PERVERSE SHEAVES, KOSZUL IC-MODULES, AND THE QUIVER FOR THE CATEGORY $\mathcal{O}$

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Abstract. For a stratified topological space we introduce the category of IC-modules, which are linear algebra devices with the relations described by the equation $d^2 = 0$. We prove that the category of (mixed) IC-modules is equivalent to the category of (mixed) perverse sheaves for flag varieties. As an application, we describe an algorithm calculating the quiver underlying the BGG category $\mathcal{O}$ for arbitrary simple Lie algebra, thus answering a question which goes back to I. M. Gelfand.

1. Introduction

1.1. Motivated by R. MacPherson's cellular perverse sheaves, W. Soergel's studies of the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$, and L. Saper's L-modules we introduce here, in Section 2, an abelian category of IC-*modules* designed to be a derived-category-free model of the category of constructible perverse sheaves if the latter is Koszul (cf. 2.8.4 for the remarks beyond the Koszul case). In Section 3 we prove the following.

Theorem A. *The category of Schubert-constructible perverse sheaves on a flag variety is equivalent to the category of* IC*-modules.*

Moreover, we suggest a relationship between perverse sheaves and IC-modules for a much wider class of stratified spaces.

1.2. There exists a number of derived-category-free models of perverse sheaves. These include $\mathcal{D}$-modules, cf. e.g. [24] (which actually predate perverse sheaves), gluing constructions [1, 29], and "quiver" presentations, cf. e.g. [30, 19, 11, 9]. We believe that our "elementary" construction has the advantage of being well suited for *mixed* perverse sheaves. We construct a natural functor from mixed perverse sheaves to mixed IC-modules which is an equivalence in our two main examples: flag varieties and simplicial complexes, cf. Section 6.

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1.3. Let G be a complex simple algebraic group, $\mathfrak{g} = \operatorname{Lie} G$ its Lie algebra, B a Borel subgroup, and $\mathfrak{b} = \operatorname{Lie} B$ its Lie algebra. Let $\mathfrak{h}$ be a Cartan of $\mathfrak{b}$, Φ the root system, and let W be the Weyl group. Let $\mathcal{O}_0$ be a regular block the BGG category $\mathcal{O}$, cf. [5, 6, 7], equivalent to Schubert-stratification constructible perverse sheaves on G/B, [2, 3]. The simple objects of $\mathcal{O}_0$ are the simple $\mathfrak{g}$-modules with the highest weights $w(\rho)-\rho$ where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. If $\mathfrak{g} = \mathfrak{sl}_2$, then $\mathcal{O}_0$ is equivalent to the following category of linear algebra data

$$\begin{tikzcd} V_0 \arrow[r, bend right, "b"'] & V_1 \arrow[l, bend right, "a"'] \end{tikzcd}$$

with the relation $b\circ a = 0$. This is the *quiver* of the category $\mathcal{O}_0$ for $\mathfrak{sl}_2$. The question of finding such a quiver for arbitrary simple Lie algebra $\mathfrak{g}$ goes back to I. M. Gelfand, cf. [18]. Seminal breakthroughs with implications for this question had been made in particular in [25, 2, 34, 4].

It is well known that the quiver has vertices labeled by the elements of the Weyl group W. If $y, w \in W$, then for every arrow from y to w there is an arrow from w to y. Moreover, the number of arrows from y to w is equal to $\mu(y, w)$ – the coefficient of the power $\frac{1}{2}(l(w) - l(y) - 1)$ of the Kazhdan-Lusztig polynomial [25]. However, the relations between arrows were not known explicitly and were difficult to obtain algorithmically, except for a few small rank cases, cf. [23, 37].

In Section 5 we use IC-modules to encode the relations in the form $d^2 = 0$ and describe a *computer-executable algorithm* to calculate them explicitly. The algorithm is a refinement of an algorithm of C. Stroppel [37]. It works for arbitrary simple Lie algebra $\mathfrak{g}$, thus settling the Gelfand's question for the category $\mathcal{O}$.

1.4. We believe that IC-modules should also be useful for the study of Harish-Chandra modules and Soergel's conjecture, cf. [36, Basic conjecture 1], as well as toric varieties, affine Grassmannians, and locally symmetric spaces, cf. Section 7.

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2. Definitions and The $d^2 = 0$ Theorem

2.1. **Setup.** Our sheaves are assumed to be sheaves of finitely generated R-modules over a ring R. Unless indicated otherwise, R is assumed to be the field $\mathbb{C}$, but many results are true for other ground rings.

Let X be a topological space equipped with a sheaf of rings and let

$$X = \bigsqcup_{S \in \mathcal{S}} S$$

be a partition of X into a finite disjoint union of locally closed strata. Let $\dim : \mathcal{S} \to \mathbb{Z}_{\geq 0}$ be a dimension function such that $\dim S \leq \dim S'$ if $\overline{S} \subseteq \overline{S}'$ and let us fix a perversity function $p : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$, cf. [22, 3].

Let L be a local system on S. Consider the IC-sheaf $\mathbf{IC}(S, L) = \mathbf{IC}_p(S, L)$, cf. [21, 22, 3] corresponding to L and the perversity p on the closure $\overline{S}$ of the stratum S, and let

$$IH^*(\overline{S}, L) = IH^*_p(\overline{S}, L) = \mathbb{H}^*(\mathbf{IC}_p(S, L))$$

be the hypercohomology of the sheaf $\mathbf{IC}_p(S, L)$. From now on we will omit perversity from the notation. Perversity is always assumed to be the middle one for algebraic varieties with algebraic stratifications. Notice that $IH^*(\overline{S}, L)$ is naturally a graded $H^*(X)$-module, where $H^*(X)$ is the cohomology of X with coefficients in R.

From now on we assume that all strata are *contractible* unless explicitly specified otherwise, e.g. in section 2.5.

2.2. **Definition.** A *mixed* IC-module V is the following data:

(1) for every stratum $S \in \mathcal{S}$, a $\mathbb{Z}$-graded R-vector space (stalk) $\oplus_{i \in \mathbb{Z}} V^i_S$ on S,
(2) for every pair of incident (i.e. one is in the closure of the other) strata S and S' such that $S \neq S'$, a boundary map

$$v(S, S') \in \mathrm{Hom}^{(1,1)}_{H^*(X)}(IH^*(\overline{S}, V^\bullet_S), IH^*(\overline{S}', V^\bullet_{S'})),$$

where $\mathrm{Hom}^{(1,1)}$ is the space of degree $(1, 1)$ morphisms of $H^*(X)$-modules $IH^*(\overline{S}, V^\bullet_S)$ bigraded by the degree of IH^* and the degree of $V^\bullet_S$. The data is subject to the

$$\text{Chain Complex Axiom:} \qquad d^2 = 0, \tag{2.2.1}$$

where d is the degree $(1, 1)$ map

$$d = \bigoplus_{S, S'} v(S, S') : \oplus_S IH^*(\overline{S}, V^\bullet_S) \to \oplus_S IH^*(\overline{S}, V^\bullet_S). \tag{2.2.2}$$

The bigraded space $\oplus_S IH^*(\overline{S}, V_S^\bullet)$ with the differential d is called the *total complex* of the IC-module V. Its cohomology is naturally bigraded.

2.3. A *non-mixed* IC module is defined exactly as above, except the stalks V_S are just (non-graded) R-vector spaces, and boundary maps

$$v(S, S') \in \mathrm{Hom}^1_{H^*(X)}(IH^*(\overline{S}), IH^*(\overline{S}')) \otimes \mathrm{Hom}_R(V_S, V_{S'})$$

and the differential d are maps of degree 1. The cohomology of the total complex has only one natural grading.

2.4. A morphism of IC modules is the collection of maps between stalks commuting with the boundary maps. The abelian category of non-mixed IC-modules of finite length (i.e. having a finite filtration with simple subquotients) will be denoted by $\mathcal{A}_\mathcal{S}(X)$.

The category of mixed IC-modules of finite length will be denoted by $\mathcal{A}^{\mathrm{mixed}}_\mathcal{S}(X)$. It is a mixed category in the sense of [4, 4.1]. A mixed IC-module V is called *pure* of weight m if it is concentrated in degree $-m$ i.e., $V_S^i = 0$ unless $i = -m$ for all $S \in \mathcal{S}$. We have the obvious functor $v : \mathcal{A}^{\mathrm{mixed}}_\mathcal{S}(X) \to \mathcal{A}_\mathcal{S}(X)$ forgetting the mixed structure.

2.5. **Local Systems.** One could further generalize the definition of an IC-module to the case of non-simply-connected strata. In general, the stalk V_S is a local system on S, and the boundary maps are

$$v(S, S') \in \mathrm{Hom}^1_{H^*(X)}(IH^*(\overline{S}, V_S), IH^*(\overline{S}', V_{S'})),$$

where Hom^1 are the degree 1 morphisms of $H^*(X)$-modules. (We consider here the non-mixed case to simplify the notation.) The Chain Complex Axiom and the total complex are precisely as above.

From now on we will consider non-mixed IC-modules unless explicitly specified otherwise, e.g. in Section 6.

2.6. **Verdier duality.** We will only consider here the middle perversity case in order to simplify the notation. For an IC-module V the stalks of its Verdier dual IC-module DV are defined as follows:

$$DV_S = V_S^*,$$

where V_S^* is the dual vector space. Notice that

$$(V_S \otimes IH^i(\overline{S}))^* = V_S^* \otimes IH^{-i}(\overline{S})$$

since according to [22, 5.3] we have $IH^i(\overline{S}) = (IH^{-i}(\overline{S}))^*$. Now the boundary maps

$$\begin{array}{cccc} dv^i(S,S'): & V_S^* \otimes IH^i(\overline{S}) & \to & V_{S'}^* \otimes IH^{i+1}(\overline{S}') \\ & || & & || \\ & V_S^* \otimes (IH^{-i}(\overline{S}))^* & \to & V_{S'}^* \otimes (IH^{-i-1}(\overline{S}'))^* \end{array}$$

of DV are $dv^i(S,S') = (v^{-i-1}(S',S))^*$, where

$$v^{-i-1}(S',S) : V_{S'} \otimes IH^{-i-1}(\overline{S}') \to V_S \otimes IH^{-i}(\overline{S})$$

are the boundary maps of V. The Chain Complex Axiom (2.2.1) for DV is obviously satisfied. We have constructed a contravariant functor $D : \mathcal{A}_{\mathcal{S}}(X) \to \mathcal{A}_{\mathcal{S}}(X)$. Clearly, $D \circ D = \mathrm{Id}$.

2.7. Denote the category of $\mathcal{S}$-constructible perverse sheaves of finite length on X by $\mathcal{P}_{\mathcal{S}}(X)$, cf. [3]. We present here a more precise formulation of Theorem A. The *raison d'être* of these notes is the following.

The $d^2 = 0$ Theorem. *If X is a flag variety stratified by Schubert cells, or a simplicial complex stratified by simplices, then*

(i) *the categories*
$$\mathcal{P}_{\mathcal{S}}(X) \simeq \mathcal{A}_{\mathcal{S}}(X)$$
are equivalent,

(ii) *the total complex of an* IC-*module calculates the hypercohomology of the corresponding perverse sheaf.*

Remark: The $d^2 = 0$ Thereom for simplicial complexes. For perverse sheaves on simplicial complexes constructible with respect to the triangulation we have

$$IH^*(\overline{\Delta}) = \mathbb{C}[\delta(\dim \Delta)],$$

where Δ is a simplex and δ is a cellular perversity, [28, 32, 39]. The IC-modules in this case are *simplicial perverse sheaves* of R. MacPherson and the statement of the The $d^2 = 0$ Thereom is a theorem due to MacPherson [28], cf. [32, 38, 39] for an alternative proof.

2.8. **Restrictions, relaxing remarks, and conjectures.** We use only a limited number of properties (some are listed below) of the category of perverse sheaves in order to prove the Theorem. It is natural to conjecture that under these restrictions The $d^2 = 0$ Theorem holds. (Note that only the statement (ii) is in question.)

2.8.1. Assume that we have only finitely many strata $S \in \mathcal{S}$. This assumption can be relaxed, but we will not pursue it here.

2.8.2. For the purposes of this text we will assume that the strata are contractible. This assumption could be relaxed: The $d^2 = 0$ Theorem with a local system version of IC-modules is a tautology for a stratification with one stratum, for example.

2.8.3. Assume that the category $\mathcal{P}_{\mathcal{S}}(X)$ has enough projectives i.e., every object has a projective resolution.

2.8.4. We assume that the category $\mathcal{P}_{\mathcal{S}}(X)$ is Koszul i.e., it is equivalent to a category of finitely generated modules over a Koszul algebra. When the category $\mathcal{P}_{\mathcal{S}}(X)$ is not Koszul one can still reconstruct it from the algebra $\mathrm{Ext}^*_{\mathcal{P}_{\mathcal{S}}(X)}(L, L)$, where L is the direct product of all simple objects in $\mathcal{P}_{\mathcal{S}}(X)$ with the canonical A_∞-structure, cf. e.g. [27, Problem 2]. One has to take this A_∞-structure into account in order to generalize the IC-modules beyond the Koszul case.

2.8.5. For any two incident strata S ans S' and two simple objects $\mathcal{L}_1 = \mathbf{IC}(S, L_S)$ and $\mathcal{L}_2 = \mathbf{IC}(S', L_{S'})$ in $\mathcal{P}_{\mathcal{S}}(X)$ we assume that we have the isomorphisms

$$\mathrm{Ext}^i_{\mathcal{P}_{\mathcal{S}}(X)}(\mathcal{L}_1, \mathcal{L}_2) = \mathrm{Ext}^i_{D(X)}(\mathcal{L}_1, \mathcal{L}_2) = \mathrm{Hom}^i_{H^*(X)}(\mathbb{H}^*(\mathcal{L}_1), \mathbb{H}^*(\mathcal{L}_2)),$$

and $\mathrm{Ext}^i_{\mathcal{P}_{\mathcal{S}}(X)}(\mathcal{L}_1, \mathcal{L}_2) = \mathrm{Ext}^i_{D(X)}(\mathcal{L}_1, \mathcal{L}_2) = 0$ for any two non-incident strata S ans S'. Here Hom^i is the space of $H^*(X)$-morphisms of degree i. In particular, the representation of the algebra $\mathrm{Ext}^*_{\mathcal{P}_{\mathcal{S}}(X)}(L, L)$ in the space $\oplus_S IH^*(\overline{S}, L_S)$ is faithful. One could generalize IC-modules using another faithful representation of the algebra $\mathrm{Ext}^*_{\mathcal{P}_{\mathcal{S}}(X)}(L, L)$.

3. Proof of Theorem A

In this section we prove The $d^2 = 0$ Theorem, and thus Theorem A, following a suggestion of W. Soergel. In fact our proof works for any stratified space with the restrictions 2.8 and one additional condition: $R_X[\dim X]$ is a simple object of $\mathcal{P}_{\mathcal{S}}(X)$.

3.1. Let G be a simple algebraic group over $\mathbb{C}$, $B \subseteq P \subseteq G$ be a Borel subgroup and a parabolic subgroup, W_P the subgroup of the Weyl group W and let G/P be the corresponding partial flag variety, stratified by Schubert cells X_w, where w is a coset in W/W_P which we identify with the set of strata $W_{\mathcal{S}}$ of X. Denote by $\mathcal{P}_{\mathrm{Schubert}}(G/P)$ the category of Schubert-stratification constructible perverse sheaves on G/P.

3.2. **Theorem.** [34, 20] *For any $y, w \in W_{\mathcal{S}}$ the natural map*

$$\mathrm{Ext}^*_{D(X)}(\mathbf{IC}_y, \mathbf{IC}_w) = \mathrm{Hom}^*_{H^*(X)}(IH^*(\overline{X}_y), IH^*(\overline{X}_w))$$

is an isomorphism of graded spaces. (Here Hom^i *denotes degree i morphisms of $H^*(X)$-modules.) Moreover, the left and right hand sides are isomorphic as graded algebras.*

3.3. **Koszul complex.** In this subsection $A = \oplus_{i \geq 0} A_i$ is an arbitrary Koszul algebra with semisimple $A_0 = k$. Let $A^!$ be the quadratic dual algebra and let us consider the Koszul complex [4]

$$\cdots A \otimes {}^*(A^!_2) \to A \otimes {}^*(A^!_1) \to A \to k. \tag{3.3.1}$$

Recall the construction of the differential from [4, 2.8]. Identify $A \otimes {}^*(A^!_i) = \mathrm{Hom}_{-k}(A^!_i, A)$. Let $\mathrm{Id}_{A_1} = \sum a'_\alpha \otimes a_\alpha \in A^!_1 \otimes A_1$ under the canonical isomorphism $\mathrm{Hom}_{-k}(A_1, A_1) = A^!_1 \otimes A_1$. Observe that a_α (resp. a'_α) is a basis in A_1 (resp. $A^!_1$). Then the differential

$$d : A \otimes {}^*(A^!_{i+1}) \to A \otimes {}^*(A^!_i)$$

is constructed as follows: $(df)(a) = \sum f(aa'_\alpha)a_\alpha$ for $f \in \mathrm{Hom}_{-k}(A^!_{i+1}, A)$, $a \in A^!_i$.

Now, the entries of the Koszul complex are A-k-bimodules. Let $k = \prod_y k_y$ be a product of fields and let e_y be the identities in these fields (local idempotents). Let us fix $e \in k$ to be such a local idempotent. We can multiply the Koszul complex by e from the right.

$$\cdots A \otimes {}^*(A^!_2)e \to A \otimes {}^*(A^!_1)e \to Ae \to k_e, \tag{3.3.2}$$

where k_e is the field corresponding to e considered as a simple left A-module. This is a projective resolution of k_e.

Let M be a left module over A. Applying $\mathrm{Hom}(\_, M)$ to the complex (3.3.2) we get the complex

$$\cdots e(A^!_2) \otimes M \leftarrow e(A^!_1) \otimes M \leftarrow eM \tag{3.3.3}$$

calculating $\mathrm{Ext}^*_A(k_e, M)$ with the differential $d : e(A^!_i) \otimes M \to e(A^!_{i+1}) \otimes M$ given by $d(ea \otimes m) = \sum eaa'_\alpha \otimes a_\alpha m$ for $a \in A^!_i$ and $m \in M$.

For a Koszul ring we have an algebra isomorphism [4, Theorem 2.10.1]

$$\mathrm{Ext}^*_A(k, k) = (A^!)^{\mathrm{opp}}$$

which induces an isomorphism of k-k-bimodules (with switched left and right action)

$$\mathrm{Ext}^i_A(k, k) = A^!_i,$$

and the isomorphism

$$\oplus_y \mathrm{Ext}^i_A(k_e, k_y) = \mathrm{Ext}^i_A(k_e, k) = eA^!_i,$$

where $k = \prod_y k_y$.

Rewriting the complex (3.3.3) we get

$$(3.3.4)\quad \cdots \oplus_y (\mathrm{Ext}^2_A(k_e, k_y) \otimes_k yM) \leftarrow \oplus_y (\mathrm{Ext}^1_A(k_e, k_y) \otimes_k yM) \leftarrow eM.$$

3.4. **Flag varieties.** Now let us apply this general machinery to our situation. Recall that perverse sheaves on $X = G/P$ are modules over a Koszul algebra A and $k = \prod_{y \in W_{\mathcal{S}}} k_y$ where $W_{\mathcal{S}}$ is the set of Schubert strata, cf. [4]. Let e be the idempotent corresponding to the open Schubert stratum. Then, as a simple perverse sheaf $k_e = \mathbb{C}_X[\dim X]$ is the shifted constant sheaf on the variety X. Due to [4, Corollary 3.3.2] we identify

$$(3.4.1)\qquad \mathrm{Ext}^i_A(k_e, M) = \mathbb{H}^{i - \dim X}(M)$$

and in particular,

$$\mathrm{Ext}^i_A(k_e, k_y) = IH^{i - \dim X}(\overline{X}_y)$$

where X_y is the Schubert cell corresponding to the idempotent $y \in k$.

Thus we can rewrite the complex (3.3.4) as follows

$$(3.4.2)\quad \cdots \bigoplus_y (IH^{1-\dim X}(\overline{X}_y) \otimes_k yM) \leftarrow \bigoplus_y (IH^{0-\dim X}(\overline{X}_y) \otimes_k yM)$$

with the differential

$$d : \bigoplus_y (IH^i(\overline{X}_y) \otimes_k yM) \to \bigoplus_y (IH^{i+1}(\overline{X}_y) \otimes_k yM)$$

obtained as follows: $d(c_y \otimes ym) = (\sum c_y \cdot a'_\alpha \otimes a_\alpha ym)$ for $c_y \in IH^i(\overline{X}_y)$, $m \in M$ and $c_y \cdot a'_\alpha$ is the action of $\mathrm{Ext}^1_A(k,k)$ on $\oplus_y IH^i(\overline{X}_y)$.

Lemma. *The category of left A-modules is equivalent to the category of left k-modules M equipped with the action of the basis a_α of A_1 in such a way that the sequence of maps (3.4.2) is a complex i.e., $d^2 = 0$.*

Proof. For an A-module M (3.4.2) is a complex by construction. Suppose we are given a left k-module M equipped with the action of the basis a_α of A_1. Consider the degree one map d from $\oplus_y (IH^*(\overline{X}_y) \otimes_k yM)$ to itself as in (3.4.2), and consider

$$d^2(c \otimes m) = \sum (c \cdot (a'_\alpha a'_\beta) \otimes a_\beta a_\alpha m)$$

for $c \in \oplus_y IH^*(\overline{X}_y)$, $m \in M$. Since $\mathrm{Ext}^2_A(k,k) = (A_1 \otimes A_1)^*/R^\perp$ where $R \subset (A_1 \otimes A_1)$ is the set of relations of A and the representation of $\mathrm{Ext}^i_A(k,k)$ in $\oplus_y IH^*(\overline{X}_y)$ is faithful by the Theorem 3.2 we see that $d^2 = 0$ if and only if $\sum a_\beta a_\alpha \in R \subset (A_1 \otimes A_1)$. □

The complex (3.4.2) calculates the hypercohomology of M considered as a perverse sheaf due to the fact that the complex (3.3.2) is a projective resolution of k_e and the formula 3.4.1. The $d^2 = 0$ Theorem is proved.

3.5. **Example:** $\mathbb{P}^1$ [31]**.** Let $X = \mathbb{P}^1$ stratified by two strata $X = S_0 \sqcup S_1$ where $S_0 = \mathbb{C}^0 = \mathrm{pt}$, and $S_1 = \mathbb{C}^1$. We have

$$IH^*(\overline{S}_0) = IH^*(\mathrm{pt}) = \mathbb{C}[0],$$

where $\mathbb{C}[0]$ is the complex with zero differential and one-dimensional space in position 0, and

$$IH^*(\overline{S}_1) = IH^*(\mathbb{P}^1) = \mathbb{C}[1] \oplus \mathbb{C}[-1],$$

where $\mathbb{C}[1] \oplus \mathbb{C}[-1]$ is the complex with zero differential and two one-dimensional spaces in positions -1 and 1. Now the spaces

$$\mathrm{Hom}^1_{H^*(\mathbb{P}^1)}(IH^*(\mathbb{P}^1), IH^*(\mathrm{pt})) \quad \text{and} \quad \mathrm{Hom}^1_{H^*(\mathbb{P}^1)}(IH^*(\mathrm{pt}), IH^*(\mathbb{P}^1))$$

are one dimensional. We will schematically depict a non-zero element of the these spaces as

$$\begin{array}{ccc} & \mathbb{C}[0] & \\ \nearrow & & \\ \mathbb{C}[1] & & \mathbb{C}[-1] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & \mathbb{C}[0] & \\ & & \searrow \\ \mathbb{C}[1] & & \mathbb{C}[-1] \end{array}$$

respectively. Now we see that an IC-module may be depicted as the following diagram

$$\begin{array}{cccc} & & V_0 & \\ & a \nearrow & & \searrow b \\ & V_1 & & V_1 \\ \text{degree}: & -1 & 0 & 1 \end{array}$$

with the relation $d^2 = 0$ i.e., $b \circ a = 0$. Thus the category of IC-modules in this case is isomorphic to the category of quiver representations

$$\begin{tikzcd} V_0 \arrow[r, bend right, "b"'] & V_1 \arrow[l, bend right, "a"'] \end{tikzcd}$$

with the relation $b \circ a = 0$.

4. Some combinatorics of Schubert varieties

We start with some well known combinatorial preliminaries. In this section $X = G/B$ is a full flag variety associated to a simple algebraic group G over $\mathbb{C}$.

4.1. **Cohomology ring.** Let us denote $C = H^*(X)$. Recall that

$$C = H^*(G/B) = S(\mathfrak{h})/S_+(\mathfrak{h})^W.$$

Let B^- be the opposite Borel. Denote by

$$Y_w = B^- wB/B$$

the opposite Schubert variety associated to $w \in W$, and denote by σ_w the class $[Y_w]$ in $H^{2l(w)}(X)$. It is well known that the ring $H^{2l(w)}(X)$ is generated by the classes

$$\sigma_{s_i}, \qquad i = 1, \ldots, r,$$

where s_i are reflections with respect to simple roots and r is the rank of $\mathfrak{g} = \operatorname{Lie} G$. Moreover, we have the following Chevalley's formula, cf. [14], for multiplying the Schubert classes in $C = H^*(X)$

$$\sigma_{s_i} \cdot \sigma_w = \sum_{\substack{\alpha \in \Phi^+ \\ l(ws_\alpha) = l(w)+1}} \langle \check{\omega}_i, \alpha \rangle \frac{(\alpha_i, \alpha_i)}{(\alpha, \alpha)} \sigma_{ws_\alpha}, \tag{4.1.1}$$

where $\check{\omega}_i$ is the i-th fundamental coweight, and α_i is the i-th simple root, and $\langle \check{\omega}_i, \alpha_j \rangle = \delta_{ij}$.

4.2. Let s_i, $i \in \{1, \ldots, r\}$ be a simple reflection in W and let C^{s_i} be the algebra of invariants of s_i. It is a subalgebra of C. We have the following description, explained to us by A. Postnikov:

$$C^{s_i} = \operatorname{span}\{\sigma_w | ws_i > w, \ w \in W\}.$$

Moreover, for any s_i, $i \in \{1, \ldots, r\}$ we have

$$C = \sigma_{s_i} \cdot C^{s_i} \oplus C^{s_i} \tag{4.2.1}$$

4.3. Now let us denote

$$\mathbb{V}_w = IH^*(\overline{X}_w).$$

It is a graded C-module. Due to Soergel, we have the following inductive description of $\mathbb{V}_w$. Take a reduced decomposition $w = s_1 \ldots s_l$ Denote

$$C_{s_1 \ldots s_l} = C \otimes_{C^{s_l}} \otimes \ldots C \otimes_{C^{s_1}} \mathbb{C}.$$

The action of C of the leftmost tensor factor C equips $C_{s_1 \ldots s_l}$ with a C-module structure. By [34, 35] we know that

$$C_{s_1 \ldots s_l} = \mathbb{V}_w \oplus \bigoplus_{y < w} \mathbb{V}_y^{n(y)} \tag{4.3.1}$$

as a C-module.

Example. Type A_2.

(1) $w = s_1 s_2$: $C_{s_1 s_2} = \mathbb{V}_w$,
(2) $w = s_1 s_2 s_1$: $C_{s_1 s_2 s_1} = \mathbb{V}_w \oplus \mathbb{V}_{s_1}$, and $C_{s_2 s_1 s_2} = \mathbb{V}_w \oplus \mathbb{V}_{s_2}$.

4.4. The C-module $C_{s_1 \dots s_l}$ has a basis

$$a_l \otimes \dots \otimes a_1 \otimes 1, \tag{4.4.1}$$

where each

$$a_i = \begin{cases} \text{either} & 1, \\ \text{or} & \sigma_{s_i}. \end{cases}$$

In particular, $\dim C_{s_1 \dots s_l} = 2^l$. This basis is homogeneous:

$$\deg a_l \otimes \dots \otimes a_1 \otimes 1 = \deg a_l + \dots + \deg a_1,$$

where $\deg 1 = -1$ and $\deg \sigma_{s_i} = 1$. We know the action of

$$\sigma_{s_i},\ i = \{1, \dots, r\}$$

on this basis by induction, the decomposition 4.2.1 and the the Chevalley's formula (4.1.1).

4.5. We can now give a description of IC-modules purely in terms of combinatorics of the ring $C = H^*(X)$. Recall that for each $w \in W$ we have a *graded* C-module $\mathbb{V}_w = IH^*(\overline{X}_w)$ whose combinatorial description is given in 4.3. Now a stalk of an IC-module M on $X = G/B$ is a vector space M_w for every $w \in W$, and for every pair of $y, w \in W$ we have boundary maps

$$m(y, w) \in \mathrm{Hom}^1_C(\mathbb{V}_y, \mathbb{V}_w) \otimes \mathrm{Hom}(M_y, M_w)$$

subject to the usual Chain Complex Axiom (2.2.1). Let us denote this category of IC-modules by $\mathcal{A}_{\text{Schubert}}(G/B)$.

4.6. **Localization.** By the Beilinson-Bernstein Localization Theorem (used in in [2] to prove the Kazhdan-Lusztig conjecture [25], cf. [26, 13]) and The $d^2 = 0$ Theorem we obtain

Theorem. *The categories*

$$\mathcal{O}_0 \simeq \mathcal{P}_{\text{Schubert}}(G/B) \simeq \mathcal{A}_{\text{Schubert}}(G/B)$$

are equivalent.

5. Quiver algorithm

Let us keep the setup of the previous section. Using Theorem 4.6 we can calculate the relations between the arrows of the quiver for the category $\mathcal{O}_0$ in the following way.

5.1. For $y, w \in W$ we have exactly $\mu(y, w)$ arrows from y to w. Let us denote them by $a^1_{y,w}, \dots, a^{\mu(y,w)}_{y,w}$. (Note that $\mu(y, w)$ could be 0.) If we have a basis in the vector space

$$\mathrm{Hom}^1_{H^*(X)}(IH^*(\overline{X}_y), IH^*(\overline{X}_w)) = \mathrm{Hom}^1_C(\mathbb{V}_y, \mathbb{V}_w)$$

given by matrices

$$A^1_{y,w}, \dots, A^{\mu(y,w)}_{y,w} \in \mathrm{Hom}^1_C(\mathbb{V}_y, \mathbb{V}_w) \tag{5.1.1}$$

with respect to some homogeneous bases in $\mathbb{V}_y$ and $\mathbb{V}_w$, then all the relations between the arrows are encoded by the equation $d^2 = 0$ or more precisely the linear relations between paths of length 2 going from y to w are the matrix elements of the matrix

$$\sum_{\substack{z \in W \\ i \in \{1, \dots, \mu(z,w)\} \\ j \in \{1, \dots, \mu(y,z)\}}} a^i_{z,w} A^i_{z,w} \circ a^j_{y,z} A^j_{y,z} = 0,$$

where $\circ$ is the matrix multiplication.

5.2. Thus, all we need to get the relations are the matrices (5.1.1) spanning

$$\mathrm{Hom}^1_C(\mathbb{V}_y, \mathbb{V}_w).$$

Algorithmically, it is enough to find homogeneous bases in $\mathbb{V}_w$ for $w \in W$ with the explicit action of the generators

$$\sigma_{s_i}, \qquad i = \{1 \dots r\}$$

of C. Then we can take all linear maps of degree 1 between $\mathbb{V}_y$ and $\mathbb{V}_w$ and solve a system of linear equations to find those linear maps which commute with the action of (the generators of) C.

5.3. We will find a homogeneous basis in $\mathbb{V}_w$ by induction on the length of the element $w \in W$.

Step 0. First of all, if $w = 1$, then $V_1 = \mathbb{C}$.

Step 1. By induction, suppose that we already have homogeneous bases with the explicit action of (the generators of) C in $\mathbb{V}_y$ for $y < w$. Take a reduced decomposition $w = s_1 \dots s_l$. It is easy to see from Theorem 3.2 that

$$\mathrm{Hom}^0_C(\mathbb{V}_y, \mathbb{V}_w) = \begin{cases} \mathbb{C}, & y = w \\ 0, & \text{otherwise} \end{cases}$$

From that and the decomposition 4.3.1 we can find matrices (with respect to the induction-assumed basis in $\mathbb{V}_y$ and the basis (4.4.1) in $C_{s_1\ldots s_l}$) spanning the space

$$\mathrm{Hom}^0_C(V_y, C_{s_1\ldots s_l})$$

by considering degree 0 maps between V_y and $C_{s_1\ldots s_l}$ and making sure they commute with the action of the generators of C by solving a system of linear equations. Applying these matrices to the induction-assumed basis in $\mathbb{V}_y$ we get a homogeneous linearly independent system of vectors $v_1, \ldots, v_m$ in $C_{s_1\ldots s_l}$ spanning the subspace

$$Y = \bigoplus_{y<w} \mathbb{V}_y^{n(y)}.$$

Step 2. Now we have a C-module $U = C_{s_1\ldots s_l}$ with the basis (4.4.1) which we will number in an arbitrary way as $x_1, \ldots, x_{2^l}$, and its C-submodule

$$Y = \bigoplus_{y<w} \mathbb{V}_y^{n(y)}$$

with the basis $v_1, \ldots, v_m$ which we know in terms of the basis $x_1, \ldots, x_{2^l}$:

$$v_i = \sum_k \alpha_i^k x_k.$$

We need to find a basis in the C-module U/Y with the explicit action of (the generators of) C. This is a problem of computational commutative algebra algorithmic solutions to which are well known, cf. e.g. [16, Chapter 15].

Step 3. Proceeding by induction we find a homogeneous basis in $\mathbb{V}_w$, for all $w \in W$.

5.4. **Remark.** In order to improve the performance of an actual computer realization of this algorithm, one should use a number of shortcuts. For example, instead of using the module $C_{s_1\ldots s_l}$ to extract a basis in $\mathbb{V}_w$, we could use $C \otimes_{C^s} \mathbb{V}_{w'}$, where $w = w's$ and $l(w) = l(w') + 1$. (I am grateful to T. Braden for this and other suggestions.) The details of the realization and computational results will appear elsewhere.

5.5. **Remark: one quiver = all quivers.** Recall the Koszul agebra A underlying the category $\mathcal{O}_0 \simeq \mathcal{P}_{\mathrm{Schubert}}(G/B)$ from [4] and 3.4. The algebra A is generated over A_0 by A_1 and it is quadratic, that is the ideal of relations is generated by an A_0-A_0-bimodule

$$\mathrm{Rel} \subset A_1 \otimes_{A_0} A_1.$$

The quiver of the category $\mathcal{O}_0$ in this language is just a choice of basis in A_1 (arrows) and a choice of basis in the subspace Rel (relations between arrows). Once we have a basis in A_1 and a basis in Rel we can obtain any other bases in A_1 and Rel by linear transformations.

6. Weight filtration and the functor $\mathcal{H}$

In this section we construct a functor $\mathcal{H}$ from mixed perverse sheaves to mixed IC-modules.

6.1. Let $\mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$ be a mixed category of mixed perverse sheaves. Examples of such categories include [4, 4.4, 4.5] and [39, 10]. Let $v : \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X) \to \mathcal{P}_{\mathcal{S}}(X)$ be the degrading functor.

By [4, Lemma 4.1.2], cf. [3], every object $A \in \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$ has a unique finite decreasing weight filtration $W_\bullet = W_\bullet A$ such that $W_m A/W_{m-1}A$ is a pure object of weight m.

6.1.1. Suppose that we have a functor $K : \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X) \to C^b(\mathrm{Vec})$ to the category of bounded complexes of (graded) R-modules such that for $A \in \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$ the cohomology of $K(A)$ is the hypercohomology of A and the weight filtration of A induces a decreasing filtration

$$K(A) = K(A) \supseteq \cdots \supseteq K(W_m A) \supseteq K(W_{m-1}A) \cdots \tag{6.1.1}$$

on $K(A)$, and moreover, the cohomology of $K(W_m A)/K(W_{m-1}A)$ is the hypercohomology of $W_m A/W_{m-1}A$.

6.2. Assuming the statement of The $d^2 = 0$ Theorem, one could take the total complex of the corresponding (non-mixed) IC-module corresponding to $v(A)$ as the complex $K(A)$ for $A \in \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$. In particular, we can do it for simplicial complexes and flag varieties. We will fix this choice of K for the rest of this section.

Consider now the first term $E^1 = E^1(A)$ of the spectral sequence associated to the filtration (6.1.1). One could look at E^1 as a mixed IC-module. Indeed, for $A \in \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$ the stalks of the corresponding IC-module $V = \mathcal{H}(A)$ are determined from the formula

$$H^i\left(K(W_m A)/K(W_{m-1}A)\right) = \bigoplus_S V_S^{-m} \otimes IH^i(\overline{S})$$

and the boundary maps of V are the differentials of E^1. The total complex of V constructed in this way is precisely the diagonal complex of E^1.

Thus we have constructed a functor $\mathcal{H} : \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X) \to \mathcal{A}_{\mathcal{S}}^{\mathrm{mixed}}(X)$. Observe that the spectral sequence degenerates at the second term and so the total complex of the IC-module $\mathcal{H}(A)$ calculates the hypercohomology of A for $A \in \mathcal{P}_S^{\mathrm{mixed}}(X)$. In other words we have the following

Theorem. *Assuming the statement of* The $d^2 = 0$ Theorem, *the total complex of the mixed* IC*-module calculates the hypercohomology of the corresponding mixed perverse sheaf.*

6.3. Let $\widetilde{\mathcal{P}}(G/B)$ be the mixed category of mixed perverse sheaves considered in [4, 4.4]. Let $\widetilde{\mathcal{A}}(G/B)$ be the subcategory of the category $\mathcal{A}^{\mathrm{mixed}}(G/B)$ with coefficients in $\mathbb{Q}_l$ consisting of objects V such that

$$V_S^i = 0 \qquad \text{unless} \qquad \dim S = i \mod 2.$$

Then the functor $\mathcal{H} : \widetilde{\mathcal{P}}(G/B) \to \widetilde{\mathcal{A}}(G/B)$ provides the mixed version of the equivalence established in Section 3.

7. Further directions and final remarks

7.1. **A combinatorial challenge.** The basis in $\mathbb{V}_w$ we constructed in 5.3 is very non-canonical: it depends on the reduced decompositions of w we choose for each w, and on the way we number our bases. However, once we have found *one* basis in $\mathbb{V}_w$ we can obtain *any other* basis by a (grading-preserving) linear transformation.

It would be very interesting to construct a *distinguished* homogeneous basis in the C-module $\mathbb{V}_w$ for all $w \in W$ independent of any choices. The basis in question should specialize to the basis of Schubert cycles when the variety $\overline{X}_w$ is smooth, and the formulas for the action of C should generalize the Chevalley's formula (4.1.1).

7.2. **Moment graphs.** In [12] T. Braden and R. MacPherson give another construction of a module structure on $IH^*(\overline{X}_w)$ and $IH_T^*(\overline{X}_w)$, where the latter is the T-equivariant intersection homology. One could use their techniques to compute with IC-modules. It would also be very interesting to define T*-equivariant* IC-modules, perhaps related to T-equivariant perverse sheaves and *singular* blocks of $\mathcal{O}$.

7.3. **Parabolic and singular case.** In Section 5 we have considered only the full flag variety and the regular block of the category $\mathcal{O}_0$. However, the algorithm goes through without change for a parabolic flag variety G/P and the corresponding regular parabolic block of the category $\mathcal{O}$. The parabolic Chevalley's formula is known, cf. [17]. The quiver for a singular block can be obtained by Koszul (and thus quadratic) duality from the corresponding parabolic quiver, cf [4].

7.4. **Harish-Chandra modules.** In fact, in [18, Section 1] I. M. Gelfand asks for a quiver describing (a block of) the category of Harish-Chandra modules over an arbitrary simple group. Harish-Chandra modules reduce to category $\mathcal{O}$ in some cases [8]. We hope that our methods will provide an answer to that question as well, and help prove Soergel's Langlands duality conjecture, cf. [36]. (There are even more general representation categories, related to arbitrary Coxeter systems, for which the theory partially goes through, cf. [15].)

7.5. **Toric varieties.** First we introduce the *relative* IC-modules. Let $D(X) = D^b_c(X)$ be the bounded derived category of constructible sheaves on X. We could define a relative version of IC-modules by using another ∂-functor $D^b_c(X) \to D^b(A)$, where A is an abelian category, instead of hypercohomology. A good example of such a functor would be the derived functor $Rf_* : D^b_c(X) \to D^b_c(Y)$ associated to a map $f : X \to Y$.

It would be interesting to look at the relative IC-modules related to the Koszul category of perverse sheaves on toric varieties studied in [10]. Let X be a toric variety with the action of the torus T and let $\mu : X \to X/T$ be the quotient map. The relative IC-modules with respect to μ would still have vector spaces V_σ as stalks at each face σ, and elements of the space

$$\mathrm{Hom}^1_{\mu_* R_X}(\mu_*\mathbf{IC}(\sigma, V_\sigma), \mu_*\mathbf{IC}(\tau, V_\tau))$$

for any two faces (T-orbits) σ and τ as boundary maps, satisfying the $d^2 = 0$ axiom. (Here μ_* is the functor between derived categories, and R_X is the constant sheaf on X.) The objects $\mu_* R_X$, $\mu_*\mathbf{IC}_\sigma$, and $\mu_*\mathbf{IC}_\tau$ have a very nice combinatorial description, cf. [10, 1.3].

7.6. **Affine Grassmannians.** The $d^2 = 0$ Theorem trivially holds for the semisimple category of $G(O)$-equivariant perverse sheaves of $\mathbb{C}$-vector spaces on the affine Grassmannian $G(K)/G(O)$. It would be interesting to prove an analogous result for a suitable Koszul category of perverse sheaves of vector spaces over a field of positive characteristic.

7.7. **Saper's L-modules.** In a remarkable recent paper [33] L. Saper introduced the notion of L-modules, in order to prove a conjecture of Rapoport and Goresky-MacPherson. It seems that L-modules should be related to IC-modules on the (compactifications of) locally symmetric spaces. It would be very interesting to understand the relationship between these structures and perverse sheaves.

7.8. **Generalized H-modules.** If we take the usual homology instead of intersection homology in the definition of IC-modules, we will get the usual triangulation-constructible sheaves on simplicial complexes, cf. [24, 28, 39]. Thus we have a machine transforming a homology theory into an abelian (or, more generally, A_∞) category of "sheaves." It would be intriguing to consider IC-modules with an arbitrary (generalized) homology theory H instead of intersection homology.

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