On analytic transformation to Birkhoff standard form

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Abstract

It is proved that each irreducible linear system of differential equations can be analytically transformed to Birkhoff standard form

1 Introduction

Consider a linear system of differential equations

$$x\frac{dy}{dx} = A(x)y, \tag{1}$$

where $A(x)$ is a matrix of size (p,p) of the form

$$A(x) = x^r \sum_{n=0}^{\infty} A_n x^{-n}, \quad A_0 \neq 0, \quad r \geq 0, \tag{2}$$

x is a complex variable, and the power series converges in some neighborhood of ∞.

Under a transformation

$$z = \Gamma(x)y \tag{3}$$

system (1) is transformed to the system

$$x\frac{dz}{dx} = B(x)z, \tag{4}$$

where

$$B(x) = x\frac{d\Gamma}{dx}\Gamma^{-1} + x\Gamma A(x)\Gamma^{-1}. \qquad (5)$$

If $\Gamma(x)$ is holomorphically invertible in some neighborhood of ∞, then such a transformation is called *analytic.* If $\Gamma(x)$ is holomorphically invertible in some punctured neighborhood of ∞, and is meromorphic at ∞, then such a transformation is called *meromorphic.*

If the matrix $B(x)$ in (4) is a polynomial in x of the smallest possible degree, then (4) is called *a Birkhoff standard form for (1).*

Birkhoff [Bi] claimed that each system (1) can be analytically transformed to a Birkhoff standard form, but Gantmacher [Ga] presented a counterexample to this statement. It turned out that Birkhoff's proof was valid only for the case when a monodromy matrix of system (1) was diagonalizable.

Let us call system (1) *reducible* if there exists a holomorphically invertible in some neighborhood of ∞ matrix $\Gamma(x)$ such that under the transformation (3) system (1) is transformed to system (4) with an lower diagonal block matrix

$$B(x) = \begin{pmatrix} B_1 & 0 \\ * & B_2 \end{pmatrix}. \qquad (6)$$

For $p = 2$ Jurkat, Lutz, Peyerimhoff [JLP], and for $p = 3$ Balser [Ba] proved that *each irreducible system (1) (generic* system in terms of Balser's paper [Ba]*) can be analytically transformed to a Birkhoff standard form.* We prove here that the analogous result is valid for arbitrary p.

Theorem 1 *Each irreducible system (1) can be analytically transformed to a Birkhoff standard form.*

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2 Technical lemmas

To prove Theorem 1 we need the following statements.

Lemma 1 . *Suppose that the matrix $W(x)$ of size $(p-l,l)$ is holomorphic, and the matrix $Y(x)$ of size (l,l) is holomorphically invertible in a neighborhood O of the point 0. For any integer-valued diagonal matrix $C = diag(c_1,\dots,c_p)$, there exists a matrix-function $\Gamma(x)$, meromorphic on the whole Riemann sphere and holomorphically invertible off the point 0, such that*

$$\Gamma(x)x^C \begin{pmatrix} Y(x) \\ W(x) \end{pmatrix} = x^{C'} \begin{pmatrix} Y(x) \\ W'(x) \end{pmatrix}, \tag{7}$$

where $C' = diag(c_1,\dots,c_l,c'_{l+1},\dots,c'_p)$, $c'_j > \min(c_1,\dots,c_l)$, $j = l+1,\dots,p$, and the matrix $W'(x)$ is holomorphic in O ([Bo2]).

Proof. By t_l denote the rows of the matrix

$$x^C \begin{pmatrix} Y(x) \\ W(x) \end{pmatrix}.$$

Let $t_m = x^{c_m}w_m(x)$ be a row of this matrix such that $m > l$, $c_m \leq \min(c_1,\dots,c_l)$, $w_m(0) \neq 0$. Since the rows $y_1(0),\dots,y_l(0)$ of the matrix $Y(0)$ are linearly independent, we have $w_m(0) = -\sum_{j=1}^{l} d_j y_j(0)$. Hence , the row vector

$$t^1_m(x) = d_1 x^{c_m-c_1}\ t_1(x) + \ldots + d_l x^{c_m-c_l}\ t_l(x) + t_m(x) = \tag{8}$$

$$x^{c_m}(d_1 y_1(x) + \cdots + d_l y_l(x) + w_m(x))$$

has the form $t^1_m(x) = x^{c^1_m}\ w^1_m(x)$, where either $w^1_m(x) \equiv 0$ or $w^1_m(0) \neq 0$, $c^1_m > c_m$. If $w^1_m(x) \equiv 0$ or $w^1_m(0) \neq 0$, $c^1_m > \min(c_1,\dots,c_l)$, then we stop the procedure. If $c^1_m \leq \min(c_1,\dots,c_l)$ and $w^1_m(0) \neq 0$, then $w^1_m(0) = -\sum_{j=1}^{l} d^1_j\ y_j(0)$ and we again can consider the corresponding polynomial

$$t^2_m(x) = d^1_1 x^{c^1_m-c_1}\ t_1(x) + \ldots + d^1_l x^{c^1_m-c_l}\ t_l(x) + t^1_m(x)$$

and so on.

In all cases after a finite number of steps, we get $t^s_m(x) = x^{c'_m}\ w'_m(x)$, where $c'_m > \min(c_1,\dots,c_l)$ with holomorphic $w'_m(x)$. We consider the polynomials

$$Q^m_j = d_j x^{c_m-c_j} + d^1_j x^{c^1_m-c_j} + \ldots + d^{s-1}_j x^{c^{s-1}_m-c_j}$$

in $\frac{1}{x}$. By construction,

$$\sum_{j=1}^{l} Q_j^m t_j(x) + t_m(x) = x^{c_m{}'}\ w'_m(x), \quad m = l+1, \ldots, p\,.$$

One should substitute

$$\Gamma(x) = \begin{pmatrix} 1 & & & & & & \\ & \cdot & & & & & \\ 0 & & \cdot & & & & \\ \cdot & & & \cdot & & & 0 \\ \cdot & & & & \cdot & & \\ Q_1^{l+1} & \cdot\ \cdot\ \cdot & & & Q_l^{l+1} & 1 & \\ \cdot & \cdot\ \cdot\ \cdot & & & \cdot & 0 & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & & & \cdot & \cdot & \cdot \\ Q_1^p & \cdot\ \cdot\ \cdot & & & Q_l^p & 0 & \cdot\ \ 1 \end{pmatrix} \tag{9}$$

for the matrix $\Gamma(x)$ in (7). This concludes the proof of the lemma.

Remark 1 *It follows from the form (9) of* $\Gamma(x)$ *that for any holomorphic in* O *matrix* $Z(x)$ *of the size* (p, m) *the matrix*

$$\Gamma(x)x^C\ Z(x)$$

is still holomorphic in O.

The following statement was proved in [Bo1] for some special case.

Lemma 2 *Let a matrix* $U(x)$ *be holomorphically invertible in a neighborhood* O *of the point* 0 *and let all the principal minors of* $U(0)$ *be nonzero. Then for any integer-valued diagonal matrix* $C = diag(c_1, \ldots, c_p)$ *with the condition* $c_1 \geq \ldots \geq c_p$ *there exist a holomorphically invertible off* 0 *matrix* $\Gamma(x)$ *(which is a matrix of polynomials in* $\frac{1}{x}$*) and a holomorphically invertible in* O *matrix* $V(x)$ *, such that*

$$\Gamma(x)x^C\ U(x) = V(x)x^C. \tag{10}$$

Proof. Rewrite the matrix $x^C\ U(x)$ as follows:

$$x^C\ U(x) = x^{C - c_p I}\ U(x)x^{c_p I}, \tag{11}$$

where I is the identity matrix and apply Lemma 1 to the matrix

$$C = C - c_p I, \quad \begin{pmatrix} Y(x) \\ W(x) \end{pmatrix} = \begin{pmatrix} U^{p-1}(x) \\ W_1(x) \end{pmatrix}, \quad l = p-1,$$

where in turn $U^l(x)$ is formed by the intersections of the rows and columns of $U(x)$ with numbers $1, \dots, l$, and $W_l(x)$ is formed by the intersections of the rows with numbers $l+1, \dots, p$ and the columns with numbers $1, \dots, l$ of U.

By Lemma 1 there exists a matrix $\Gamma_1(x)$ of form (9) (with $l = p-1$), such that

$$\Gamma_1(x) x^{C - c_p I} \begin{pmatrix} U^{p-1}(x) \\ W_1(x) \end{pmatrix} = x^{C_1} \begin{pmatrix} U^{p-1}(x) \\ W_1'(x) \end{pmatrix}, \tag{12}$$

where $C_1 = \mathrm{diag}(c_1 - c_p, \dots, c_{p-1} - c_p, c_p^1)$, $c_p^1 > c_{p-1} - c_p$. Therefore,

$$x^{C_1} \begin{pmatrix} U^{p-1}(x) \\ w_p'(x) \end{pmatrix} = x^{C_1 - (c_{p-1} - c_p)I} \begin{pmatrix} U^{p-1}(x) \\ w_p'(x) \end{pmatrix} x^{(c_{p-1} - c_p)I_{p-1}},$$

where I_{p-1} is the identity matrix of the size $(p-1, p-1)$. It follows from (11) and the latter formula that the following factorization holds:

$$\Gamma_1(x) x^C U(x) = x^{C_1 - (c_{p-1} - c_p)I} \left(\begin{matrix} U^{p-2}(x) \\ W_2(x) \end{matrix} \middle| \, Z_1 \right) x^{D_1}, \tag{13}$$

where W_2 is a matrix of the size $(2, p-2)$, holomorphic in O , $D_1 = \mathrm{diag}(c_{p-1}, \dots, c_{p-1}, c_p)$, and by Remark 1 Z_1 is holomorphic in O too. Let apply Lemma 1 to the matrices

$$C = C_1 - (c_{p-1} - c_p)I, \quad \begin{pmatrix} Y(x) \\ W(x) \end{pmatrix} = \begin{pmatrix} U^{p-2}(x) \\ W_2(x) \end{pmatrix}, \quad l = p-2.$$

By Lemma 1 there exists $\Gamma_2(x)$ such that

$$\Gamma_2(x) x^{C_1 - (c_{p-1} - c_p)I} \begin{pmatrix} U^{p-2}(x) \\ W_2(x) \end{pmatrix} = x^{C_2} \begin{pmatrix} U^{p-2}(x) \\ W_2'(x) \end{pmatrix}, \tag{14}$$

where $C_2 = \mathrm{diag}(c_1 - c_{p-1}, \dots, c_{p-2} - c_{p-1}, c_{p-1}', c_p'')$, $c_{p-1}' > c_{p-2} - c_{p-1}$, $c_p'' > c_{p-2} - c_{p-1}$. Therefore,

$$x^{C_2} \begin{pmatrix} U^{p-2}(x) \\ W_2'(x) \end{pmatrix} = x^{C_2 - (c_{p-2} - c_{p-1})I} \begin{pmatrix} U^{p-2}(x) \\ W_2'(x) \end{pmatrix} x^{(c_{p-2} - c_{p-1})I_{p-2}}, \tag{15}$$

where I_{p-2} is the identity matrix of the size $(p-2, p-2)$. From (14) and (15) we get

$$\Gamma_2\Gamma_1(x)x^C U(x) = x^{C_2-(c_{p-2}-c_{p-1})I} \left(\begin{matrix} U^{p-3}(x) \\ W_3(x) \end{matrix} \middle| Z_2 \right) x^{D_2},$$

where W_3 is a matrix of the size $(3, p-3)$, W_3, Z_2 are holomorphic in O, $D_2 = \mathrm{diag}(c_{p-2}, \dots, c_{p-2}, c_{p-1}, c_p)$. And so on.

As a result after $p-1$ steps (the first two of which were described above) we obtain a matrix $\Gamma(x) = \Gamma_{p-1} \cdot \ldots \cdot \Gamma_1$, such that (10) holds with some holomorphic in O matrix $V(x)$.

Since

$$\det V(0) = \lim_{x\to 0} \det \Gamma(x) \det U(0) = \det U(0) \neq 0,$$

we obtain that $V(x)$ is holomorphically invertible at 0. (Here we used form (9) of each $\Gamma_i(x)$, which implies $\det \Gamma_i(x) \equiv 1$).

Lemma 3 *Let a matrix $U(x)$ be holomorphically invertible in a neighborhood $O$of the point 0. Then for any integer-valued diagonal matrix $C = diag(c_1, \dots, c_p)$ there exist a holomorphically invertible off 0 matrix $\Gamma(x)$ and a holomorphically invertible in O matrix $V(x)$, such that*

$$\Gamma(x)x^C U(x) = V(x)x^D, \tag{16}$$

where $D = diag(d_1, \dots, d_p)$ is obtained by some permutation of diagonal elements of the matrix C.

Proof. First let the diagonal elements of C be nonincreasing. With help of some constant nondegenerated matrix S we transpose the columns of the matrix $U(x)$ so that all principal minors of the new matrix $U' = US$ are not equal to zero. Applying Lemma 2 to U', we obtain

$$\Gamma(x)x^C U'(x) = V'(x)x^C,$$

therefore ,

$$\Gamma(x)x^C U(x) = \Gamma(x)x^C U'(x)S^{-1} = V'(x)x^C S^{-1} =$$

$$= V'(x)S^{-1}x^{SCS^{-1}} = V(x)x^D.$$

If the elements $c_1, \ldots, c_p$ are not ordered, then there exists a constant matrix S', such that $(S')^{-1}CS' = C'$, where $C' = \mathrm{diag}(c'_1, \ldots, c'_p)$ and $c'_1, \ldots, c'_p$ already form a nonincreasing sequence. For the matrix $x^{C'}\ (S')^{-1}U(x)$ consider the corresponding matrix $\Gamma'(x)$. In this case one can take the matrix $\Gamma = \Gamma'(S')^{-1}$ for the matrices C and $U(x)$ in (16).

3 Analytic transformation of an irreducible system

Now we have all we need to prove Theorem 1.

Proof of Theorem 1. Consider a fundamental matrix $Y(x)$ of system (1) of the form

$$Y(x) = M(x)x^E, \tag{17}$$

where $M(x)$ is a single-valued matrix function with nonvanishing $\det M(x)$ in some punctured neighborhood K of ∞, $E = \frac{1}{2\pi i}\log G$ has a Jordan normal form, G is a monodromy matrix of (1) in the basis of the columns of $Y(x)$.

Let F be arbitrary integer-valued diagonal matrix $F = \mathrm{diag}(f_1, \ldots, f_p)$ such that

$$f_1 \geq \cdots \geq f_p. \tag{18}$$

Treat the matrix $M(x)x^{-F}$ as the transition function of some vector bundle on the Riemann sphere P^1 with the coordinate neighborhoods $K \cup \{\infty\}$ and $P^1 \setminus \{\infty\}$. By Birkhoff-Grothendieck's theorem [OSS] there exist a holomorphically invertible in some neighborhood of ∞ matrix $T(x)$ and holomorphically invertible in complex plane matrix $U(x)$ such that

$$T(x)M(x)x^{-F} = x^C U(x), \tag{19}$$

where $C = \mathrm{diag}(c_1, \ldots, c_p)$, $c_i \in Z$, $c_1 \geq \cdots \geq c_p$. (This follows also from Sauvage's lemma, cf. [Ha]).

Proposition 1 *If system (1),(2) is irreducible, then for arbitrary integer-valued diagonal matrix F with condition (18) the following inequalities hold for the elements of the corresponding matrix C from (19):*

$$c_i - c_{i+1} \leq r, \quad i = 1, \ldots, p-1. \tag{20}$$

Proof. Assume that for some $k = 1, \dots, p-1$:

$$c_k - c_{k+1} > r. \tag{21}$$

Consider the system with the fundamental matrix

$$Y'(x) = x^C U(x) x^F x^E. \tag{22}$$

This system has only two singular points 0 and ∞ on the whole Riemann sphere (since $U(x)$ is holomorphically invertible everywhere except ∞) and its coefficient matrix $A' = x\frac{dY'}{dx}(Y')^{-1}$ has a pole of order r at ∞. The last statement follows from the fact that this system is obtained from the original system (1) by a transformation $T(x)$, which is analytic in some neighborhood of ∞.

On the other hand from (22) it follows that

$$A'(x) = C + x^C \left[\frac{dU}{dx}U^{-1} + U(F+L)U^{-1}\right] x^{-C}, \tag{23}$$

where $L = x^F E x^{-F}$. It follows from (18) and the fact that E has an upper-triangular form, that L is holomorphic on the whole complex plane. Therefore, the matrix in square brackets in (23) is holomorphic everywhere except the point ∞. Denote this matrix by $W(x)$. Then

$$A'(x) = C + x^C W(x) x^{-C}. \tag{24}$$

Since an element a'_{ij} of the matrix A' and an element w_{ij} of the matrix $W(x)$ are connected as follows

$$a'_{ij} = x^{c_i - c_j} w_{ij}, \quad i \neq j,$$

we obtain from assumption (21) that for $i = 1, \dots, k,\ j = k+1, \dots, p$ the following inequalities hold:

$$c_i - c_j > r.$$

Therefore, for every pair of such i, j the element a'_{ij} has a zero of order $m > r$ at 0 while an order of its pole at ∞ is less or equal to r. This means that

$$a'_{ij} \equiv 0, \quad i = 1, \dots, k,\ j = k+1, \dots, p$$

and therefore, the original system (1) is reducible . But this contradicts the assumption of the proposition. This contradiction means that equalities (20) hold true.

Let us continue the proof of Theorem 1. Consider a matrix F from (18) such that

$$f_i - f_{i+1} > r(p-1), \quad i = 1, \ldots, p-1 \tag{25}$$

and consider the corresponding matrices $T(x), C$ from (19). By Lemma 3, applyed to the matrix $x^C U(x)$ from (22) there exists a holomorphically invertible off ∞ matrix $\Gamma(x)$ such that (16) holds true.

Under the analytic (in some neighborhood of ∞) transformation

$$z = \Gamma(x)T(x)y \tag{26}$$

our original system (1) is transformed to system (4) with the fundamental matrix

$$Z(x) = \Gamma(x)T(x)Y(x) = V(x)x^{D+F}x^E, \tag{27}$$

where $V(x)$ is a matrix holomorphically invertible on the whole Riemann sphere except the point ∞.

It follows from Lemma 3 and Proposition 1 that for elements d_j of the integer-valued diagonal matrix D the following inequalities hold:

$$d_j - d_{j+1} \leq r(p-1), \quad j = 1, \ldots, p-1. \tag{28}$$

Indeed,

$$|d_j - d_{j+1}| \leq \max_j d_j - \min_i d_i = c_1 - c_p =$$

$$(c_1 - c_2) + (c_2 - c_3) + \cdots + (c_{p-1} - c_p) \leq r(p-1),$$

since D is obtained by some permutation of diagonal elements of C.

Thus, from (28) and (25) we get that the diagonal elements of the matrix $D+F$ are in nonincreasing order. Since the matrix E is upper-triangular, we again obtain that the matrix

$$L' = x^{D+F}Ex^{-D-F}$$

is the entire matrix function . Therefore ,

$$B(x) = x\frac{dZ}{dx}Z^{-1} = x\frac{dV}{dx}V^{-1} + V(D+F+L')V^{-1}$$

is entire matrix function too. This completes the proof of the theorem.

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