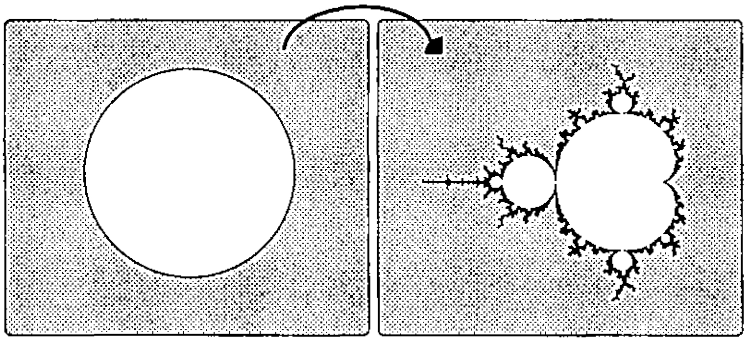
Computing the Laurent Series of the Map $\Psi: \mathbb{C}\text{-}\bar{D} \rightarrow \mathbb{C}\text{-}M$



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Abstract : We discuss several interesting properties of the Laurent series of $\Psi: C-\bar{D} \to C-M$, the inverse of the uniformizing map of the Mandelbrot set $M = \{c \in C : c, c^2+c, (c^2+c)^2+c, \ldots \not\to \infty \text{ as } n \to \infty\}$. Continuity of the Laurent series on ∂D implies local connectivity of M,which is an open question. We show how the coefficients of the series can be easily computed by following Hubbard and Douady's construction of the uniformizing map for M. As as result, we show that the coefficients are rational with powers of 2 in their denominator and that many are zero. Furthermore, if the series is continuous on ∂D, we show that it is not Hölder continuous. We also include several empirical observations made by Don Zagier on the growth of the power of 2 in the denominator.

Douady and Hubbard [DH], in demonstrating the connectedness of the Mandelbrot set, construct a conformal isomorphism $\Phi : C-M \to C - \bar{D}$, where M is the Mandelbrot set, $\bar{D}$ is the closed unit disk, and C is the complex plane. While visiting the Max Planck Institut für Mathematik in Bonn, F. Hirzebruch asked us if anything was known about the coefficients of the Laurent series of this map. Motivated by this question, we discuss several interesting properties of the Laurent series of the inverse map $\Phi^{-1} = \Psi: C - \bar{D} \to C-M$, including how its coefficients can be easily computed. It should be noted that convergence of this Laurent series on ∂D implies the conjecture that M is locally connected. With this in mind we discuss the rate of growth of the coefficients of the Laurent series. It is interesting to note that while M is surely the most complicated set ever studied and the square $S = \{z=x+iy : |x| \leq 1, |y| \leq 1\}$, for example, is one of the simplest , the Laurent series for Ψ is far easier to compute than the equivalent virtually intractable map for C-S .

In section 1 we describe Hubbard and Douady's proof that the Mandelbrot set is connected. The computation in section 2 is a more detailed version of the analysis found in [J]. Jungreis observed that many coefficients b_i of the Laurent series of Ψ are zero, and he proved that $b_{2^n} = 0$ for $n \geq 2$. We prove that $b_{(2k+1)2^n} = 0$ when $k \leq 2^n-3$ and that if P is any polynomial of degree d, then $P(\Psi(z))$ has no $1/z^{(2k+1)2^n}$ term when $d+k \leq 2^n-2$.

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§1 - Preliminaries.

We begin with definitions and an outline of Hubbard and Douady's proof of the connectedness of the Mandelbrot set. We then derive several simple estimates based on their proof.

For $c \in C$, let $P_c(z) = z^2+c$, $P_c^{\circ n}(z) = P_c(P_c(\ldots P_c(z) \ldots))$, n times, be the n-th iterate of $P_c(z)$, with $P_c^{\circ 0}(z) = z$. Also define $A_n(c) = P_c^{\circ n}(c)$ with $A_0(c) = c$. This implies

$$A_{n+1}(c) = [A_n(c)]^2 + c\,. \tag{1.1}$$

The Mandelbrot set is defined by $M = \{c \in C : | A_n(c) | \not\rightarrow \infty \text{ as } n \rightarrow \infty\}$. It is not hard to show that M is closed, has a trivial fundamental group and is contained in the closed disk of radius 2.

Theorem 1.0 (Douady and Hubbard) *There exists a conformal isomorphism* Φ:C-M $\rightarrow$ C - $\overline{D}$.

Proof. If X and Y are Riemann surfaces and $f:X \rightarrow Y$ is an analytic proper map of degree 1, then f is an isomorphism. We will construct a map Φ satisfying these conditions.

For a fixed c, we first find a map $\varphi_c(z)$ which will conjugate $P_c(z)$ to $z \rightarrow z^2$ near infinity in $\overline{C}$, the Riemann sphere. That is,

$$\varphi_c(P_c(z)) = [\varphi_c(z)]^2 \tag{1.2}$$

for z near ∞. We will then show that $\varphi_c(z)$ can be extended to $z = c$, and define $\Phi(c) = \varphi_c(c)$.

Equation (1.2) determines a formal Laurent series for $\varphi_c(z)$ which is unique if we restrict $\varphi_c(z)$ to be tangent to the identity at ∞. Proving convergence of the formal Laurent series is complicated, so we attempt to solve (1.2) via a scattering theory argument motivated by

$$\varphi_c(z) = \lim_{n \rightarrow \infty} [P_c^{\circ n}(z)]^{2^{-n}} \,. \tag{1.3}$$

If such a limit were well defined, it would clearly satisfy (1.2). However, the $^1/_{2^n}$-th root is not unique, so we define $\varphi_c(z)$ as follows, Write $T_0 = z$ and

$$T_n = \left(1 + \frac{c}{[P_c^{\circ(n-1)}(z)]^2}\right)^{\frac{1}{2^n}} \qquad \text{for } n = 1, 2, 3, \ldots$$

Claim 1.1 There exists a neighborhood U of ∞ such that for z in U the product $\prod T_i$ is well defined and converges to a function which is tangent to the identity at ∞.

It is clear that if $|z|$ is sufficiently large, each term in the product is 1 + something small, so that the root has a natural definition as the principal branch. It is easy to show that $\log |T_n|$ converges to 0 rapidly enough to insure convergence of the product, for instance $\log |T_n| = O(2^{-n})$.

We now define

$$\varphi_c(z) = \prod_{i=0}^{\infty} T_i \qquad \text{for } z \in U.$$

Claim 1.2 $\varphi_c(z)$ depends analytically on c, satisfies Eqn. (1.2) and is tangent to the identity at ∞.

If $S_n = \prod_{i=0\ldots n} T_i$, then

$$[S_n]^{2^n} = [T_0 T_1 \ldots T_n]^{2^n} = z^{2^n} (1 + c/z^2)^{2^{n-1}} (1 + c/(z^2+c)^2)^{2^{n-2}} \ldots$$
$$= z^{2^n} \left((z^2+c)/z^2 \right)^{2^{n-1}} \left(((z^2+c)^2+c)/(z^2+c)^2 \right)^{2^{n-2}} \ldots$$
$$= z^{2^n} \left(P_c^{\circ 1}(z)/z^2 \right)^{2^{n-1}} \left(P_c^{\circ 2}(z)/(P_c^{\circ 1}(z))^2 \right)^{2^{n-2}} \ldots$$

which is a telescoping product reducing to

$$= P_c^{\circ n}(z)$$

Hence $\varphi_c(z) = \prod T_i$ is a well defined version of Eqn. (1.3).

We now show that $\varphi_c(z)$ can be defined in a neighborhood of ∞ which contains c. Let

$$h_c(z) = \begin{cases} \log |\varphi_c(z)| & z \in U \\ 1/2^n \log |\varphi_c(P_c^{\circ n}(z))| & \text{when } P_c^{\circ n}(z) \in U \\ 0 & \text{when } P_c^{\circ n}(z) \notin U \text{ for all n.} \end{cases}$$

Claim 1.3 $h_c(z)$ is continuous.

Claim 1.4 $\varphi_c(z)$ can be extended analytically to a neighborhood of ∞ containing c if $c \in$ C-M .

Proof. Because $\varphi_c(z)$ is tangent to the identity near ∞, there exists a positive real number K such that for every $K' > K$ the set $\{z : h_c(z) > K'\}$ is a simply connected neighborhood of ∞ which is contained in U.

For $c \in$ C-M, $P_c^{\circ n}(0) \to \infty$ as $n \to \infty$. So there exists $K' = h_c(P_c^{\circ N}(0)) > K$ for some positive integer N. Let $U_0 = \{z : h_c(z) > K'\}$ and $U_1 = \{z : P_c(z) \in U_0\} = \{z : h_c(P_c(z)) > K'\}$. In general, define

$U_i = \{z : P_c^{\circ i}(z) \in U_0\} = \{z : h_c(P_c^{\circ i}(z)) > K'\}$. Then $U_N = \{z : h_c(z) > h_c(0) = {}^1/_{2^N} K'\}$. See figure 1.

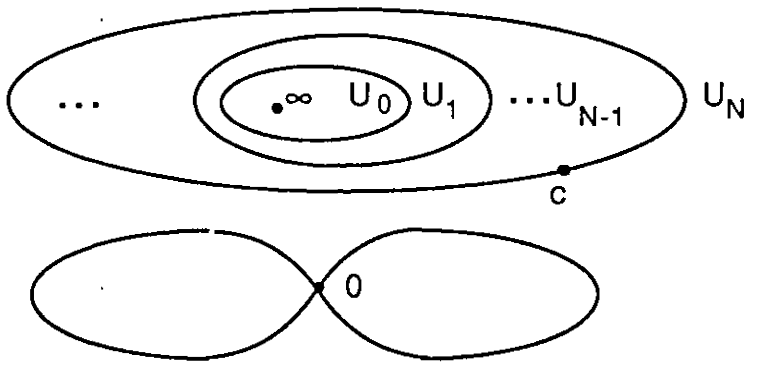
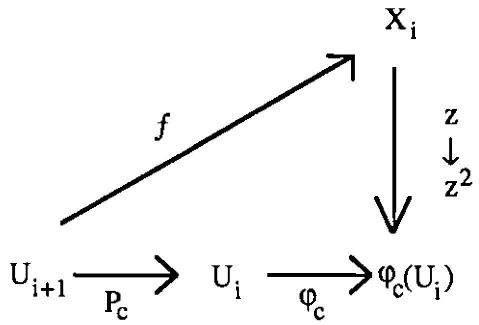


Figure 1

Zero and ∞ are the critical points of $P_c(z)$, so if $0 \notin U_{i+1}$ then $P_c : U_{i+1} \to U_i$ is a covering space ramified only at ∞. Thus if U_i is simply connected then U_{i+1} will also be simply connected. Let the map $z \to z^2 : X_i \to \varphi_c(U_i)$ be a double cover ramified at ∞. If we define $f(z)$ as a lift of the following diagram with base point $x \in U_i$ such that $\varphi_c(x) = f(x)$, then $\varphi_c(z) = f(z)$ for $z \in U_i \cap U_{i+1}$. In particular, $f(z)$ is tangent to the identity at ∞ and conjugates $z \to z^2$ with $P_c(z)$. Thus we define $\varphi_c(z) = f(z)$ on U_{i+1} .



Now $0 \notin U_i$ for $i \le N$, and so $\varphi_c(z)$ is well defined on U_N, and $c \in U_N$.

QED Claim 1.4

We now define the map Φ:C-M $\to$ C - $\overline{D}$ by $\Phi(c) = \varphi_c(c)$. Φ(C-M) is contained in C - $\overline{D}$, that is $|\varphi_c(c)| > 1$, because $|\varphi_c(c)|^{2^n} = |\varphi_c(P_c^{\circ n}(c))|$ which goes to infinity for $c \in$ C-M. Notice that near ∞, the map $\Phi(c)$ can be written $\prod \tau_i$ for i = 0,1,... where $\tau_0 = c$ and

$$\tau_n = \left(1 + \frac{c}{[A_{n-1}(c)]^2}\right)^{\frac{1}{2^n}} \quad \text{for } n = 1, 2, 3, \ldots$$

where the 2^{-n}-th root has a natural definition as the principal branch of the root. It is clear that the product converges near infinity since the terms rapidly converge to 1.

Claim 1.5 Φ : C-M $\to$ C - $\overline{D}$ is an analytic proper map.

Proof. The set $L = \{(c,z) : h_c(z) \le h_c(0)\}$ is closed. On C^2-L, $(c,z) \to \varphi_c(z)$ is a determination of the root $\varphi_c(P_c^{\circ n}(z))^{2^{-n}}$. This determination is analytic for sufficiently large n, and hence $\Phi(c)$ is analytic.

To show that Φ is proper we show that for any ε there exists a neighborhood V_ε of M such that for any $c \in V_\varepsilon$-M, $|\Phi(c)| < 1+\varepsilon$. Choose N such that $(312)^{2^{-N}} < 1+\varepsilon$, and let $V_\varepsilon = \{c : |P_c^{\circ N-1}(c)| < 10\}$ which contains M since $|P_c^{\circ N-1}(c)| \le 2$ for $c \in$ M.

Claim 1.6 If $c \in$ C-M with $|c| < 4$ and $|z| > 10$ then $|\varphi_c(z)| < 3|z|$.

This is a simple consequence of crude estimates based on the product expansion of $\varphi_c(z)$.

Now for every $c \in V_\varepsilon$-M there exists $n \geq N$ such that $|P_c^{on-1}(c)| < 10$ and $|P_c^{on}(c)| \geq 10$. Therefore,

$$10 \leq |P_c^{on}(c)| \leq |P_c^{on-1}(c)|^2 + |c| < 10^2 + 4.$$

And

$$|\varphi_c(P_c^{on}(c))| = |\varphi_c(c)|^{2^n} = |\Phi(c)|^{2^n} < 3\ |P_c^{on}(c)| < 3(10^2 + 4) = 312.$$

So $|\Phi(c)| < 1+\varepsilon$.

QED Claim 1.5

Finally, since $\Phi(c)$ is a proper map, it has a degree. The Laurent series of $\Phi(c) = \prod \tau_i$ begins with c + ... , so the degree of $\Phi(c)$ is 1. Thus $\Phi(c)$ is one-to-one.

QED Theorem 1.0

Based on this definition of $\Phi(c) = \prod \tau_i$ we now derive several estimates on its Laurent series. The map $\Phi(c)$ is naturally a limit of functions $\Phi_n(c) = \tau_0\tau_1 \ldots \tau_n$.

Claim 1.7 $\tau_n = 1 + {}^1/(2^n\, c^{(2^n-1)}) + \ldots$

Claim 1.8 The Laurent series expansions of $\Phi(c)$ and $\Phi_n(c)$ near ∞ have identical terms $c + a_0 + a_1/c + a_2/c^2 + \ldots + a_{k(n)}/c^{k(n)}$, with $k(n) = 2^{n+1}-3$.

Proof. $\Phi_n(c) = \tau_0\tau_1 \ldots \tau_n$ and $\Phi_{n+1}(c) = \tau_0\tau_1 \ldots \tau_{n+1}$ have this many terms in common, as is immediate from claim 1.7 .

QED Claim 1.8

§2 - Computing the Coefficients of Φ^{-1}.

In this section we show how the coefficients of the Laurent series of Φ^{-1} can be easily computed. Let $\Phi^{-1} = \Psi$: C - $\bar{D} \rightarrow$ C-M . We can compute the first k(n) terms of $\Psi(c)$ by inverting the first k(n) terms of $\Phi(c)$ or $\Phi_n(c)$. The latter lends itself to an easy algorithm :

We can write

$$[\Phi_n(c)]^{2^n} = [\tau_0\tau_1\ldots\tau_n]^{2^n} = c^{2^n}(1+{}^1/_c)^{2^{n-1}}(1+{}^c/_{(c^2+c)^2})^{2^{n-2}}\ldots$$
$$= c^{2^n}(\,{}^{(c^2+c)}/_{c^2}\,)^{2^{n-1}}(\,{}^{((c^2+c)^2+c)}/_{(c^2+c)^2}\,)^{2^{n-2}}\ldots$$
$$= c^{2^n}(\,{}^{A_1(c)}/_{c^2}\,)^{2^{n-1}}(\,{}^{A_2(c)}/_{(A_1(c))^2}\,)^{2^{n-2}}\ldots$$

which is a telescoping product leaving only one term $A_n(c)$. Then

$$[\Phi_n(c)]^{2^n} = A_n(c) = c^{2^n} + 2^{n-1}c^{2^{n-1}} + \ldots$$

and $\Phi(\Psi(z)) = z$ implies $\Phi_n(\Psi(z)) = z + O({}^1/_{z^{1+k(n)}})$ as $z \rightarrow \infty$.
So

$$[\Phi_n(\Psi(z))]^{2^n} = z^{2^n} + O({}^1/_{z^{(2^n-1)}}) = A_n(\Psi(z)) = \Psi(z)^{2^n} + 2^{n-1}\Psi(z)^{2^{n-1}} + \ldots \qquad (2.1)$$

We want to compute $\Psi(z) = z + b_0 + b_1/z + b_2/z + \ldots$

Suppose we know the constants $b_0, b_1, \ldots, b_{j-1}$ then substituting

$$\Psi(z) \approx z + b_0 + {}^{b_1}/_z + \ldots + {}^{b_j}/_{z^j}$$

into the right side of (2.1) gives a term ($2^n b_j$ + terms involving only $b_0, b_1, \ldots, b_{j-1}$) z^{2^n-j-1}. Since for $j \leq k(n)$ the coefficient of z^{2^n-j-1} is zero on the right side of (2.1), we can solve for b_j in terms of $b_0, b_1, \ldots, b_{j-1}$. Moreover, this shows that all the coefficients of the Laurent series of $\Psi(z)$ are rational with powers of 2 in their denominators.

As an example, we compute the first two terms of the Laurent series expansion. Since $k(1) = 2^{1+1}-3 = 1$, $n = 1$ is sufficient for this computation. Writing $\Psi(z) \approx z + b_0 + {}^{b_1}/_z$ and substituting this into the right hand side of (2.1), we get

$$P_\Psi(\Psi) = \Psi^2 + \Psi = z^2 + (2b_0+1)z + (2b_1 + (b_0)^2 + b_0) + \ldots = z^2 + O(1/z).$$

Showing that $b_0 = {}^{-1}/_2$ and $b_1 = {}^1/_8$.

In this way one can compute as many coefficients of Ψ as one wants, beginning

$$\Psi(z) = z - \frac{1}{2} + \frac{1}{8z} - \frac{1}{4z^2} + \frac{15}{128z^3} + \frac{0}{z^4} - \frac{47}{1024z^5} - \frac{1}{16z^6} + \frac{987}{32768z^7} + \frac{0}{z^8} + \cdots.$$

We know that all of the coefficients have denominators which are powers of 2, and it is reasonable to look at the powers of 2 occurring. Write $d_m = -v(a_m)$, where a_m is the coefficient of z^{-m} in $\Psi(z)$ and $v(\cdot)$ denotes 2-adic valuation. If $a_m = 0$, we set $d_m = -\infty$. The first values of d_m are given by the following table:

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
d_n	0	1	3	2	7	-∞	10	4	15	-∞	18	5	22	-∞	25	9	31	-∞	34	11	38	6	41

n	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40 ...
d_n	11	46	-∞	49	17	53	8	56	19	63	-∞	66	20	70	10	73	24	78	-∞

Based on more extensive data (up to about $m = 1000$), Don Zagier has made several empirical observations about the numbers d_m. To state them, we write $m = 2^n m_0$ with $n \geq 0$, m_0 odd. We use $A(k)$ to denote $v(k!)$; which can be computed recursively by $A(k) = [k/2] + A([k/2])$. Then Zagier's observations are as follows:

(i) $a_m = 0 \Leftrightarrow m_0 \leq 2^{n+1} - 5$; the first two non-zero coefficients with a given n (> 0) differ by a factor of 4 (except for $n = 0$). Thus

$$\begin{aligned} & a_2 = -\frac{1}{2^2}, \quad a_6 = -\frac{1}{2^4}, \\ a_4 = a_{12} = 0, \quad & a_{20} = -\frac{1}{2^6}, \quad a_{28} = -\frac{1}{2^8}, \\ a_8 = a_{24} = \cdots = a_{88} = 0, \quad & a_{104} = -\frac{33}{2^{14}}, \quad a_{120} = -\frac{33}{2^{16}}, \\ a_{16} = a_{48} = \cdots = a_{432} = 0, \quad & a_{464} = -\frac{334305}{2^{30}}, \quad a_{496} = -\frac{334305}{2^{32}}, \\ a_{32} = a_{96} = \cdots = a_{1888} = 0, \quad & a_{1952} = -\frac{238436656373197}{2^{62}}, \quad a_{2016} = -\frac{238436656373197}{2^{64}}. \end{aligned}$$

(ii) $d_m \leq A(2m+2)$ for all m with equality exactly when m is odd. Equivalently, $(2m+2)!a_m$ is always an integer and is congruent to m modulo 2.

(iii) As well as the closed formula $d_m = A(2m+2)$ for $n = 0$, one has (conjecturally) the following complete formula for $n = 1$ and partial formula for $n = 2$:

$$n = 1: \qquad d_m = A\left(\left[\frac{2m+2}{3}\right]\right) + \begin{cases} 0 & \text{if } m_0 \equiv -1 \pmod{12}, \\ 1 & \text{otherwise,} \end{cases}$$

$$n = 2: \qquad d_m = A\left(\left[\frac{2m-25}{7}\right]\right) + \epsilon(m_0)$$

(m_0 mod 28)	1	3	5	7	9	11	13	15	17	19	21	23	25	27
$\varepsilon(m_0)$	5	6	5	5	6	5	5	?	?	5	5	6	5	4

More generally, for each n there is apparently a partial periodicity with period $2(2^{n+1}-1)$ in m_0 (and hence $2^{n+1}(2^{n+1}-1)$ in m). In particular, if we write m_0 (uniquely) as $2(2^{n+1}-1)k + l$ with $k \geq 0$ and l odd, $1 \leq l \leq 2^{n+2} - 3$, then

$$d_m = A(2^{n+2}k) + \begin{cases} l - 1 & \text{if } l = 2^{n+2} - 3, \ k \text{ odd}, \\ l & \text{if } l = 2^{n+2} - 3, \ k \text{ even}, \\ l + 1 & \text{if } 2^{n+1} - 3 \leq l \leq 2^{n+2} - 5. \end{cases}$$

These formulas cover more than half of the values of m for given n.

We will prove half of statement (i), viz., the statement $a_{2^n m_0}$ for $m_0 \leq 2^{n+1} - 5$, below. It would probably also not be difficult to prove half of statement (ii), namely, the assertion that $(2m+2)!a_m$ is always integral.

§3 - Properties of the Laurent Series of Ψ.

In this section we prove that many of the coefficients of the Laurent series of Ψ are zero, which is a special case of the following more general theorem.

Theorem 3.0 If P(x) is a polynomial of degree d, then $P(\Psi(z))$ has no $1/z^{(2j+1)2^n}$ term when $d + j \leq 2^n-2$.

When P(x) = x, an immediate consequence is that the $(2j+1)2^n$-th coeficient of $\Psi(z)$ is zero when $0 \leq j \leq 2^n-3$.

The proof proceeds in several steps. The main idea is simple, and we demonstrate it first for j = 0 and P(x) = x. An observation which we will use repeatedly is that the derivative of a Laurent series has no 1/z term. We will denote the coefficient of the 1/z term in a Laurent series $\chi(z)$ by $\mathrm{Res}(\chi(z))$. Also we will denote $d/_{dz}P(z)$ by [P(z)]' or P'(z) .

Claim 3.1 If P(x) and Q(x) are polynomials and $\chi(z) = z + a_0 + a_1/z + \ldots$, then $Q(\chi(z))\cdot[P(\chi(z))]'$ has no 1/z term. That is $\mathrm{Res}(Q(\chi(z))\cdot[P(\chi(z))]') = 0$.

The following simple proposition demonstrates the ideas we use in the proof of Theorem 3.1 .

Proposition 3.2 $b_{2^n} = 0$.

Proof. By (2.1), $A_n(\Psi(z))\cdot\Psi'(z) = [z^{2^n} + O(1/z^{(2^n-1)})]\cdot\Psi'(z)$. And from claim 3.1 we have, $\mathrm{Res}(A_n(\Psi(z))\cdot\Psi'(z)) = 0$. But

$$\mathrm{Res}([z^{2^n} + O(1/z^{(2^n-1)})]\cdot\Psi'(z)) = -2^n\, b_{2^n}$$

when $n \geq 2$.

QED Proposition 3.2

Claim 3.3 $A_n(\Psi(z))$ satisfies

$$A_n(\Psi(z)) = z^{2^n} + {B_1(\Psi(z))}/{z^{2^n}} + {B_2(\Psi(z))}/{z^{3\cdot 2^n}} + \ldots + {B_j(\Psi(z))}/{z^{(2j-1)\cdot 2^n}} + \ldots$$

where $B_1(x) = -x/2$,
$B_2(x) = 1/2\,[B_1 - (B_1)^2]$,

$$B_3(x) = - B_1 \cdot B_2 ,$$
$$B_4(x) = {}^1/_2 [B_2 - (B_2)^2] - B_1 \cdot B_3 ,$$
$$B_5(x) = - B_1 \cdot B_4 - B_2 \cdot B_3$$

and in general

$$B_{2i} = {}^1/_2 [B_i - (B_i)^2] - B_1 \cdot B_{2i-1} - B_2 \cdot B_{2i-2} - \dots - B_{i-1} \cdot B_{i+1}$$

$$B_{2i+1} = - B_1 \cdot B_{2i} - B_2 \cdot B_{2i-1} - \dots - B_i \cdot B_{i+1}$$

Proof. Because $P_c(z)=P_c(-z)$, φ_c^{-1} is odd. Thus, for z large $\varphi_c^{-1}(z) = z + {}^{B_1(c)}/_z + {}^{B_2(c)}/_{z^3} + \dots$ for some analytic functions $B_i(c)$.

From $P_c(\varphi_c^{-1}(z))=\varphi_c^{-1}(z^2)$ we obtain

$$(z + {}^{B_1(c)}/_z + {}^{B_2(c)}/_{z^3} + \dots)^2 + c = z^2 + {}^{B_1(c)}/_{z^2} + {}^{B_2(c)}/_{z^6} + \dots$$

Solving for $B_i(c)$ gives the polynomials defined above.

Also, $z = \Phi(\Psi(z)) = \varphi_\Psi(\Psi(z))$ implies $\Psi(z) = \varphi_\Psi^{-1}(z)$. Substituting $c = \Psi(z)$ into

$$P_c^{\circ n}(\varphi_c^{-1}(z)) = \varphi_c^{-1}(z^{2^n}) = z^{2^n} + {}^{B_1(c)}/_{z^{2^n}} + {}^{B_2(c)}/_{z^{3 \cdot 2^n}} + \dots$$

yields

$$P_\Psi^{\circ n}(\varphi_\Psi^{-1}(z)) = \varphi_\Psi^{-1}(z^{2^n}) = z^{2^n} + {}^{B_1(\Psi)}/_{z^{2^n}} + {}^{B_2(\Psi)}/_{z^{3 \cdot 2^n}} + \dots$$
$$= P_\Psi^{\circ n}(\Psi(z))$$
$$= A_n(\Psi(z)).$$

QED Claim 3.3

Proof of Theorem 3.0 If $P(z) = az^d + \dots$, write

$$P(\Psi(z)) = az^d + \dots + {}^{c_0}/_{z^{2^n}} + \dots + {}^{c_j}/_{z^{(2j+1)2^n}} + \dots$$

We prove the theorem by induction on j.

For j=0 , (2.1) and claim 3.1 imply that

$$A_n(\Psi(z))\cdot[P(\Psi)]' = [\, z^{2^n} + O(1/z^{(2^n-1)})\,]\cdot[P(\Psi)]'$$

has no 1/z term. But when $d \le 2^n-2$,

$$\mathrm{Res}([z^{2^n} + O(1/z^{(2^n-1)})]\cdot[P(\Psi)]') = -2^n c_0 .$$

So $c_0 = 0$ when $d \le 2^n-2$.

Assuming that the hypothesis of the theorem holds for $j \le k-1$, we observe that $\mathrm{Res}(\, [A_n(\Psi(z))]^{2k+1} \cdot [P(\Psi)]'\,) = 0$, because

$$\begin{aligned}[A_n(\Psi(z))]^{2k+1} &= [z^{2^n} + B_1(\Psi(z))/z^{2^n} + \dots + B_k(\Psi(z))/z^{(2k-1)\cdot 2^n} + \dots\,]^{2k+1} \\ &= [z^{2^n} + B_1(\Psi(z))/z^{2^n} + B_2(\Psi(z))/z^{3\cdot 2^n} + \dots + O(1/z^{((2k+1)\cdot 2^n-k-1)})]^{2k+1} \\ &= z^{(2k+1)2^n} + z^{(2k-1)2^n}\, Q_1(\Psi) + z^{(2k-3)2^n}\, Q_2(\Psi) + \dots + z^{2^n}\, Q_k(\Psi) + O(1/z^{(2^n-k-1)})\end{aligned}$$

where $Q_i(\Psi)$ is a polynomial in $\Psi(z)$ of degree i arrising from the cross terms of the $B_j(\Psi)$.

We now compute the 1/z term of the last expression multiplied by $[P(\Psi)]'$

$$z^{(2k+1)2^n}\, [P(\Psi)]' + z^{(2k-1)2^n}\, Q_1(\Psi)\, [P(\Psi)]' + z^{(2k-3)2^n}\, Q_2(\Psi)\, [P(\Psi)]' + \dots + z^{2^n}\, Q_k(\Psi)\, [P(\Psi)]' + O(1/z^{(2^n-k-1)})\, [P(\Psi)]' ,$$

which is

$$\left.\begin{aligned}&\mathrm{Res}(\, z^{(2k+1)2^n}\, [P(\Psi)]'\,) + \mathrm{Res}(\, z^{(2k-1)2^n}\, Q_1(\Psi)\, [P(\Psi)]'\,) + \\ &\quad \mathrm{Res}(z^{(2k-3)2^n}\, Q_2(\Psi)\, [P(\Psi)]'\,) + \dots + \mathrm{Res}(\, z^{2^n}\, Q_k(\Psi)\, [P(\Psi)]'\,) + \\ &\quad \mathrm{Res}(\, O(1/z^{(2^n-k-1)})\, [P(\Psi)]'\,) .\end{aligned}\right\} \quad (3.6)$$

Now,

$$\mathrm{Res}(z^{(2k+1)2^n} [P(\Psi)]') = -(2k+1)\cdot 2^n c_k$$

and

$$\mathrm{Res}(z^{(2(k-i)+1)2^n} Q_i(\Psi) [P(\Psi)]') = 0 \qquad \text{for } i = 1, \ldots, k$$

because $Q_i(\Psi)\ [P(\Psi)]' = [R(\Psi)]'$ where R is some polynomial of degree i+d, and by induction, $R(\Psi)$ has no $1/z^{(2(k-i)+1)2^n}$ term. Finally,

$$\mathrm{Res}(O(1/z^{(2^n-k-1)})\ [P(\Psi)]') = 0$$

when $d \leq 2^n-2-k$.

Since (3.6) is zero, $c_k = 0$ when $d \leq 2^n-2-k$.

QED Proposition
QED Theorem 3.0

§4 - Ψ is not Hölder continuous.

Figure 2 shows a plot of the natural logarithm of the absolute value of the first 8000 coefficients of the Laurent expansion of $\Psi(z)$. At first glance, the coefficients appear to be bounded not only by $^1/_n$ but by $^1/_{n^{1+\varepsilon}}$. Such a bound would imply absolute convergence of the series and thus that the Mandelbrot set is locally connected.

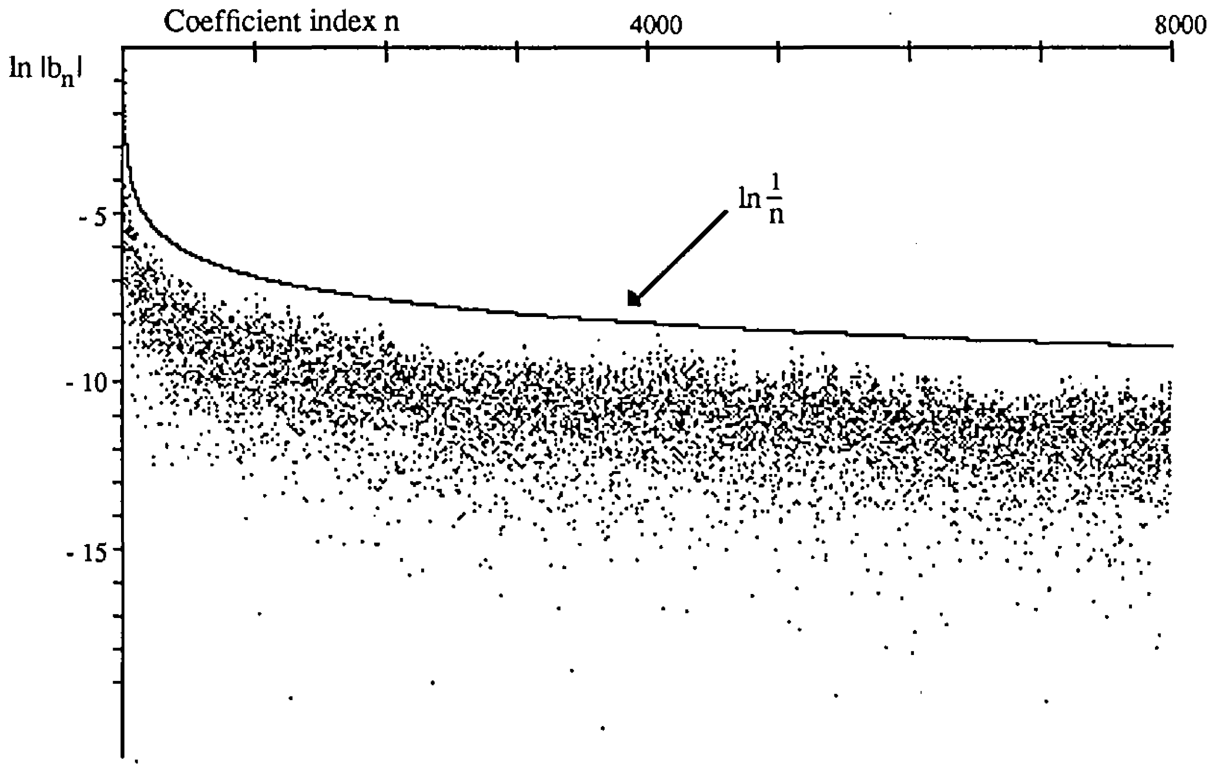


Figure 2. A log graph of the absolute value of the first 8000 coefficients of the Laurent series for $\Psi(z)$.

We now show that if $\Psi(z)$ extends continuously to ∂D then it is not Hölder continuous there. In particular, this implies that simple bounds on the coefficients of the Laurent series of $\Psi(z)$, such as $|b_n| < K/n^{1+\varepsilon}$, (K a positive constant) will fail because they imply Hölder continuity.

Claim 4.0 The points $c_n = {}^1\!/_4\, e^{2\pi i/n} - 1$ are contained in M and are the radial limits of the image by $\Psi(z)$ of the rays $re^{i\theta_n}$, $r > 1$, where $\theta_n = 2\pi({}^1\!/_3 + {}^1\!/_{(4^n-1)})$.

This is a consequence of several (not at all transparent) theorems which we will not prove here. The interested reader is referred to the articles by Douady and Hubbard [D],[DH].

If $\Psi(z)$ extends continuously to ∂D then if $\theta_\infty = {}^{2\pi}\!/_3$, we have

$$| \Psi(e^{i\theta_n}) - \Psi(e^{i\theta_\infty}) | = | c_n - {}^{-3}/_4 | = {}^1/_4 \, | e^{2\pi i/n} - 1 | > {}^{\pi}/_{4n} > {}^{-C}/_{\log |\theta_n - \theta_\infty|}$$

for some positive constant C. Specifically, the coefficients of the Laurent series $\Psi(z) = z + b_0 + {}^{b_1}/_z + {}^{b_2}/_z + \ldots$ cannot satisfy $|b_n| < {}^K/_{n^{1+\varepsilon}}$, since it is not hard to show that this condition on the coefficients implies that $| \Psi(e^{i\theta_n}) - \Psi(e^{i\theta_\infty}) | < C \, | \theta_n - \theta_\infty |^{\varepsilon}$ for some $\varepsilon > 0$ and some constant C.

Remark : We conjecture that $\Psi(z)$ extends continuously to ∂D and that

$$| \Psi(e^{i\theta_1}) - \Psi(e^{i\theta_2}) | < {}^{-C'}/_{\log |\theta_1 - \theta_2|}$$

is the modulus of continuity of $\Psi(z)$, for some positive constant C'. If $|b_n| < {}^K/_{n \log^2 n}$ then the series would converge absolutely with this modulus of continuity, but computation of the coefficients suggests that no such bound holds.

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