1(\mathcal{C}_o^{n-2}(\mathbb{C}^{②})) & \text{if } X=\mathbb{C}\,,\\ (\mathbb{Z}/2\mathbb{Z})\times \pi_1(\mathcal{C}_o^{n-3}(\mathbb{C}^{②})) & \text{if } X=\mathbb{CP}^1\,.\end{cases}\end{aligned}\tag{3.15}$$

The fundamental group $\pi_1(\mathcal{C}_o^{n-2}(\mathbb{C}^{②}))$ is one of the important objects of the classical braid theory, which usually appears by the following way. The projection

$$\mathcal{C}_o^n(\mathbb{C})\ni q=(q_1,...,q_n)\mapsto q_n\in\mathbb{C}$$

is a smooth fiber bundle with the fiber

$$\mathcal{C}_o^{n-1}(\mathbb{C}^{①})=\{(q_1,...,q_{n-1})\in\mathbb{C}\mid q_i\ne 0\ \forall i,\ q_i\ne q_j\ \forall i\ne j\}\,.$$

Since the base $\mathbb{C}$ is contractible, the exact homotopy sequence of this fibration shows that

$$\pi_k(\mathcal{C}_o^n(\mathbb{C}))\cong \pi_k(\mathcal{C}_o^{n-1}(\mathbb{C}^{①}))\ \ \forall\, k\,.$$

The next projection

$$\mathcal{C}_o^{n-1}(\mathbb{C}^{②})\ni (q_1,...,q_{n-1})\mapsto q_{n-1}\in\mathbb{C}\setminus\{0\}=\mathbb{C}^{①}$$

is a smooth fiber bundle as well and the fiber is

$$\mathcal{C}_o^{n-2}(\mathbb{C}^{②})=\{(q_1,...,q_{n-2})\in\mathbb{C}\mid q_i\ne 0,1\ \forall i,\ q_i\ne q_j\ \forall i\ne j\}\,.$$

Since the base $\mathbb{C}^{①}=\mathbb{C}\setminus\{0\}$ is aspherical, we have

$$\pi_k(\mathcal{C}_o^{n-1}(\mathbb{C}^{①}))\cong \pi_k(\mathcal{C}_o^{n-2}(\mathbb{C}^{②}))\ \ \forall\, k\ge 1\,.$$

Moreover, the final segment of the corresponding exact homotopy sequence is

$$1\to \pi_1(\mathcal{C}_o^{n-2}(\mathbb{C}^{②}))\to \pi_1(\mathcal{C}_o^{n-1}(\mathbb{C}^{①}))\to \pi_1(\mathbb{C}\setminus\{0\})\to 0$$

may be written as

$$1\to PB_n^{(2)}\to PB_n^{(1)}\to\mathbb{Z}\to 0\,,$$

where $PB_n^{(2)}\overset{\text{def}}{=\!=}\pi_1(\mathcal{C}_o^{n-2}(\mathbb{C}^{②}))$ and $PB_n^{(1)}\overset{\text{def}}{=\!=}\pi_1(\mathcal{C}_o^{n}(\mathbb{C}))=PB_n$. This projecting process may be continued; say the next step leads to a fiber bundle with the fiber

$$\mathcal{C}_o^{n-3}(\mathbb{C}^{③})=\{(q_1,...,q_{n-3})\in\mathbb{C}\mid q_i\ne 0,1,2\ \forall i,\ q_i\ne q_j\ \forall i\ne j\}$$

[6]We invite the reader to write down these mappings for $X=\mathbb{C}$ and $X=\mathbb{CP}^1$ separately.

over the aspherical base $\mathbb{C}^{②} = \mathbb{C} \setminus \{0, 1\}$, the relations

$$\pi_k(\mathcal{C}_o^{n-3}(\mathbb{C}^{③})) \cong \pi_k(\mathcal{C}_o^{n-2}(\mathbb{C}^{②})) \ \ \forall\, k > 1$$

and the exact sequence

$$1 \to PB_n^{(3)} \to PB_n^{(2)} \to \mathbb{F}_2 \to 1\,,$$

where $PB_n^{(3)} \stackrel{\text{def}}{=\!=} \pi_1(\mathcal{C}_o^{n-3}(\mathbb{C}^{③}))$ and $\mathbb{F}_2$ stands for a free group of rank 2. This inductive procedure eventually leads to the following important result going back to E. Artin [**Art25**], A. Markov [**Mar45**] (at least part (***c***) and the determination of the fundamental groups in part (***a***)) and also to E. Fadell, R. H. Fox and L. Neuwirth [**FadNeu62, FoxNeu62**] (asphericity in parts (***a,b***)).

Theorem 3.1 (Artin-Markov-Fadell-Fox-Neuwirth)**.** ***a***) *The spaces $\mathcal{C}_o^n(\mathbb{C})$ and $\mathcal{C}^n(\mathbb{C})$ are Eilenberg-MacLane $K(\pi, 1)$ spaces for the pure braid group PB_n and the braid group B_n respectively, i. e., these spaces are aspherical and have the fundamental groups that are pointed out.*[7]

b) *All configuration spaces $\mathcal{C}_o^m(\mathbb{C}^{ⓚ})$ are aspherical.*

c) *The pure braid group PB_n possesses a finite series of normal subgroups*

$$\{1\} = PB_n^{(n)} \subset PB_n^{(n-1)} \subset \cdots \subset PB_n^{(2)} \subset PB_n^{(1)} = PB_n \tag{3.16}$$

such that $PB_n^{(r)}/PB_n^{(r+1)} \cong \mathbb{F}_r$ (the free group of rank r), $1 \leq r \leq n-1$. $\square$

The normal series (3.16) will be referred to as the *standard* one.

Replacing n in (3.16) with $n-1$, we see that the fundamental group $\pi_1(\mathcal{C}_o^{n-3}(\mathbb{C}^{②}))$, which participate in the formula for $\pi_1(\mathcal{C}_o^n(\mathbb{CP}^1)) = PB_n(S^2)$ given by (3.15), coincides with second term $PB_{n-1}^{(2)} \lhd PB_{n-1}$ of the standard normal series for PB_{n-1}. Together with (3.11), (3.13) and asphericity of $\mathcal{C}_o^{n-3}(\mathbb{C}^{②})$, this implies the following result.

Theorem 3.2. ***a***) *The pure sphere braid group $PB_3(S^2) \cong \mathbb{Z}/2\mathbb{Z}$. For $n > 3$ the group $PB_n(S^2) \cong (\mathbb{Z}/2\mathbb{Z}) \times PB_{n-1}^{(2)}$, where $PB_{n-1}^{(2)}$ possesses a series of normal subgroups*

$$\{1\} = PB_{n-1}^{(n-1)} \subset PB_{n-1}^{(n-2)} \subset \cdots \subset PB_{n-1}^{(3)} \subset PB_{n-1}^{(2)} \tag{3.17}$$

such that $PB_{n-1}^{(r)}/PB_{n-1}^{(r+1)} \cong \mathbb{F}_r$ (the free group of rank r), $1 \leq r \leq n-2$.

b) *The higher homotopy groups of $\mathcal{C}^n(\mathbb{CP}^1)$ and $\mathcal{C}_o^n(\mathbb{CP}^1)$ are as follows:*

$$\begin{aligned} \pi_k(\mathcal{C}^n(\mathbb{CP}^1)) &= \pi_k(\mathcal{C}_o^n(\mathbb{CP}^1)) = \pi_k(\mathbf{PSL}(2, \mathbb{C})) \\ &= \pi_k(\mathbf{SL}(2, \mathbb{C})) = \pi_k(S^3) = \begin{cases} 0 & \text{if } k = 2\,, \\ \pi_k(S^2) & \text{if } k > 2\,. \end{cases} \end{aligned} \tag{3.18}$$

The following theorem, which contains also Theorem 2.6 formulated in Section 2.2, is an immediate consequence of part (***c***) of Artin-Markov-Fadell-Fox-Neuwirth Theorem, part (***a***) of Theorem 3.2 and Commutator Theorem, which we still need to prove.

[7] This certainly follows from (3.11), (3.13) and (3.14); but the standard argument exhibited above is more preferable since it is quite elementary.

Theorem 3.3. *Perfect groups do not admit non-trivial homomorphisms into the pure braid group $PB_k(X)$. In particular, for $n > 4$ the commutator subgroup $B_n'(X)$ of the braid group $B_n(X)$ does not possess non-trivial homomorphisms to the pure braid group $PB_k(X)$.* ○

4. Proof of Invariance Theorem

The main aim of this section is to prove Invariance Theorem; we follow the way outlined in Section 2.

4.1. Canonical presentations of B_n and $B_n(S^2)$. The inclusion $\mathbb{C} \hookrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$ provides the inclusions $i_o\colon \mathcal{C}_o^n(\mathbb{C}) \hookrightarrow \mathcal{C}_o^n(\mathbb{CP}^1)$ and $i\colon \mathcal{C}^n(\mathbb{C}) \hookrightarrow \mathcal{C}^n(\mathbb{CP}^1)$, which, in turn, induce the epimorphisms of the fundamental groups

$$i_{o*}\colon PB_n = \pi_1(\mathcal{C}_o^n(\mathbb{C})) \to \pi_1(\mathcal{C}_o^n(\mathbb{CP}^1)) = PB_n(S^2) \tag{4.1}$$

and

$$i_*\colon B_n = \pi_1(\mathcal{C}^n(\mathbb{C})) \to \pi_1(\mathcal{C}^n(\mathbb{CP}^1)) = B_n(S^2)\,. \tag{4.2}$$

Recall that the canonical presentation of the Artin braid group $B_n = \pi_1(\mathcal{C}^n(\mathbb{C}))$ involves $n-1$ generators $\sigma_1, .., \sigma_{n-1}$ and the defining system of relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|i-j| \geq 2)\,, \tag{4.3}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad (1 \leq i \leq n-2)\,. \tag{4.4}$$

It is well known that the kernel of the epimorphism $i_*\colon B_n \to B_n(S^2)$ is the normal subgroup $S \lhd B_n$ generated (as a normal subgroup) by the single element

$$s = \sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_2\sigma_1 \in PB_n\,, \tag{4.5}$$

so that $S \lhd PB_n$. Thus, involving also the monomorphisms p_* that correspond to the coverings $p\colon \mathcal{C}_o^n(\mathbb{C}) \to \mathcal{C}^n(\mathbb{C})$ and $p\colon \mathcal{C}_o^n(\mathbb{CP}^1) \to \mathcal{C}^n(\mathbb{CP}^1)$, we obtain the following commutative diagram with exact lines and rows:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & S \arrow[r] \arrow[d] & PB_n \arrow[r, "i_{o*}"'] \arrow[d, "p_*"'] & PB_n(S^2) \arrow[r] \arrow[d, "p_*"] & 1 \\
1 \arrow[r] & S \arrow[r] \arrow[d] & B_n \arrow[r, "i_*"'] \arrow[d, "\mu"'] & B_n(S^2) \arrow[r] \arrow[d, "\mu"] & 1 \\
 & 1 \arrow[r] & \mathbf{S}(n) \arrow[r, "\mathrm{id}"'] \arrow[d] & \mathbf{S}(n) \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd} \tag{4.6}$$

(compare to (1.4)). It follows that the sphere braid group $B_n(S^2) = \pi_1(\mathcal{C}^n(\mathbb{CP}^1))$ admits a presentation (we call it the *canonical* one) with $n-1$ generators $\sigma_1, .., \sigma_{n-1}$

and the defining system of relations

$$\begin{aligned} &\sigma_i\sigma_j = \sigma_j\sigma_i && (|i-j| \geq 2)\,, \\ &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} && (1 \leq i \leq n-2)\,, \\ &\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_2\sigma_1 = 1\,. \end{aligned} \tag{4.7}$$

4.2. Proof of Commutator Theorem. A finite presentation of the commutator subgroup B_n' found in [**GorLin69**] implies immediately that for $n > 4$ the abelianization $B_n'/[B_n', B_n']$ of B_n' is trivial; that is, B_n' is perfect. Since $B_n(S^2)$ is a quotient group of B_n, its commutator subgroup $B_n'(S^2)$ is a quotient group of B_n'. A quotient group of a perfect group is obviously perfect; hence for $n > 4$ the group $B_n'(S^2)$ is perfect as well. □

***Remark* 4.1.** This remark allows to avoid the quotation of [**GorLin69**] in the above proof of Commutator Theorem.

About 15 years after our joint paper [**GorLin69**] was published, E. A. Gorin discovered the following simple and very beautiful relation, which holds for any $n \geq 4$:

$$\sigma_3\sigma_1^{-1} = (\sigma_1\sigma_2)^{-1} \cdot \left[\sigma_3\sigma_1^{-1}, \sigma_1\sigma_2^{-1}\right] \cdot (\sigma_1\sigma_2)\,, \tag{4.8}$$

where $\left[\sigma_3\sigma_1^{-1}, \sigma_1\sigma_2^{-1}\right] = \left(\sigma_3\sigma_1^{-1}\right)^{-1} \cdot \left(\sigma_1\sigma_2^{-1}\right)^{-1} \cdot \left(\sigma_3\sigma_1^{-1}\right) \cdot \left(\sigma_1\sigma_2^{-1}\right)$ is the commutator of the elements $g_1 = \sigma_3\sigma_1^{-1}$ and $g_2 = \sigma_1\sigma_2^{-1}$. Clearly, $g_1, g_2 \in B_n'$, and (4.8) shows that $\sigma_3\sigma_1^{-1} \in [B_n', B_n']$. Hence, the normal subgroup $N \lhd B_n$ generated (as a normal subgroup) by the element $\sigma_3\sigma_1^{-1}$ is contained in $[B_n', B_n']$. For $n > 4$ it follows readily from (4.3) and (4.4) that N contains the whole commutator subgroup B_n'. Indeed, the presentation of B_n/N involves the generators $\sigma_1, ..., \sigma_{n-1}$ and the defining system of relations consisting of (4.3), (4.4) and the additional relation $\sigma_3\sigma_1^{-1} = 1$; since $n > 4$ relations (4.3) and $\sigma_3\sigma_1^{-1} = 1$ imply $\sigma_3\sigma_4 = \sigma_4\sigma_3$; in view of (4.4), this shows that $\sigma_3 = \sigma_4$ and, finally, $\sigma_1 = \sigma_2 = ... = \sigma_{n-1}$. Thus, $B_n/N \cong \mathbb{Z}$ and $N \supseteq B_n'$. Thereby $B_n' \subseteq N \subseteq [B_n', B_n']$. ○

4.3. Proof of Surjectivity Theorem 2.7. Since $B_n(S^2)$ is a quotient group of B_n, it suffices to prove the theorem for B_n. This was done in [**Lin96b**] (Lemma 2.7). It turns out that this theorem is a very convenient technical tool. Since I still did not take care of publication of [**Lin96b**], I decided to outline the proof here.

First, let me recall (in a convenient form) some classical results of E. Artin [**Art47b**].

***Definition* 4.2.** Two group homomorphisms $\phi, \psi\colon G \to H$ are said to be *conjugate* if there is an element $h \in H$ such that $\psi(g) = h\phi(g)h^{-1}$ for all $g \in G$. If this is the case, we write $\phi \sim \psi$; otherwise, we write $\phi \nsim \psi$.

***Definition* 4.3.** A group homomorphism $\psi\colon G \to \mathbf{S}(n)$ is said to be *transitive* if its image $\psi(G)$ is a transitive subgroup of the symmetric $\mathbf{S}(n)$ (meaning the standard action of $\mathbf{S}(n)$ on $\mathbf{\Delta}_n \overset{\text{def}}{=\!=} \{1, ..., n\}$).

Recall that the elements σ_1 and $\alpha = \sigma_1 \cdots \sigma_{n-1}$ generate the whole Artin braid group B_n; this follows from the relations

$$\begin{aligned}\alpha\sigma_j &= \sigma_1 \cdots \sigma_{j-1} \cdot (\sigma_j\sigma_{j+1}\sigma_j) \cdot \sigma_{j+2} \cdots \sigma_{n-1} \\ &= \sigma_1 \cdots \sigma_{j-1} \cdot (\sigma_{j+1}\sigma_j\sigma_{j+1}) \cdot \sigma_{j+2} \cdots \sigma_{n-1} = \sigma_{j+1}\alpha \quad (1 \le j \le n-2)\end{aligned} \tag{4.9}$$

(the same relations hold true in $B_n(S^2)$). Thus, a homomorphism $\psi\colon B_n \to H$ is uniquely defined by its values $\psi(\sigma_1)$, $\psi(\alpha)$.

Artin Theorem. *Let $\psi\colon B_n \to \mathbf{S}(n)$ be a non-cyclic transitive homomorphism.*

***a**) If $n \neq 4$ and $n \neq 6$ then the homomorphism ψ is conjugate to the standard epimorphism $\mu\colon B_n \to \mathbf{S}(n)$.*

***b**) If $n = 6$ and $\psi \not\sim \mu$ then ψ is conjugate to the homomorphism ν_6 defined by*

$$\nu_6(\sigma_1) = (1,2)(3,4)(5,6), \qquad \nu_6(\alpha) = (1,2,3)(4,5).$$

***c**) If $n = 4$ and $\psi \not\sim \mu$ then ψ is conjugate to one of the following three homomorphisms $\nu_{4,1}$, $\nu_{4,2}$, $\nu_{4,3}$:*

$$\begin{array}{lll} \nu_{4,1}(\sigma_1) = (1,2,3,4), & \nu_{4,1}(\alpha) = (1,2); & \\ \nu_{4,2}(\sigma_1) = (1,3,2,4), & \nu_{4,2}(\alpha) = (1,2,3,4); & [\nu_{4,2}(\sigma_3) = \nu_{4,2}(\sigma_1^{-1})] \\ \nu_{4,3}(\sigma_1) = (1,2,3), & \nu_{4,3}(\alpha) = (1,2)(3,4); & [\nu_{4,3}(\sigma_3) = \nu_{4,3}(\sigma_1)]. \end{array}$$

***d**) Except of the case $n = 4$ and $\psi \sim \nu_{4,3}$, the homomorphism ψ is surjective. In the exceptional case when $\psi \sim \nu_{4,3}$ the image of ψ coincides with the alternating subgroup $\mathbf{A}(4) \subset \mathbf{S}(4)$.* ○

The following lemma is the heart of Artin's methods developed in [**Art47b**].[8]

Artin Lemma. *Let $n > 4$ and m be natural numbers. Suppose that there is a prime $p > 2$ such that*

$$m/2 < p \le n - 2\,. \tag{4.10}$$

Then for every non-cyclic transitive homomorphism $\psi\colon B_n \to \mathbf{S}(m)$ the permutation $\widehat{\sigma}_1 = \psi(\sigma_1)$ has at least $n-2$ fixed points; in particular, such a homomorphism cannot exist if $m < n$. ○

We use Artin Lemma to prove the following useful addition to Artin Theorem (see [**Lin72a, Lin74, Lin79**], and also [**Lin96b**], Theorem 2.1).

Theorem 4.4. *For $n \neq 4$ and $n > k$ any homomorphism $\psi\colon B_n(X) \to \mathbf{S}(k)$ is cyclic.*

Proof. As $B_n(S^2)$ is a quotient group of B_n it suffices to prove the theorem for homomorphisms $\psi\colon B_n \to \mathbf{S}(k)$.

For $n \le 3$ all is trivial, since $\mathbf{S}(2) \cong \mathbb{Z}/2\mathbb{Z}$. Suppose that $n > \max\{4, k\}$ and there is a non-cyclic homomorphism $\psi\colon B_n \to \mathbf{S}(k)$. Let $H = \operatorname{Im}\psi \subseteq \mathbf{S}(k)$. For each orbit $O \subseteq \mathbf{\Delta}_k = \{1, ..., k\}$ of H, define the *reduction* $\psi_O\colon B_n \to \mathbf{S}(O)$ of ψ to

[8] In fact, this lemma is not formulated explicitly in [**Art47b**]; moreover, Artin treats only the special case $n = m$. However, the corresponding part of the proof of Lemma 6 in [**Art47b**], after an evident minor modification, leads to the desired result. See [**Lin96b**] for more details.

O as follows: $\psi_O(g) = \psi(g)|O$ for all $g \in B_n$, where $\psi(g)|O$ denotes the restriction of the permutation $\psi(g) \in H \subseteq \mathbf{S}(k)$ to the H-orbit O. Clearly, all such reductions are transitive (meaning that Im ψ_O is a transitive subgroup of the symmetric group $\mathbf{S}(O)$), and for at least one orbit $O_\circ$ the corresponding reduction $\psi_{O_\circ}\colon B_n \to \mathbf{S}(O_\circ) \cong \mathbf{S}(m)$ ($m = \#O_\circ$) must be non-cyclic (otherwise, ψ itself would be abelian and hence cyclic). Since $n > 4$ and $m \leq k < n$, it follows from the famous Chebyshev Theorem that there is a prime $p > 2$ such that $m/2 < p \leq n - 2$. In view of the above inequalities $m \leq k < n$, Artin Lemma implies that any transitive homomorphism $B_n \to \mathbf{S}(m)$ must be cyclic, which contradicts our choice of the orbit $O_\circ$. $\square$

We are now in a position to prove Theorem 2.7.

PROOF. Let $n \neq 4$ and let $\psi\colon B_n(X) \to \mathbf{S}(n)$ be a non-cyclic homomorphism. We must prove that ψ is surjective.

We have already noticed that it suffices to deal with the Artin braid group B_n. If ψ is transitive the statement follows directly from Artin Theorem. Assume that ψ is intransitive. Then $\#O < n$ for any (Im ψ)-orbit $O \subset \mathbf{\Delta}_n$ and Theorem 4.4 implies that the reduction of ψ to any such orbit is cyclic. Hence ψ itself is cyclic, which contradicts our assumption. $\square$

4.4. Proof of Composition Theorem. Suppose, on the contrary, that the composition $\psi = \mu \circ \varphi\colon B_n(X) \xrightarrow{\varphi} B_k(X) \xrightarrow{\mu} \mathbf{S}(k)$ is cyclic. Then $\mu(\varphi(B_n'(X))) = \{1\}$ and hence $\varphi(B_n'(X)) \subseteq \operatorname{Ker}\mu = PB_k(X)$. By Commutator Theorem, the commutator subgroup $B_n'(X)$ is perfect; thus, the inclusion $\varphi(B_n'(X)) \subseteq PB_k(X)$ and Theorem 3.3 imply $\varphi(B_n'(X)) = \{1\}$, which contradicts the assumption that φ is non-cyclic. $\square$

4.5. Homomorphisms $B_n(X) \to B_k(X)$ for $n > k$. Theorem 4.4 and Composition Theorem imply the following result.

Theorem 4.5. *For $n \neq 4$ and $n > k$ any homomorphism $\varphi\colon B_n(X) \to B_k(X)$ is cyclic.*

PROOF. For $n = 3$ all is trivial since $B_2(X)$ is cyclic. Let $n > 4$. Were φ non-cyclic, then, by Composition Theorem, the composition $\psi = \mu \circ \varphi\colon B_n(X) \xrightarrow{\varphi} B_k(X) \xrightarrow{\mu} \mathbf{S}(k)$ would be non-cyclic, which contradicts Theorem 4.4. $\square$

4.6. Proof of Equivariance Theorem. As we mentioned in Section 2, Equivariance Theorem follows immediately from Composition Theorem. $\square$

4.7. Proof of Kernel Theorem. Let $n > 4$ and let $\psi\colon B_n(X) \to \mathbf{S}(n)$ be a non-cyclic homomorphism. By Theorem 2.7, ψ is surjective and, all the more, *transitive*[9].

All non-cyclic transitive homomorphisms $\psi\colon B_n \to \mathbf{S}(n)$ were described by E. Artin [**Art47b**] (see also [**Lin96b**], Section 0.5); it follows from this description that $\operatorname{Ker}\psi = PB_n$.

[9]That is, its image $\psi(B_n(X))$ is a transitive permutation group.

Finally, for a non-cyclic homomorphism $\psi\colon B_n(S^2)\to\mathbf{S}(n)$ of the sphere braid group, we apply the above result to the composition

$$\widetilde{\psi}=\psi\circ i_*\colon B_n\xrightarrow{i_*}B_n(S^2)\xrightarrow{\psi}\mathbf{S}(n)$$

and conclude that $(i_*)^{-1}(\operatorname{Ker}\psi)=\operatorname{Ker}(\psi\circ i_*)=\operatorname{Ker}\widetilde{\psi}=PB_n$. Since the homomorphism $i_*\colon B_n\to B_n(S^2)$ is surjective, $i_*((i_*)^{-1}(H))=H$ for any $H\subseteq B_n(S^2)$; hence $\operatorname{Ker}\psi=i_*((i_*)^{-1}(\operatorname{Ker}\psi))=i_*(PB_n)=PB_n(S^2)$ (see (4.6)). □

4.8. Proof of Invariance Theorem. We prove the following stronger statement:

Let $n>4$. Then $\varphi^{-1}(PB_n(X))=PB_n(X)$ for every non-cyclic endomorphism $\varphi\colon B_n(X)\to B_n(X)$.

By Composition Theorem, the homomorphism $\psi=\mu\circ\varphi\colon B_n(X)\xrightarrow{\varphi}B_n(X)\xrightarrow{\mu}\mathbf{S}(n)$ is non-cyclic, and Kernel Theorem shows that $\operatorname{Ker}\psi=PB_n(X)$. Thus,

$$PB_n(X)=\operatorname{Ker}\psi=\varphi^{-1}(\operatorname{Ker}\mu)=\varphi^{-1}(PB_n(X))\,.$$

Of course, this implies $\varphi(PB_n(X))=\varphi(\varphi^{-1}(PB_n(X)))\subseteq PB_n(X)$. □

As we mentioned in Section 2.1, Lifting Theorem follows immediately from Invariance Theorem. □

5. Holomorphic functions omitting two values

In this section we prove Theorem 2.15, that is, determine explicitly all non-constant holomorphic functions $\mathcal{C}_o^n(\mathbb{C})\to\mathbb{C}\setminus\{0;\,1\}$ and $\mathcal{C}_o^n(\mathbb{CP}^1)\to\mathbb{C}\setminus\{0;\,1\}$.

Furthermore, we establish that the set $L(Z)$ of all non-constant holomorphic functions $Z\to\setminus\{0;\,1\}$ on a quasi-projective algebraic variety Z carries a natural structure of a finite simplicial complex. For $Z=\mathcal{C}_o^n(\mathbb{C})$ and $Z=\mathcal{C}_o^n(\mathbb{CP}^1)$ we study this structure in more details, which provides us with a very useful tool for the proof of Coherence Theorem.

5.1. Proof of Theorem 2.15. For non-zero complex functions f,g of the same variables, we write $f\approx g$ whenever $f=\gamma g$ for some $\gamma\in\mathbb{C}^*$. The following lemma is, in a sense, a multivariate analog of so-called *"abc-lemma"* for polynomials of one variables.

Lemma 5.1. *Let $P,Q,R\in\mathbb{C}[z_1,...,z_n]$ be pairwise coprime polynomials that do not vanish outside of the "big diagonal" $\mathcal{D}=\bigcup_{i\neq j}\{z_i=z_j\}$. Assume that at least one of them is non-constant and*

$$P+Q+R=0\,. \tag{5.1}$$

Then either

$$P\approx z_q-z_r\,,\qquad Q\approx z_r-z_p\,,\qquad R\approx z_p-z_q \tag{5.2}$$

or

$$P\approx(z_q-z_r)(z_s-z_p)\,,\quad Q\approx(z_r-z_p)(z_s-z_q)\,,\quad R\approx(z_p-z_q)(z_s-z_r)\,. \tag{5.3}$$

PROOF. Let a_i, b_i, c_i denote the degrees of P, Q, R with respect to the variable z_i and let $d_i = \max\{a_i, b_i, c_i\}$; the condition (5.1) implies that at least two of the three numbers a_i, b_i, c_i must coincide with d_i.

Every polynomial non-vanishing outside of $\mathcal{D}$ is of the form $\gamma \prod_{i\neq j}(z_i - z_j)^{s_{ij}}$, where $\gamma \in \mathbb{C}^*$ and all $s_{ij} \in \mathbb{Z}_+$. Hence, for every $i_\circ$ we have representations

$$P = P_{i_\circ} \prod_{j\neq i_\circ}(z_{i_\circ} - z_j)^{a_{i_\circ j}}\,, \quad Q = Q_{i_\circ} \prod_{j\neq i_\circ}(z_{i_\circ} - z_j)^{b_{i_\circ j}}\,, \quad R = R_{i_\circ} \prod_{j\neq i_\circ}(z_{i_\circ} - z_j)^{c_{i_\circ j}}\,, \tag{5.4}$$

where

$$P_{i_\circ} \approx \prod_{\substack{i\neq j\\ i,j\neq i_\circ}} (z_i - z_j)^{a_{i_\circ;i,j}}\,, \quad Q_{i_\circ} \approx \prod_{\substack{i\neq j\\ i,j\neq i_\circ}} (z_i - z_j)^{b_{i_\circ;i,j}}\,, \quad R_{i_\circ} \approx \prod_{\substack{i\neq j\\ i,j\neq i_\circ}} (z_i - z_j)^{c_{i_\circ;i,j}} \tag{5.5}$$

are pairwise coprime polynomials not depending on $z_{i_\circ}$, and the products

$$\prod_{j\neq i_\circ}(z_{i_\circ} - z_j)^{a_{i_\circ j}}\,, \quad \prod_{j\neq i_\circ}(z_{i_\circ} - z_j)^{b_{i_\circ j}}\,, \quad \prod_{j\neq i_\circ}(z_{i_\circ} - z_j)^{c_{i_\circ j}}$$

are pairwise coprime polynomials in the variable $z_{i_\circ}$ of degrees $a_{i_\circ}, b_{i_\circ}, c_{i_\circ}$, respectively.

We consider the following two mutual complementary cases:

a) there exists a value $i = p$ such that the triple $\{a_p, b_p, c_p\}$ contains at least two distinct numbers;

b) $a_i = b_i = c_i$ for every i.

a) We may assume that $a_p < b_p = c_p$. Taking into account (5.4) (with $i_\circ = p$) and comparing the leading terms with respect to the variable z_p in identity (5.1), we obtain $Q_p + R_p = 0$. As Q_p and R_p are coprime, they are non-zero constants. Hence

$$Q \approx \prod_{j\neq p}(z_p - z_j)^{b_{pj}}\,, \qquad R \approx \prod_{j\neq p}(z_p - z_j)^{c_{pj}}$$

and identity (5.1) may be written as

$$P_p \prod_{j\neq p}(z_p - z_j)^{a_{pj}} + B \prod_{j\neq p}(z_p - z_j)^{b_{pj}} + C \prod_{j\neq p}(z_p - z_j)^{c_{pj}} = 0\,, \tag{5.6}$$

where $B, C \in \mathbb{C}^*$.

Each exponent a_{pj} in (5.6) must be zero. Indeed, each factor $z_p - z_j$ that actually occurs in the first summand of (5.6) cannot occur in the second and the third ones; were $a_{pj_\circ} > 0$ for some $j_\circ$, the left hand side would be a non-trivial polynomial in the variable $z_{j_\circ}$, which contradicts (5.6).

This means that $a_p = \sum_{j\neq p} a_{pj} = 0$, the polynomial P does not depend on z_p, $P = P_p$, and identity (5.6) takes the form

$$P + B \prod_{j\neq p}(z_p - z_j)^{b_{pj}} + C \prod_{j\neq p}(z_p - z_j)^{c_{pj}} = 0\,.$$

For the particular value $z_p = 0$ the latter identity shows that

$$P + B \prod_{j\neq p}(-z_j)^{b_{pj}} + C \prod_{j\neq p}(-z_j)^{c_{pj}} = 0\,,$$

where the polynomial $P = P_p$ is given by (5.5) (with $i_\circ = p$), and the products

$$\prod_{j\neq p}(-z_j)^{b_{pj}}\,, \qquad \prod_{j\neq p}(-z_j)^{c_{pj}}$$

are coprime. One can readily see that this may only happen if P, Q, R satisfy (5.2) with some distinct $q, r \neq p$, which complete the proof in case (***a***).

b) Since $a_i = b_i = c_i$ for any i, and at least one of the polynomials P, Q, R is non-constant, there is a value $i = s$ such that z_s actually occurs in P, Q, and R. Taking into account (5.4) and (5.5) (with $i_\circ = s$), we can write (5.1) as

$$P_s \prod_{j\neq s}(z_s - z_j)^{a_{sj}} + Q_s \prod_{j\neq s}(z_s - z_j)^{b_{sj}} + R_s \prod_{j\neq s}(z_s - z_j)^{c_{sj}} = 0\,. \tag{5.7}$$

By our choice of s,

$$\sum_{j\neq s} a_{sj} = \sum_{j\neq s} b_{sj} = \sum_{j\neq s} c_{sj} > 0\,;$$

hence there is $p \neq s$ such that $a_{sp} > 0$. As P is coprime with Q, R and contains the factor $(z_s - z_p)^{a_{sp}}$, we have $b_{sp} = c_{sp} = 0$. Comparing the leading terms with respect to z_s in (5.7), we see that

$$P_s + Q_s + R_s = 0. \tag{5.8}$$

Let a'_{sp}, b'_{sp}, c'_{sp} be the degrees of the polynomials P_s, Q_s, R_s with respect to z_p. Clearly,

$$a'_{sp} = a_p - a_{sp} < a_p\,, \quad b'_{sp} = b_p - b_{sp} = b_p = a_p\,, \quad c'_{sp} = c_p - c_{sp} = c_p = a_p\,;$$

hence $a'_{sp} < b'_{sp} = c'_{sp}$; in particular, Q_s and R_s are non-constant. In view of (5.8), this means that the triple $\{P_s, Q_s, R_s\}$ is of the type considered in case (***a***). Therefore,

$$P_s \approx (z_q - z_r)\,, \quad Q_s \approx (z_r - z_p)\,, \quad R_s \approx (z_p - z_q)$$

for certain distinct q, r such that $q \neq p, s$ and $r \neq p, s$. Consequently, identity (5.7) may be written as

$$\begin{gathered} A(z_q - z_r)(z_s - z_p)^{a_{sp}} \prod_{j\neq s,p}(z_s - z_j)^{a_{sj}} \\ + B(z_r - z_p) \prod_{j\neq s,p}(z_s - z_j)^{b_{sj}} + C(z_p - z_q) \prod_{j\neq s,p}(z_s - z_j)^{c_{sj}} = 0 \end{gathered} \tag{5.9}$$

(with certain $A, B, C \in \mathbb{C}^*$). In particular, on the hyperplane $z_p = z_q$ we have the identity

$$A(z_q - z_r)(z_s - z_q)^{a_{sp}} \prod_{j\neq s,p}(z_s - z_j)^{a_{sj}} + B(z_r - z_q) \prod_{j\neq s,p}(z_s - z_j)^{b_{sj}} = 0\,,$$

or

$$A(z_s - z_q)^{a_{sp}} \prod_{j\neq s,p}(z_s - z_j)^{a_{sj}} - B \prod_{j\neq s,p}(z_s - z_j)^{b_{sj}} = 0\,. \tag{5.10}$$

The products $\prod_{j\neq s,p}(z_s - z_j)^{a_{sj}}$ and $\prod_{j\neq s,p}(z_s - z_j)^{b_{sj}}$ must be coprime; hence identity (5.10) can hold true only if

$$A = B, \qquad a_{sj} = 0 \ \text{ for all } \ j \neq s, p\,, \quad b_{sj} = 0 \ \text{ for all } \ j \neq s, p, q\,, \quad \text{and} \quad a_{sp} = b_{sq}\,.$$

Under these conditions we have

$$P \approx (z_q - z_r)(z_s - z_p)^{a_{sp}},\ Q \approx (z_r - z_p)(z_s - z_q)^{a_{sp}},\ R \approx (z_p - z_q) \prod_{j \neq s,p,q} (z_s - z_j)^{c_{sj}}, \tag{5.11}$$

and identity (5.9) takes the form

$$A(z_q - z_r)(z_s - z_p)^{a_{sp}} + A(z_r - z_p)(z_s - z_q)^{a_{sp}} + CR = 0\,. \tag{5.12}$$

Taking into account the conditions $a_q = b_q = c_q$ and $a_r = b_r = c_r$, we obtain from (5.12) and (5.11) that $a_{sp} = 1$, $c_{sj} = 0$ for all $j \neq r$, and $c_{sr} = 1$; hence $P \approx (z_q - z_r)(z_s - z_p)$, $Q \approx (z_r - z_p)(z_s - z_q)$, $R \approx (z_p - z_q)(z_s - z_r)$. □

Now we are ready to prove Theorem 2.15.

PROOF. ***a***) Let $\ell \subset \mathbb{C}^n$ be any com lex straight line such that $\ell \not\subseteq \mathcal{D}$; by Great Picard Theorem, the restriction $\mu|_{\mathcal{C}_o^n(\mathbb{C})\cap\ell}$ is a regular function. Thus, the function μ itself is rational on $\mathbb{C}^n$ and regular on $\mathcal{C}_o^n(\mathbb{C})$. That is, $\mu = -P/R$, where polynomials $P, R \in \mathbb{C}[q_1, ..., q_n]$ are coprime, do not vanish on $\mathcal{C}_o^n(\mathbb{C})$, and at least one of them is non-constant. The polynomial $Q = -P - R$ is coprime to P and R; it has no zeros in $\mathcal{C}_o^n(\mathbb{C})$, as $-Q/R = 1 - \mu$ and the function μ does not assume the value 1. Lemma 5.1 implies the desired result.

b) By (***a***), the restriction of μ to the Zariski open subset $\mathcal{C}_o^n(\mathbb{C}) \subset \mathcal{C}_o^n(\mathbb{CP}^1)$ must be either a simple ratio sr_{ijk} or a cross ratio cr_{ijkl}. However, the function $sr_{ijk} = (q_k - q_i)/(q_k - q_j)$ has a pole in $\mathcal{C}_o^n(\mathbb{CP}^1)$ along the divisor $q_i = \infty$, and only the cross ratios survive when passing from $\mathcal{C}_o^n(\mathbb{C})$ to $\mathcal{C}_o^n(\mathbb{CP}^1)$. □

***Remark* 5.2.** We have seen in Section 3.1 that for any $m \geq 1$ there are a natural identification

$$\mathcal{D}^m(\mathbb{CP}^1) \ni (q_1, ..., q_m, 0, 1, \infty) \overset{\cong}{\longleftrightarrow} (q_1, ..., q_m) \in \mathcal{C}_o^m(\mathbb{C}^{②})\,. \tag{5.13}$$

and a biholomorphic isomorphism

$$j_{\mathbb{CP}^1, m+3}\colon \mathcal{C}_o^{m+3}(\mathbb{CP}^1) \xrightarrow{\cong} \mathbf{PSL}(2, \mathbb{C}) \times \mathcal{D}^m(\mathbb{CP}^1) = \mathbf{PSL}(2, \mathbb{C}) \times \mathcal{C}_o^m(\mathbb{C}^{②})\,; \tag{5.14}$$

$j_{\mathbb{CP}^1, m+3}$ and its inverse $j^{-1}_{\mathbb{CP}^1, m+3}$ are defined by

$$\begin{gathered} j_{\mathbb{CP}^1, m+3}(q) = (\Theta(q),\ (\Theta(q))^{-1}q)\,, \qquad j^{-1}_{\mathbb{CP}^1, m+3}(A, z) = Az \\ (q \in \mathcal{C}_o^{m+3}(\mathbb{CP}^1)\,,\ \ A \in \mathbf{PSL}(2, \mathbb{C})\,,\ \ z \in \mathcal{D}^m(\mathbb{CP}^1))\,, \end{gathered} \tag{5.15}$$

where $\Theta\colon \mathcal{C}_o^{m+3}(\mathbb{CP}^1) \to \mathbf{PSL}(2, \mathbb{C})$ denote the morphism whose value at any point $q = (q_1, ..., q_{m+3}) \in \mathcal{C}_o^{m+3}(\mathbb{CP}^1)$ is a unique element $\Theta(q) \in \mathbf{PSL}(2, \mathbb{C})$ that carries $(0, 1, \infty)$ to $(q_{m+1}, q_{m+2}, q_{m+3})$. Due to the Picard theorem any holomorphic function $\mathcal{C}_o^{m+3}(\mathbb{CP}^1) \to \mathbb{C} \setminus \{0;\, 1\}$ is constant on each fiber of the projection $\mathcal{C}_o^{m+3}(\mathbb{CP}^1) \to \mathcal{C}_o^m(\mathbb{C}^{②})$; therefore there is a one-to-one correspondence between the set $L(\mathcal{C}_o^m(\mathbb{C}^{②}))$ of all non-constant holomorphic functions $\mathcal{C}_o^m(\mathbb{C}^{②}) \to \mathbb{C}\setminus\{0;\, 1\}$ and the set $L(\mathcal{C}_o^{m+3}(\mathbb{CP}^1))$ of all non-constant holomorphic functions $\mathcal{C}_o^{m+3}(\mathbb{CP}^1) \to \mathbb{C} \setminus \{0;\, 1\}$. Due to Theorem 2.15 we know all functions of the latter set; this provides us with an explicit description of all functions $h \in L(\mathcal{C}_o^{m+3}(\mathbb{CP}^1))$, which was also obtained by S. Kaliman in a different way.

Similarly, using an isomorphism $\mathcal{C}_o^m(\mathbb{C}^*)) \cong \mathbb{C}^* \times \mathcal{C}_o^{m-1}(\mathbb{C}^{②})$ we may identify the set $L(\mathcal{C}_o^m(\mathbb{C}^*))$ first with $L(\mathcal{C}_o^{m-1}(\mathbb{C}^{②}))$ and then with $L(\mathcal{C}_o^{m+2}(\mathbb{CP}^1))$, which provides an explicit description of all non-constant holomorphic functions $\mathcal{C}_o^m(\mathbb{C}^*) \to \mathbb{C} \setminus \{0;\ 1\}$ obtained in [**Zin77**] by a different way. ◯

5.2. Simplicial complex of holomorphic functions omitting two values. Here I will describe an almost obvious combinatorial structure, which occurs amazingly useful for the proof of Coherence Theorem. I came to it very recently when the first version of this paper was already written. In this renewed version the proof does not become much shorter but its main ideas are now absolutely clear. Strangely enough I paid no attention to this structure 30 years ago. But seemingly nobody did so...

For an irreducible quasi-projective variety Z we denote by $L(Z)$ the set of all non-constant holomorphic functions $\lambda\colon Z \to \mathbb{C} \setminus \{0;1\}$. It is well known that all $\lambda \in L(Z)$ are regular and $L(Z)$ is finite.[10] Clearly $\lambda^{-1} \in L(Z)$ whenever $\lambda \in L(Z)$ and $\iota\colon L(Z) \ni \lambda \mapsto \iota(\lambda) \overset{\text{def}}{=\!=} \lambda^{-1} \in L(Z)$ is an involution. The set $L(Z)$ carries a natural structure of a finite simplicial complex $L_\vartriangle(Z)$ described in the following definition.

***Definition* 5.3.** Let $\mu, \nu \in L(Z)$; we say that ν is *a proper divisor of* μ and write $\nu \mid \mu$ if the quotient $\lambda = \mu : \nu \in L(Z)$, i. e., $\lambda \neq \text{const}$ and $\lambda(z) \neq 1$ for all $z \in Z$; otherwise, we write $\nu \nmid \mu$. Clearly, $\nu \mid \mu$ is equivalent to $\mu \mid \nu$.

A non-empty subset (respectively, a non-empty ordered subset) $\Delta \subseteq L(Z)$ is said to be a *simplex* (respectively, an *ordered simplex*) of dimension $m = \#\Delta - 1$ if $\nu \mid \mu$ for each pair of distinct $\mu, \nu \in L(Z)$; the functions $\mu \in \Delta$ are said to be the *vertices* of Δ. Any non-empty subset of an (ordered) simplex is an (ordered) simplex. We define the *boundary* $\partial\Delta$ of an ordered simplex $\Delta = [\mu_0, ..., \mu_m] \subseteq L(Z)$ by the standard formula

$$\partial\Delta \overset{\text{def}}{=\!=} \sum_{i=0}^{m} (-1)^i [\mu_0, ..., \widehat{\mu_i}, ..., \mu_m]\,. \tag{5.16}$$

Thereby $L(Z)$ becomes a finite simplicial complex $L_\vartriangle(Z)$ with the simplicial involution ι.

When we need to emphasize that $L(Z)$ is regarded as a simplicial complex, i. e., the collection of all simplices $\Delta \subset L(Z)$ rather than the set of all vertices $\mu \in L(Z)$ we write $L_\vartriangle(Z)$ instead of $L(Z)$ and $\Delta \in L_\vartriangle(Z)$ instead of $\Delta \subset L(Z)$. ◯

Lemma 5.4. *Let $f\colon Z \to Y$ be a holomorphic mapping of irreducible quasi-projective varieties. Suppose that for each $\lambda \in L(Y)$ the composition*

$$f^*(\lambda) \overset{\text{def}}{=\!=} \lambda \circ f\colon Z \xrightarrow{f} Y \xrightarrow{\lambda} \mathbb{C} \setminus \{0;1\}$$

is non-constant (this is certainly the case whenever f is dominant). Then

$$f^*\colon L(Y) \ni \lambda \mapsto \lambda \circ f \in L(Z) \tag{5.17}$$

is a simplicial mapping whose restriction to each simplex $\Delta \subseteq L(Y)$ is injective. Consequently, f^* preserves dimension of simplices. Moreover, $f^*(\iota\lambda) = \iota f^*(\lambda)$ for all $\lambda \in L(Y)$.

[10] In what follows we are interested only in the case when $L(Z) \neq \varnothing$.

PROOF. Let μ, ν be two distinct vertices of a simplex $\Delta \in L_\vartriangle(Y)$ and let $\lambda = \mu : \nu$. By Definition 5.3, $\lambda \in L(Y)$. By our assumption, $f^*(\mu) : f^*(\nu) = f^*(\lambda) \neq \text{const}$ and $\mu(f(z)) : \nu(f(z)) = \lambda(f(z)) \neq 1$ for all $z \in Z$. Hence $f^*(\nu) \mid f^*(\mu)$. This shows that $f^*(\Delta) \in L_\vartriangle(Z)$ and the restriction of f^* to the set of all vertices of Δ is injective. The last statement of Lemma is evident. $\square$

5.3. Complexes of simple and cross ratios. Let, as usual, $X = \mathbb{C}$ or $X = \mathbb{CP}^1$ and let $Z = \mathcal{C}_o^n(X)$. Due to Theorem 2.15, where we met all non-constant holomorphic functions $Z \to \mathbb{C} \setminus \{0; 1\}$ face to face, certain properties of the simplicial complex $L(Z)$ may be described in a more explicit manner.

Notation. It is convenient to start with a little bit more formal situation. Let $\mathbf{q} = \{q_i \mid i \in \mathbb{N}\}$ be a countable set of independent variables. For any three distinct $i, j, k \in \mathbb{N}$ and any four distinct $i, j, k, l \in \mathbb{N}$ the rational functions[11]

$$\begin{aligned} sr_{ijk} &= sr(q_i, q_j, q_k) \stackrel{\text{def}}{=\!=} \frac{q_k - q_i}{q_k - q_j} \quad \text{and} \\ cr_{ijkl} &= cr(q_i, q_j, q_k, q_l) \stackrel{\text{def}}{=\!=} \frac{q_l - q_i}{q_l - q_j} : \frac{q_i - q_k}{q_j - q_k} = \frac{(q_l - q_i)(q_j - q_k)}{(q_l - q_j)(q_i - q_k)} \end{aligned} \tag{5.18}$$

are referred to as *simple* and *cross* ratios, respectively. The *unordered* set of variables q that actually participate in such a function μ is called its *support*; we denote it by $\operatorname{supp} \mu$; the differences of these variables in the numerator and the denominator of μ are said to be the *factors* of μ. There are 6 distinct simple ratios with support $\{q_i, q_j, q_k\}$ and 6 distinct cross ratios with support $\{q_i, q_j, q_k, q_l\}$. (Permutations of i, j, k, l formally lead to 24 cross ratios with the same support, but only 6 of them are distinct, since Kleinian permutations 1, $(i,j)(k,l)$, $(i,k)(j,l)$, $(i,l)(j,k)$ do not change the rational function cr_{ijkl}.) The sets of all simple and all cross ratios are denoted by $SR(\mathbf{q})$ and $CR(\mathbf{q})$, respectively. We set $L(\mathbf{q}) = SR(\mathbf{q}) \cup CR(\mathbf{q})$. The following definition is similar to Definition 5.3.

***Definition* 5.5.** Let $\mu, \nu \in L(\mathbf{q})$ and $\mu \neq \nu$. We say that ν is a *proper divisor* of μ and write $\nu \mid \mu$ if the quotient $\mu : \nu \in L(\mathbf{q})$; otherwise, we write $\nu \nmid \mu$. As $\lambda \in L(\mathbf{q})$ is equivalent to $\lambda^{-1} \in L(\mathbf{q})$, $\nu \mid \mu$ is equivalent to $\mu \mid \nu$. The *set* (λ, μ) *of all common proper divisors* of elements $\lambda, \mu \in L(\mathbf{q})$ consists of all $\nu \in L(\mathbf{q})$ such that $\nu \mid \lambda$ and $\nu \mid \mu$.

A non-empty finite subset $\Delta \subset L(\mathbf{q})$ is called a *simplex* (respectively, an *ordered simplex* when Δ is ordered) of dimension $\dim \Delta = \#\Delta - 1$ if $\mu \mid \nu$ for every two distinct $\mu, \nu \in \Delta$.[12] The elements $\mu \in \Delta$ are said to be the *vertices* of Δ and the union $\operatorname{supp} \Delta \stackrel{\text{def}}{=\!=} \bigcup_{\mu \in \Delta} \operatorname{supp} \mu$ is called the *support* of Δ. Any non-empty subset of an (ordered) simplex is an (ordered) simplex. The boundary $\partial\Delta$ of an ordered simplex $\Delta = [\mu_0, ..., \mu_m] \subseteq L(\mathbf{q})$ is defined by formula (5.16). Thereby, $L(\mathbf{q})$ becomes a simplicial complex with the simplicial involution $\iota \colon \lambda \mapsto \lambda^{-1}$.

[11]The ground field $\mathbf{k}$ is not important as soon as it is commutative and of characteristic 0; for instance, $\mathbf{k} = \mathbb{C}$ and $\mathbf{k} = \mathbb{Q}$ both are suitable.

[12]The latter definition *does not* require that the quotients $\nu : \mu$, $\mu : \nu$ must be elements of Δ.

As in Section 5.2, we use also the notation $L_\vartriangle(\mathbf{q})$ instead of $L(\mathbf{q})$ to emphasize that we deal with a simplicial complex, i. e., the collection of all simplices $\Delta \subset L(\mathbf{q})$ rather than just the set of all vertices $\mu \in L(\mathbf{q})$. There are three types of simplices in $L_\vartriangle(\mathbf{q})$: two types of *"pure"* ones, which belong either to $SR_\vartriangle(\mathbf{q})$ (they are referred to as *"simple"* ones) or to $CR_\vartriangle(\mathbf{q})$ (they are referred to as *"cross"* ones), and *"mixed"* ones, which have vertices in both $SR(\mathbf{q})$ and $CR(\mathbf{q})$.

We denote by $SR(\mathbf{q}_n) \subset SR(\mathbf{q})$, $CR(\mathbf{q}_n) \subset CR(\mathbf{q})$ and $L(\mathbf{q}_n) \subset L(\mathbf{q})$ the subsets consisting of all vertices of the corresponding type whose supports are contained in the set $\mathbf{q}_n = \{q_1, ..., q_n\} \subset \mathbf{q}$ of the first n variables. The subcomplexes $SR_\vartriangle(\mathbf{q}_n)$, $CR_\vartriangle(\mathbf{q}_n)$ and $L_\vartriangle(\mathbf{q}_n)$ consist of all simplices whose vertices are in $SR(\mathbf{q}_n)$, $CR(\mathbf{q}_n)$ and $L(\mathbf{q}_n)$, respectively.[13] ○

The main purpose of the following three lemmas is to estimate dimension of simplices of a certain type provided their supports are contained in $\mathbf{q}_n$.

Lemma 5.6. ***a***) *If $\mu, \nu \in SR(\mathbf{q})$ and $\nu \mid \mu$ then either μ and ν have the same denominator or they have the same numerator; in particular, $\#(\mathrm{supp}\ \mu \cap \mathrm{supp}\ \nu) = 2$.*

b) *Each proper divisor $\mu \in SR(\mathbf{q})$ of sr_{ijk} is obtained from sr_{ijk} by replacing of one of the indices i, j with some $l \neq i, j, k$.*

In fact, up to a choice of $l \neq i, j, k$ and the order of multipliers, the formula

$$sr_{ijk} = sr_{ilk} \cdot sr_{ljk}\,. \tag{5.19}$$

provides us with a unique possible decompositions of sr_{ijk} to a product of two other simple ratios.

c) *For every four distinct $i, j, k, l \in \mathbb{N}$*

$$\begin{aligned} (sr_{ijk}, sr_{ilk}) \cap SR(\mathbf{q}) &= \{sr_{imk} \mid m \neq i, j, k, l\}\,, \\ (sr_{ijk}, sr_{ljk}) \cap SR(\mathbf{q}) &= \{sr_{mjk} \mid m \neq i, j, k, l\}\,. \end{aligned} \tag{5.20}$$

d) *If $\Delta = \{\lambda, \mu, \nu\} \subset SR(\mathbf{q})$ is a simplex, then either λ, μ, ν have the same denominator or they all have the same numerator; in any case $\#\,\mathrm{supp}\ \Delta = 5$.*

e) *Let $n \geq 3$ and $\Delta \in SR_\vartriangle(\mathbf{q}_n)$. Then either all vertices of Δ have the same denominator or they all have the same numerator. Moreover, $\dim \Delta \leq n - 3$, i. e., $\dim SR_\vartriangle(\mathbf{q}_n) \leq n - 3$.*

Proof. (***a***) is evident; the proof of (***b***) is straightforward. (***c***) is a simple consequence of (***b***), and (***d***) follows from (***c***). The first statement of (***e***) follows immediately from (***d***), and the second statement follows from the first one. □

Lemma 5.7. ***a***) *If $\mu, \nu \in CR(\mathbf{q})$ and $\nu \mid \mu$ then $\#(\mathrm{supp}\ \mu \cap \mathrm{supp}\ \nu) = 3$.*

b) *Each proper divisor $\mu \in CR(\mathbf{q})$ of cr_{ijkl} is obtained from cr_{ijkl} by replacing of one of the indices i, j, k, l with some $m \neq i, j, k, l$.*

In fact, up to a choice of $m \neq i, j, k, l$ and the order of multipliers, the formulae

$$cr_{ijkl} = cr_{ijkm} \cdot cr_{ijml} = cr_{imkl} \cdot cr_{mjkl} \tag{5.21}$$

[13] $L(\mathbf{q}_2) = CR(\mathbf{q}_3) = \varnothing$. It should be emphasized that $SR_\vartriangle(\mathbf{q}_n)$, $CR_\vartriangle(\mathbf{q}_n)$ and $L_\vartriangle(\mathbf{q}_n)$ *are not* the n-skeletons of the corresponding complexes.

provide us with all decompositions of cr_{ijkl} to a product of two other cross ratios.[14]

c) *For every five distinct $i, j, k, l, m \in \mathbb{N}$*

$$\begin{aligned} (cr_{ijkl}, cr_{ijkm}) \cap CR(\mathbf{q}) &= \{cr_{ijkn} \mid n \neq i, j, k, l, m\}\,, \\ (cr_{ijkl}, cr_{ijml}) \cap CR(\mathbf{q}) &= \{cr_{ijnl} \mid n \neq i, j, k, l, m\}\,, \\ (cr_{ijkl}, cr_{imkl}) \cap CR(\mathbf{q}) &= \{cr_{inkl} \mid n \neq i, j, k, l, m\}\,, \\ (cr_{ijkl}, cr_{mjkl}) \cap CR(\mathbf{q}) &= \{cr_{njkl} \mid n \neq i, j, k, l, m\}\,. \end{aligned} \tag{5.22}$$

d) *If $\Delta = \{\lambda, \mu, \nu\} \subset CR(\mathbf{q})$ is a simplex then $\#\operatorname{supp}\Delta = 6$.*

e) *Let $n \geq 4$. Then $\dim CR_{\vartriangle}(\mathbf{q}_n) \leq n - 4$, that is, $\dim \Delta = \#\Delta - 1 \leq n - 4$ for every cross simplex Δ whose vertices are supported in $\{q_1, ..., q_n\}$.*

PROOF. ***a***) The quotient $\mu : \nu$ formally contains 8 factors; mutual cancellation of two pairs of them cannot happen unless $\#(\operatorname{supp}\mu \cap \operatorname{supp}\nu) \geq 3$. A straightforward verification shows that $\nu \nmid \mu$ whenever $\operatorname{supp}\mu = \operatorname{supp}\nu$.

b) Let $cr_{ijkl} = \mu' \cdot \mu''$ for some two cross ratios μ', μ''; due to (***a***), we may also assume that supp μ' is one of the sets $\{q_i, q_j, q_k, q_m\}$, $\{q_i, q_j, q_l, q_m\}$, $\{q_i, q_k, q_l, q_m\}$, $\{q_j, q_k, q_l, q_m\}$. There are precisely $4 \cdot 6 = 24$ cross ratios μ' with such a support, and one may check them all, one by one; this leads to the desired result.

c) By (***b***), to get a common proper divisor $\nu \in CR(\mathbf{q})$ of cr_{ijkl} and cr_{ijkm}, one must replace one of the indices in each of the latter two cross ratios with some $n \neq i, j, k, l, m$ so that the results coincide with each other. Only the cross ratio $\nu = cr_{ijkn}$ appears in such a way; this proves the first of relations (5.22). As Kleinian permutations $(i, j)(k, l)$, $(i, k)(j, l)$ and $(i, l)(j, k)$ do not change the rational function cr_{ijkl}, and for any $s \neq i, j, k, l$ we have $cr_{jils} = cr_{ijsl}$, $cr_{klis} = cr_{iskl}$ and $cr_{lkjs} = cr_{sjkl}$, three other relations follow from the first one.

d) Let $\lambda = cr_{ijkl}$. By (***b***), there is $m \neq i, j, k, l$ such that μ coincides with one of the four cross ratios cr_{ijkm}, cr_{ijml}, cr_{imkl}, cr_{mjkl}. In each of these cases one of relations (5.22) shows that $\#(\operatorname{supp}\lambda \cup \operatorname{supp}\mu \cup \operatorname{supp}\nu) = 6$.

e) By (***a***), statement (***e***) is valid for $n = 4$. Suppose that for some $n \geq 4$ we have $\dim CR_{\vartriangle}(\mathbf{q}_n) \leq n - 4$ and consider any simplex $\Delta \in CR_{\vartriangle}(\mathbf{q})$ with support supp $\Delta \subseteq \{q_1, ..., q_{n+1}\}$. We must prove that $\dim \Delta \leq n - 3$, i. e., $\#\Delta \leq n - 2$.

Take two distinct $\lambda, \mu \in \Delta$. According to (***a***), $\#(\operatorname{supp}\lambda \cap \operatorname{supp}\mu) = 3$; thus, supp $\lambda = \{q_i, q_j, q_k, q_l\}$ and supp $\mu = \{q_i, q_j, q_k, q_m\}$ for some distinct $i, j, k, l, m \in \{1, ..., n + 1\}$.

Let $\Delta(\widehat{m}) = \{\tau \in \Delta \mid q_m \notin \operatorname{supp}\tau\}$; clearly, $\lambda \in \Delta(\widehat{m})$ and $\mu \notin \Delta(\widehat{m})$. By the induction hypothesis, $\dim \Delta = \#\Delta(\widehat{m}) - 1 \leq n - 4$. Let us show that actually $\Delta \setminus \Delta(\widehat{m}) = \{\mu\}$; this will imply $\#\Delta = \#\Delta(\widehat{m}) + 1 \leq (n-3) + 1 = n-2$ and complete the step of induction.

Suppose to the contrary that $\nu \in \Delta \setminus \Delta(\widehat{m})$ and $\nu \neq \mu$; clearly, $q_m \in \operatorname{supp}\nu$. As $\lambda \in \Delta(\widehat{m})$, we have $\nu \neq \lambda$. Thereby, $\{\lambda, \mu, \nu\}$ is a simplex; by (***d***), $\#\operatorname{supp}\Delta = \#(\operatorname{supp}\lambda \cup$

[14] As $cr_{imkl} = cr_{klim}$ and $cr_{mjkl} = cr_{klmj}$, the Kleinian permutation $(i, k)(j, l)$, which does not change cr_{ijkl}, transforms these decompositions to each other.

$\operatorname{supp}\mu \cup \operatorname{supp}\nu) = 6$. Since $\operatorname{supp}\lambda \cup \operatorname{supp}\mu = \{q_i, q_j, q_k, q_l, q_m\}$, the set $\operatorname{supp}\nu$ must contain at least one additional variable, say q_p, distinct from all q_i, q_j, q_k, q_l, q_m. As $\operatorname{supp}\lambda = \{q_i, q_j, q_k, q_l\}$ and $q_m, q_p \in \operatorname{supp}\nu$, we see that $\#(\operatorname{supp}\lambda \cap \operatorname{supp}\nu) \leq 2$, which contradicts (***a***). $\square$

Lemma 5.8. ***a***) *Each mixed simplex Δ contains precisely one vertex $\mu \in SR(\mathbf{q})$.*

b) $\dim\Delta \leq n-3$ *for any mixed simplex with support* $\operatorname{supp}\Delta \subseteq \{q_1, ..., q_n\}$.

Proof. ***a***) As Δ is mixed, there are $\mu \in \Delta \cap SR(\mathbf{q})$ and $\nu \in \Delta \cap CR(\mathbf{q})$. We may assume that $\mu = sr_{ijk}$; then it is easily seen that $\nu = cr_{ijlk}$ for a certain $l \neq i, j, k$. Up to an order of multipliers, there exist exactly two distinct decompositions of $\nu = cr_{ijlk}$ to a product of two simple ratios:

$$cr_{ijlk} = sr_{ijk} \cdot sr_{jil} = sr_{kli} \cdot sr_{lkj}\,.$$

Each of the three simple ratios sr_{jil}, sr_{kli}, sr_{lkj} is not a proper divisor of $\mu = sr_{ijk}$; hence Δ does not contain any simple ratio $\lambda \neq \mu$.

b) By (***a***), there is a single vertex $\mu \in \Delta \cap SR(\mathbf{q})$. Thus, $\Delta' \stackrel{\text{def}}{=\!=} \Delta\setminus\{\mu\}$ is a simplex contained in $CR(\mathbf{q})$ with support in $\{q_1, ..., q_n\}$. By Lemma 5.7(***e***) $\dim\Delta' \leq n-4$, which implies $\dim\Delta \leq n-4$. $\square$

Example* 5.9.** ***a) Let $n \geq 3$. Then $\Delta = \{sr_{3,2,1}, sr_{4,2,1}, ..., sr_{n,2,1}\} \subset SR(\mathbf{q})$ is a simplex of dimension $n-3$ with support in $\{q_1, ..., q_n\}$.

b) Let $s \geq 4$. Then $\Delta = \{cr_{1,2,3,4}, cr_{1,2,3,5}, ..., cr_{1,2,3,n}\} \subset CR(\mathbf{q})$ is a simplex of dimension $n-4$ with support in $\{q_1, ..., q_n\}$.

c) Let $n \geq 3$. Then $\Delta = \{sr_{1,2,3}; cr_{1,2,4,3}, cr_{1,2,5,3}, ..., cr_{1,2,n,3}\}$ is a mixed simplex of dimension $n-3$ with support in $\{q_1, ..., q_n\}$.

These examples show that the upper bounds of $\dim\Delta$ established in Lemma 5.6(***e***), Lemma 5.7(***e***) and Lemma 5.8(***b***) are the best possible ones. $\bigcirc$

Due to Theorem 2.15 the complexes $L_\vartriangle(\mathcal{C}_o^n(\mathbb{C}))$ and $L_\vartriangle(\mathcal{C}_o^n(\mathbb{CP}^1))$ defined according to Section 5.2 may be identified to the complex $L_\vartriangle(\mathbf{q}_n)$ and to the subcomplex $CR_\vartriangle(\mathbf{q}_n) \subset L_\vartriangle(\mathbf{q}_n)$, respectively. When it is convenient we use also the notation

$$\begin{aligned} &SR(\mathcal{C}_o^n(\mathbb{C}))\,,\ SR_\vartriangle(\mathcal{C}_o^n(\mathbb{C}))\,,\ CR(\mathcal{C}_o^n(\mathbb{C})) = CR(\mathcal{C}_o^n(\mathbb{CP}^1))\\ &\text{and}\ \ CR_\vartriangle(\mathcal{C}_o^n(\mathbb{C})) = CR_\vartriangle(\mathcal{C}_o^n(\mathbb{CP}^1))\end{aligned}$$

instead of $SR(\mathbf{q}_n)$, $SR_\vartriangle(\mathbf{q}_n)$, $CR(\mathbf{q}_n)$ and $CR_\vartriangle(\mathbf{q}_n)$.

Lemma 5.6, Lemma 5.7 and Example 5.9 imply the following corollary.

Corollary 5.10. ***a***) $\dim SR_\vartriangle(\mathbf{q}_n) = n-3$ *and* $\dim CR_\vartriangle(\mathbf{q}_n) = n-4$.

b) $\dim L_\vartriangle(\mathcal{C}_o^n(\mathbb{C})) = \dim SR_\vartriangle(\mathcal{C}_o^n(\mathbb{C})) = \dim SR_\vartriangle(\mathbf{q}_n) = n-3$ *and* $\dim L_\vartriangle(\mathcal{C}_o^n(\mathbb{CP}^1)) = \dim CR_\vartriangle(\mathcal{C}_o^n(\mathbb{CP}^1)) = \dim CR_\vartriangle(\mathbf{q}_n) = n-4$. $\bigcirc$

5.4. $\mathbf{S}(n)$ action in $L(\mathbf{q}_n)$. The natural $\mathbf{S}(n)$ action in $\{1, ..., n\}$ induces the left $\mathbf{S}(n)$ action in $L(\mathbf{q}_n)$; in more details, for $\mu \in L(\mathbf{q}_n)$ and $\sigma \in \mathbf{S}(n)$ the rational function $\sigma\mu$ is defined by $(\sigma\mu)(q) = \mu(\sigma^{-1}q)$, i. e., $(\sigma\mu)(q_1, ..., q_n) = \mu(q_{\sigma(1)}, ..., q_{\sigma(n)})$.

This action is transitive on the sets of vertices $SR(\mathbf{q}_n)$ and $CR(\mathbf{q}_n)$ (for $\mathbf{S}(n)$ action in $\{1, ..., n\}$ is n transitive).

Furthermore, if $\mu, \nu \in L(\mathbf{q}_n)$ and $\nu \mid \mu$ then for each $\sigma \in \mathbf{S}(n)$ we have $(\sigma\mu)(q) : (\sigma\nu)(q) = \mu(\sigma^{-1}q) : \nu(\sigma^{-1}q) = (\mu : \nu)(\sigma^{-1}q)$, which shows that $\sigma\nu \mid \sigma\mu$. Hence for each simplex $\Delta = \{\mu_0, ..., \mu_s\} \in L_\vartriangle(\mathbf{q}_n)$ and each $\sigma \in \mathbf{S}(n)$ the set $\sigma\Delta = \{\sigma\mu_0, ..., \sigma\mu_s\}$ is the simplex of the same type and dimension. Thus we have the (left) dimension preserving $\mathbf{S}(n)$ action in $SR_\vartriangle(\mathbf{q}_n)$ and $CR_\vartriangle(\mathbf{q}_n)$.

We denote by $SR_{\vartriangle^m}(\mathbf{q}_n) \subseteq SR_\vartriangle(\mathbf{q}_n)$ and $CR_{\vartriangle^m}(\mathbf{q}_n) \subseteq CR_\vartriangle(\mathbf{q}_n)$ the $\mathbf{S}(n)$ invariant subsets consisting of all m simplices of the corresponding type.

***Definition* 5.11.** We define the following three *normal forms* of m simplices:

$$\Delta_S^m \stackrel{\text{def}}{=\!=} \{sr_{3,2,1},\ sr_{4,2,1}, ..., sr_{m+3,2,1}\} \in SR_\vartriangle(\mathbf{q}_n) \quad (0 \le m \le n-3)\,, \tag{5.23}$$

$$\Delta_S^{-m} \stackrel{\text{def}}{=\!=} \{sr_{2,3,1},\ sr_{2,4,1}, ..., sr_{2,m+3,1}\} \in SR_\vartriangle(\mathbf{q}_n) \quad (0 \le m \le n-3) \tag{5.24}$$

and

$$\Delta_C^m \stackrel{\text{def}}{=\!=} \{cr_{1,2,3,4},\ cr_{1,2,3,5}, ..., cr_{1,2,3,m+4}\} \in CR_\vartriangle(\mathbf{q}_n) \quad (0 \le m \le n-4)\,. \tag{5.25}$$

A simplex in a normal form we call *normal.*

The following lemma shows that the $\mathbf{S}(n)$ action in $L(\mathbf{q}_n)$ allows to carry any simplex to a normal form.

Lemma 5.12. ***a***) *For $m \ge 1$ the set of all simple m simplices $SR_{\vartriangle^m}(\mathbf{q}_n)$ is the disjoint union of two $\mathbf{S}(n)$ orbits $\mathbf{S}(n)\Delta_S^m$ and $\mathbf{S}(n)\Delta_S^{-m}$ of the normal m simplices Δ_S^m and Δ_S^{-m}. That is, any simple m simplex can be carried to one of the normal forms* (5.23), (5.24) *by a permutation $\sigma \in \mathbf{S}(n)$.*

b) *The $\mathbf{S}(n)$ action on $CR_{\vartriangle^m}(\mathbf{q}_n)$ is transitive, that is, any cross m simplex can be brought to the normal form* (5.25) *by a permutation $\sigma \in \mathbf{S}(n)$.*

PROOF. ***a***) Let $\Delta = \{\mu_0, ..., \mu_m\} \in SR_{\vartriangle^m}(\mathbf{q}_n)$. As permutations are admitted we may assume that $\mu_0 = sr_{3,2,1}$. By Lemma 5.6(***b***), μ_1 must be one of the functions $sr_{l,2,1}$, $sr_{3,l,1}$ with a certain $l \ne 1, 2, 3$. A suitable permutation of $\{4, ..., n\}$ keeps μ_0 and carries μ_1 to one of the functions $sr_{4,2,1}$ and $sr_{3,4,1}$.

In the first case $\{\mu_0,\ \mu_1\} = \{sr_{3,2,1},\ sr_{4,2,1}\}$; in the second case we apply the transposition $(2, 3)$ to all vertices of Δ and get $\{\mu_0,\ \mu_1\} = \{sr_{2,3,1},\ sr_{2,4,1}\}$.

If $m = 1$ we are done. For $m \ge 2$ the simplex Δ has at least one more vertex μ_2; so either

$$(i)\ \ \Delta = \{\mu_0,\ \mu_1,\ \mu_2, ..., \mu_m\} = \{sr_{3,2,1},\ sr_{4,2,1},\ \mu_2, ..., \mu_m\} \ \text{ or}$$
$$(ii)\ \ \Delta = \{\mu_0,\ \mu_1,\ \mu_2, ..., \mu_m\} = \{sr_{2,3,1},\ sr_{2,4,1},\ \mu_2, ..., \mu_m\}\,.$$

In case (i), μ_0 and μ_1 have the same denominator $q_1 - q_2$. Lemma 5.6(***e***) shows that all $\mu_2, ..., \mu_m$ also have the same denominator $q_1 - q_2$. By Lemma 5.6(***c***), they must be of the form $\mu_2 = sr_{k_2,2,1}, ..., \mu_m = sr_{k_m,2,1}$, where all $k_2, ..., k_m \in \{5, ..., n\}$ are distinct. A suitable permutation of $\{5, ..., n\}$ keeps the forms of μ_0, μ_1 and carries Δ to the normal form (5.23).

In case (ii), μ_0 and μ_1 have the same numerator $q_1 - q_2$ and Lemma 5.6(***e***) shows that so do all other vertices $\mu_2, ..., \mu_m$ of Δ. As above, we conclude that they must be of the form $\mu_2 = sr_{2,k_2,1}, ..., \mu_m = sr_{2,k_m,1}$, where all $k_2, ..., k_m \in \{5, ..., n\}$ are distinct. A suitable permutation of $\{5, ..., n\}$ does not change μ_0, μ_1 and brings Δ to the normal form (5.24).

b) Let $\Delta = \{\mu_0, ..., \mu_m\} \in CR_{\Delta^m}(\mathbf{q}_n)$. We may assume that $\mu_0 = cr_{1,2,3,4}$. By Lemma 5.7(***b***), μ_1 is one of the functions $cr_{1,2,3,l},\ cr_{1,2,l,4},\ cr_{1,l,3,4},\ cr_{l,2,3,4}$ for a certain $l \neq 1, 2, 3, 4$. A suitable permutation of $\{5, ..., n\}$ keeps μ_0 and carries μ_1 to one of the functions $cr_{1,2,3,5},\ cr_{1,2,5,4},\ cr_{1,5,3,4},\ cr_{5,2,3,4}$. The Kleinian permutation $(1, 3)(2, 4)$ does not change the rational function $cr_{1,2,3,4}$ and carries the pair $cr_{1,2,3,5},\ cr_{1,2,5,4}$ to the pair $cr_{1,5,3,4},\ cr_{5,2,3,4}$; hence we may restrict ourselves to the following two cases: either $\{\mu_0,\ \mu_1\} = \{cr_{1,2,3,4},\ cr_{1,2,3,5}\}$ or $\{\mu_0,\ \mu_1\} = \{cr_{1,2,3,4},\ cr_{1,2,5,4}\}$. In fact, the Kleinian permutation $(1, 2)(3, 4)$ keeps $cr_{1,2,3,4}$ and brings $cr_{1,2,5,4}$ to $cr_{1,2,3,5}$; thus we are left with the case when $\Delta = \{\mu_0,\ \mu_1, ..., \mu_m\} = \{cr_{1,2,3,4},\ cr_{1,2,3,5}, ..., \mu_m\}$.

If $m = 1$ we are done. For $m \geq 2$ the first of relations (5.22) in Lemma 5.7(***c***) shows that $\mu_2, ..., \mu_m$ must be of the form $\mu_2 = cr_{1,2,3,k_2}, ..., \mu_m = cr_{1,2,3,k_m}$, where all $k_2, ..., k_m \in \{6, ..., n\}$ are distinct. A suitable permutation of $\{6, ..., n\}$ keeps μ_0, μ_1 and brings Δ to the normal form (5.25). $\square$

6. Coherence Theorem

In this section we prove Coherence Theorem formulated in Section 2.4, which completes the proof of Tame Map Theorem.

It is clear that all components of a strictly equivariant endomorphism

$$f = (f_1, ..., f_n)\colon \mathcal{C}^n_o(X) \to \mathcal{C}^n_o(X)$$

are non-constant holomorphic mappings $\mathcal{C}^n_o(X) \to X$. The following lemma shows that Lemma 5.4 applies to strictly equivariant endomorphisms.

Lemma 6.1. *Let $f = (f_1, ..., f_n)\colon \mathcal{C}^n_o(X) \to \mathcal{C}^n_o(X)$ be a strictly equivariant endomorphism and $\mu\colon \mathcal{C}^n_o(X) \to \mathbb{C} \setminus \{0;\, 1\}$ be a non-constant holomorphic function. Then the composition $\mu \circ f\colon \mathcal{C}^n_o(X) \xrightarrow{f} \mathcal{C}^n_o(X) \xrightarrow{\mu} \mathbb{C} \setminus \{0;\, 1\}$ is non-constant.*

Proof. Suppose to the contrary that $\mu \circ f = c = \text{const} \neq 0, 1$. Then $(\mu \circ f)(\sigma q) \equiv c$ for all $\sigma \in \mathbf{S}(n)$; as f is strictly equivariant, this implies $(*)$ $\mu(\sigma f(q)) \equiv c$ for each $\sigma \in \mathbf{S}(n)$.

First we notice that $c = -1$. Indeed, either $\mu \in SR(\mathbf{q}_n)$ or $\mu \in CR(\mathbf{q}_n)$ (for $X = \mathbb{CP}^1$ the first case does not occur). $\mathbf{S}(n)$ is transitive on the sets $SR(\mathbf{q}_n)$ and $CR(\mathbf{q}_n)$, which are invariant under the involution $\mu \mapsto \mu^{-1}$; thus $\mu^{-1} = s\mu$ for a certain $s \in \mathbf{S}(n)$. Let $\sigma = \alpha^{-1}(s^{-1})$, where α is the automorphism of $\mathbf{S}(n)$ related to f; then $c^{-1} \equiv \mu^{-1}(f(q)) \equiv s\mu((q)) \equiv \mu(s^{-1}f(q)) \equiv \mu(f(\sigma q)) \equiv c$. Thus $c = -1$ and $(*)$ takes the form

$(**)$ $\mu(\sigma f(q)) \equiv -1$ for all $\sigma \in \mathbf{S}(n)$.

From this point we treat the cases $X = \mathbb{C}$ and $X = \mathbb{CP}^1$ separately.

a) $X = \mathbb{C}$. Then either

(SR) $\mu = sr_{ijk} = (q_k - q_i)/(q_k - q_j)$ and
$\mu \circ f = sr(f_i, f_j, f_k) = (f_k - f_i)/(f_k - f_j) = -1$ or

(CR) $\mu = cr_{ijkl} = [(q_l - q_i)(q_j - q_k)]/[(q_l - q_j)(q_i - q_k)]$ and
$\mu \circ f = cr(f_i, f_j, f_k, f_l) = [(f_l - f_i)(f_j - f_k)]/[(f_l - f_j)(f_i - f_k)] = -1$.

In case (SR) we have $2f_k = f_i + f_j$. In view of $(**)$, we may take any other ordered triple of distinct indices; in particular, we have also $2f_j = f_k + f_i$; subtracting this equality from the previous one, we obtain $3(f_k - f_j) = 0$, which is impossible.

In case (CR) we have $(f_l - f_i)(f_j - f_k) = -(f_l - f_j)(f_i - f_k)$; in view of $(**)$, we may apply the transposition (j, k), which gives $(f_l - f_i)(f_k - f_j) = -(f_l - f_k)(f_i - f_j)$. These relations and the identity

$$(a - b)(d - c) - (a - c)(d - b) + (d - a)(b - c) = 0$$

with $a = f_i,\ b = f_j,\ c = f_k,\ d = f_l$ imply

$$\begin{aligned} 3(f_l - f_i)(f_j - f_k) &= (f_l - f_i)(f_j - f_k) + (f_l - f_i)(f_j - f_k) + (f_l - f_i)(f_j - f_k) \\ &= (f_i - f_j)(f_l - f_k) - (f_i - f_k)(f_l - f_j) + (f_l - f_i)(f_j - f_k) = 0\,, \end{aligned}$$

which is impossible.

b) $X = \mathbb{CP}^1$. In this case we have $\mu = cr_{ijkl}$ and $\mu \circ f = cr(f_i, f_j, f_k, f_l) = [(f_l - f_i)(f_j - f_k)]/[(f_l - f_j)(f_i - f_k)] = -1$. Each component f_m is a non-constant holomorphic map $f_m\colon \mathcal{C}_o^n(X) \to \mathbb{CP}^1$; therefore, the pole set $P(f) = \bigcup\limits_{m=1}^{n} f_m^{-1}(\infty)$ is an analytic subset of $\mathcal{C}_o^n(\mathbb{CP}^1)$ of complex dimension $\le n - 1$. Its complement $R(f) = \mathcal{C}_o^n(\mathbb{CP}^1) \setminus P(f)$ is a non-empty $\mathbf{S}(n)$ invariant Zariski open subset of $\mathcal{C}_o^n(\mathbb{CP}^1)$, the restriction $f' = f|_{R(f)}$ of f to $R(f)$ is a holomorphic map from $R(f)$ to $\mathcal{C}_o^n(\mathbb{C})$, and its components $f'_i = f_i|_{R(f)}$ are just holomorphic functions on $R(f)$ that satisfy the strict equivariance condition $f'_m(\sigma q) = f'_{[\alpha(\sigma)]^{-1}(m)}(q)$ for all $m = 1, ..., n$, $q \in R(f)$, and $\sigma \in \mathbf{S}(n)$. In particular, $[(f'_l - f'_i)(f'_j - f'_k)]/[(f'_l - f'_j)(f'_i - f'_k)] = -1$ on $R(f)$. Now we can repeat word by word what has been done above in case (CR) and obtain $3(f'_l - f'_i)(f'_j - f'_k) = 0$ on $R(f)$, which is impossible, since $R(f) \ne \varnothing$ and $f'(q) = (f'_1(q), ..., f'_n(q)) \in \mathcal{C}_o^n(\mathbb{C})$ for all $q \in R(f)$. $\square$

Recall the notation $t(X) = 2$ for $X = \mathbb{C}$ and $t(X) = 3$ for $X = \mathbb{CP}^1$. The following result is just an evident combination of Theorem 2.15, Lemma 6.1 and Lemma 5.4.

Corollary 6.2. *Let $n > t(X)$*[15] *and let f be a strictly equivariant endomorphism of $\mathcal{C}_o^n(X)$. The induced map $f^*\colon L(\mathcal{C}_o^n(X)) \ni \mu \mapsto \mu \circ f \in L(\mathcal{C}_o^n(X))$ is simplicial.* $\square$

Lemma 6.3. *Let f be a strictly equivariant endomorphism of $\mathcal{C}_o^n(X)$ and let α be the automorphism of $\mathbf{S}(n)$ related to f* (see (2.8) and Definition 2.8).

a) *The map $f^*\colon L(\mathcal{C}_o^n(X)) \to L(\mathcal{C}_o^n(X))$ is strictly equivariant meaning that*

$$f^*(\sigma\mu) = \alpha^{-1}(\sigma) f^*(\mu) \text{ for all } \sigma \in \mathbf{S}(n) \text{ and all } \mu \in L(\mathcal{C}_o^n(\mathbb{C}))\,. \tag{6.1}$$

Consequently, for any vertex $\mu \in L(\mathcal{C}_o^n(X))$ the f^*-image of its $\mathbf{S}(n)$ orbit coincides with the $\mathbf{S}(n)$ orbit of the vertex $f^*(\mu)$.

[15]See Notation 1.11.

b) *If $X = \mathbb{C}$ then $f^*(SR(\mathcal{C}_o^n(\mathbb{C}))) = SR(\mathcal{C}_o^n(\mathbb{C}))$.*

PROOF. ***a***) Let $s = \alpha^{-1}(\sigma)$. The proof of (6.1) is straightforward:

$$\begin{aligned}[f^*(\sigma\mu)](q) &= (\sigma\mu)(f(q)) = (\alpha(s)\mu)(f(q)) = \mu(\alpha(s^{-1})f(q)) \\ &= \mu(f(s^{-1}q)) = (f^*(\mu))(s^{-1}q) = [sf^*(\mu)](q) = [\alpha^{-1}(\sigma)f^*(\mu)](q)\,.\end{aligned}$$

The permutation s runs over the whole $\mathbf{S}(n)$ whenever σ does so, and vice versa. Hence (6.1) implies $f^*(\mathbf{S}(n)\mu) = \mathbf{S}(n)f^*(\mu)$ for each $\mu \in L(\mathcal{C}_o^n(X))$.

b) By Lemma 5.17, the map f^* preserves dimension of simplices; taking into account Corollary 5.10, we have $\dim f^*(SR_\Delta(\mathcal{C}_o^n(\mathbb{C}))) = \dim SR_\Delta(\mathcal{C}_o^n(\mathbb{C})) = n-3$ and $\dim CR_\Delta(\mathcal{C}_o^n(\mathbb{C})) = n-4$, which shows that $f^*(SR_\Delta(\mathcal{C}_o^n(\mathbb{C}))) \neq CR_\Delta(\mathcal{C}_o^n(\mathbb{C}))$. Thus $f^*(SR_\Delta(\mathcal{C}_o^n(\mathbb{C}))) = SR_\Delta(\mathcal{C}_o^n(\mathbb{C}))$, which is certainly equivalent to $f^*(SR(\mathcal{C}_o^n(\mathbb{C}))) = SR(\mathcal{C}_o^n(\mathbb{C}))$. □

***Remark* 6.4.** It follows from Coherence Theorem (which is not proved yet!) that $f^*(CR(\mathcal{C}_o^n(\mathbb{C}))) = CR(\mathcal{C}_o^n(\mathbb{C}))$; we do not need this fact at the moment. ○

6.1. Proof of Coherence Theorem for $X = \mathbb{C}$. Let $n > 4$ and let f be a strictly equivariant endomorphism of $\mathcal{C}_o^n(\mathbb{C})$. In view of what has been done above, in this case Coherence Theorem is equivalent to the following

Claim 1. *Let $\Delta = \Delta_S^{n-3} \in SR_\Delta(\mathcal{C}_o^n(\mathbb{C}))$ be the normal simplex of the form* (5.23)*, that is, the simplex with the vertices $\mu_i = sr_{i,2,1}$, $i = 3, ..., n$. Then $f^*(\Delta) \in \mathbf{S}(n)\Delta$.*

PROOF. By Lemma 6.3(***b***), $f^*(\Delta) \in SR_{\Delta^{n-3}}(\mathcal{C}_o^n(\mathbb{C}))$. According to Lemma 5.12(***a***), $SR_{\Delta^{n-3}}(\mathcal{C}_o^n(\mathbb{C}))$ is the disjoint union of the orbits $\mathbf{S}(n)\Delta_S^{n-3}$ and $\mathbf{S}(n)\Delta_S^{-(n-3)}$. So it suffices to show that $f^*(\Delta) \notin \mathbf{S}(n)\Delta_S^{-(n-3)}$; in view of Lemma 6.3(***a***), the latter is equivalent to $f^*(\mathbf{S}(n)\Delta) \not\ni \Delta_S^{-(n-3)}$.

Suppose to the contrary that $f^*(s\Delta) = \Delta_S^{-(n-3)}$ for some $s \in \mathbf{S}(n)$. Then

$$\frac{f_1(\sigma q) - f_i(\sigma q)}{f_1(\sigma q) - f_2(\sigma q)} = \frac{q_1 - q_2}{q_1 - q_i}$$

for all $i = 3, ..., n$, where $\sigma = \alpha^{-1}(s^{-1})$ and α is the automorphism of $\mathbf{S}(n)$ related to f. This implies that for all distinct $j, k \in \{3, ..., n\}$.

$$\begin{aligned}\frac{f_k(\sigma q) - f_j(\sigma q)}{f_1(\sigma q) - f_2(\sigma q)} &= \frac{f_1(\sigma q) - f_j(\sigma q)}{f_1(\sigma q) - f_2(\sigma q)} - \frac{f_1(\sigma q) - f_k(\sigma q)}{f_1(\sigma q) - f_2(\sigma q)} \\ &= \frac{q_1 - q_2}{q_1 - q_j} - \frac{q_1 - q_2}{q_1 - q_k} = \frac{(q_1 - q_2)(q_j - q_k)}{(q_1 - q_j)(q_1 - q_k)}\,.\end{aligned}$$

Consequently, for distinct $i, j, k \in \{3, ..., n\}$ we have

$$[f^*(sr_{ijk})](\sigma q) = sr_{ijk}(f(\sigma q)) = \frac{f_k(\sigma q) - f_i(\sigma q)}{f_k(\sigma q) - f_j(\sigma q)}$$
$$= \frac{f_k(\sigma q) - f_i(\sigma q)}{f_1(\sigma q) - f_2(\sigma q)} : \frac{f_k(\sigma q) - f_j(\sigma q)}{f_1(\sigma q) - f_2(\sigma q)}$$
$$= \frac{(q_1 - q_2)(q_i - q_k)}{(q_1 - q_i)(q_1 - q_k)} : \frac{(q_1 - q_2)(q_j - q_k)}{(q_1 - q_j)(q_1 - q_k)} = \frac{q_k - q_i}{q_k - q_j} : \frac{q_i - q_1}{q_j - q_1} = cr_{i,j,1,k}(q)\,,$$

which contradicts Lemma 6.3(***b***). □

6.2. Proof of Coherence Theorem for $X = \mathbb{CP}^1$. Let $n \geq 4$ and let $f = (f_1, ..., f_n)$ be a strictly equivariant endomorphism of $\mathcal{C}_o^n(\mathbb{CP}^1)$. In this case Coherence Theorem is equivalent to the following

Claim 2. *Let $\Delta = \Delta_C^{n-4} \in CR_{\Delta^{n-4}}(\mathcal{C}_o^n(\mathbb{CP}^1))$ be the normal simplex of the form* (5.25)*, that is, the simplex with the vertices $\mu_i = cr_{1,2,3,i}$, $i = 4, ..., n$. Then $f^*(\Delta)$ is a $n-4$ simplex contained in the $\mathbf{S}(n)$ orbit $\mathbf{S}(n)\Delta$ of Δ itself.*

PROOF. By Lemma 6.1 and Lemma 5.4, $f^*(\Delta)$ is a $n-4$ simplex in $L((\mathcal{C}_o^n(\mathbb{CP}^1))$. By Theorem 2.15, $L((\mathcal{C}_o^n(\mathbb{CP}^1))) = CR(\mathbf{q}_n)$; hence $f^*(\Delta) \in CR_{\Delta^{n-4}}(\mathbf{q}_n)$. Finally, Lemma 5.12(***b***) asserts that $\mathbf{S}(n)$ is transitive on each $CR_{\Delta^m}(\mathbf{q}_n)$. This completes the proof of Coherence Theorem and thereby the proof of Tame Map Theorem as well. □

6.3. Proof of Corollary 1.6.

6.4. Proof of Corollary 1.7.

7. Linked Map Theorem

The main aim of this section is to prove Linked Map Theorem 1.12. In fact there are several similar results, which we also intend to discuss. In all cases the key points of the proofs are the the "explicit" description of the holomorphic universal covering $\tau\colon \mathbf{T}(0, m+3) \to \mathcal{C}_o^m(\mathbb{C}^{②})$ presented in Section 3.1 and the following remarkable property of the universal Teichmüller families established first by J. H. Hubbard [**Hub72, Hub76**] for compact Riemann surfaces and then generalized to all Riemann surfaces of finite type by C. J. Earle and I. Kra [**EarKra74, EarKra76**]:

Theorem 7.1 (HUBBARD-EARL-KRA)**.** *The universal punctured Teichmüller family $\mathcal{T}(g, m) :=\ \rho\colon \mathbf{V}'(g, m) \to \mathbf{T}(g, m)$ has no holomorphic sections $\mathbf{T}(g, m) \to \mathbf{V}'(g, m)$ whenever $\dim_{\mathbb{C}} \mathbf{T}(g, m) > 1$.* □

We start with the following definition similar to Definition 1.10.

Definition 7.2. A map $h\colon \mathcal{C}_o^n(X) \to \mathcal{C}^k(X)$ is said to be *linked* if there is a point $q = (q_1, ..., q_n) \in \mathcal{C}_o^n(X)$ such that $\{q_1, ..., q_n\} \cap h(q) \neq \varnothing$, and *disjoint* otherwise. A map $h\colon \mathcal{C}_o^m(\mathbb{C}^{②}) \to \mathcal{C}^k(X)$ is said to be *linked* if there is $z = (z_1, ..., z_m) \in \mathcal{C}_o^m(\mathbb{C}^{②})$ such that $\{z_1, ..., z_m, 0, 1, \infty\} \cap h(z) \neq \varnothing$, and *disjoint* otherwise. ◯

7.1. Linked Map Theorem for $\mathcal{C}_o^m(\mathbb{C}^{②})$. Among all the forms of Linked Map Theorem that we were able to prove there is the strongest one, which implies all others as simple consequences; it will be proved in this section.

Theorem 7.3 (LINKED MAP THEOREM FOR $\mathcal{C}_o^m(\mathbb{C}^{②})$). *For $m > 1$ and $k \geq 1$ every holomorphic map $h\colon \mathcal{C}_o^m(\mathbb{C}^{②}) \to \mathcal{C}^k(X)$ is linked.* ◯

Our proof of this theorem is based on Lemma 7.4, which directly involves the universal punctured Teichmüller family $\mathbf{V}'(0, m+3) \to \mathbf{T}(0, m+3)$, and on the analytic fact stated in Theorem 7.5 below. The latter theorem, in turn, is an easy consequence of Hubbard-Earl-Kra Theorem 7.1 formulated above.

Define

$$\begin{aligned} &\mathcal{E}_o^{m+1}(\mathbb{C}^{②}) = \{(\zeta, z) \in \mathbb{C} \times \mathcal{C}_o^m(\mathbb{C}^{②}) \mid \zeta \neq z_1, ..., z_m, 0, 1\}\,, \\ &\Pi := \ \pi\colon \mathcal{E}_o^{m+1}(\mathbb{C}^{②}) \ni (\zeta, z) \mapsto \pi(\zeta, z) \stackrel{\text{def}}{=\!=} z \in \mathcal{C}_o^m(\mathbb{C}^{②})\,. \end{aligned} \tag{7.1}$$

Then $\mathcal{E}_o^{m+1}(\mathbb{C}^{②})$ is a complex manifold and Π is a smooth locally trivial fiber bundle with the fiber diffeomorphic to $\mathbb{C} \setminus \{m+2 \text{ points}\}$. Of course, the projection π is holomorphic and Π may be also regarded as a holomorphic family of curves of type $(0, m+3)$. Set

$$\tau^*(\mathcal{E}_o^{m+1}(\mathbb{C}^{②})) = \{(\mathbf{t}, (\zeta, z)) \in \mathbf{T}(0, m+3) \times \mathcal{E}_o^{m+1}(\mathbb{C}^{②}) \mid \tau(\mathbf{t}) = z\}\,; \tag{7.2}$$

then the fibration

$$\tau^*(\Pi) := \ \tau^*(\pi)\colon \tau^*(\mathcal{E}_o^{m+1}(\mathbb{C}^{②})) \ni (\mathbf{t}, (\zeta, z)) \mapsto \mathbf{t} \in \mathbf{T}(0, m+3) \tag{7.3}$$

is the pullback of the fibration Π along the map $\tau\colon \mathbf{T}(0, m+3) \to \mathcal{C}_o^m(\mathbb{C}^{②})$ and we have the commutative diagram

$$\begin{tikzcd} \tau^*(\mathcal{E}_o^{m+1}(\mathbb{C}^{②})) \arrow[r] \arrow[d, "\tau^*(\pi)"'] & \mathcal{E}_o^{m+1}(\mathbb{C}^{②}) \arrow[d, "\pi"] \\ \mathbf{T}(0, m+3) \arrow[r, "\tau"'] & \mathcal{C}_o^m(\mathbb{C}^{②})\,, \end{tikzcd} \tag{7.4}$$

where the top horizontal map is just the restriction of the projection of the product $\mathbf{T}(0, m+3) \times \mathcal{E}_o^{m+1}(\mathbb{C}^{②})$ to the second factor.

Lemma 7.4. *The family $\tau^*(\Pi) := \ \tau^*(\pi)\colon \tau^*(\mathcal{E}_o^{m+1}(\mathbb{C}^{②})) \to \mathbf{T}(0, m+3)$ is isomorphic to the universal Teichmüller family $\rho\colon \mathbf{V}'(0, m+3) \to \mathbf{T}(0, m+3)$, that is, there is a commutative diagram*

$$\begin{tikzcd} \mathbf{V}'(0, m+3) \arrow[r, "\lambda"] \arrow[dr, "\rho"] & \tau^*(\mathcal{E}_o^{m+1}(\mathbb{C}^{②})) \arrow[r] \arrow[d, "\tau^*(\pi)"] & \mathcal{E}_o^{m+1}(\mathbb{C}^{②}) \arrow[d, "\pi"] \\ & \mathbf{T}(0, m+3) \arrow[r, "\tau"] & \mathcal{C}_o^m(\mathbb{C}^{②})\,, \end{tikzcd} \tag{7.5}$$

where λ is a biholomorphic fiber isomorphism.

PROOF. Keeping in mind that a point $(\zeta, \mathbf{t}) \in \mathbb{C} \times \mathbf{T}(0, m+3)$ belongs to $\mathbf{V}'(0, m+3)$ *iff* $\zeta \neq s_1(\mathbf{t}), ..., s_m(\mathbf{t}), 0, 1$, while a point $(\mathbf{t}, \zeta, z) \in \mathbf{T}(0, m+3) \times \mathbb{C} \times \mathcal{C}_o^m(\mathbb{C}^{②})$ belongs to $\tau^*(\mathcal{E}_o^{m+1}(\mathbb{C}^{②}))$ *iff* $\rho(\mathbf{t}) = z = (z_1, ..., z_m) \in \mathcal{C}_o^m(\mathbb{C}^{②})$ and $\zeta \neq z_1, ..., z_m, 0, 1$, we define λ by

$$\begin{aligned}\lambda\colon \mathbf{V}'(0, m+3) \ni \mathbf{v} = (\zeta, \rho(\mathbf{v})) &\mapsto (\mathbf{t}, \zeta, s_1(\mathbf{t}), ..., s_m(\mathbf{t})) \\ &\stackrel{\text{def}}{=\!=} (\rho(\mathbf{v}), \zeta, s_1(\rho(\mathbf{v})), ..., s_m(\rho(\mathbf{v}))) \in \tau^*(\mathcal{E}_o^{m+1}(\mathbb{C}^{②}))\,.\end{aligned} \tag{7.6}$$

The map λ^{-1} is defined by

$$\lambda^{-1}\colon \tau^*(\mathcal{E}_o^{m+1}(\mathbb{C}^{②})) \ni (\mathbf{t}, (\zeta, z)) \mapsto (\zeta, \mathbf{t}) \in \mathbf{V}'(0, m+3)\,. \tag{7.7}$$

These λ and λ^{-1} are holomorphic fiber mappings inverse to each other. $\square$

Theorem 7.5. *For $m > 1$ the restriction $\pi|_A\colon A \to \mathcal{C}_o^m(\mathbb{C}^{②})$ of the projection $\pi\colon \mathcal{E}_o^{m+1}(\mathbb{C}^{②}) \to \mathcal{C}_o^m(\mathbb{C}^{②})$ to an analytic subset $A \subset \mathcal{E}_o^{m+1}(\mathbb{C}^{②})$ cannot be an unbranched covering over the whole manifold $\mathcal{C}_o^m(\mathbb{C}^{②})$.*

PROOF. Suppose, on the contrary, that an analytic subset $A \subset \mathcal{E}_o^{m+1}(\mathbb{C}^{②})$ covers the whole base $\mathcal{C}_o^m(\mathbb{C}^{②})$. The pullback $\tau^*(\pi|_A)\colon \tau^*(A) \to \mathbf{T}(0, m+3)$ of the covering $\pi|_A\colon A \to \mathcal{C}_o^m(\mathbb{C}^{②})$ along the mapping τ is a holomorphic unbranched covering as well. It must be trivial since the Teichmüller space $\mathbf{T}(0, m+3)$ is contractible; hence there exists a holomorphic section $s\colon \mathbf{T}(0, m+3) \to \tau^*(A) \subset \tau^*(\mathcal{E}_o^{m+1}(\mathbb{C}^{②}))$. As λ in diagram (7.5) is an isomorphism, the latter section provides a holomorphic section $S\colon \mathbf{T}(0, m+3) \to \mathbf{V}'(0, m+3)$ of the universal Teichmüller family $\mathcal{T}(0, m+3) := \rho\colon \mathbf{V}'(0, m+3) \to \mathbf{T}(0, m+3)$. Since $\dim_{\mathbb{C}} \mathbf{T}(0, m+3) = m > 1$, this contradicts the Hubbard-Earl-Kra Theorem 7.1. $\square$

We are now in a position to prove Linked Map Theorem for $\mathcal{C}_o^m(\mathbb{C}^{②})$ (Theorem 7.3).

PROOF. Suppose, on the contrary, that for some $m > 1$ and $k \geq 1$ there is a disjoint holomorphic map $h\colon \mathcal{C}_o^m(\mathbb{C}^{②}) \to \mathcal{C}^k(X)$.

Notice first that for each point $z = (z_1, ..., z_m) \in \mathcal{C}_o^m(\mathbb{C}^{②})$ its image $h(z) \subset X$ is in fact contained in $\mathbb{C}$, i. e., $\infty \notin h(z)$. In the case $X = \mathbb{C}$ this is self evident; in the case $X = \mathbb{CP}^1$ this follows from the condition $\{z_1, ..., z_m, 0, 1, \infty\} \cap h(z) = \varnothing$ (see Definition 7.2). Let $P_k(t, h(z))$ be the polynomial in t with the leading coefficient 1 and with the set of the roots $h(z)$; it may clearly be regarded as an analytic function on $\mathbb{C} \times \mathcal{C}_o^m(\mathbb{C}^{②})$. Let $A \subset \mathbb{C} \times \mathcal{C}_o^m(\mathbb{C}^{②})$ be the analytic subset defined by

$$A = \{(t, z) \in \mathbb{C} \times \mathcal{C}_o^m(\mathbb{C}^{②}) \mid t \in h(z)\} = \{(t, z) \in \mathbb{C} \times \mathcal{C}_o^m(\mathbb{C}^{②}) \mid P_k(t, h(z)) = 0\}\,.$$

As $\{z_1, ..., z_m, 0, 1, \infty\} \cap h(z) = \varnothing$, the image $h(z)$ of any point $z \in \mathcal{C}_o^m(\mathbb{C}^{②})$ contains no punctures of the fiber $\pi^{-1}(z)$ of the family $\pi\colon \mathcal{E}_o^{m+1}(\mathbb{C}^{②}) \to \mathcal{C}_o^m(\mathbb{C}^{②})$ defined in (7.1). That is, all k distinct roots $t_1, ..., t_k \in h(z)$ of the polynomial $P_k(t, h(z))$ are in $\pi^{-1}(z)$. Hence $A \subseteq \mathcal{E}_o^{m+1}(\mathbb{C}^{②})$ and the restriction $\pi|_A\colon A \to \mathcal{C}_o^m(\mathbb{C}^{②})$ of the projection $\pi\colon \mathcal{E}_o^{m+1}(\mathbb{C}^{②}) \to \mathcal{C}_o^m(\mathbb{C}^{②})$ to A is a holomorphic unbranched covering over the whole base $\mathcal{C}_o^m(\mathbb{C}^{②})$, which contradicts Theorem 7.5. $\square$

7.2. Linked Map Theorem for $\mathcal{C}^n_o(X)$. If for a map $f\colon \mathcal{C}^n_o(X) \to \mathcal{C}^k(X)$ and a subset $M \subseteq \mathcal{C}^n_o(X)$ there is $q = (q_1, ..., q_n) \in M$ such that $\{q_1, ..., q_n\} \cap f(q) \neq \varnothing$ then f is certainly linked. Hence the following form of Linked Map Theorem, which is certainly stronger that the original one formulated in Section 1.5 (Theorem 1.12), follows immediately from Linked Map Theorem for $\mathcal{C}^m_o(\mathbb{C}^{②})$ (Theorem 7.3) proved above.

Theorem 7.6 (Linked Map Theorem for $\mathcal{C}^n_o(X)$)**.** *For $n > t(X) + 1$ and $k \geq 1$ every holomorphic map $h\colon \mathcal{C}^n_o(X) \to \mathcal{C}^k(X)$ is linked.* $\square$

***Remark* 7.7.** Let $n > t(X)$ and $k \geq 1$. The above theorem implies that

a) *any holomorphic map $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ is linked, i. e., there is $Q \in \mathcal{C}^n(X)$ such that $Q \cap F(Q) \neq \varnothing$;*

b) *any holomorphic map $f\colon \mathcal{C}^n_o(X) \to \mathcal{C}^k_o(X)$ is linked, i. e., there is $q \in \mathcal{C}^n_o(X)$ such that $p(q) \cap p(f(q)) \neq \varnothing$;*

c) *any holomorphic map $g\colon \mathcal{C}^n(X) \to \mathcal{C}^k_o(X)$ is linked, i. e., there is $Q \in \mathcal{C}^n(X)$ such that $Q \cap p(g(Q)) \neq \varnothing$.*

To prove these statements, one just take $h = F \circ p$ in case (a), $h = p \circ f$ in case (b) and $h = p \circ g \circ p$ in case (c) and applies Theorem 7.6 to the map h. $\bigcirc$

8. Morphisms $\mathcal{C}^n(X) \to \mathcal{C}^k(X)$ and $\operatorname{Aut} X$ orbits. Cyclic and orbit-like maps

In this section, using the direct decomposition (3.11), we show that any morphism $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ respects $\operatorname{Aut} X$ orbits, that is, $F((\operatorname{Aut} X)Q) \subseteq (\operatorname{Aut} X)F(Q)$ for all $Q \in \mathcal{C}^n(X)$. Then we try to clarify, as far as possible, the structure of cyclic morphisms.

8.1. Liouville and Picard spaces. Here we exhibit a certain class of holomorphic maps $Z \to \mathcal{C}^k(X)$ whose images must always be contained in an $\operatorname{Aut} X$ orbit. We will use these results in Section 8.2 below. All complex spaces under consideration are assumed to be reduced.

***Definition* 8.1.** A connected complex space Z is said to be *Liouville* if it carries no non-constant bounded holomorphic functions. We say that Z is a *Picard space* if it carries no non-constant holomorphic functions omitting the values 0 and 1; Z is called *ultra-Picard* whenever every its connected unbranched finite cover is Picard. $\bigcirc$

Lemma 8.2. *Let Y be a connected complex space and $p\colon Y \to Z$ be a holomorphic finite unbranched covering map onto a Liouville space Z. Then Y is Liouville.*[16]

Proof. Let h be a bounded holomorphic function on Y. The coefficients of the polynomial $P(t, z) = \prod_{y \in p^{-1}(z)} (t - h(y))$ are bounded holomorphic functions on Z and hence constant. Since Y is connected this implies $h = \text{const}$. $\square$

[16] Of course, this is a very well-known elementary result. See [**Lin88, LinZai98**] and the references therein for more interesting results about the Liouville property.

Proposition 8.3. *Let Z be a connected complex space and $F\colon Z \to \mathcal{C}^m(X)$ be a holomorphic map. Suppose that either (i) Z is an ultra-Picard space or (ii) Z is a Liouville space and the homomorphism $F_*\colon \pi_1(Z) \to \pi_1(\mathcal{C}^m(X))$ is solvable. Then there is a point $Q^* \in \mathcal{C}^m(X)$ such that $F(Z) \subseteq (\operatorname{Aut} X)Q^*$.*

PROOF. Let $F^*(p)\colon F^*(\mathcal{C}_o^m(X)) \to Z$ be the pullback of the covering $p\colon \mathcal{C}_o^m(X) \to \mathcal{C}^m(X)$ along the mapping F and let Y be a connected component of $F^*(\mathcal{C}_o^m(X))$. Then we have the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\nu"'] & \mathcal{C}_o^m(X) \arrow[r, "j_{X,m}", "\cong"'] \arrow[d, "p"] & (\operatorname{Aut} X) \times \mathcal{D}^{m-t(X)}(X) \arrow[r, "\xi"] & \mathcal{D}^{m-t(X)}(X) \\ Z \arrow[r, "F"'] & \mathcal{C}^m(X)\,, \end{tikzcd} \tag{8.1}$$

where $\xi\colon (\operatorname{Aut} X)\times\mathcal{D}^{m-t(X)}(X) \to \mathcal{D}^{m-t(X)}(X)$ is the projection onto the second factor and ν is the restriction of $F^*(p)$ to Y (see (3.11) and (3.12)). The coordinate functions $z_1, ..., z_{m-t(X)}$ on $\mathcal{D}^{m-t(X)}(X) = \mathcal{C}_o^{m-t(X)}(\mathbb{C}^{②})$ are holomorphic and do not assume the values 0 and 1; hence for each i, $1 \le i \le m - t(X)$, the function $h_i = z_i \circ \xi \circ j_{X,m} \circ f$ is holomorphic on Y and does not take the values 0 and 1.

In case (i), Y is a Picard space. Thus, $\xi \circ j_{X,m} \circ f = \mathrm{const}$ and $f(Y)$ is contained in the $\operatorname{Aut} X$ orbit of some point $q^* \in \mathcal{C}_o^m(X)$, which implies $F(Z) \subseteq (\operatorname{Aut} X)p(q^*)$.

In case (ii) for each $i = 1, ..., m - t(X)$ the image $G_i = (h_i)_*(\pi_1(Y))$ of the homomorphism $(h_i)_*\colon \pi_1(Y) \to \pi_1(\mathbb{C} \setminus \{0,1\})$ is a solvable subgroup of $\pi_1(\mathbb{C} \setminus \{0,1\}) \cong \mathbb{F}_2$. Thus, either G_i is trivial or $G_i \cong \mathbb{Z}$. Let $\nu_i\colon \Gamma_i \to \mathbb{C} \setminus \{0,1\}$ be the covering corresponding to the subgroup G_i so that either $\Gamma_i \cong \mathbb{D} = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ or $\Gamma_i \cong K_{r_i} = \left\{\zeta \in \mathbb{C} \mid r_i^{-1} < |\zeta| < r_i\right\}$ for some $r_i > 1$. By the covering map theorem, there is a holomorphic map $\widetilde{h}_i\colon Y \to \Gamma_i$ that fits into the commutative diagram

$$\begin{tikzcd} & \Gamma_i \arrow[d, "\nu_i"] \\ Y \arrow[ur, "\widetilde{h}_i"] \arrow[r, "h_i"'] & \mathbb{C} \setminus \{0,1\}\,. \end{tikzcd}$$

By Lemma 8.2, the space Y is Liouville; since both $\mathbb{D}$ and K_{r_i} are bounded domains, $\widetilde{h}_i = \mathrm{const}$, which implies $h_i = \mathrm{const}$. Thus, $\xi \circ j_{X,m} \circ f = \mathrm{const}$, $f(Y)$ is contained in the $\operatorname{Aut} X$ orbit of some point $q^* \in \mathcal{C}_o^m(X)$ and $F(Z) \subseteq (\operatorname{Aut} X)p(q^*)$. □

Any connected complex Lie group $\mathcal{L}$ satisfies both (i) and (ii) since the union of the images of all holomorphic homomorphisms $\mathbb{C} \to \mathcal{L}$ contains an open neighborhood of the unity and, moreover, $\pi_1(\mathcal{L})$ is abelian. Thus, we have:

Corollary 8.4. *Let $\mathcal{L}$ be a connected complex Lie group. Then for every holomorphic map $\Phi\colon \mathcal{L} \to \mathcal{C}^m(X)$ there is a point $Q^* \in \mathcal{C}^m(X)$ such that $\Phi(\mathcal{L}) \subseteq (\operatorname{Aut} X)Q^*$.* □

8.2. Morphisms $\mathcal{C}^n(X) \to \mathcal{C}^k(X)$ and $\operatorname{Aut} X$ orbits. Here we prove the statement about morphisms of configuration spaces mentioned in Remark 1.2.

Proposition 8.5. *Every morphism $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ respects $\operatorname{Aut} X$ orbits, that is, $F((\operatorname{Aut} X)Q) \subseteq (\operatorname{Aut} X)F(Q)$ for all $Q \in \mathcal{C}^n(X)$.*

Proof. Let $Q \in \mathcal{C}^n(X)$ and let $\Psi\colon \operatorname{Aut} X \to \mathcal{C}^n(X)$ be the holomorphic map defined by $\Psi(A) = AQ$ for each $A \in \operatorname{Aut} X$. Set $\Phi = F \circ \Psi$; clearly $\Phi(\mathrm{id}_X) = F(Q)$. By Corollary 8.4, the image $F((\operatorname{Aut} X)Q) = F(\Psi(\operatorname{Aut} X)) = \Phi(\operatorname{Aut} X)$ is contained in an $\operatorname{Aut} X$ orbit, which certainly is the orbit of the point $F(Q)$. $\square$

8.3. Stabilizers of points in $\mathcal{C}^k(X)$. In what follows we need to use the classification of finite subgroups $G \subset \mathbf{SO}(3)$; this is a very classical matter, which goes back at least to F. Klein's "Vorlesungen über das Ikosaeder" [**Kle**]. The following list exhibits all such subgroups[17]:

a) cyclic groups $\mathbb{Z}/m\mathbb{Z}$, $m \geq 2$;

b) $(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z})$;

c) dihedral groups $(\mathbb{Z}/m\mathbb{Z}) \leftthreetimes (\mathbb{Z}/2\mathbb{Z})$, $m > 2$, having two generators a, b and the defining system of relations $a^m = 1$, $b^2 = 1$, $bab^{-1} = a^{-1}$;

d) the rotation groups of Platonic solids, that is, the symmetric group $\mathbf{S}(4)$ (rotation groups of the cube and the octahedron), the alternating groups $\mathbf{A}(4)$ (the rotation group of the tetrahedron) and $\mathbf{A}(5)$ (rotation groups of the icosahedron and the dodecahedron), and their subgroups.

Notice that all groups of this list except of $\mathbf{A}(5)$ are solvable.

Proposition 8.6. *For $k \geq t(X)$ the stabilizer $St_{Q^*} = \{A \in \operatorname{Aut} X \mid AQ^* = Q^*\}$ of any point $Q^* \in \mathcal{C}^k(X)$ is a finite group which is either solvable or isomorphic to the alternating group $\mathbf{A}(5)$.*

Proof. Let $Q^* = \{q_1^*, ..., q_k^*\}$. Any element $A \in St_{Q^*}$ is either an affine or a Möbius transformation permuting the points $q_1^*, ..., q_k^*$; A is uniquely determined by its values Aq_i^*, $1 \leq i \leq t(X)$, which shows that St_{Q^*} is finite.

If $X = \mathbb{C}$ then St_{Q^*} is a finite subgroup of $\mathbf{Aff}\ \mathbb{C}$. Every such subgroup is conjugate to a finite subgroup of the maximal compact subgroup $\mathbb{T} \subset \mathbf{Aff}\ \mathbb{C}$, which consists of all rotations $z \mapsto e^{it}z$, $t \in \mathbb{R}$. Every finite subgroup of the latter group is cyclic.

If $X = \mathbb{CP}^1$ then St_{Q^*} is a finite subgroup of $\mathbf{PSL}(2,\mathbb{C})$. Every such subgroup is conjugate to a finite subgroup of the maximal compact subgroup $K \subset \mathbf{PSL}(2,\mathbb{C})$, which is isomorphic to $\mathbf{SO}(3)$ and consists of all Möbius transformations of the form

$$z \mapsto \frac{az - \bar{c}}{cz + \bar{a}}\,, \quad |a|^2 + |c|^2 = 1\,.$$

Thus, in this case St_{Q^*} is isomorphic to one of the groups exhibited in the above list, which proves the proposition. $\square$

8.4. Cyclic Map Theorem. Recall that a continuous map $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ is said to be orbit-like if its image $F(\mathcal{C}^n(X))$ is contained in some orbit of $\operatorname{Aut} X$ action in $\mathcal{C}^k(X)$, that is, there is a point $Q^* \in \mathcal{C}^k(X)$ such that $F(\mathcal{C}^n(X)) \subseteq (\operatorname{Aut} X)Q^*$. We have already mentioned that the stabilizer St_{Q^*} of a generic point $Q^* \in \mathcal{C}^k(X)$ is trivial; thereby a morphism F whose image is contained in the orbit of such a point

[17] We take no care about possible repetitions in this list. See, for instance, [**Fo51**], Chapter VI, Sections 51-57, for more details.

Q^* is degenerate tame, that is, there is a morphism $T\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ such that $F(Q) = T(Q)Q^*$ for all $Q \in \mathcal{C}^n(X)$ (see Definition 1.1 and Remark 1.2).

Theorem 8.7 (Cyclic Map Theorem)**.** *Let $n > 4$ and $k \geq t(X)$. Then*

a) every cyclic morphism $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ is orbit-like;

b) every continuous orbit-like map $F\colon \mathcal{C}^n(X) \to \mathcal{C}^k(X)$ is cyclic.

Statement (***a***) follows from Proposition 8.3(ii) since $\mathcal{C}^n(X)$ is a connected algebraic manifold and therefore Liouville, whereas F is supposed to be cyclic.

Statement (***b***) is an immediate consequence of the following lemma.

Lemma 8.8. *Let $n > 4$ and $k \geq t(X)$. Then for each point $Q^* \in \mathcal{C}^k(X)$ every homomorphism $\varphi\colon B_n(X) \to \pi_1((\operatorname{Aut} X)Q^*)$ is cyclic.*

Proof. If the stabilizer St_{Q^*} is trivial then $(\operatorname{Aut} X)Q^* \cong \operatorname{Aut} X$, $\pi_1((\operatorname{Aut} X)Q^*) \cong \pi_1(\operatorname{Aut} X)$ is cyclic and φ is cyclic.

Suppose that St_{Q^*} is non-trivial; then the natural map $h\colon \operatorname{Aut} X \to (\operatorname{Aut} X)Q^*$ is a finite regular covering with the fiber St_{Q^*} and we have the commutative diagram

$$\begin{tikzcd} & & B_n(X) \arrow[d, "\varphi"'] \arrow[dr, "\delta\circ\varphi"] & & \\ 1 \arrow[r] & \pi_1(\operatorname{Aut} X) \arrow[r, "h_*"'] & \pi_1((\operatorname{Aut} X)Q^*) \arrow[r, "\delta"'] & St_{Q^*} \arrow[r] & 1\,, \end{tikzcd} \tag{8.2}$$

where the horizontal line is the exact sequence corresponding to the covering h.

Consider the restriction of the homomorphisms φ and $\delta\circ\varphi$ to the commutator subgroup $B'_n(X)$, which is perfect since $n > 4$.

By Proposition 8.6, St_{Q^*} is either solvable or isomorphic to $\mathbf{A}(5)$.

If St_{Q^*} is solvable then the restriction of $\delta \circ \varphi$ to $B'_n(X)$ is trivial and hence

$$\varphi(B'_n(X)) \subseteq \operatorname{Ker}\delta = h_*(\pi_1(\operatorname{Aut} X)) \cong \pi_1(\operatorname{Aut} X)\,.$$

The group $\pi_1(\operatorname{Aut} X)$ is cyclic, which shows that $\varphi(B'_n(X))$ is trivial and φ is cyclic.

Thus, we are left with the case $St_{Q^*} \cong \mathbf{A}(5) \subset \mathbf{S}(5)$. If $n > 5$ then, by Theorem 4.4, the homomorphism $\delta \circ \varphi$ is cyclic, which implies that $(\delta \circ \varphi)(B'_n(X)) = \{1\}$ and, as above, $\varphi(B'_n(X)) \subseteq \operatorname{Ker}\delta = \pi_1(\operatorname{Aut} X)$, $\varphi(B'_n(X)) = \{1\}$ and φ is cyclic. Finally, if $n = 5$ then the homomorphism $\delta\circ\varphi$ maps $B_5(X)$ into the proper subgroup $\mathbf{A}(5) \subset \mathbf{S}(5)$ and, by Surjectivity Theorem 2.7, it is cyclic again; as above, we conclude that φ is cyclic. This proves the lemma and completes the proof of Cyclic Map Theorem. ☐

8.5. Holomorphic maps to the space of balanced configurations. We start with the following

***Definition* 8.9.** The *balanced configuration space* $\mathcal{C}_b^{m-1}(\mathbb{C})$ is the $(m-1)$-dimensional subspace of $\mathcal{C}^m(\mathbb{C})$ consisting of all sets $Q = \{q_1, ..., q_n\} \in \mathcal{C}^m(\mathbb{C})$ having its mass centers at the origin:

$$\mathcal{C}_b^{m-1}(\mathbb{C}) = \{Q = \{q_1, ..., q_m\} \in \mathcal{C}^m(\mathbb{C}) \mid q_1 + ... + q_m = 0\}\,.$$

We set also

$$\mathcal{C}^{m-1}_{ob}(\mathbb{C}) = \{q = (q_1, ..., q_m) \in \mathcal{C}^m_o(\mathbb{C}) \mid q_1 + ... + q_m = 0\}\,,$$
$$\mathcal{C}^{m-2}_{ob,1}(\mathbb{C}) = \{q = (q_1, ..., q_m) \in \mathcal{C}^n_o(\mathbb{C}) \mid q_1 + ... + q_m = 0, \quad q_1 - q_2 = 1\}\,. \qquad \bigcirc$$

The multiplicative group $\mathbb{C}^*$ acts in $\mathcal{C}^{m-1}_b(\mathbb{C})$ and $\mathcal{C}^{m-1}_{ob}(\mathbb{C})$ via

$$\mathbb{C}^* \ni \zeta: \ \{q_1, ..., q_m\} \mapsto \{\zeta q_1, ..., \zeta q_m\} \quad \text{and} \quad (q_1, ..., q_m) \mapsto (\zeta q_1, ..., \zeta q_m) \tag{8.3}$$

respectively. The $\mathbb{C}^*$-orbit of any point $q = (q_1, ..., q_m) \in \mathcal{C}^{m-1}_{ob}(\mathbb{C})$ intersects the subspace $\mathcal{C}^{m-2}_{ob,1}(\mathbb{C}) \subset \mathcal{C}^{m-1}_{ob}(\mathbb{C})$ at a single point $\widetilde{q} = (q_1/(q_1 - q_2), ..., q_m/(q_1 - q_2))$; this provides the biholomorphic direct decomposition $\mathcal{C}^{m-1}_{ob}(\mathbb{C}) = \mathbb{C}^* \times \mathcal{C}^{m-2}_{ob,1}(\mathbb{C})$,

$$\mathcal{C}^{m-1}_{ob}(\mathbb{C}) \ni q = (q_1, ..., q_m) \mapsto (q_1 - q_2; \, \widetilde{q}) \in \mathbb{C}^* \times \mathcal{C}^{m-2}_{ob,1}(\mathbb{C})\,.$$

On the other hand, the functions $\phi_i(q) = (q_1 - q_i)/(q_1 - q_2)$, $i = 3, ..., n$, are holomorphic on $\mathcal{C}^{m-1}_{ob}(\mathbb{C})$, separate points of $\mathcal{C}^{m-2}_{ob,1}(\mathbb{C})$ and do not take the values 0 and 1. This implies that

(i) *for every holomorphic map $\Phi\colon \mathcal{L} \to \mathcal{C}^m(\mathbb{C})$ of a connected complex Lie group $\mathcal{L}$ there is a point $Q^* \in \mathcal{C}^{m-1}_b(\mathbb{C})$ such that $\Phi(\mathcal{L}) \subseteq \mathbb{C}^* Q^*$* and

(ii) *any holomorphic map $F\colon \mathcal{C}^{n-1}_b(\mathbb{C}) \to \mathcal{C}^{k-1}_b(\mathbb{C})$ respects $\mathbb{C}^*$-orbits, that is, $F(\mathbb{C}^* Q) \subseteq \mathbb{C}^* F(Q)$ for all $Q \in \mathcal{C}^{n-1}_b(\mathbb{C})$.*

The proof of the following analog of Proposition 8.3 is clear:

Proposition 8.10. *Let Z be a connected complex space and $F\colon Z \to \mathcal{C}^{m-1}_b(\mathbb{C})$ be a holomorphic map. Suppose that either Z is an ultra-Picard space or the homomorphism $F_*\colon \pi_1(Z) \to \pi_1(\mathcal{C}^{m-1}_b(\mathbb{C}))$ is solvable and Z is Liouville. Then there is a point $Q^* \in \mathcal{C}^{m-1}_b(\mathbb{C})$ such that $F(Z) \subseteq \mathbb{C}^* Q^*$.* $\square$

In what follows, instead of $\mathcal{C}^m(\mathbb{C})$ and its balanced subspace $\mathcal{C}^{m-1}_b(\mathbb{C})$, it is convenient to deal with the space of polynomials with simple roots $\mathbf{G}_m$ and its subspace

$$\mathbf{G}^0_m = \{w = (w_1, ..., w_m) \in \mathbf{G}_m \mid w_1 = 0\} \tag{8.4}$$

that corresponds to $\mathcal{C}^{m-1}_b(\mathbb{C})$ under the identification $\mathcal{C}^m(\mathbb{C}) = \mathbf{G}_m$ (see Remark 1.9). The $\mathbb{C}^*$ action in $\mathbf{G}_m$ corresponding to the $\mathbb{C}^*$ action (8.3) in $\mathcal{C}^{m-1}_b(\mathbb{C})$ look as follows:

$$\mathbb{C}^* \ni \zeta\colon \mathbf{G}^0_m \ni (0, w_2, ..., w_m) \mapsto \mathcal{U}_\zeta w = (0, \zeta^2 w_2, ..., \zeta^m w_m) \in \mathbf{G}^0_m\,. \tag{8.5}$$

***Notation* 8.11.** For $w = (0, w_2, ..., w_m) \in \mathbf{G}^0_m$ set $\mathcal{Z}(w) = \{j \mid j \geq 2, \ w_j = 0\}$, $\mathcal{NZ}(w) = \{j \mid j \geq 2, \ w_j \neq 0\}$ and denote by $\mathcal{D}(w)$ the greatest common divisor of all $j \in \mathcal{NZ}(w)$. Since $\mathcal{Z}(w)$ cannot contain both $m - 1$ and m, the number $\mathcal{D}(w)$ divides $m(m-1)$. As usual, $\mathcal{O}(Z)$ denotes the algebra of all holomorphic functions on Z; $\mathcal{O}^*(Z)$ is the multiplicative group of all non-vanishing functions $f \in \mathcal{O}(Z)$ and its subgroup $\exp \mathcal{O}(Z) \subseteq \mathcal{O}^*(Z)$ consists of all functions e^f, $f \in \mathcal{O}(Z)$.

The following evident lemma provides a complete description of $\mathbb{C}^*$ orbits in the space of balanced configurations $\mathcal{C}^{m-1}_b(\mathbb{C}) = \mathbf{G}^0_m$; it will be used in next sections.

Lemma 8.12. *Let $w^\circ = (0, w_2^\circ, ..., w_m^\circ) \in \mathbf{G}_m^0$ and $\mathcal{D} = \mathcal{D}(w^\circ)$. Then the stabilizer $St_{w^\circ} \subset \mathbb{C}^*$ of w° is the cyclic subgroup $\{\zeta \in \mathbb{C}^* \mid \zeta^{\mathcal{D}} = 1\} \cong \mathbb{Z}/\mathcal{D}\mathbb{Z}$, and the $\mathbb{C}^*$-orbit $\mathbb{C}^* w^\circ \cong \mathbb{C}^*/St_{w^\circ} \cong \mathbb{C}^*$ of w° admits the following biholomorphic parameterization:*

$$f\colon \mathbb{C}^* \xrightarrow{\cong} \mathbb{C}^* w^\circ\,, \quad f(\zeta) = (0, \zeta^{2/\mathcal{D}} w_2^\circ, ..., \zeta^{m/\mathcal{D}} w_m^\circ) \in \mathbb{C}^* w^\circ \subset \mathbf{G}_m^0 \subset \mathbf{G}_m \tag{8.6}$$

(*notice that the set $\{j/\mathcal{D} \mid w_j^\circ \neq 0\}$ consists of mutually co-prime integers*). $\square$

8.6. Solvable holomorphic maps of Liouville spaces to $\mathcal{C}^m(\mathbb{C}) = \mathbf{G}_m$.

Proposition 8.13. *Let Z be a connected complex space and $F\colon Z \to \mathbf{G}_m$ be a holomorphic map. Suppose that either (i) Z is an ultra-Picard space or (ii) the homomorphism $F_*\colon \pi_1(Z) \to \pi_1(\mathbf{G}_m)$ is solvable and Z is Liouville. Then:*

a) *There exist $w^\circ = (0, w_2^\circ, ..., w_m^\circ) \in \mathbf{G}_m^0$, $\zeta \in \mathcal{O}^*(Z)$ and $c, \eta_2, ..., \eta_m \in \mathcal{O}(Z)$ such that $\eta_j = 0$ for $j \in \mathcal{Z}(w^\circ)$, $\eta_j = \zeta^{j/\mathcal{D}(w^\circ)}$ for $j \in \mathcal{NZ}(w^\circ)$ and*

$$F(z) = (t + c(z))^m + \sum_{j=2}^m w_j^\circ \eta_j(z)(t + c(z))^{m-j}\,, \quad z \in Z\,. \tag{8.7}$$

b) *If $w_m^\circ \neq 0$ then the map (8.7) is homotopic to the model map*

$$\mathcal{A}_a\colon Z \ni z \mapsto (0, ..., 0, 0, a(z)) = t^m + a(z) \in \mathbf{G}_m^0 \subset \mathbf{G}_m\,, \tag{8.8}$$

where $a(z) = [\zeta(z)]^{m/\mathcal{D}(w^\circ)}$.

c) *If $w_m^\circ = 0$ then $w_{m-1}^\circ \neq 0$ and the map (8.7) is homotopic to the model map*

$$\mathcal{B}_b\colon Z \ni z \mapsto (0, ..., 0, b(z), 0) = t^m + b(z)t \in \mathbf{G}_m^0 \subset \mathbf{G}_m\,, \tag{8.9}$$

where $b(z) = [\zeta(z)]^{(m-1)/\mathcal{D}(w^\circ)}$.

PROOF. ***a***) Let $F(z) = (F_1(z), F_2(z), ..., F_m(z))$; consider a holomorphic map $F^0 = (0, F_2^0, ..., F_m^0)\colon Z \to \mathbf{G}_m^0$ with the components $F_2^0, ..., F_m^0$ defined by the condition

$$t^m + \sum_{j=2}^m F_j^0(z) t^{m-j} \equiv (t - F_1(z)/m)^m + \sum_{j=1}^m F_j(z)(t - F_1(z)/m)^{m-j}\,. \tag{8.10}$$

By Proposition 8.10, $F^0(Z) \subseteq \mathbb{C}^* w^\circ$ for a certain point $w^\circ = (0, w_2^\circ, ..., w_m^\circ) \in \mathbf{G}_m^0$. It follows from Lemma 8.12 that there are $\zeta \in \mathcal{O}^*(Z)$ and $\eta_2, ..., \eta_m \in \mathcal{O}(Z)$ such that $\eta_j = 0$ for $j \in \mathcal{Z}(w^\circ)$, $\eta_j = \zeta^{j/\mathcal{D}(w^\circ)}$ for $j \in \mathcal{NZ}(w^\circ)$ and

$$F^0(z) = (0, w_2^\circ \eta_2(z), ..., w_m^\circ \eta_m(z)) = t^m + \sum_{j=2}^m w_j^\circ \eta_j(z) t^{m-j}\,, \quad z \in Z\,. \tag{8.11}$$

Let $c(z) = F_1(z)/m$; then (8.10) and (8.11) show that

$$\begin{aligned} F(z) = t^m + \sum_{j=1}^m F_j(z) t^{m-j} &= (t + c(z))^m + \sum_{j=2}^m F_j^0(z)(t + c(z))^{m-j} \\ &= (t + c(z))^m + \sum_{j=2}^m w_j^\circ \eta_j(z)(t + c(z))^{m-j}\,, \end{aligned} \tag{8.12}$$

which proves (***a***).

(***b,c***) By (***a***), F is of the form (8.7). The homotopy

$$F(z,s)=(t+sc(z))^m+\sum_{j=2}^m w_j^\circ\eta_j(z)(t+sc(z))^{m-j}\,,\quad z\in Z\,,\quad 0\le s\le 1\,,\qquad (8.13)$$

joins $F=F(\cdot,1)$ to the map $F(\cdot,0)$, $F(z,0)=t^m+\sum_{j=2}^m w_j^\circ\eta_j(z)t^{m-j}$ in the class of holomorphic maps $Z\to\mathbf{G}_m$. At least one of the numbers $m-1,m$ is not in $\mathcal{Z}$ and the vector subspace

$$V(w^\circ)\stackrel{\text{def}}{=\!=}\{w=(0,w_2,...,w_m)\in\mathbb{C}^m\mid w_j=0\ \text{ whenever } j\in\mathcal{Z}\}$$

is of positive dimension. The discriminant $d_m(w)$ cannot vanish identically on $V(w^\circ)$, for $w^\circ\in\mathbf{G}_m^0\cap V(w^\circ)$; thus, the intersection

$$\mathbf{G}_m^0\cap V(w^\circ)=V(w^\circ)\setminus\{w=(0,w_2,...,w_m)\in\mathbb{C}^m\mid d_m(w)=0\}$$

is arcwise connected. Set $w'=(0,...,0,0,1)=t^m+1$ when $w_m^\circ\ne 0$ and $w'=(0,...,0,1,0)=t^m+t$ otherwise; notice that in both cases $w'\in\mathbf{G}_m^0\cap V(w^\circ)$. Let $w(s)=(0,w_2(s),...,w_m(s))$, $0\le s\le 1$, be a path in $\mathbf{G}_m^0\cap V(w^\circ)$ joining the point $w(0)=w^\circ$ to the point $w(1)=w'$. It is easily seen that

$$d_m(0,w_2(s)\eta_2(z),...,w_m(s)\eta_m(z))=[\zeta(z)]^{m(m-1)/\mathcal{D}(w^\circ)}d_m(0,w_2(s),...,w_m(s))\,;$$

therefore the formula $h_s(\zeta)=t^m+\sum_{j=2}^m w_j(s)\eta_j(z)t^{m-j}$ provides a well-defined homotopy joining the map $F(\cdot,0)$ to a model map $\mathcal{A}_a$ in case (***b***) and to a model map $\mathcal{B}_b$ in case (***c***). □

***Remark* 8.14.** Let $a,b,a_i,b_i\in\mathcal{O}^*(Z)$, $i=1,2$. Then[18] $\mathcal{A}_{a_1}\sim\mathcal{A}_{a_2}$ (respectively $\mathcal{B}_{b_1}\sim\mathcal{B}_{b_2}$) if and only if $a_1/a_2\in\exp\mathcal{O}(Z)$ (respectively if and only if $b_1/b_2\in\exp\mathcal{O}(Z)$). Moreover, $\mathcal{A}_a\sim\mathcal{B}_b$ if and only if $a^{m-1}/b^m\in\exp\mathcal{O}(Z)$. ○

***Definition* 8.15.** Define the points $w_\circ^{(m)},w_\circ^{(m-1)}\in\mathbf{G}_m$ as

$$w_\circ^{(m)}=(0,...,0,0,-1)=t^m-1\quad\text{and}\quad w_\circ^{(m-1)}=(0,...,0,-1,0)=t^m-t$$

respectively. For any $r\in\mathbb{Z}$ we define the *model holomorphic maps*

$$\begin{aligned}\mathcal{A}_r\colon\mathbb{C}^*\ni\zeta\mapsto\mathcal{U}_{\zeta^r}w_\circ^{(m)}\quad&=(0,...,0,0,-\zeta^r)=t^m-\zeta^r\in\mathbf{G}_m^0\subset\mathbf{G}_m=\mathcal{C}^m(\mathbb{C})\,,\\ \mathcal{B}_r\colon\mathbb{C}^*\ni\zeta\mapsto\mathcal{U}_{\zeta^r}w_\circ^{(m-1)}&=(0,...,0,-\zeta^r,0)=t^m-\zeta^r t\in\mathbf{G}_m^0\subset\mathbf{G}_m=\mathcal{C}^m(\mathbb{C})\,.\end{aligned}\qquad(8.14)$$

Notice that $\mathcal{A}_p\sim\mathcal{B}_q$ if and only if $(m-1)p=mq$. ○

Corollary 8.16. *Any holomorphic map $\mathbb{C}^*\to\mathcal{C}^m(\mathbb{C})=\mathbf{G}_m$ is homotopic to one of the model maps $\mathcal{A}_r$, $\mathcal{B}_r$ $(r\in\mathbb{Z})$.* □

[18] Here "$\sim$" means free homotopy.

8.7. Cyclic morphisms $\mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$. Notice that any non-vanishing holomorphic function $f(Q)$ on $\mathcal{C}^n(\mathbb{C})$ is of the form $f(Q) = [d_n(Q)]^l e^{\varphi(Q)}$, where $l \in \mathbb{Z}$ and $\varphi \in \mathcal{O}(\mathcal{C}^n(\mathbb{C}))$. Since any connected algebraic variety is a Liouville space, Proposition 8.13 applies to cyclic morphisms $\mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$:

Theorem 8.17. *Let $F\colon \mathcal{C}^n(\mathbb{C}) = \mathbf{G}_n \to \mathbf{G}_k = \mathcal{C}^k(\mathbb{C})$ be a cyclic holomorphic map. Then:*

a) *There exist $w^\circ = (0, w_2^\circ, ..., w_k^\circ) \in \mathbf{G}_k^0 = \mathcal{C}_b^{k-1}(\mathbb{C})$, $l \in \mathbb{Z}$ and $c, \varphi \in \mathcal{O}(\mathcal{C}^n(\mathbb{C}))$ such that*

$$F(Q) = (t + c(Q))^k + \sum_{j=2}^{k} w_j^\circ [d_n(Q)]^{jl/\mathcal{D}(w^\circ)} e^{jl\varphi(Q)} (t + c(Q))^{k-j}\,, \quad Q \in \mathcal{C}^n(\mathbb{C})\,. \tag{8.15}$$

b) *If $w_k^\circ \neq 0$ then $kl/\mathcal{D}(w^\circ) \in \mathbb{Z}$ and the map* (8.15) *is homotopic to the model map*

$$\mathcal{A}_l\colon \mathcal{C}^n(\mathbb{C}) \ni Q \mapsto (0, ..., 0, 0, [d_n(Q)]^{kl/\mathcal{D}(w^\circ)}) = t^k + [d_n(Q)]^{kl/\mathcal{D}(w^\circ)}\,. \tag{8.16}$$

c) *If $w_k^\circ = 0$ then $w_{k-1}^\circ \neq 0$, $(k-1)l/\mathcal{D}(w^\circ) \in \mathbb{Z}$ and the map* (8.15) *is homotopic to the model map*

$$\mathcal{B}_l\colon \mathcal{C}^n(\mathbb{C}) \ni Q \mapsto (0, ..., 0, [d_n(Q)]^{(k-1)l/\mathcal{D}(w^\circ)}, 0) = t^k + [d_n(Q)]^{(k-1)l/\mathcal{D}(w^\circ)} t\,. \tag{8.17}$$

d) *$\mathcal{A}_{l_1} \sim \mathcal{A}_{l_2}$ (respectively $\mathcal{B}_{l_1} \sim \mathcal{B}_{l_2}$) if and only if $l_1 = l_2$. Moreover, $\mathcal{A}_{l_1} \sim \mathcal{B}_{l_2}$ if and only if $(k-1)l_1 = kl_2$.*

In particular, every cyclic holomorphic map $F\colon \mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$ homotopically splits, that is, there exists a holomorphic map $G\colon \mathbb{C}^* \to \mathcal{C}^k(\mathbb{C})$ such that F is homotopic to the composition

$$G \circ d_n\colon \mathcal{C}^n(\mathbb{C}) \xrightarrow{d_n} \mathbb{C}^* \xrightarrow{G} \mathcal{C}^k(\mathbb{C})\,. \qquad \square$$

9. Morphisms $\mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$ and special homomorphisms $B_n \to B_k$

Due to Theorem 4.5, for $n \neq 4$ and $n > k$ every continuous map (and all the more every morphism) $\mathcal{C}^n(X) \to \mathcal{C}^k(X)$ must be cyclic. On the other hand, for $3 \leq n \leq k$ it is easy to construct non-cyclic continuous maps $\mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$. It is not so in the case $X = \mathbb{CP}^1$. To see this, one may use a theorem of Kunio Murasugi [**Mur82**], which provides a complete description of the torsion in the sphere braid group $B_m(S^2)$ and in the quotient group B_m/CB_m of the Artin braid group B_m by its center CB_m.

Let us set $\alpha = \sigma_1 \cdots \sigma_{m-1}$, $\beta = \sigma_1 \cdots \sigma_{m-1}\sigma_1$ (both in B_m and $B_m(S^2)$), and $\gamma = \sigma_1 \cdots \sigma_{m-2}\sigma_1$ in $B_m(S^2)$; furthermore, let us denote by $\hat{\alpha}$, $\hat{\beta}$ the images of the elements $\alpha, \beta \in B_m$ in the quotient group B_m/CB_m.

MURASUGI THEOREM. ***a***) *Each element of finite order in B_m/CB_m is conjugate to an element of the form $\hat{\alpha}^p$ or $\hat{\beta}^p$, where $p \in \mathbb{Z}$. The elements $\hat{\alpha}, \hat{\beta} \in B_m/CB_m$ are of the order m and $m-1$ respectively.*

b) *Each element of finite order in $B_m(S^2)$ is conjugate to an element of the form α^p, β^p or γ^p, where $p \in \mathbb{Z}$. The elements $\alpha, \beta, \gamma \in B_m(S^2)$ are of the order $2m$, $2(m-1)$ and $2(m-2)$ respectively.*

For any homomorphism $\varphi\colon B_n(S^2) \to B_k(S^2)$ we have $\varphi(\operatorname{Tors} B_n(S^2)) \subseteq \operatorname{Tors} B_k(S^2)$. Hence part (***b***) of Murasugi Theorem implies that φ cannot be non-cyclic unless the number $n(n-1)(n-2)$ divides $k(k-1)(k-2)$.

A homomorphism $\varphi\colon B_n \to B_k$ (cyclic or not) must not induce a homomorphism of quotient groups $B_n/CB_n \to B_k/CB_k$. Hence Murasugi Theorem puts no restraints in homomorphisms $B_n \to B_k$. In fact, for $k > n \geq 3$ there is plenty of non-cyclic homomorphisms $\varphi\colon B_n \to B_k$ and each of them is induced by a certain non-cyclic continuous map $\mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$.

For holomorphic maps the situation is completely different. In fact I do not know any example of a non-cyclic holomorphic map $\mathcal{C}^n(X) \to \mathcal{C}^k(X)$ for $4 < n < k$ (compare to Linked Map Theorem); most likely they do not exist at all. Being unable to prove this I present here a rather restricted result, which says that in the case $X = \mathbb{C}$ such a map cannot exist unless the number $n(n-1)$ divides $k(k-1)$. In order to prove this, we will show that a homomorphism $F_*\colon B_n \to B_k$ induced by a holomorphic map $\mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$ must satisfy a strong algebraic condition.

9.1. Special presentations and special homomorphisms. Recall that the elements $\alpha = \sigma_1 \cdots \sigma_{m-1}$ and $\beta = \alpha\sigma_1$ generate the whole braid group $B_m(X)$ (see 4.9).

***Definition* 9.1.** A pair of elements $a, b \in B_m$ is said to be a *special system of generators* in B_m if there exists an automorphism ψ of B_m such that $\psi(\alpha) = a$ and $\psi(\beta) = b$. If $\{a, b\}$ is such a system of generators then the elements

$$s_i = \psi(\sigma_i) = a^{i-2}ba^{-(i-1)} \qquad (1 \leq i \leq m-1)$$

also form a system of generators of B_m that satisfy relations (4.3), (4.4); we call such a system of generators *standard.* ○

***Definition* 9.2.** Given a special system of generators $\{a, b\}$ in B_m, we denote by $\mathcal{H}_m(a, b)$ the set of all elements of the form $g^{-1}a^pg$ or $g^{-1}b^pg$, where g runs over B_m and p runs over $\mathbb{Z}$. A homomorphism $\varphi\colon B_n \to B_k$ is said to be *special* if $\varphi(\mathcal{H}_n(a, b)) \subseteq \mathcal{H}_k(a', b')$ for some choice of special systems of generators $a, b \in B_n$ and $a', b' \in B_k$. ○

Part (***a***) of Murasugi Theorem quoted above shows that *the set $\mathcal{H}_m = \mathcal{H}_m(a, b)$ does not depend on a choice of a special system of generators $a, b \in B_m$, and a homomorphism $\varphi\colon B_n \to B_k$ is special if and only if for any element $g \in B_m$ of finite order modulo CB_n its image $\varphi(g) \in B_k$ is an element of finite order modulo CB_k* (we do not use this fact below; it is convenient however to keep it in mind).

***Remark* 9.3.** It is easily seen that for $n \geq 3$ every *endomorphism* of the group B_n is special. On the other hand, for $k > n \geq 3$ the standard embedding $B_n \hookrightarrow B_k$ can never be a special homomorphism. ○

The following result stated in [**Lin71**] and [**Lin79**] is the crucial one for our purposes.

Theorem 9.4. *For every holomorphic map $F\colon \mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$, every point $Q^\circ \in \mathcal{C}^n(\mathbb{C})$ and any choice of isomorphisms $B_n \cong \pi_1(\mathcal{C}^n(\mathbb{C}), Q^\circ)$ and $\pi_1(\mathcal{C}^n(\mathbb{C}), F(Q^\circ)) \cong B_k$ the induced homomorphism*

$$F_*\colon B_n \cong \pi_1(\mathcal{C}^n(\mathbb{C}), Q^\circ) \to \pi_1(\mathcal{C}^n(\mathbb{C}), F(Q^\circ)) \cong B_k \tag{9.1}$$

is special. ◯

To prove this theorem we will use the description of holomorphic maps $\mathbb{C}^* \to \mathcal{C}^m(\mathbb{C})$ given in Corollary 8.16. But first I want to present a general concept of holomorphic elements of homotopy groups introduced in [**Lin71**].

9.2. Holomorphic elements of homotopy groups. The idea of this section is to choose for each $i \in \mathbb{N}$ an appropriate *test complex manifold* $\mathcal{M}_i$ homotopy equivalent to the i-dimensional sphere S^i and define corresponding *holomorphic elements* in the homotopy groups $\pi_i(Y)$ of connected complex spaces Y as those admitting holomorphic representatives $M_i \to Y$. Doing so one may sometimes come to certain non-trivial algebraic restraints imposed by the condition that a homomorphism $\pi_i(Y) \to \pi_i(Z)$ can be induced by a holomorphic map $Y \to Z$, since such a homomorphism must carry holomorphic elements of $\pi_i(Y)$ to holomorphic elements of $\pi_i(Z)$. What are "test manifolds" depends on the problem under consideration; for our purpose the following choice is an appropriate one just because it works.

***Definition* 9.5.** Let Y be a connected complex space. We define the *holomorphic part* $\boldsymbol{hol}\,\pi_i(Y, y_\circ)$ of the homotopy group $\pi_i(Y, y_\circ)$ as follows. In the space $\mathbb{C}^{i+1}$ with coordinates $\zeta_0, ..., \zeta_i$ consider the quadric

$$\mathbb{C}S^i = \left\{\zeta = (\zeta_0, ..., \zeta_i) \in \mathbb{C}^{i+1} \mid \zeta_0^2 + ... + \zeta_i^2 = 1\right\} .$$

Its intersection $S^i = \mathbb{C}S^i \cap \mathbb{R}^{i+1}$ with the real subspace $\mathbb{R}^{i+1} \subset \mathbb{C}^{i+1}$ is the standard unit sphere in $\mathbb{R}^{i+1}$; moreover, S^i is a deformation retract of $\mathbb{C}S^i$ (for $\mathbb{C}S^i$ is diffeomorphic to the total space of the tangent bundle TS^i). Therefore choosing $\zeta_\circ = (1, 0, ..., 0) \in \mathbb{C}S^i$ as a basepoint we may regard any element $\gamma \in \pi_i(Y, y_\circ)$ as a homotopy class of continuous mappings $(\mathbb{C}S^i, \zeta_\circ) \to (Y, y_\circ)$. Let us say that an element $\gamma \in \pi_i(Y, y_\circ)$ is *holomorphic* if there exists a holomorphic mapping $h\colon \mathbb{C}S^i \to Y$ freely homotopic to some (and hence to any) map contained in γ. The set of all holomorphic elements in $\pi_i(Y, y_\circ)$ is denoted by $\boldsymbol{hol}\,\pi_i(Y, y_\circ)$; in general, it is not a subgroup; the subgroup of $\pi_i(Y, y_\circ)$ generated by $\boldsymbol{hol}\,\pi_i(Y, y_\circ)$ is denoted by $\boldsymbol{Hol}\,\pi_i(Y, y_\circ)$. ◯

Proposition 9.6. *The subset $\boldsymbol{hol}\,\pi_i(Y, y_\circ)$ and the subgroup $\boldsymbol{Hol}\,\pi_i(Y, y_\circ)$ have the following properties:*

i) If Y is a homogeneous space of a complex Lie group then $\boldsymbol{hol}\,\pi_i(Y, y_\circ) = \pi_i(Y, y_\circ)$ for every natural i. In particular, the complex orthogonal group $\mathbf{O}(n+1, \mathbb{C})$ acts transitively on $\mathbb{C}S^m$ and hence

$$\boldsymbol{hol}\,\pi_i(\mathbb{C}S^m, \zeta_\circ) = \pi_i(\mathbb{C}S^m, \zeta_\circ) \cong \pi_i(S^m)$$

for all natural i and m.

ii) If $\gamma \in \boldsymbol{hol}\,\pi_i(Y, y_\circ)$ then the whole cyclic subgroup generated by γ is contained in $\boldsymbol{hol}\,\pi_i(Y, y_\circ)$.

iii) *The isomorphism* $\chi_i\colon \pi_i(Y, y_1) \to \pi_i(Y, y_0)$ *induced by a path* $\chi\colon [0,1] \to Y$ *joining points* $y_0, y_1 \in Y$ *carries* $\boldsymbol{hol}\, \pi_i(Y, y_1)$ *into* $\boldsymbol{hol}\, \pi_i(Y, y_0)$.

iv) *The homomorphism* $f_{*i}\colon \pi_i(Y, y_\circ) \to \pi_i(Z, z_\circ)$ *induced by a holomorphic map* $f\colon (Y, y_\circ) \to (Z, z_\circ)$ *carries* $\boldsymbol{hol}\, \pi_i(Y, y_\circ))$ *into* $\boldsymbol{hol}\, \pi_i(Z, z_\circ)$.

v) $\boldsymbol{Hol}\, \pi_1(Y, y_\circ)$ *is a normal subgroup of the fundamental group* $\pi_1(Y, y_\circ)$.

Proof. Property (i) follows from the Cartan–Grauert–Ramspott theorem (see [**Car40, Gra58, Rams65**]). It follows from (i) that $\boldsymbol{hol}\, \pi_i(\mathbb{C}S^i, \zeta_\circ) = \pi_i(\mathbb{C}S^i, \zeta_\circ) \cong \pi_i(S^i, \zeta_\circ) \cong \mathbb{Z}$. Hence if $h\colon \mathbb{C}S^i \to X$ is a holomorphic representative of a class $\gamma \in \boldsymbol{hol}\, \pi_i(Y, y_\circ)$ and $f^{(q)}\colon \mathbb{C}S^i \to \mathbb{C}S^i$ is a holomorphic map homotopic to a map $S^i \to S^i$ of degree $q \in \mathbb{Z}$ then the composition $\mathbb{C}S^i \xrightarrow{f^{(q)}} \mathbb{C}S^i \xrightarrow{h} Y$ is a holomorphic representative of the class $q\gamma$ (for $i = 1$ we use the multiplicative notation γ^q); this proves (ii). Properties (iii) and (iv) follow directly from the definitions and (v) is a consequence of (iii). □

9.3. Holomorphic elements of $\pi_i(\mathcal{C}^m(\mathbb{C}))$**.** It follows from Proposition 8.3 that the image of any holomorphic map $f\colon \mathbb{C}S^i \to \mathcal{C}^m(\mathbb{C})$ is contained in the $\operatorname{Aut} X$-orbit of some point $Q^* \in \mathcal{C}^m(\mathbb{C})$.

First, let $i \geq 2$. Then $\boldsymbol{hol}\, \pi_i(\mathcal{C}^m(\mathbb{C})) = \pi_i(\mathcal{C}^m(\mathbb{C})) = 0$ (moreover, it is easily shown that for such i any holomorphic map $\mathbb{C}S^i \to \mathcal{C}^m(\mathbb{C})$ is contractible in the class of holomorphic maps). Thus, on this way no difference occurs in the behavior of higher homotopy group homomorphisms induced by holomorphic maps and continuous ones.

Let us compute holomorphic elements of $\pi_1(\mathcal{C}^m(\mathbb{C}))$; we use the concept of model holomorphic maps $\mathcal{A}_r, \mathcal{B}_r$ introduced in Definition 8.15.

Proposition 9.7. ***a***) *For any* $Q^* \in \mathcal{C}^m(\mathbb{C}) = \mathbf{G}_m$ *there is a special system of generators* $\{a, b\}$ *in* $\pi_1(\mathcal{C}^m(\mathbb{C})) \cong B_m$ *such that the set* $\boldsymbol{hol}\, \pi_1(\mathcal{C}^m(\mathbb{C}), Q^*)$ *of all holomorphic elements in* $\pi_1(\mathcal{C}^m(\mathbb{C}), Q^*)$ *coincides with* $\mathcal{H}_m(a, b)$.

b) *The set* $\boldsymbol{hol}\, B_m$ *of all elements that for some choice of a basepoint* $Q^* \in \mathcal{C}^m(\mathbb{C})$ *and an isomorphism* $B_m \cong \pi_1(\mathcal{C}^m(\mathbb{C}), Q^*)$ *admit holomorphic representatives* $\mathbb{C}^* \to \mathcal{C}^m(\mathbb{C})$ *coincides with the set of all elements of finite order modulo the center* CB_m.

Proof. For any element $h \in \pi_1(\mathcal{C}^m(\mathbb{C}), Q^*)$ let us denote by $[h]$ the free homotopy class of continuous mappings $\mathbb{C}^* \to \mathcal{C}^m(\mathbb{C})$ containing some (and hence each one) representative of h. A straightforward geometric consideration shows that a standard system of generators $s_1, ..., s_{m-1}$ in the group $\pi_1(\mathcal{C}^m(\mathbb{C}), Q^*) = \pi_1(\mathbf{G}_m, w^*) \cong B_m$ may be chosen in such a way that the model holomorphic map $\mathcal{A} = \mathcal{A}_1\colon \mathbb{C}^* \ni \zeta \mapsto (0, ..., 0, 0, -\zeta) = t^m - \zeta$ belongs to the free homotopy class $[a]$ corresponding to the element $a = s_1 \cdots s_{m-1}$ and the map $\mathcal{B} = \mathcal{B}_1\colon \mathbb{C}^* \ni \zeta \mapsto (0, ..., 0, -\zeta, 0) = t^m - \zeta t$ belongs to the free homotopy class $[gbg^{-1}]$ corresponding to the element gbg^{-1}, where $b = as_1 = s_1 \cdots s_{m-1} s_1$ and g is a suitable element of $\pi_1(\mathbf{G}_m^0, w_*)$. Then it is clear that $\mathcal{A}_r \in [a^r]$ and $\lfloor_r \in g[b^r g^{-1}]$. On the other hand, by Corollary 8.16, any holomorphic map $\mathbb{C}^* \to \mathcal{C}^m(\mathbb{C}) = \mathbf{G}_m$ is homotopic to one of the model maps $\mathcal{A}_r$, $\mathcal{B}_r$ $(r \in \mathbb{Z})$; this completes the proof of statement (***a***). In view of the Murasugi theorem quoted above, statement (***b***) is just a consequence of (***a***) □

Now we are ready to prove Theorem 9.4.

PROOF OF THEOREM 9.4. Let $F\colon \mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$ be a holomorphic map. By Proposition 9.6(*iv*), the homomorphism $F_*\colon \pi_1(\mathcal{C}^n(\mathbb{C}), Q^\circ) \to \pi_1(\mathcal{C}^k(\mathbb{C}), F(Q^\circ))$ of the fundamental groups induced by F carries $\boldsymbol{hol}\,\pi_1(\mathcal{C}^n)$ into $\boldsymbol{hol}\,\pi_1(\mathcal{C}^k)$. According to Proposition 9.7($\boldsymbol{a}$), this implies that $F_*(\mathcal{H}_n(a,b)) \subseteq \mathcal{H}_k(a,b)$ and F_* is special. □

9.4. Special homomorphisms $B_n \to B_k$ and morphisms $\mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$. The following result proved in [**Lin96b**] (Theorem 8.1) clarifies the arithmetical consequences of the assumption that the Artin braid group B_n admits a non-cyclic special homomorphism to the Artin braid group B_k and, in view of Theorem 9.4, imposes the corresponding restraints on the possible existence of non-cyclic morphisms $\mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$.

Theorem 9.8. *If $n \neq 4$ and there is a non-cyclic special homomorphism $B_n \to B_k$ then $n(n-1)$ divides $k(k-1)$.* ○

Theorems 9.4 and 9.8 imply the following

Corollary 9.9. *If $n \neq 4$ and $n(n-1)$ does not divide $k(k-1)$ then every holomorphic map $\mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^k(\mathbb{C})$ is cyclic.* □

***Remark* 9.10.** The group B_3 possesses non-cyclic special homomorphisms $B_3 \to B_k$ for any k that is not forbidden by Theorem 9.8. Moreover, let $\alpha, \beta \in B_3$ and $a, b \in B_4$ be special systems of generators; then the map $\alpha \mapsto b$, $\beta \mapsto a^2$ extends uniquely to a surjective (and thereby non-cyclic) homomorphism $B_4 \to B_3$; hence, if 6 divides $k(k-1)$ then non-cyclic special homomorphisms $B_4 \to B_k$ do exist.

Suppose that $n > 4$ and k is not forbidden by Theorem 9.8, i.e., $n(n-1)$ divides $k(k-1)$. Then non-cyclic special homomorphisms $B_n \to B_k$ certainly exist whenever n divides k (see [**Lin96b**], Section 8.1, Theorem 8.3); however, I do not know whether such homomorphisms can exist when n does not divide k. ○

10. Dimension of the image

Let $n > 4$ and let $F\colon \mathcal{C}^n(X) \to \mathcal{C}^n(X)$ be a morphism. If F is cyclic then, by Cyclic Map Theorem 8.7, it is orbit-like and hence $\dim_{\mathbb{C}} F(\mathcal{C}^n(X)) \le \dim_{\mathbb{C}} \operatorname{Aut} X = t(X)$ (see Notation 1.11). If F is non-cyclic then it is tame and hence $F((\operatorname{Aut} X)Q) \subseteq (\operatorname{Aut} X)Q$ for all $Q \in \mathcal{C}^n(X)$; thus, each $\operatorname{Aut} X$ orbit in $\mathcal{C}^n(X)$ contains at least one point of its own image and therefore $\dim_{\mathbb{C}} F(\mathcal{C}^n(X)) \ge n - \dim_{\mathbb{C}} \operatorname{Aut} X = n - t(X)$. The next theorem strengthens this estimate.

Theorem 10.1. *Let $n > 4$ and let $F\colon \mathcal{C}^n(X) \to \mathcal{C}^n(X)$ be a non-cyclic morphism. Then $\dim_{\mathbb{C}} F(\mathcal{C}^n(X)) \ge n - t(X) + 1$.*

This theorem follows immediately from Tame Map Theorem, the existence of points $Q \in \mathcal{C}^n(X)$ with non-trivial stabilizers St_Q and the following local result.

Lemma 10.2. *Let $T\colon U \to \operatorname{Aut} X$ be a holomorphic map of a non-empty open $\operatorname{Aut} X$ invariant subset $U \subseteq \mathcal{C}^n(X)$ and let $F\colon U \to U$ be the map defined by $F(u) = T(u)u$*

for all $u \in U$. Suppose that the stabilizer St_{u^*} of some point $u^* \in U$ is non-trivial. There is a non-empty open $\operatorname{Aut} X$ *invariant subset $V \subseteq U$ such that*

$(*)$ *$F((\operatorname{Aut} X)v)$ contains a subset biholomorphic to the open disc $\mathbb{D}$*

for each point $v \in V$ Moreover, for such a set V we have $\dim_{\mathbb{C}} F(V) \geq n - t(X) + 1$.

PROOF. Notice that $\dim_{\mathbb{C}}(\operatorname{Aut} X)u = \dim_{\mathbb{C}} \operatorname{Aut} X = t(X)$ for each $u \in U$; consider the holomorphic map

$$f_u\colon \operatorname{Aut} X \ni A \mapsto T(Au)Au = F(Au) \in (\operatorname{Aut} X)u\,.$$

Let us show that the differential $d_A f_u\colon \mathbb{C}^{t(X)} \to \mathbb{C}^{t(X)}$ of f_u at A cannot be zero identically on $U \times (\operatorname{Aut} X)$. Since $\operatorname{Aut} X$ is connected, the assumption that $d_A f_u = 0$ on $U \times (\operatorname{Aut} X)$ implies $f_u = \text{const}$ for each $u \in U$. I. e., $T(Au)Au$ does not depend on A. Hence $T(Au)Au = T(u)u$ and $(T(u))^{-1}T(Au)A \in St_u$ on $U \times (\operatorname{Aut} X)$. Since St_u is finite, $(T(u))^{-1}T(Au)A$ cannot depend on A and hence $(T(u))^{-1}T(Au)A = (T(u))^{-1}T(u) = 1$ for all $(u, A) \in U \times (\operatorname{Aut} X)$. For $u = u^*$ there is a non-trivial $A \in St_{u^*}$ and we have $1 = (T(u^*))^{-1}T(Au^*)A = (T(u^*))^{-1}T(u^*)A = A$, which is impossible.

Take $(u_0, A_0) \in U \times (\operatorname{Aut} X)$ such that $d_{A_0} f_{u_0} \neq 0$; then $\operatorname{rank} d_{A_0} f_{u_0} \geq 1$ and, by semi-continuity, $\operatorname{rank} d_A f_u \geq 1$ in some open neighborhood $U_0 \times \mathcal{A}_0$ of (u_0, A_0). The implicit map theorem implies that for each $u \in U_0$ the image $f_u(\operatorname{Aut} X)$ contains a subset biholomorphic to $\mathbb{D}$.

Clearly $F((\operatorname{Aut} X)u) = f_u(\operatorname{Aut} X)$; hence each point v of the non-empty open $\operatorname{Aut} X$ invariant set $V = (\operatorname{Aut} X)U_0$ satisfies $(*)$, which implies $\dim_{\mathbb{C}} F(V) \geq n - t(X) + 1$. □

***Remark* 10.3.** The statement of the above lemma may fail if $St_u = \{1\}$ for all $u \in U$. Indeed, let $u_0 \in \mathcal{C}^n(X)$ and $St_{u_0} = \{1\}$; then $St_u = \{1\}$ for all u in a small open ball B centered at u_0. Let D be a very small open complex disc of dimension $n - t(X)$ contained in B, centered at u_0 and transversal to the orbit $(\operatorname{Aut} X)u_0$ at u_0. Then $[(\operatorname{Aut} X)v] \cap D = \{v\}$ for each $v \in D$ and the correspondence $(A, v) \mapsto Av$ gives rise to the biholomorphic isomorphism $f\colon (\operatorname{Aut} X) \times D \xrightarrow{\cong} (\operatorname{Aut} X)D \stackrel{\text{def}}{=\!=} U$. The composition $\tau = \pi_1 \circ f^{-1}\colon U \xrightarrow{f^{-1}} (\operatorname{Aut} X) \times D \xrightarrow{\pi_1} \operatorname{Aut} X$ of the inverse isomorphism f^{-1} and the projection π_1 to the first factor is a holomorphic surjection, the map $T\colon U \ni u \mapsto (\tau(u))^{-1} \in \operatorname{Aut} X$ is holomorphic, and the map $F\colon U \ni u \mapsto T(u)u$ carries U to D. Indeed, for each $u \in U$, $u = f(A, v) = Av \in U$, we have $\tau(u) = (\pi_1 \circ f^{-1})(u) = \pi_1(f^{-1}(f(A, v))) = \pi_1(A, v) = A$, $T(u) = (\tau(u))^{-1} = A^{-1}$ and, finally, $F(u) = T(u)u = A^{-1}Av = v \in D$ so that $\dim_{\mathbb{C}} F(U) = n - t(X)$. ○

The linear map $L = (L_1, ..., L_n)\colon \mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^n(\mathbb{C})$ defined by

$$L_i(\{q_1, ..., q_n\}) = q_i - (q_1 + ... + q_n)/n\,, \quad i = 1, ..., n\,,$$

carries the whole space $\mathcal{C}^n(\mathbb{C})$ to its $(n-1)$-dimensional submanifold $\mathcal{C}_b^{n-1}(\mathbb{C})$ consisting of all balanced configurations (see Definition 8.9). Thus, in the case $X = \mathbb{C}$ the lower bound $\dim_{\mathbb{C}} F(\mathcal{C}^n(\mathbb{C})) \geq n - t(\mathbb{C}) + 1 = n - 1$ is the best possible one.

Let $L = (L_1, ..., L_n) \colon \mathcal{C}^n(\mathbb{C}) \to \mathcal{C}_b^{n-1}(\mathbb{C})$ be the above linear map; the following theorem provides a more precise result about images of non-cyclic endomorphisms, which strengthens Theorem 10.1 in the case $X = \mathbb{C}$.

Theorem 10.4. *Let $n > 4$ and let $F \colon \mathcal{C}^n(\mathbb{C}) \to \mathcal{C}^n(\mathbb{C})$ be a non-cyclic holomorphic map. Then the linear map L caries the image $F(\mathcal{C}^n(\mathbb{C}))$ of F on the whole $\mathcal{C}_b^{n-1}(\mathbb{C})$; more precisely,*

$$L(F(\mathcal{C}^n(\mathbb{C}))) = L(F(\mathcal{C}_b^{n-1}(\mathbb{C}))) = \mathcal{C}_b^{n-1}(\mathbb{C})\,.$$

Proof. By Tame Map Theorem, $F(Q) = T(Q)Q$, where $T \colon \mathcal{C}^n(\mathbb{C}) \to \mathbf{Aff}\ \mathbb{C}$ is a holomorphic map, that is, $T(Q) \colon \mathbb{C} \ni z \mapsto A(Q)z + B(Q) \in \mathbb{C}$ with certain holomorphic functions $A \colon \mathcal{C}^n(\mathbb{C}) \to \mathbb{C}^*$ and $B \colon \mathcal{C}^n(\mathbb{C}) \to \mathbb{C}$. The cohomology class $d_n^* \in H^1(\mathcal{C}^n(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$ provided by the discriminant map $d_n \colon \mathcal{C}^n(\mathbb{C}) \to \mathbb{C}^*$ generates $H^1(\mathcal{C}^n(\mathbb{C}), \mathbb{Z})$; hence $A(Q) = (d_n(Q))^s \exp \varphi(Q)$, where $s \in \mathbb{Z}$ and φ is a holomorphic function on $\mathcal{C}^n(\mathbb{C})$. Take any $Q^\circ = \{q_1^\circ, ..., q_n^\circ\} \in \mathcal{C}_b^{n-1}(\mathbb{C})$ and consider the holomorphic function $a \colon \mathbb{C}^* \ni \zeta \mapsto a(\zeta) = \zeta A(\zeta Q^\circ) \in \mathbb{C}^*$. Then $a(\zeta) = \zeta A(\zeta Q^\circ) = \zeta \cdot (d_n(\zeta Q^\circ))^s \exp \varphi(\zeta Q^\circ) = \zeta^{sn(n-1)+1}(d_n(Q^\circ))^s \exp \varphi(\zeta Q^\circ)$, which shows that $a \neq \mathrm{const}$. Since a does not take the value 0, it follows from the Picard theorem that $a(\zeta^\circ) = 1$ for some $\zeta^\circ \in \mathbb{C}^*$. A straightforward computation shows that $G_i(\zeta^\circ Q^\circ) = L_i(F(\zeta^\circ Q^\circ) = \zeta^\circ A(\zeta^\circ Q^\circ) q_i^\circ = a(\zeta^\circ) q_i^\circ = q_i^\circ$ for all $i = 1, ..., n$, that is, $G(\zeta^\circ Q^\circ) = Q^\circ$, which proves the theorem. $\square$

***Remark* 10.5.** I do not know any example of a non-cyclic holomorphic endomorphism F of $\mathcal{C}^n(\mathbb{CP}^1)$ such that $\dim_{\mathbb{C}} F(\mathcal{C}^n(\mathbb{CP}^1)) < n$. $\bigcirc$

11. Some applications to algebraic functions

The n-valued algebraic function $[x : y] = U_n([z])$ defined by the equation

$$\begin{aligned} &\phi_n(x, y; z) = z_0 x^n + z_1 x^{n-1} y + ... + z_n y^n = 0\,, \\ &z = (z_0, ..., z_n) \in \mathbb{C}^{n+1}\,, \quad [z] = [z_0 : ... : z_n] \in \mathbb{CP}^n\,, \end{aligned} \tag{11.1}$$

and the n-valued *entire* algebraic function $t = u_n(w)$ defined by the equation

$$p_n(t, w) = t^n + w_1 t^{n-1} + ... + w_n = 0\,, \quad w = (w_1, ..., w_n) \in \mathbb{C}^n\,, \tag{11.2}$$

are said to be the *universal* ones. It is easily seen that any n-valued algebraic function on $\mathbb{CP}^m$ (respectively any n-valued entire algebraic function on $\mathbb{C}^m$) is a composition of a rational map $\mathbb{CP}^m \to \mathbb{CP}^n$ (respectively a polynomial map $\mathbb{C}^m \to \mathbb{C}^n$) and the corresponding universal function. The branch loci of the universal functions are also called universal ones.

In this section we study algebraic functions whose branch loci coincide with a universal one and apply the obtained results to a version of the 13th Hilbert problem for algebraic functions.

11.1. Algebraic functions with the universal branch locus. To avoid the doubling related to our wish to consider both "general" and "entire" algebraic functions, we will use the following notation.

***Notation* 11.1.** We consider the following two cases:

(***a***) $X = \mathbb{CP}^1$, $X^n/\mathbf{S}(n) = \mathbb{CP}^n$, $\mathcal{U}_n = U_n$ is the universal n-valued algebraic function on $\mathbb{CP}^n$ defined by (11.1) with the branch locus

$$\Sigma_{\mathcal{U}_n} = \Sigma_{U_n} = \{[z] \in \mathbb{CP}^n \mid D_{\phi_n}([z]) = 0\}\,, \qquad (a')$$

and $(X^n/\mathbf{S}(n)) \setminus \Sigma_{\mathcal{U}_n} = \mathbb{CP}^n \setminus \Sigma_{\mathcal{U}_n} \cong \mathcal{C}^n(X) = \mathcal{C}^n(\mathbb{CP}^1)$;

(***b***) $X = \mathbb{C}$, $X^n/\mathbf{S}(n) = \mathbb{C}^n$, $\mathcal{U}_n = u_n$ is the universal n-valued entire algebraic function on $\mathbb{C}^n$ defined by (11.2) with the branch locus

$$\Sigma_{\mathcal{U}_n} = \Sigma_{u_n} = \{w \in \mathbb{C}^n \mid d_n(w) = 0\}\,, \qquad (b')$$

and $(X^n/\mathbf{S}(n)) \setminus \Sigma_{\mathcal{U}_n} = \mathbb{C}^n \setminus \Sigma_{\mathcal{U}_n} \cong \mathcal{C}^n(X) = \mathcal{C}^n(\mathbb{C}^1)$.

In case (***a***) the term "an m-valued algebraic function $\mathcal{F}$ on $X^n/\mathbf{S}(n)$" will mean an algebraic function $[x : y] = \mathcal{F}([z])$ on $\mathbb{CP}^n$ defined by an equation

$$\varphi(x, y; z) = a_0(z)x^m + a_1(z)x^{m-1}y + ... + a_m(z)y^m = 0\,,$$

where $\varphi(x, y; z) = a_0(z)x^m + a_1(z)x^{m-1}y + ... + a_m(z)y^m$ is a *homogeneous* binary form whose coefficients $a_0, ..., a_m$ are coprime *homogeneous* polynomials in $z_0, ..., z_n$ *of the same positive degree.*[19] The form $\varphi(x, y; z)$ is called the *defining form* of the algebraic function F. The zero set $\Sigma_\varphi = \{z = (z_0, ..., z_n) \in \mathbb{C}^{n+1} \mid D_\varphi(z) = 0\}$ of the discriminant $D_\varphi(z)$ is said to be *degeneration locus* of the form $\varphi(x, y; z)$; in particular, Σ_φ contains all common zeros of the coefficients $a_0, ..., a_m$. The quotient space $\Sigma_F = \Sigma_\varphi/\mathbb{C}^* \subset \mathbb{CP}^n$ is the *branch locus* of the function F (it may happen that $D_\varphi = 0$ and $\Sigma_F = \mathbb{CP}^n$).

For two algebraic functions $[x : y] = G([z])$ and $[x : y] = F([z])$ with the defining forms $\psi(x, y; z)$ and $\varphi(x, y; z)$ we say that G is a (multivalued) *branch* of F and write $G \subseteq F$ if there is a form $\omega(x, y; z)$ such that $\varphi(x, y; z) = \psi(x, y; z) \cdot \omega(x, y; z)$. We write $F = G$ if $\varphi(x, y; z) = c\psi(x, y; z)$ for some $c \in \mathbb{C}^*$.

Let us notice that if $t = f(w)$ is an algebraic function of $w \in \mathbb{C}^n$ defined by an algebraic equation $P(t, w) = A_0(w)t^m + A_1(w)t^{m-1} + ... + A_m(w) = 0$ with coefficients $A_i \in \mathbb{C}[w]$ then one may replace t with x/y, each of w_i with z_i/z_0, and rewrite the function as $[x : y] = F([z])$ and the defining equation as $\varphi(x, y; z) = 0$, where $\varphi(x, y; z)$ is a binary form of the type described above.

In case (***b***) the term "an m-valued algebraic function $\mathcal{F}$ on $X^n/\mathbf{S}(n)$" will mean an entire algebraic function $t = \mathcal{F}(w)$ on $\mathbb{C}^n$ defined by an equation

$$P(t; w) = t^m + a_1(w)t^{m-1} + ... + a_m(w) = 0\,, \qquad (b')$$

where all $a_i \in \mathbb{C}[w]$. For two entire algebraic functions $t = g(w)$ and $t = f(w)$ with the defining polynomials $Q(t, w)$ and $P(t, w)$ we say that g is a (multivalued) *branch* of f and write $g \subseteq f$ if there is a polynomial $R(t, w)$ such that $P = Q \cdot R$. We write $g = f$ if $P = Q$.

The following result is in fact equivalent to Tame Map Theorem for regular endomorphisms.

[19]In this section, all binary forms under consideration are supposed to be of such kind. It is possible that some coefficients $a_i = 0$.

Theorem 11.2. *Let $n > 4$ and let $\mathcal{F}$ be an n-valued algebraic function on $X^n/\mathbf{S}(n)$ with the branch locus $\Sigma_{\mathcal{F}} = \Sigma_{\mathcal{U}_n}$ and a non-cyclic monodromy representation*

$$\pi_1((X^n/\mathbf{S}(n)) \setminus \Sigma_{\mathcal{U}_n}) \to \mathbf{S}(n)\,. \tag{11.3}$$

Then there exists a rational map $T\colon X^n/\mathbf{S}(n) \to \operatorname{Aut} X$ regular on $(X^n/\mathbf{S}(n)) \setminus \Sigma_{\mathcal{U}_n}$ such that $\mathcal{F}(z) = T(z)\mathcal{U}_n(z)$ for all $z \in X^n/\mathbf{S}(n)$.

Proof. The algebraic function $\mathcal{F}$ is the composition $\mathcal{F}(z) = \mathcal{U}_n(A(z))$ of the rational map $A\colon X^n/\mathbf{S}(n) \to X^n/\mathbf{S}(n)$ given by the coefficients of the equation defining $\mathcal{F}$ and the universal function $\mathcal{U}_n$. The assumption $\Sigma_{\mathcal{F}} = \Sigma_{\mathcal{U}_n}$ implies that A is regular on $(X^n/\mathbf{S}(n)) \setminus \Sigma_{\mathcal{U}_n}$ and carries the latter set into itself. Thus, under the identification of the defining equations to the sets of their roots established in Remark 1.9, A may be regarded as a regular endomorphism A' of the configuration space $\mathcal{C}^n(X)$. Since the monodromy representation (11.3) is non-cyclic, A' is non-cyclic as well. By Tame Map Theorem, there is a regular map $T'\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ such that $A'(Q) = T'(Q)Q$ for all $Q \in \mathcal{C}^n(X)$. Denoting by T the regular morphism $(X^n/\mathbf{S}(n)) \setminus \Sigma_{\mathcal{U}_n} \to \operatorname{Aut} X$ corresponding to T', we obtain that $\mathcal{F}(z) \equiv \mathcal{U}_n(A(z)) \equiv T(z)\mathcal{U}_n(z)$ meaning that each value of $\mathcal{F}$ at $z \in (X^n/\mathbf{S}(n)) \setminus \Sigma_{\mathcal{U}_n}$ is obtained from the corresponding value of $\mathcal{U}_n$ at z with help of the Möbius transformation $T(z)$. $\square$

The following result is just a simple consequence of Linked Map Theorem (for $X = \mathbb{C}$ it has been proved in [**Lin96b**]).

Theorem 11.3. *Let $n > t(X) + 1$ and let $\mathcal{F}$ be an N-valued algebraic function on $X^n/\mathbf{S}(n)$. If $\mathcal{U}_n \subseteq \mathcal{F}$ then either $N = n$ and $\mathcal{F} = \mathcal{U}_n$ or $N > n$ and $\mathcal{F}$ has branch points that are not branch points of $\mathcal{U}_n$.*

Proof. We consider here only the case $X = \mathbb{CP}^1$. Let $\varphi(x, y; z)$ be the defining form of the function F. The condition $U_n \subseteq F$ means that there is a binary form $\omega(x, y; z)$ of degree $k = N - n \geq 0$ such that $\varphi(x, y; z) = \phi_n(x, y; z) \cdot \omega(x, y; z)$; in this case $D_{\phi_n}(z) = 0$ implies $D_\varphi(z) = 0$, which means that $\Sigma_F \supseteq \Sigma_{U_n}$.

Assume that $k > 0$ and $\Sigma_F = \Sigma_{U_n}$. Then for every $z \in \mathbb{CP}^n \setminus \Sigma_{U_n} = \mathcal{C}^n(\mathbb{CP}^1)$ the form $\varphi(x, y; z)$ is non-degenerate; therefore $\omega(x, y; z)$ is non-degenerate as well and has no common roots with $\phi_n(x, y; z)$. This means that the correspondence $z \mapsto \omega(x, y; z)$ defines a disjoint morphism $\mathcal{C}^n(\mathbb{CP}^1) \cong \mathcal{F}^n \to \mathcal{F}^k \cong \mathcal{C}^k(\mathbb{CP}^1)$, which contradicts Linked Map Theorem 1.12. Thus, Σ_{U_n} must be a proper subset of Σ_F, which shows that F has branch points that are not branch points of U_n.

Finally, if $k = 0$ then $N = n$, $\omega = \text{const} \neq 0$ and $F = U_n$. $\square$

The problem about existence of an algebraic function with the preassigned branch locus and monodromy group goes back to B. Riemann (in one variable), F. Enriques and O. Zariski (for functions of two variables; see [**Enr23, Zar29**]). It was discovered that in two variables there are obstacles related to the fundamental group of the complement of the branch curve.

We deal with very distinguished subvarieties $\Sigma_{U_n} \subset \mathbb{CP}^n$ and $\Sigma_{u_n} \subset \mathbb{C}^n$ and we know in advance that there exist n valued algebraic functions, namely the universal ones U_n

and u_n, with these very branch loci and the standard monodromy representations

$$\mu\colon \pi_1(\mathbb{CP}^n \setminus \Sigma_{U_n}) = \pi_1(\mathcal{C}^n(\mathbb{CP}^1)) \cong B_n(S^2) \to \mathbf{S}(n)\,,$$
$$\mu\colon \pi_1(\mathbb{C}^n \setminus \Sigma_{u_n}) = \pi_1(\mathbf{G}_n) = \pi_1(\mathcal{C}^n(\mathbb{C})) \cong B_n \to \mathbf{S}(n)\,.$$

Theorem 11.2 provides a reasonable description of all non-cyclic n-valued algebraic functions of n variables $z \in X^n/\mathbf{S}(n)$ with the universal branch locus $\Sigma_{\mathcal{U}_n}$ in terms of the universal function $\mathcal{U}_n(z)$ and automorphisms of the target domain $X = \mathbb{C}$ or $X = \mathbb{CP}^1$ depending regularly on a point $z \in (X^n/\mathbf{S}(n)) \setminus \Sigma_{\mathcal{U}_n}$. On the other hand, Theorem 11.3 tells that the universal function cannot be a proper branch of an algebraic function with the same branch locus.

11.2. Compositions of multivalued functions. One possible form of the question posed in the 13th Hilbert problem is as follows:

What is the minimal natural number m for which the function u_n *can be represented as a composition of algebraic functions of* m *variables*?

Since u_n is multivalued, the emphasized expression may have quite different meanings. It has been proved in [**Arn70c, Lin76, Lin96a**] that for certain possible interpretations of the problem the answer is $m = n-1$. Here we discuss the corresponding problem for the function $[x:y] = U_n([z])$ and show that in this case $m \geq n-2$.

Following the papers quoted above, we explain first some formalities related to multivalued maps; we do this in a manner convenient for working with non-entire algebraic functions.

Given two topological spaces Y and Z, we use the notation $f\colon Z \xrightarrow{n} Y$ for a map defined and continuous on an open dense subset $Z_f \subseteq Z$ and carrying it into the space $\operatorname{Sym}^m Y = Y^m/\mathbf{S}(m)$; such a map is said to be an *m-valued map of Z to Y*. The points $y_1, ..., y_m \in f(z)$ are said to be the *values of f at* $z \in Z_f$. The subset $\operatorname{Im} f \subseteq Y$ consisting of all values of f at all points $z \in Z_f$ is called the *image of f*. The subset $\Gamma_f \subseteq Z \times Y$ consisting of all pairs (z, y) such that y is a value of f at $z \in Z_f$ is called the *graph* of g.

Let $f\colon Z \xrightarrow{m} Y$ and $g\colon Y \xrightarrow{n} W$ be multivalued maps; suppose that there are open dense subsets $Z' \subseteq Z_f$ and $Y' \subseteq Y_g$ such that all values of f at any point $z \in Z'$ belong to Y'. The *composition*

$$g \circ f\colon Z \xrightarrow{mn} W$$

is the (mn)-valued map whose values at a point $z \in Z'$ are all mn values of g at the m points of Y' which are the values of f at z. The *complete composition*

$$g * f\colon Z \xrightarrow{mn} Y \times W$$

is the (mn)-valued map whose values at a point $z \in Z'$ are all pairs (y, w) such that y is one of the values of f at z and w is one of the values of g at y.

Let k, m, M be natural numbers. Assume that for each $i = 1, \ldots, M$ we are given with an n_i-valued algebraic function $[x:y] = g_i([w])$, $[w] = [w_0 : ... : w_m] \in \mathbb{CP}^m$, defined by an equation $\varphi_i(x, y; [w]) = a_0(w)x^{n_i} + a_1(w)x^{n_i-1}y + ... + a_{n_i}(w)y^{n_i} = 0$, where $a_0, ..., a_{n_i}$ are co-prime homogeneous polynomials of the same positive degree. Assuming that the discriminant $D_{\varphi_i}(w)$ of φ_i is not identically zero, we may regard g_i as an n_i-valued map $g_i\colon \mathbb{CP}^m \xrightarrow{n_i} \mathbb{CP}^1$, which is surely defined on the non-empty Zariski open set $W_i = \{w \in \mathbb{CP}^m \mid D_{\varphi_i}(w) \neq 0\}$.

Furthermore, for each $i = 1, \ldots, M$ let $R_i\colon \mathbb{CP}^k \times (\mathbb{CP}^1)^{i-1} \to \mathbb{CP}^m$ be a rational map such that its regularity domain contains a non-empty Zariski open subset V_i with the image $R_i(V_i) \subseteq W_i$.

Let $F_0\colon \mathbb{CP}^k \to \mathbb{CP}^k$ denote the identity map. Using the compositions $\circ$ and $*$ of multivalued functions defined above, we construct algebraic functions

$$G_i = g_i \circ R_i\colon \mathbb{CP}^k \times (\mathbb{CP}^1)^{i-1} \xrightarrow{R_i} \mathbb{CP}^m \xrightarrow{n_i} \mathbb{CP}^1\,, \quad i = 1, ..., M\,,$$

$$\begin{aligned}
&F_1 = G_1 * F_0\colon \mathbb{CP}^k \xrightarrow{n_1} \mathbb{CP}^k \times \mathbb{CP}^1\,,\\
&F_2 = G_2 * F_1\colon \mathbb{CP}^k \xrightarrow{n_1 n_2} \mathbb{CP}^k \times (\mathbb{CP}^1)^2\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&F_i = G_i * F_i\colon \mathbb{CP}^k \xrightarrow{n_1 n_2 \cdots n_i} \mathbb{CP}^k \times (\mathbb{CP}^1)^i\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&F_{M-1} = G_{M-1} * F_{M-2}\colon \mathbb{CP}^k \xrightarrow{n_1 n_2 \cdots n_{M-1}} \mathbb{CP}^k \times (\mathbb{CP}^1)^{M-1}\,,
\end{aligned}$$

and, finally,

$$F = F_M = G_M \circ F_{M-1}\colon \mathbb{CP}^k \xrightarrow{n_1 n_2 \cdots n_M} \mathbb{CP}^1\,.$$

F is a well-defined $N = n_1 \cdots n_M$-valued algebraic function on $\mathbb{CP}^k$ whenever Im $F_i \subseteq V_{i+1}$ for all $i = 1, ..., M-1$. If this is the case then the collection $\mathfrak{A} = \{V_i, W_i, R_i, g_i \mid i = 1, ..., M\}$ is said to be a *tame algebraic* (k, m)*-ensemble in* $\mathbb{CP}^k$, and the algebraic function $F_{\mathfrak{A}} = F$ is called the *function generated by the ensemble* $\mathfrak{A}$. Two tame algebraic (k, m)-ensembles $\mathfrak{A}$ and $\mathfrak{A}'$ in $\mathbb{CP}^k$ are said to be *equivalent* if the functions $F_{\mathfrak{A}}$ and $F_{\mathfrak{A}'}$ generated by them coincide.

Some of the algebraic functions g_i appearing in a tame ensemble $\mathfrak{A}$ may be single-valued; the number of those ones which are actually multivalued is called the *rank of the ensemble* $\mathfrak{A}$.

***Example* 11.4.** For $M = 2$ we have $F(z) = g_2\{R_2[z, g_1(R_1(z))]\}$. If $\mathfrak{A}$ is an ensemble of rank 1 then $F_{\mathfrak{A}} = R_2(z, g(R_1(z)))$, where $[x : y] = g(w)$ is a multivalued algebraic function and R_1, R_2 are rational. ◯

***Definition* 11.5.** We say that an algebraic function g in $\mathbb{CP}^k$ *can be represented as a composition of algebraic functions of* m *variables* if there exists a tame algebraic (k, m)-ensemble $\mathfrak{A}$ such that $g \subseteq F_{\mathfrak{A}}$, that is, g is a (multivalued) branch of $F_{\mathfrak{A}}$. The function $F_{\mathfrak{A}}$ is called a *representing function-composition for* g; clearly $\Sigma_g \subseteq \Sigma_{F_{\mathfrak{A}}}$. If the ensemble $\mathfrak{A}$ can be chosen so that $g = F_{\mathfrak{A}}$ we say that g can be *faithfully* represented as a composition of algebraic functions of m variables; in this case $\Sigma_g = \Sigma_{F_{\mathfrak{A}}}$. In general, the conditions $g \subseteq F_{\mathfrak{A}}$ and $\Sigma_g = \Sigma_{F_{\mathfrak{A}}}$ do not imply $g = F_{\mathfrak{A}}$. ◯

In the following theorem we deal with a representation of the universal algebraic function U_n as a composition $U_n \subseteq F_{\mathfrak{A}}$, which a priori is not supposed to be faithful but satisfies the formally weaker condition $\Sigma_{U_n} = \Sigma_{F_{\mathfrak{A}}}$. We will see however that the conditions $U_n \subseteq F_{\mathfrak{A}}$ and $\Sigma_{U_n} = \Sigma_{F_{\mathfrak{A}}}$ imply $U_n = F_{\mathfrak{A}}$.

Theorem 11.6. *For $n > \max\{4, m+2\}$ the universal n-valued algebraic function $[x : y] = U_n([z])$ cannot be represented as a composition of algebraic functions of m variables in such a way that a representing function-composition $F = F_{\mathfrak{A}}$ and U_n itself have the same branch loci.*

Proof. Suppose to the contrary that there exists a tame algebraic (n, m)-ensemble $\mathfrak{A} = \{U_i, V_i, R_i, g_i \mid i = 1, ..., M\}$ in $\mathbb{CP}^n$ such that $U_n \subseteq F = F_{\mathfrak{A}}$ and the branch locus Σ_F of the N-valued algebraic function-composition $F = F_{\mathfrak{A}}$ defined by $\mathfrak{A}$ coincides with the branch locus Σ_{U_n}. Then, by Theorem 11.3, $N = n$ and $U_n = F$. In particular, the monodromy group of F coincides with the monodromy group $\mathbf{S}(n)$ of the function U_n. This implies that the unbranched n-covering $p\colon \Gamma_F \to \mathbb{CP}^n \setminus \Sigma_{U_n}$ defined by the function F is *indecomposable*, that is, the projection π cannot be represented as a composition

$$p = q_1 \circ q_2 \colon \Gamma_F \xrightarrow{q_2} E \xrightarrow{q_1} \mathbb{CP}^n \setminus \Sigma_{U_n}$$

of two non-univalent unbranched coverings q_1,q_2 (see [**Arn70c, Lin76, Lin96a**]). This property implies that the ensemble $\mathfrak{A}$ must be of rank 1, that is, it cannot contain more than one multivalued algebraic function and thereby the corresponding representation of the function U_n as a superposition must be of the form $U_n(z) = F(z) = R_2(z, g(R_1(z))$, where $[x : y] = g(w)$ is an n-valued algebraic function on $\mathbb{CP}^m$ with a defining binary form

$$\varphi(x, y; w) = a_0(w)x^n + a_1(w)x^{n-1}y + ... + a_n(w)y^n \,, \tag{11.4}$$

$R_1\colon \mathbb{CP}^n \to \mathbb{CP}^m$ is a rational map regular on $\mathbb{CP}^n \setminus \Sigma_{U_n}$ with the image

$$R_1(\mathbb{CP}^n \setminus \Sigma_{U_n}) \subseteq \{w \in \mathbb{CP}^m \mid D_\varphi(w) \neq 0\} \,,$$

and $R_2\colon \mathbb{CP}^n \times \mathbb{CP}^1 \to \mathbb{CP}^1$ is a rational function regular at each point $(z, [x : y]) \in (\mathbb{CP}^n \setminus \Sigma_{U_n}) \times \mathbb{CP}^1$ such that $\varphi(x, y; R_1(z)) = 0$. Let $A = (a_1, \ldots, a_n)\colon \mathbb{CP}^m \to \mathbb{CP}^n$ be the rational map defined by the coefficients of the binary form (11.4). The identity $U_n(z) \equiv R_2(z, g(R_1(z))$ shows that A must be regular on $\{w \in \mathbb{CP}^m \mid D_\varphi(w) \neq 0\}$ and the A-image $A(R_1(\mathbb{CP}^n \setminus \Sigma_{U_n}))$ of the set $R_1(\mathbb{CP}^n \setminus \Sigma_{U_n})$ must be contained in $\mathbb{CP}^n \setminus \Sigma_{U_n}$. Thus, the composition $f = A \circ R_1$,

$$f\colon \mathcal{C}^n(\mathbb{CP}^1) = \mathbb{CP}^n \setminus \Sigma_{U_n} \xrightarrow{R_1} \{w \in \mathbb{CP}^m \mid D_\varphi(w) \neq 0\} \xrightarrow{A} \mathbb{CP}^n \setminus \Sigma_{U_n} = \mathcal{C}^n(\mathbb{CP}^1) \,,$$

is a regular map. This map is certainly non-cyclic, for otherwise the monodromy group of the algebraic function $g((R_1(z))$ would be cyclic, as well as the monodromy group of the function $R_2(z, g(R_1(z)) = U_n(z)$. Since $m < n - 2$, we have $\dim_{\mathbb{C}}(f(\mathcal{C}^n(\mathbb{CP}^1))) \leq m < n - 2$, which contradicts Theorem 10.1. $\square$

***Remark* 11.7. *a*)** The corresponding result for the universal entire algebraic function u_n on $\mathbb{C}^n$ has been proved in [**Lin96a**]:

Theorem 11.8. *For $n > \max\{4, m+1\}$ the universal n-valued entire algebraic function $t = u_n(z)$ cannot be represented as a composition of entire algebraic functions of m variables in such a way that a representing function-composition $F = F_{\mathfrak{A}}$ and u_n itself have the same branch loci.*

The proof is also based on Theorems 1.12, 10.1 and 11.3, which in the case $X = \mathbb{C}$ have been proved in the paper quoted above.

b) The compositions of *entire* algebraic functions treated in Theorem 11.8 are defined similarly to what has been done above. Just in the definition of an algebraic (k, m)-ensemble one should consider affine spaces instead of projective ones and work with polynomials instead of rational functions; that is, *the operation of division is in fact prohibited*. V. I. Arnold many times expressed his conviction that *whenever rational functions are admitted u_n cannot be faithfully represented as a composition of algebraic functions of less than $n-2$ variables*. Seemingly, Theorem 11.6 confirms this Arnold's opinion (and even for a formally wider class of representations *without superfluous branch points* instead of the *faithful* ones). ○

12. Automorphisms

12.1. How to catch automorphisms? By Tame Map Theorem, every non-cyclic endomorphisms F of $\mathcal{C}^n(X)$ is of the form $F = F_T$, $F_T(Q) = T(Q)Q$, where $T\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ is a morphism uniquely defined by F. It is reasonable to ask the following question: *for which morphism $T\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ the corresponding tame endomorphism F_T is an automorphism?* It is easy to give the following "answer":

F_T is an automorphism if and only if there exists a morphism $S\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ such that $T(S(Q)Q)S(Q) = S(T(Q)Q)T(Q) = I$ (the unity of the group $\operatorname{Aut} X$) *for all $Q \in \mathcal{C}^n(X)$.*

It is unclear how ensure that T satisfies such a condition. It is naturally to look for a relationship between the group multiplication in $\operatorname{Aut} X$ and the composition of tame maps. In general, these operations know nothing about each other; however in the special case of $\operatorname{Aut} X$ *invariant* maps $\mathcal{C}^n(X) \to \operatorname{Aut} X$ they are in a close connection.

***Definition* 12.1.** As usually, the pointwise product $ST\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ of two maps $S, T\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ is defined by $(ST)(Q) = S(Q)T(Q)$, where $S(Q)T(Q)$ denotes the group product of the elements $S(Q), T(Q) \in \operatorname{Aut} X$.

A map $S\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ is said to be $\operatorname{Aut} X$ *invariant* if $S(AQ) = S(Q)$ for all $A \in \operatorname{Aut} X$ and $Q \in \mathcal{C}^n(X)$. ○

Lemma 12.2. *Let $F_S, F_T\colon \mathcal{C}^n(X) \to \mathcal{C}^n(X)$ be the tame maps induced by two maps $S, T\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$. If S is $\operatorname{Aut} X$ invariant then $F_{ST} = F_S \circ F_T$.*

PROOF. It suffices to know that $S(Q) = S(T(Q)Q)\ \forall\, Q = \{q_1, ..., q_n\} \in \mathcal{C}^n(X)$; indeed, $F_{ST}(Q) = \big\{\big((ST)(Q)\big)q_1, ..., \big((ST)(Q)\big)q_n\big\} = \{S(Q)T(Q)q_1, ..., S(Q)T(Q)q_n\} = \{S(Q)Aq_1, ..., S(Q)Aq_n\} = \{S(AQ)Aq_1, ..., S(AQ)Aq_n\} = F_S(AQ) = F_S(T(Q)Q)) = F_S(F_T(Q))$, that is, $F_{ST} = F_S \circ F_T$. □

Corollary 12.3. *The tame map $F_T\colon \mathcal{C}^n(X)\to\mathcal{C}^n(X)$ induced by an* Aut X *invariant morphism $T\colon \mathcal{C}^n(X)\to$* Aut X *is an automorphism of $\mathcal{C}^n(X)$.*

Proof. Set $S=T^{-1}$, that is, $S(Q)=(T(Q))^{-1}$ for all $Q\in\mathcal{C}^n(X)$. Then S is an Aut X invariant morphism and, by Lemma 12.2, $F_S\circ F_T=F_{ST}=F_I=\mathrm{Id}_{\mathcal{C}^n(X)}=F_{TS}=F_T\circ F_S$. □

***Remark* 12.4.** If T satisfies $T((T(Q))^{-1}Q)=T(Q)$ for all $Q\in\mathcal{C}^n(X)$ then F_T is an automorphism whose inverse is $F_{T^{-1}}$. ○

12.2. Some examples. Corollary 12.3 gives rise to some simple examples.

***Example* 12.5.** Take $c\in\mathbb{C}^*$ and an $\mathbf{Aff}\ \mathbb{C}$ invariant regular function $b\colon\mathcal{C}^n(\mathbb{C})\to\mathbb{C}$; then the map

$$F\colon \mathcal{C}^n(\mathbb{C})\ni Q\mapsto cQ+b(Q)\cdot\mathbb{1}\in\mathcal{C}^n(\mathbb{C})\,,\quad \mathbb{1}=\{1,...,1\}\in\mathbb{C}^n\,, \tag{12.1}$$

is a biregular automorphism of $\mathcal{C}^n(\mathbb{C})$. The inverse map is given by

$$\mathcal{C}^n(\mathbb{C})\ni Q\mapsto c'Q+b'(Q)\cdot\mathbb{1}\in\mathcal{C}^n(\mathbb{C})\,,$$

where $c'=c^{-1}$ and $b'(Q)=-c^{-1}b(Q)$. For instance, one may take a homogeneous polynomial P of degree $rn(n-1)/2$ in $n(n-1)/2$ variables $\xi=\{\xi_{ij}\mid 1\le i<j\le n\}$ and for each $Q=\{q_1,...,q_n\}\in\mathcal{C}^n(\mathbb{C})$ set $\xi_{ij}(Q)=q_i-q_j$, $b(Q)=P(\xi(Q))/d_n^r(Q)$.

A similar example of a biregular automorphism of $\mathcal{C}^n(\mathbb{CP}^1)$ looks as follows: $Q=\{q_1,...,q_n\}\mapsto\{Cq_1+B(Q),...,Cq_n+B(Q)\}$, where $C\in\mathbb{C}^*$ and $B\colon\mathcal{C}^n(\mathbb{CP}^1)\to\mathbb{C}$ is a regular $\mathbf{PSL}(2,\mathbb{C})$ invariant function. ○

***Example* 12.6.** The map

$$F_T\colon \mathcal{C}^n(\mathbb{C})\ni Q\mapsto cQ+\lambda a(Q)\cdot\mathbb{1}\in\mathcal{C}^n(\mathbb{C})\,,\quad \mathbb{1}=\{1,...,1\}\in\mathbb{C}^n\,,$$

where $a(Q)=\sum\limits_{\zeta\in Q}\zeta$, $c\in\mathbb{C}^*$ and $\lambda\in\mathbb{C}$, $\lambda\ne -c/n$, provides an example of a tame biregular automorphism F_T of $\mathcal{C}^n(\mathbb{C})$ corresponding to a *non-invariant* morphism

$$T\colon \mathcal{C}^n(\mathbb{C})\ni Q\mapsto T(Q)\in\mathbf{Aff}\ \mathbb{C}\,,\ \ T(Q)w=cw+\lambda a(Q)\,,\ w\in\mathbb{C}\,.$$

The inverse map is given by

$$\mathcal{C}^n(\mathbb{C})\ni Q\mapsto c'Q+\lambda' a(Q)\cdot\mathbb{1}\in\mathcal{C}^n(\mathbb{C})\,,$$

where $c'=1/c$ and $\lambda'=-\lambda/[c(c+n\lambda)]$. ○

***Example* 12.7.** Let $A(z)$,$B(z)$ be homogeneous polynomials in $z=(z_0,...,z_n)\in\mathbb{C}^{n+1}$ of degrees $2a(n-1)$ and $2b(n-1)$ respectively (one or both of A,B may be 0). As in Remark 1.9, let $Q\in\mathcal{C}^n(\mathbb{CP}^1)$ be the zero set of a binary projective form $\phi(x,y;z)=z_0x^n+z_1x_{n-1}y+...+z_ny^n$ with the discriminant $D_n(z)$. Set $h(z)=\dfrac{A(z)\exp[B(z)/(D_n(z))^b]}{(D_n(z))^a}$ and define the holomorphic map $T_{A,B}\colon\mathcal{C}^n(\mathbb{CP}^1)\to\mathbf{SL}(2,\mathbb{C})$ and the corresponding tame holomorphic endomorphism $F_{T_{A,B}}\colon\mathcal{C}^n(\mathbb{CP}^1)\to\mathcal{C}^n(\mathbb{CP}^1)$ by

$$T_{A,B}(Q)=\begin{pmatrix} h(z), & h(z)+1\\ h(z)-1, & h(z)\end{pmatrix}\,,\quad F_{T_{A,B}}(Q)=T_{A,B}(Q)Q\,. \tag{12.2}$$

It is easily seen that $F_{T_{A,B}}$ is homotopic to the automorphism

$$\mathcal{C}^n(\mathbb{CP}^1) \ni Q = \{q_1, ..., q_n\} \mapsto \{-q_1^{-1}, ..., -q_n^{-1}\} \stackrel{\text{def}}{=\!=} -Q^{-1} \in \mathcal{C}^n(\mathbb{CP}^1)$$

in the class of holomorphic endomorphisms of $\mathcal{C}^n(\mathbb{CP}^1)$. ○

13. Certain coverings of $\mathcal{C}^n(X)$

The main objective of this section is describe all unbranched connected k-coverings of the configuration spaces $\mathcal{C}^n(X)$ for $k \leq 2n$. We will use the notation introduced in Remark 1.9.

13.1. Cyclic coverings. The following unbranched holomorphic cyclic coverings will be referred to as the standard ones:

$$E^n_\infty(\mathbb{C}) = \{(\zeta, w) \in \mathbb{C} \times \mathbf{G}_n \mid e^\zeta = d_n(w)\},\ p_\infty\colon E^n_\infty(\mathbb{C}) \ni (\zeta, w) \mapsto w \in \mathbf{G}_n; \quad (13.1)$$

$$E^n_k(\mathbb{C}) = \{(\zeta, w) \in \mathbb{C}^* \times \mathbf{G}_n \mid \zeta^k = d_n(w)\},\ p_k\colon E^n_k(\mathbb{C}) \ni (\zeta, w) \mapsto w \in \mathbf{G}_n; \quad (13.2)$$

$$\begin{aligned} &E^n_{2(n-1)}(\mathbb{CP}^1) = \{z = (z_0, ..., z_n) \in \mathbb{C}^{n+1} \mid D_n(z) = 1\},\\ &\qquad p_{2(n-1)}\colon E^n_{2(n-1)}(\mathbb{CP}^1) \ni z \mapsto \psi([x:y], z) \in \mathcal{F}^n. \end{aligned} \quad (13.3)$$

For a non-trivial divisor m of $2(n-1)$ set $k = 2(n-1)/m$; the cyclic subgroup $\mathbb{Z}/m\mathbb{Z} \subset \mathbb{Z}/2(n-1)\mathbb{Z}$ acts freely on $E^n_{2(n-1)}(\mathbb{CP}^1)$ via

$$\begin{aligned} r(z_0, ..., z_n) =& (e^{i\pi r/(n-1)} z_0, ..., e^{i\pi r/(n-1)} z_n)\\ &\text{for all } \ r \in \mathbb{Z}/m\mathbb{Z} \ \text{ and all } \ z = (z_0, ..., z_n) \in E^n_{2(n-1)}(\mathbb{CP}^1)\,, \end{aligned} \quad (13.4)$$

and the quotient space

$$E^n_k(\mathbb{CP}^1) = [E^n_{2(n-1)}(\mathbb{CP}^1)]/(\mathbb{Z}/m\mathbb{Z}) \quad (13.5)$$

is an unbranched holomorphic $\mathbb{Z}/k\mathbb{Z}$ covering of $\mathcal{C}^n(\mathbb{CP}^1)$, which is also referred to as the standard one.

Since the abelianization $B_n(X)/B'_n(X)$ is a cyclic group, the equivalence classes of cyclic coverings over $\mathcal{C}^n(X)$ are in the one-to-one correspondence with the subgroups of $B_n(X)/B'_n(X)$. This implies the following result.

Theorem 13.1. *Every unbranched cyclic covering of $\mathcal{C}^n(X)$ is equivalent to one of the standard cyclic coverings exhibited above.* □

***Remark* 13.2. *a*)** The $\mathbb{Z}$ cover $E^n_\infty(\mathbb{C})$ defined by (13.1) is in fact biholomorphic to the algebraic manifold $\widetilde{E}^n_\infty(\mathbb{C}) = \mathbb{C} \times \{v \in \mathbb{C}^n \mid d_n(v) = 1\}$. The isomorphism is given by

$$\begin{aligned} &E^n_\infty(\mathbb{C}) \ni (\zeta; w_1, w_2, ..., w_n)\\ &\qquad \mapsto (\zeta; e^{-\zeta/n(n-1)} w_1, e^{-2\zeta/n(n-1)} w_2, ..., e^{-n\zeta/n(n-1)} w_n) \in \widetilde{E}^n_\infty(\mathbb{C}) \end{aligned}$$

and the covering map $\widetilde{E}^n_\infty(\mathbb{C}) \to \mathbf{G}_n = \mathcal{C}^n(\mathbb{C})$ is defined by

$$\begin{aligned} &\widetilde{E}^n_\infty(\mathbb{C}) \ni (\zeta; v_1, v_2, ..., v_n)\\ &\qquad \mapsto t^n + e^{\zeta/n(n-1)} v_1 t^{n-1} + e^{2\zeta/n(n-1)} v_2 t^{n-2} + ... + e^{n\zeta/n(n-1)} v_n \in \mathbf{G}_n\,. \end{aligned}$$

Moreover, $\pi_1(E^n_\infty(\mathbb{C})) = \pi_1(\widetilde{E}^n_\infty(\mathbb{C})) \cong B'_n$ and $E^n_\infty(\mathbb{C})$ is the maximal abelian cover of $\mathcal{C}^n(\mathbb{C})$.

b) If $k \equiv 1 \mod n(n-1)$ then $E^n_k(\mathbb{C})$ is isomorphic to $\mathcal{C}^n(\mathbb{C})$. The biregular isomorphism $\mathbf{G}_n \ni w = (w_1, ..., w_n) \mapsto (\zeta, v) = (\zeta; v_1, ..., v_n) \in E^n_k(\mathbb{C})$ is given by

$$\zeta = d_n(w), \quad v_1 = (d_n(w))^m w_1, \quad v_2 = (d_n(w))^{2m} w_2, ..., v_n = (d_n(w))^{nm} w_n\,, \tag{13.6}$$

where $m = (k-1)/n(n-1)$. In other words, for $m \in \mathbb{N}$ and $k = mn(n-1)+1$ the regular endomorphism[20]

$$\begin{aligned} p\colon \mathcal{C}^n(\mathbb{C}) \ni Q = \{q_1, ..., q_n\} \\ \mapsto \{(d_n(Q))^m q_1, ..., (d_n(Q))^m q_n\} = (d_n(Q))^m Q \in \mathcal{C}^n(\mathbb{C}) \end{aligned} \tag{13.7}$$

is an unbranched analytic cyclic covering of order k, which is equivalent to the standard cyclic $\mathbb{Z}/k\mathbb{Z}$ covering defined by (13.2). The induced monomorphism $p_*\colon B_n \to B_n$ is given by $\sigma_i \mapsto \sigma_i \cdot c^m$ $(i = 1, ..., n-1)$, where $c = a^n = (\sigma_1 \cdots \sigma_{n-1})^n$ generates the center $CB_n \cong \mathbb{Z}$ of the group B_n. The image of p_* consists of all $g \in B_n$ such that $\chi(g) \in (mn(n-1)+1)\mathbb{Z}$, where $\chi\colon B_n \to \mathbb{Z}$ is the abelianization epimorphism.

c) $\pi_1(E^n_{2(n-1)}(\mathbb{CP}^1)) \cong B'_n(S^2)$ and $E^n_{2(n-1)}(\mathbb{CP}^1)$ is the maximal abelian cover of $\mathcal{C}^n(\mathbb{CP}^1)$. $\bigcirc$

In the rest of this section, assuming that $n > 6$ and $k \le 2n$, we describe all connected unbranched non-cyclic k covers of $\mathcal{C}^n(X)$. The monodromy concept provides the one-to-one correspondence between the equivalence classes of such covers and the conjugacy classes of transitive non-cyclic homomorphisms $B_n(X) \to \mathbf{S}(k)$, which are described below.

13.2. Transitive homomorphisms $B_n(X) \to \mathbf{S}(k)$, $k \le 2n$. Artin Theorem quoted in Section 4.3 and Theorem 4.4 give the complete description of transitive homomorphisms $B_n(X) \to \mathbf{S}(k)$ for $n \ge k$. Here we present the case $n < k \le 2n$.

***Definition* 13.3.** For $i = 1, ..., n-1$ set

$$\varphi_1(\sigma_i) = \underbrace{(2i-1, 2i+2, 2i, 2i+1)}_{\text{4-cycle}};$$

$$\begin{aligned}\varphi_2(\sigma_i) = (1,2)\cdots(2i-3, 2i-2)\underbrace{(2i-1, 2i+1)(2i, 2i+2)}_{\text{two transpositions}}\times \\ \times (2i+3, 2i+4)\cdots(2n-1, 2n);\end{aligned}$$

$$\begin{aligned}\varphi_3(\sigma_i) = (1,2)\cdots(2i-3, 2i-2)\underbrace{(2i-1, 2i+2, 2i, 2i+1)}_{\text{4-cycle}}\times \\ \times (2i+3, 2i+4)\cdots(2n-1, 2n).\end{aligned}$$

These formulae determine three non-cyclic transitive homomorphisms $\varphi_j\colon B_n \to \mathbf{S}(2n)$ $(j = 1, 2, 3)$. Since $\varphi_2(\sigma_1 \cdots \sigma_{n-1}\sigma_{n-1} \cdots \sigma_1) = 1$, the homomorphism φ_2 gives also rise

[20] Here $d_n(Q) = d_n(w(Q))$ denotes the discriminant of the polynomial $p_n(t; w(Q)) = t^n + w_1(Q)t^{n-1} + ... + w_n(Q) = \prod_{\zeta \in Q}(t - \zeta)$ corresponding to a point $Q \in \mathcal{C}^n(\mathbb{C})$.

to the homomorphism $\varphi_2\colon B_n(S^2)\to \mathbf{S}(2n)$.[21] All these homomorphisms are referred to as the *standard non-cyclic transitive homomorphisms* $B_n(X)\to \mathbf{S}(2n)$. $\bigcirc$

For the braid group B_n the next theorem has been proved in [**Lin96b**] (part (***b***) for $n>8$ only; the cases $n=7$ and $n=8$ were treated by S. Orevkov [**Ore98**]). For the sphere braid group $B_n(S^2)$ both (***a***) and (***b***) are easy consequences of the corresponding properties of B_n and the presentation of $B_n(S^2)$ exhibited in Section 4.1.

Theorem 13.4. ***a***) *Let* $6<n<k<2n$. *Then every transitive homomorphism* $B_n(X)\to \mathbf{S}(k)$ *is cyclic.*

b) *For* $n>6$ *every non-cyclic transitive homomorphism* $B_n(X)\to \mathbf{S}(2n)$ *is conjugate to one of the standard homomorphisms.*

Remark **13.5.** For any $n\ge 3$ and any $r\ge 2$ there exist non-cyclic transitive homomorphisms $B_n\to \mathbf{S}(rn)$; let us present some of them.

For any $l\in\mathbb{Z}$ and $t\in\mathbb{N}$ we denote by $|l|_t$ the image of l along the natural epimorphism $\mathbb{Z}\to\mathbb{Z}/t\mathbb{Z}$. We identify the set $\mathbf{\Delta}_n=\{1,...,n\}$ to the group $\mathbb{Z}/n\mathbb{Z}$ via $\{1,...,n\}\ni m\leftrightarrow |m|_n-1$ and identify the symmetric groups $\mathbf{S}(n)$ and $\mathbf{S}(\mathbb{Z}/n\mathbb{Z})$. Furthermore, we identify the set $\mathbf{\Delta}_{rn}=\{1,...,rn\}$ to the direct product $\mathcal{D}(r,n)=(\mathbb{Z}/r\mathbb{Z})\times(\mathbb{Z}/n\mathbb{Z})$ via

$$\mathbf{\Delta}_{rn}\ni m\mapsto (R(m),N(m))\in(\mathbb{Z}/r\mathbb{Z})\times(\mathbb{Z}/n\mathbb{Z})\,,$$
$$\text{where } R(m)=|m-1|_r\in\mathbb{Z}/r\mathbb{Z}\ \text{ and }\ N(m)=|(m-1-R(m))/r|_n\in\mathbb{Z}/n\mathbb{Z}$$

and regard $\mathbf{S}(rn)$ as the symmetric group $\mathbf{S}(\mathcal{D}(r,n))$.

For each $x,y\in\mathbb{Z}/r\mathbb{Z}$ and any $i=1,...,n-1$ let $s_i=s_{x,y;i}\in\mathbf{S}(\mathcal{D}(r,n))$ denote the permutation that acts on elements $(R,N)\in\mathcal{D}(r,n)=(\mathbb{Z}/r\mathbb{Z})\times(\mathbb{Z}/n\mathbb{Z})$ as follows:

$$s_i(R,N)=\begin{cases}(R+y,\,N) & \text{if } N\ne i-1,i;\\ (R,\,N+1) & \text{if } N=i-1;\\ (R+x,\,N-1) & \text{if } N=i;\end{cases}\qquad (1\le i\le n-1)\,.$$

It is easily shown that the map $\sigma_i\to s_{x,y;i}$, $i=1,...,n-1$, extends to a uniquely defined non-cyclic homomorphism $\varphi_{x,y}\colon B_n\to\mathbf{S}(\mathcal{D}(r,n))=\mathbf{S}(rn)$. This homomorphism $\varphi_{x,y}$ is transitive if and only if the elements $x,y\in\mathbb{Z}/r\mathbb{Z}$ generate the whole group $\mathbb{Z}/r\mathbb{Z}$, or, which is the same, if and only if x and y are co-prime. However, $\varphi_{x,y}$ never can be primitive. For more details see [**Lin96b**]. $\bigcirc$

13.3. Non-cyclic coverings of $\mathcal{C}^n(X)$. We start with some simple examples.

Example **13.6.** The unbranched connected analytic covering

$$\mathcal{E}_n(X)=\{(\zeta,Q)\in X\times\mathcal{C}^n(X)\mid \zeta\in Q\}\,,\quad \nu_n\colon\mathcal{E}_n(X)\ni(\zeta,Q)\mapsto Q\in\mathcal{C}^n(X)\quad (13.8)$$

is referred to as the *standard non-cyclic n covering of* $\mathcal{C}^n(X)$. In the case $X=\mathbb{C}$ it may also be described as

$$\mathcal{E}_n(\mathbb{C})=\{(\zeta,w)\in\mathbb{C}\times\mathbf{G}_n\mid p_n(\zeta,w)=0\}\,,\quad \nu_n\colon\mathcal{E}_n(\mathbb{C})\ni(\zeta,w)\mapsto w\in\mathbf{G}_n\,,\quad (13.9)$$

[21] This is not the case for φ_1 and φ_3.

and in the case $X = \mathbb{CP}^1$ as

$$\begin{aligned} &\mathcal{E}_n(\mathbb{CP}^1) = \{([x:y], z) \in \mathbb{CP}^1 \times \mathcal{F}^n \mid \varphi([x:y], z) = 0\}\,, \\ &\nu_n\colon \mathcal{E}_n(\mathbb{CP}^1) \ni ([x:y], z) \mapsto z = [z_0 : \dots : z_n] \in \mathcal{F}^n \subset \mathbb{CP}^n\,. \end{aligned} \tag{13.10}$$

The monodromy representation corresponding to the covering (13.8) coincides with the standard epimorphism $\mu\colon B_n(X) \to \mathbf{S}(n)$. ◯

***Example* 13.7.** Here is a generalization of the previous example. For $m \le n$ set

$$\begin{aligned} &\mathcal{E}_{\binom{n}{m}}(X) = \{(O', Q) \in \mathcal{C}^m(X) \times \mathcal{C}^n(X) \mid Q' \subseteq Q\}\,, \\ &\nu_{\binom{n}{m}}\colon \mathcal{E}_{\binom{n}{m}}(X) \ni (Q', Q) \mapsto Q \in \mathcal{C}^n(X)\,. \end{aligned} \tag{13.11}$$

This is an unbranched connected non-cyclic $\binom{n}{m}$-covering of $\mathcal{C}^n(X)$. Its monodromy representation is the composition $B_n(X) \xrightarrow{\mu} \mathbf{S}(n) \xrightarrow{\tau^n_m} \mathbf{S}(\binom{n}{m})$, where τ^n_m stands for the transitive representation corresponding to the natural $\mathbf{S}(n)$ action on the set of all m point subsets $M \subset \{1, \dots, n\}$. ◯

***Example* 13.8.** Set $p'_n(t, w) = nt^{n-1} + (n-1)w_1 t^{n-2} + \dots + w_{n-1}$. Define the following non-trivial unbranched analytic 2 coverings of the space $\mathcal{E}_n(\mathbb{C})$:

$$\begin{aligned} &\nu^{(1)}\colon \mathcal{E}^{(1)}_n(\mathbb{C}) = \big\{ (\xi; (\zeta, w)) \in \mathbb{C}^* \times \mathcal{E}_n(\mathbb{C}) |\ \xi^2 = p'_n(\zeta, w) \big\} \to \mathcal{E}_n(\mathbb{C}), \\ &\nu^{(2)}\colon \mathcal{E}^{(2)}_n(\mathbb{C}) = \big\{ (\xi; (\zeta, w)) \in \mathbb{C}^* \times \mathcal{E}_n(\mathbb{C}) |\ \xi^2 = d_n(w) \big\} \to \mathcal{E}_n(\mathbb{C}), \\ &\nu^{(3)}\colon \mathcal{E}^{(3)}_n(\mathbb{C}) = \big\{ (\xi; (\zeta, w)) \in \mathbb{C}^* \times \mathcal{E}_n(\mathbb{C}) |\ \xi^2 = d_n(w) p'_n(\zeta, w) \big\} \to \mathcal{E}_n(\mathbb{C}), \end{aligned} \tag{13.12}$$

all with the same natural projection $(\xi; (\zeta, w)) \mapsto (\zeta, w)$. Any non-trivial 2 covering of $\mathcal{E}_n(\mathbb{C})$ is equivalent to one of the coverings $\nu^{(j)}\colon \mathcal{E}^{(j)}_n(\mathbb{C}) \to \mathcal{E}_n(\mathbb{C})$, $j = 1, 2, 3$ (see [**Lin96b**] for the proof; we will not use this fact in what follows).

Let us construct a non-trivial unbranched analytic 2 covering of the space $\mathcal{E}_n(\mathbb{CP}^1)$. To this end, take the standard $\mathbb{Z}/2n(n-1)\mathbb{Z}$ covering $p_{2(n-1)}\colon E^n_{2(n-1)}(\mathbb{CP}^1) \to \mathcal{F}^n$ described in (13.3). The subgroup $\mathbb{Z}/n(n-1)\mathbb{Z} \subset \mathbb{Z}/2n(n-1)\mathbb{Z}$ acts freely on $E^n_{2(n-1)}(\mathbb{CP}^1)$ by the transformations $r(z_0, \dots, z_n) = (e^{i\pi r/(n-1)} z_0, \dots, e^{i\pi r/(n-1)} z_n)$, where $r \in \mathbb{Z}/n(n-1)\mathbb{Z}$ and $z = (z_0, \dots, z_n) \in E^n_{2(n-1)}(\mathbb{CP}^1)$. This provides us with the non-trivial unbranched analytic 2 covering

$$\lambda\colon E^n_2(\mathbb{CP}^1) = [E^n_{2(n-1)}(\mathbb{CP}^1)]/(\mathbb{Z}/n(n-1)\mathbb{Z}) \to \mathcal{F}^n = \mathcal{C}^n(\mathbb{CP}^1)\,. \tag{13.13}$$

The desired 2-covering

$$\nu^{(2)} = \nu^*_n(\lambda)\colon \mathcal{E}^{(2)}_n(\mathbb{CP}^1) \to \mathcal{E}_n(\mathbb{CP}^1) \tag{13.14}$$

is obtained as the pullback $\nu^{(2)} = \nu^*_n(\lambda)$ of the covering (13.13) along the mapping $\nu_n\colon \mathcal{E}_n(\mathbb{CP}^1) \to \mathcal{C}^n(\mathbb{CP}^1)$ defined by (13.10).

The 2-coverings $\nu^{(j)}\colon \mathcal{E}^{(j)}_n(\mathbb{C}) \to \mathcal{E}_n(\mathbb{C})$ $(j = 1, 2, 3)$ and $\nu^{(2)}\colon \mathcal{E}^{(2)}_n(\mathbb{CP}^1) \to \mathcal{E}_n(\mathbb{CP}^1)$ defined by (13.8) and (13.14) are referred to as the *standard* 2*-coverings of* $\mathcal{E}_n(X)$. ◯

***Example* 13.9.** We define $2n$-coverings $\mu_{2n}^{(j)}(X)\colon \mathcal{E}_n^{(j)}(X) \to \mathcal{C}_n(X)$ as compositions of the 2-coverings $\nu^{(j)}\colon \mathcal{E}_n^{(j)}(X) \to \mathcal{E}_n(X)$ and the n-covering $\nu_n\colon \mathcal{E}_n(X) \to \mathcal{C}_n(X)$ exhibited in Examples 13.12 and 13.6 respectively. That is,

$$\begin{aligned}&\mu_{2n}^{(j)}(\mathbb{C}) = \nu_n \circ \nu^{(j)}\colon \mathcal{E}_n^{(j)}(\mathbb{C}) \xrightarrow{\nu^{(j)}} \mathcal{E}_n(\mathbb{C}) \xrightarrow{\nu_n} \mathcal{C}_n(\mathbb{C})\,, \quad j=1,2,3\,,\\ &\mu_{2n}^{(2)}(\mathbb{CP}^1) = \nu_n \circ \nu^{(2)}\colon \mathcal{E}_n^{(2)}(\mathbb{CP}^1) \xrightarrow{\nu^{(2)}} \mathcal{E}_n(\mathbb{CP}^1) \xrightarrow{\nu_n} \mathcal{C}_n(\mathbb{CP}^1)\,.\end{aligned} \tag{13.15}$$

These covering are referred to as the *standard non-cyclic* $2n$*-coverings of* $\mathcal{C}_n(X)$.

The monodromy representations $\varphi_j = \mu_{2n}^{(j)*}\colon B_n(X) \to \mathbf{S}(2n)$ corresponding to the above $2n$-coverings of $\mathcal{C}_n(X)$ are conjugate to the eponymous standard non-cyclic transitive homomorphisms $\varphi_j\colon B_n(X) \to \mathbf{S}(2n)$ introduced in Definition 13.3. ◯

The description of transitive homomorphisms $B_n(X) \to \mathbf{S}(k)$ provided by Theorem 4.4, Artin Theorem (quoted in Section 4.3) and Theorem 13.4, together with Theorem 13.1 and the latter example, leads the following complete description of unbranched connected k coverings of $\mathcal{C}^n(X)$ for $k \le 2n$.

Theorem 13.10. *Let $\mathcal{E} = \{p\colon E \to \mathcal{C}^n(X)\}$ be an unbranched connected k-covering.*

a) *If $n > \max\{k, 4\}$ then $\mathcal{E}$ is equivalent to one of the cyclic k-coverings exhibited in Section* 13.1.

b) *If $6 < n \le k \le 2n$ then either $\mathcal{E}$ is cyclic and then equivalent to one of the cyclic k-coverings exhibited in Section* 13.1 *or $k = n$ and $\mathcal{E}$ is equivalent to the standard non-cyclic n-covering defined by* (13.8) *or $k = 2n$ and $\mathcal{E}$ is equivalent to one of the standard non-cyclic $2n$-coverings exhibited in Example* 13.9. *In the latter case $\mathcal{E}$ splits to the composition of a 2-covering and the standard non-cyclic n-covering.* □

14. Configuration spaces of small dimension and some other examples

Configuration spaces of small dimension admit certain exceptional morphisms prohibited by results stated above. In this section we present some of such examples. We will also describe the Eisenstein automorphism mentioned in Section 1.6.

14.1. Ferrari map. The Ferrari map (L. Ferrari, 1545), which carries each polynomial of degree 4 to its cubic resolvent, provides the oldest and undoubtedly the most famous example of a non-cyclic morphism of configuration spaces. This morphism $Fe\colon \mathcal{C}^4(\mathbb{C}) \to \mathcal{C}^3(\mathbb{C})$ arises as follows. For every four distinct $q_1, q_2, q_3, q_4 \in \mathbb{C}$ the points $z_1 = (q_1-q_2-q_3+q_4)^2/4$, $z_2 = (q_1-q_2+q_3-q_4)^2/4$, $z_3 = (q_1+q_2-q_3-q_4)^2/4$ are distinct and a permutation of q_1, q_2, q_3, q_4 leads to a permutation of z_1, z_2, z_3. Thus, we obtain polynomial maps $fe\colon \mathcal{C}_o^4(\mathbb{C}) \to \mathcal{C}_o^3(\mathbb{C})$ and $Fe\colon \mathcal{C}^4(\mathbb{C}) \to \mathcal{C}^3(\mathbb{C})$. The morphism Fe induces the epimorphism of braid groups $Fe_*\colon B_4 \to B_3$, which carries both σ_1,σ_3 to σ_1 and keeps σ_2.

14.2. Disjoint morphisms $\mathcal{C}^3(\mathbb{C}) \to \mathcal{C}^6(\mathbb{C})$ and $\mathcal{C}^3(\mathbb{C}) \to \mathcal{C}^9(\mathbb{CP}^1)$. For distinct $q_1, q_2, q_3 \in \mathbb{C}$ define six points $\mu_1^{\pm}, \mu_2^{\pm}, \mu_3^{\pm} \in \mathbb{C}$ by the quadratic equations

$$(\mu_1^{\pm}-q_1)^2 = (q_1-q_2)(q_1-q_3),\ (\mu_2^{\pm}-q_2)^2 = (q_2-q_3)(q_2-q_1),\ (\mu_3^{\pm}-q_3)^2 = (q_3-q_1)(q_3-q_2).$$

All nine points q_i, $\mu_j^{\pm}$, $1 \le i, j \le 3$, are distinct and a permutation of q's leads to a permutation of μ's. Thus, we obtain a disjoint polynomial map

$$L\colon \mathcal{C}^3(\mathbb{C}) \ni \{q_1, q_2, q_3\} \mapsto \{\mu_1^+, \mu_1^-, \mu_2^+, \mu_2^-, \mu_3^+, \mu_3^-\} \in \mathcal{C}^6(\mathbb{C})\,,$$

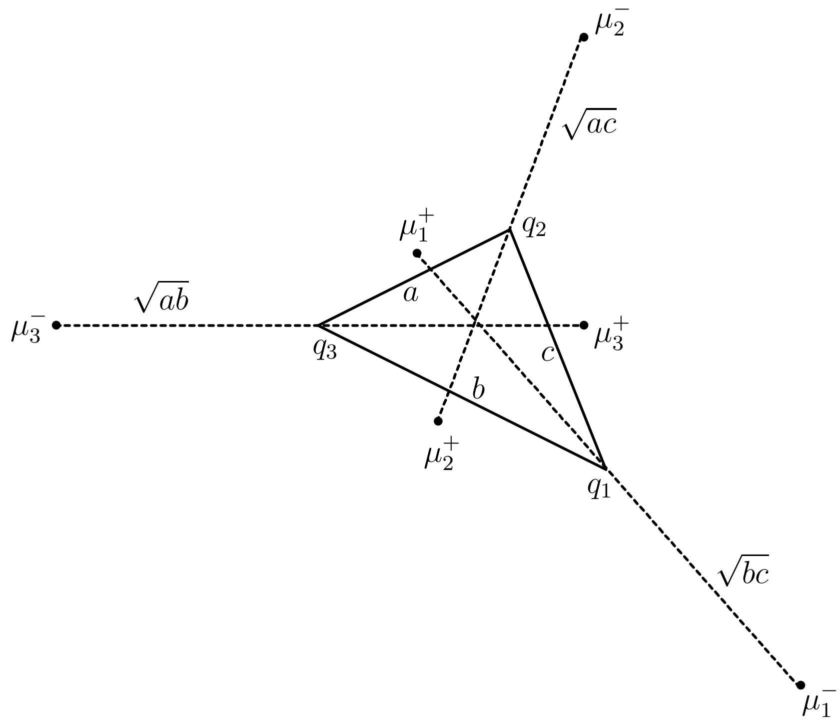
which may also be viewed as a polynomial map $L\colon \mathbf{G}_3 \to \mathbf{G}_6$ that carries any polynomial $p_3(t,z) = t^3 + z_1 t^2 + z_2 t + z_3 \in \mathbf{G}_3$ to the polynomial $p_6(t, L(z)) \in \mathbf{G}_6$, which is co-prime to $p_3(t,z)$ and whose coefficients $L_1(z), \dots, L_6(z)$ are as follows:

$$\begin{aligned}
L_1(z) &= 2z_1; & L_4(z) &= 20z_1z_3 - 5z_2^2;\\
L_2(z) &= 5z_2; & L_5(z) &= 8z_1^2z_3 - 2z_1z_2^2 - 4z_2z_3;\\
L_3(z) &= 20z_3; & L_6(z) &= 4z_1z_2z_3 - z_2^3 - 8z_3^2.
\end{aligned}$$

The first three of these formulae show that L is an embedding; one can check that it is non-cyclic and that the following relation between the discriminants holds:

$$d_6(L(z)) = -4^9 \cdot [d_3(z)]^5.$$

The following simple geometric interpretation of the map L due to E. A. Gorin (personal communication). Let $\Delta(Q) \subset \mathbb{C}$ be the triangle with the vertices at the points q_1, q_2, q_3 (it may degenerate to a straightline segment with a distinguished interior point). For each vertex $A = q_i$, let b and c denote the lengths of the sides of $\Delta(Q)$ intersecting at A, and let $\ell(A)$ be the bisector line of the inner angle of $\Delta(Q)$ at A. Take two points on $\ell(A)$ which are $\sqrt{bc}$-distant from A. Thus, we obtain six points, which coincide with the six points $\mu_i^{\pm}$ defined above (see the picture below).



On the other hand, Yoel Feler has recently noticed that the same morphism L may be defined via the formulae

$$\left(\frac{\mu_1^{\pm}-q_2}{\mu_1^{\pm}-q_3}\right)^2=\frac{q_1-q_2}{q_1-q_3}\,,\quad \left(\frac{\mu_2^{\pm}-q_3}{\mu_2^{\pm}-q_1}\right)^2=\frac{q_2-q_3}{q_2-q_1}\,,\quad \left(\frac{\mu_3^{\pm}-q_1}{\mu_3^{\pm}-q_2}\right)^2=\frac{q_3-q_1}{q_3-q_2}\,.$$

This observation let him to construct a disjoint[22] morphism

$$F\colon \mathcal{C}^3(\mathbb{C})\ni\{q_1,q_2,q_3\}\mapsto\{\mu_1^{(1)},\mu_1^{(2)},\mu_1^{(3)},\mu_2^{(1)},\mu_2^{(2)},\mu_2^{(3)},\mu_3^{(1)},\mu_3^{(2)},\mu_3^{(3)}\}\in\mathcal{C}^9(\mathbb{CP}^1)\,,$$

where $\mu_i^{(j)}$ $(1\le i,j\le 3)$ are defined by the cubic equations

$$\begin{aligned}\left(\frac{\mu_1^{(j)}-q_2}{\mu_1^{(j)}-q_3}\right)^3&=\left(\frac{q_1-q_2}{q_1-q_3}\right)^2\,,\\ \left(\frac{\mu_2^{(j)}-q_3}{\mu_2^{(j)}-q_1}\right)^3&=\left(\frac{q_2-q_3}{q_2-q_1}\right)^2\,,\\ \left(\frac{\mu_3^{(j)}-q_1}{\mu_3^{(j)}-q_2}\right)^3&=\left(\frac{q_3-q_1}{q_3-q_2}\right)^2\,.\end{aligned}$$

In terms of polynomials $p(t,q)=(t-q_1)(t-q_2)(t-q_3)\in\mathbf{G}_3$ and binary projective forms $\phi([x:y],F(q))\in\mathcal{F}^9$, the map F may be defined by the formula

$$\begin{aligned}\phi([x:y],F(q))=&((x-yq_1)^3(q_2-q_3)^2-(x-yq_2)^3(q_3-q_1)^2)\\ &\times((x-yq_2)^3(q_3-q_1)^2-(x-yq_3)^3(q_1-q_2)^2)\\ &\times((x-yq_3)^3(q_1-q_2)^2-(x-yq_1)^3(q_2-q_3)^2)\,.\end{aligned}$$

A straightforward computation of the discriminant shows that

$$D_\phi=3^{27}[(q_1-q_2)(q_2-q_3)(q_3-q_1)]^{56}\,.$$

14.3. Eisenstein automorphism. Here we present three different constructions of the Eisenstein automorphism: the original one [**Eis1844**], the Cayley construction (1845) and also the new one in the form of a tame morphism $\mathcal{C}^3(\mathbb{CP}^1)\to\mathcal{C}^3(\mathbb{CP}^1)$ found by Yoel Feler in 2001.

The discriminant D_ϕ of a binary cubic form $\phi(x,y)=z_0x^3+3z_1x^2y+3z_2xy^2+z_3y^3$ is given by

$$D_\phi=z_0^2z_3^2-3z_1^2z_2^2-6z_0z_1z_2z_3+4z_0z_2^3+4z_1^3z_3\,.$$

For every binary cubic form ϕ, Eisenstein considers the binary cubic form $(\mathcal{E}\phi)(x,y)=w_0x^3+3w_1x^2y+3w_2xy^2+w_3y^3$ whose coefficients w_i are homogeneous cubic polynomials

[22]Meaning that each set $Q\in\mathcal{C}^3(\mathbb{C})$ is disjoint to its image $F(Q)\in\mathcal{C}^9(\mathbb{CP}^1)$.

in $z_0, ..., z_3$ defined by[23]

$$w_0 = \frac{1}{2}\frac{\partial D_\phi}{\partial z_3} = 2z_1^3 - 3z_0z_1z_2 + z_0^2z_3\,, \quad w_1 = \frac{1}{6}\frac{\partial D_\phi}{\partial z_2} = 2z_0z_2^2 - z_0z_1z_3 - z_1^2z_2\,,$$
$$w_2 = \frac{1}{6}\frac{\partial D_\phi}{\partial z_1} = 2z_1^2z_3 - z_0z_2z_3 + z_1z_2^2\,, \quad w_3 = \frac{1}{2}\frac{\partial D_\phi}{\partial z_0} = 2z_2^3 - 3z_1z_2z_3 + z_0z_3^2\,.$$

He proves the identities $\mathcal{E}(\mathcal{E}\phi) = (D_\phi)^2 \cdot \phi$ and

$$\begin{aligned} D_{\mathcal{E}\phi} &= w_0^2w_3^2 - 3w_1^2w_2^2 - 6w_0w_1w_2w_3 + 4w_0w_2^3 + 4w_1^3w_3 \\ &= (z_0^2z_3^2 - 3z_1^2z_2^2 - 6z_0z_1z_2z_3 + 4z_0z_2^3 + 4z_1^3z_3)^3 = (D_\phi)^3\,. \end{aligned}$$

Thus Eisenstein's correspondence $f \mapsto \mathcal{E}\phi$ determines a *birational involution* of the space $\mathbb{CP}^3$ of all projective cubic binary forms. Moreover, its restriction to the space $\mathcal{F}_3$ of non-degenerate cubic forms is an *involutive biregular automorphism* of the latter space:

$$\begin{tikzcd} \mathcal{F}_3 \arrow[d, "\mathcal{E}"', "\cong"] \arrow[dr, "\mathrm{id}"] & \\ \mathcal{F}_3 \arrow[r, "\cong", "\mathcal{E}"'] & \mathcal{F}_3\,. \end{tikzcd} \tag{14.1}$$

14.4. Cayley's form of the Eisenstein automorphism. Take a complex binary cubic form[24]

$$\phi(x, y) = z_0x^3 + 3z_1x^2y + 3z_2xy^2 + z_3y^3\,.$$

Its Hessian

$$\psi(x, y) = \det\begin{pmatrix} \partial^2\phi/\partial x^2\,, & \partial^2\phi/\partial x\partial y \\ \partial^2\phi/\partial y\partial x\,, & \partial^2\phi/\partial y^2 \end{pmatrix}$$

is a binary quadratic form. The Jacobian

$$J(x, y) = \det\begin{pmatrix} \partial\phi/\partial x\,, & \partial\phi/\partial y \\ \partial\psi/\partial x\,, & \partial\psi/\partial y \end{pmatrix}$$

is a binary cubic form.

Eisenstein's map: $\quad \mathcal{E}\colon \phi \mapsto \mathcal{E}\phi = J.$

14.5. Feler's form of the Eisenstein automorphism. For any $Q \in \mathcal{C}^3(\mathbb{CP}^1)$ let $z(Q) = [z_0(Q) : z_1(Q) : z_2(Q) : z_3(Q)]$ denote the coefficients of the projective binary cubic form $\phi = \phi([x : y], Q)$ whose zero set

$$Z_\phi = \{[x : y] \in \mathbb{CP}^1 \mid \phi([x : y]) = 0\}$$

coincides with Q, and let D_ϕ denote the discriminant of ϕ. The Eisenstein's automorphism of $\mathcal{C}^3(\mathbb{CP}^1)$ coincides with the tame map

$$F_T\colon \mathcal{C}^3(\mathbb{CP}^1) \ni Q \mapsto T(Q)Q \in \mathcal{C}^3(\mathbb{CP}^1)$$

[23]Eisenstein has not pointed out that the discriminant D_ϕ plays the part of a potential function for the coefficients of the form $\mathcal{E}\phi$; I have learned this fact from M. Gizatullin.

[24]There is a minor difference between the automorphism described here and the original Eisenstein form: one has just to interchange x and y in one of the examples.

that corresponds to the morphism $T\colon \mathcal{C}^3(\mathbb{CP}^1) \to \mathbf{PSL}(2,\mathbb{C})$, where

$$T(Q)\colon \mathbb{CP}^1 \ni \zeta \mapsto \frac{a(Q)\zeta + b(Q)}{c(Q)\zeta + d(Q)} \in \mathbb{CP}^1 \tag{14.2}$$

is the Möbius transformation with the coefficients

$$\begin{aligned} a(Q) &= (z_1 z_2 - z_0 z_3)/\sqrt{-D_\phi}\,; \\ b(Q) &= 2(z_2^2 - z_1 z_3)/\sqrt{-D_\phi}\,; \\ c(Q) &= 2(z_0 z_2 - z_1^2)/\sqrt{-D_\phi}\,; \\ d(Q) &= -a(Q) = -(z_1 z_2 - z_0 z_3)/\sqrt{-D_\phi}\,. \end{aligned} \tag{14.3}$$

It is easily seen that $T(Q) \in \mathbf{PSL}(2,\mathbb{C})$ is well defined by (14.2) and (14.3).

Bibliography

[Arn70a] V. I. Arnold, *Cohomology classes of algebraic functions invariant under Tschirnhausen transformation*, Funct. Anal. Appl. **4**, n. 1 (1970), 74–75.

[Arn70b] V. I. Arnold, *On certain topological invariants of algebraic functions*, Trudy Mosk. Mat. Obshch. **21** (1970), 27–46 (in Russian). *English translation:*

[Arn70c] V. I. Arnold, *Topological invariants of algebraic functions,* II, Funkt. Analiz i Priloz. **4**, no. 2 (1970), 1–9 (in Russian). *English translation:* Funct. Anal. Appl. **4**, no. 2 (1970), 91–98.

[Art25] E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg **4** (1925), 47–72.

[Art47a] E. Artin, *Theory of braids*, Annals of Math. **48**, no. 1 (1947), 101–126.

[Art47b] E. Artin, *Braids and permutations*, Ann. of Math. **48**, no. 3 (1947), 643–649.

[Bir74] J. S. Birman, Braids, links and mapping class groups. Annals of Math. Studies **82**. Princeton Univercity Press, Princeton, N. J., 1974.

[BorNar67] A. Borel, R. Narasimhan, *Uniqueness conditions for certain holomorphic mappings*, Invent. Math. **2** (1967), 247–255.

[Bohn47] F. Bohnenblust, *The algebraic braid group*, Annals of Math. **48**, no. 1 (1947), 127–136.

[Bro76] F. E. Browder, *Problems of Present Day Mathematics*, Mathematical Developments Arising from Hilbert Problems, Proc. Symp. Pure Math. **28** (F. E. Browder, ed.) Amer. Math. Soc., 1976, 35–80.

[Bur32] W. Burau, *Über Zopfinvarianten*, Abh. Math. Sem. Univ. Hamburg **9** (1932), 117–124.

[Buh96] J. Buhler, Z. Reichstein *On the essential dimension of a finite group*, Composito Math. **106**, no. 2 (1997), 159–179.

[Car40] H. Cartan, *Sur les matrices holomorphes de n variables complexes*, J. Math. Pures et Appl. **19** (1940), 1–26.

[Chow48] W.-L. Chow, *On the algebraic braid group*, Annals of Math. **49**, no. 3 (1948), 654–658.

[DanGiz72] V. I. Danilov and M. H. Gizatullin, *Automorphisms of affine surfaces* II, Math. USSR Izvestija **41**, no. 2 (1977),

[Del72] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17**, no. 4 (1972),273–302.

[DyeGro81] J. Dyer and E. Grossman, *The automorphism groups of the braid groups*, Amer. J. Math. **103**, no. 6 (1981), 1151–1169.

[Eis1844] G. Eisenstein, *Über eine merkwürdige identische Gleichung*, Journal für die reine und angewandte Mathematik **27** (1844), 105–106 (see also: Gotthold Eisenstein, Mathematische Werke, Band I, Chelsea, New York, 1975,1989, pp. 26–27).

[EarKra74] C. J. Earle and I. Kra *On holomorphic mappings between Teichmüller spaces*, in book: Contributions to Analysis. A collection of papers dedicated to L. Bers. L. V. Ahlfors etc., ed., Academic Press, New-York 1974, 107–124.

[EarKra76] C. J. Earle and I. Kra *On selections of some families of closed Riemann surfaces*, Acta Math. **137** (1976), 49–79

[Enr23] *Sulla costrucione delle funcioni algebriche di due variabili possedenti una data curva di diramazione*, Annali di matematica pure ed applicata, Ser. 4, **1** (1923), 185–198.

[FadNeu62] E. Fadell and L. Neuwirth, *On Configuration spaces*, Math. Scand. **10**, no. 1 (1962), 111–118.

[FadHus01] E. R. Fadell and S. Y. Husseini, Geometry and topology of configuration spaces. Springer, 2001.

[FeiZie99] E. M. Feichtner, M. Ziegler, *The integral cohomology algebras of ordered configuration spaces of spheres*, Documenta Math. **5** (2000), 115-139.

[Fin45] P. Finsler, Über die Primzahlen zwischen n und $2n$. Speiser-Festschrift, Orell–Füssli, Zürich, 1945.

[Fo51] L. R. Ford, Automorphic Functions, Chelsea, 1951.

[FoxNeu62] R. H. Fox and L. Neuwirth, *The braid groups*, Math. Scand. **10**, no. 1 (1962), 119–126.

[Fuks70] D. B. Fuks, *Cohomologies of a braid group mod 2*, Funct. Anal. Appl. **4**, no. 2 (1970), 91–98.

[Gar] F. Garside, *The braid group and other groups*, Quart. J. Math. Oxford **20**, n. 78 (1969), 235–254.

[Gol74] D. L. Goldsmith, *Homotopy of braids in answer to a question of E. Artin*, Topology Conference, Lect. Notes Math. **375** (R. F. Dickman, Jr. and P. Fletcher, eds.), Springer, 1974, 91–96.

[GorLin69] E. A. Gorin and V. Ya. Lin, *Algebraic equations with continuous coefficients and some problems of the algebraic theory of braids*, Matem. Sbornik **78 (120)**, no. 4 (1969), 579–610 (in Russian). *English translation:* Math. USSR Sbornik **7** (1969), 569–596.

[GorLin74] E. A. Gorin and V. Ya. Lin, *On separable polynomials over commutative Banach algebras*, Dokl. Akad. Nauk SSSR **218**, no. 3 (1974), 505–508 (in Russian). *English translation:* Soviet Math. Dokl. **15**, no. 5 (1974), 1357–1361.

[Gra58] H. Grauert, *Analytische Faserungen über holomorphvollständigen Räumen*, Math. Ann. **135** (1958), 263–273.

[Hall] M. Hall, The theory of groups. McMillan Co., New York, 1959.

[Hub72] J. H. Hubbard, *Sur la non-existence de sections analytiques à la courbe universelle de Teichmüller*, C.R. Acad. Sci. Paris Sér. A-B **274** (1972), A978–A979.

[Hub76] J. H. Hubbard, *Sur les sections analytiques de la courbe universelle de Teichmüller*, Mem. Amer. Math. Soc. **4** (1976), no.166, ix+137pp.

[Kal75] S. I. Kaliman, *Holomorphic universal covering of the space of polynomials without multiple roots*, Funkt. Analiz i Priloz. **9**, no. 1 (1975), 71 (in Russian). *English translation:* Funct. Anal. Appl. **9**, no. 1 (1975), 67–68.

[Kal76] S. I. Kaliman, *Holomorphic endomorphisms of the manifold of complex polynomials with discriminant* 1, Uspehi Mat. Nauk **31**, no. 1 (1976), 251–252 (in Russian).

[Kal93] Sh. I. Kaliman *The holomorphic universal covers of the spaces of polynomials without multiple roots*, Selected translations. Selecta Math. Soviet. **12**, no. 4 (1993), 395–405.

[Kle] F. Klein, Vorlesungen über das Ikosaeder und die Aufl" osung der Gleichungen vom f" unften Grade, Birkhäuser Verlag & B. G. Teubner Verlagsgesellschaft, 1993.

[Kurth97] A. Kurth, *SL_2-equivariant polynomial automorphisms of binary forms*, Ann. Inst. Fourier, Grenoble **47**, no. 2 (1997), 585–597.

[Lin71] V. Ya. Lin, *Algebroidal functions and holomorphic elements of homotopy groups of complex manifolds*, Dokl. Akad. Nauk SSSR **201:1** (1971), 28–31 (in Russian). *English translation:* Soviet Math. Dokl. **12**, n. 6 (1971), 1608–1612.

[Lin72a] V. Ya. Lin, *On representations of the braid group by permutations*, Uspehi Matem. Nauk **27**, n. 3 (1972), 192.

[Lin72b] V. Ya. Lin, *Algebraic functions with universal discriminant manifolds*, Funkt. Analiz i Priloz. **6**, no. 1 (1972), 81–82 (in Russian). *English translation:* Funct. Anal. Appl. **6**, n. 1 (1972), 73–75.

[Lin72c] V. Ya. Lin, *On superpositions of algebraic functions*, Funkt. Analiz i Priloz. **6**, no. 3 (1972), 77–78 (in Russian). *English translation:* Funct. Anal. Appl. **6**, n. 3 (1972), 240–241.

[Lin74] V. Ya. Lin, *Representations of braids by permutations*, Uspehi Mat. Nauk **29**, n. 1 (1974), 173–174 (in Russian).

[Lin76] V. Ya. Lin, *Superpositions of algebraic functions*, Funkt. Analiz i Priloz. **10**, no. 1 (1976), 37–45 (in Russian). *English translation:* Funct. Anal. Appl. **10**, n. 1 (1976), 32–38.

[Lin79] V. Ya. Lin, *Artin braids and the groups and spaces connected with them*, Itogi Nauki i Tekhniki, Algebra, Topologiya, Geometriya **17**, VINITI, Moscow, 1979, pp. 159–227 (in Russian). *English translation:* Journal of Soviet Math. **18** (1982), 736–788.

[Lin88] V. Ya. Lin, *Liouville coverings of complex spaces and amenable groups*, Matem. Sbornik **132 (174)**, no. 2 (1987), 202–224 (in Russian). *English translation:* Math. USSR Sbornik **60**, n. 1 (1988), 197–216.

[Lin96a] V. Ya. Lin, *Around the 13th Hilbert problem for algebraic functions*, Israel Math. Conference Proc. **9** (1996), 307–327.

[Lin96b] V. Ya. Lin, *Braids, permutations, polynomials*–I, Preprint MPI 96-118, Max-Planck-Institut für Mathematik in Bonn, August 1996, 112pp.

[LinZai98] V. Ya. Lin and M. Zaidenberg, *Liouville and Carathéodory coverings in Riemannian and complex geometry* in book: Voronezh Winter Mathematical School. P. Kuchment & V. Lin eds., Amer. Math. Soc. Transl. (2), **184** (1998), 111–130.

[Mal40] A. I. Malcev, *On an isomorphic representation of infinite groups of matrices*, Matemat. Sbornik **4** (1940), 405–422 (in Russian).

[Mar45] A. A. Markov, *Foundations of the algebraic theory of tresses*, Proc. Steklov Math. Institute **16**, 1945 (in Russian).

[Mur82] K.Murasugi, *Seifert fiber spaces and braid groups*, Proc. London Math. Soc. **44** (1982), 71–84.

[MurKur99] K. Murasugi, B. I. Kurpita, A study of braids. Kluwer, 1999.

[HNeum] H. Neumann, Varieties of groups. Springer, 1967.

[Ore98] S. Orevkov, *Ttransitive homomorphisms* $B_7 \to \mathbf{S}(14)$ *and* $B_8 \to \mathbf{S}(16)$, private letter, October 4, 1998.

[Rams65] K. J. Ramspott, *Stetige und holomorphe Schnitte in Bündeln mit homogener Faser*, Math. Z. **89**, no. 3 (1965), 234–246.

[Trost] E. Trost, Primzahlen. Birkhäuser Verlag, Basel–Stuttgart, 1968.

[Vas88] V. A. Vassiliev, *Braid group cohomologies and algorithm complexity*, Funct. Anal. Appl. **22**, no. 3 (1988), 182–189.

[Vas98] V. A. Vassiliev, Complements of Discriminants of Smooth Maps: Topology and Applications. Translations of Mathematical Monographs **98**, Amer. Math. Soc., Providence, Rhode Island, 1992.

[Wi64] H. Wielandt, Finite permutation groups. Academic Press, 1964.

[Zar29] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929).

[Zar36] O. Zariski, *On the Poincaré group of rational plane curves*, Amer. J. Math. **58** (1936), 607–619.

[Zar37a] O. Zariski, *The topological discriminant group of a Riemann surface of genus p*, Amer. J. Math. **59** (1937), 335–358.

[Zar37b] O. Zariski, *A theorem on the Poincaré group of an algebraic hypersurface*, Ann. Math. **38** (1937).

[Zin77] V. Zinde, *Holomorphic maps of regular orbit spaces of Coxeter groups of series B and D*, Sibirsk. Math. J. **18**, no. 5 (1977), 1016–1026 (in Russian). *English translation:*

Index