"On Finite Domination and Simple Homotopy Type of Nonsimply–Connected G–spaces"

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Introduction. Suppose W is a compact manifold with $\pi_1(W) = \pi$ and G is a finite group, and we wish to construct a free G–action on W with certain desired properties. In the relative version, we may be given, in addition, a submanifold $V \subset W$ which has a G–action $\varphi : G \times V \longrightarrow V$ and we require that the G–action $\psi : G \times W \longrightarrow W$ restricts to φ on V, i.e. $\psi | G \times V = \varphi$. This is the extension problem, considered in [AV], for example. (See [Wr] for a survey and further examples and applications.) In [AV] and [Wr], the extra condition is that G should act trivially on homology, and $\pi_1(W/G) = \pi \times G$.

One systematic approach to construct such group actions is the following. Using some homotopy theoretic tools, one constructs a space X (with $\pi_1(X) = \Gamma$, where Γ fits into an exact sequence $1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$) such that the regular covering space of X

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with $\pi_1(\tilde{X}) = \pi$, is homotopy equivalent to the given W. Thus G acts freely on a space $\tilde{X}$ homotopy equivalent to W and the problem reduces to the following:

(a) Find a finitely dominated space Y which is homotopy equivalent to X.
(b) Find a finite complex K homotopy equivalent to Y and such that the induced covering $\tilde{K}$, with $\pi_1(\tilde{K}) = \pi$, is π–simple homotopy equivalent to W.

Once K is found, the problem is reduced to surgery theory; that is, to find an appropriate normal invariant and to show that the surgery obstruction vanishes for a suitable choice of the normal invariant. Of course, it could happen that at some stage there is an obstruction and one does not succeed to carry out this procedure. The question arises then, as to how to measure such obstructions, and how to express them in terms of the topological or other invariants of W.

In this paper, we consider problems (a) and (b) above in a fairly general setting. Namely, in Section 1, we discuss the problem of finite domination of nonsimply–connected free G–spaces for any finite group G, and we show that the question reduces to the case of $G = \mathbb{Z}/p\mathbb{Z}$, where p is a prime. This reduction is a significant step in computations, and we illustrate this by an application.

Next, we address the problem (b) above in Section 2, and formulate the appropriate obstruction group $Wh_1^T(\pi \longrightarrow \Gamma)$ from an algebraic point of view, following our earlier treatment in [AV] for the case $\Gamma = \pi \times G$ and $Wh_1^T(\pi \subset \pi \times G)$. As in [AV], this abelian group is defined as the Grothendieck group of a certain category of projective modules. Wh_1^T is closely related to the functor Wh_1 and $\tilde{K}_0$ via a five–term exact sequence involving transfer homomorphisms. It is fair to say that Wh_1^T plays the same role in the nonsimply–connected cases that $\tilde{K}_0$ does in the simply–connected ones (at least in construction and classification problems of group actions).

This generalization of Wh_1^T as a functor of extensions $1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$

(rather than pairs of groups $\pi \subset \pi \times G$) suggests a further generalization. Namley, to define Wh_1^T on the space level, as it has been the current trend ever since Hatcher's "Whitehead spaces" [H]. The recent developments in geometric topology, in particular surgery theory, have put more emphasis on spaces rather than their corresponding algebraic objects, and this has proved quite fruitful. Moreover, the topological construction has a wider domain of definition than the algebraic one, and naturally it is expected to have more topological applications. Thus we have included a discussion of this point of view in Section 3, based on Hatcher's and Waldhausen's theories [H] [W] with suitable modifications. While the semi-simplicial language is more natural in this context, we have chosen to informally discuss the matters in the topological category leaving the details and applications for a future opportunity.

SECTION ONE. FINITE DIMENSIONALITY AND FINITE DOMINATION. Recall that a topological space is called finitely dominated if there exists a finite CW complex K and a map $f: K \longrightarrow X$ which has a right homotopy inverse $r: X \longrightarrow K$. A CW complex X is said to be of finite type if every finite dimensional skeleton of X is a finite complex, i.e. X has finitely many cells in each dimension. It is easy to see that a finitely dominated complex is homotopy equivalent to a finite dimensional CW complex with finitely generated total homology. An algebraic criterion for finite dimensionality of complexes can be formulated with the help of the following result of [Wall II]:

1.0. Theorem ([Wall II] p. 137, Theorem 6). A projective positive chain complex C_* is chain hmotopy equivalent to an n–dimensional complex if and only if $H_i(C_*) = 0$ for $i > n$ and $\mathrm{Im}\,(d : C_{n+1} \longrightarrow C_n)$ is a projective module.

With the help of ([Wall II] Theorem 2) as well as related results of [Wall I], one can translate the above mentioned finite dimensionality criterion of [Wall II] into the following: A CW complex X with bounded $H_*(X;\mathbb{Z}\pi_1(X))$ is homotopy equivalent to a finite dimensional complex if and only if for $C_* \equiv C_*(X;\mathbb{Z}\pi_1(X))$ (= cellular chain complex of the universal covering space $\tilde{X}$ of X) and some sufficiently large n, $\mathrm{Ker}(d : C_{n+1} \longrightarrow C_n)$ is a projective $\mathbb{Z}\pi_1(X)$–module. Clearly if the latter condition holds for some large n, then it holds for all $m \geq n$.

The passage from finite dimensionality to finite domination for spaces is technical in general. However, for applications to manifolds etc., the results of [Bieri–Eckmann] and especially, Browder's theorem ([Browder] Corollary 2) are quite useful. Namely, if X is a Poincaré space with finitely presented fundamental group, then X is finitely dominated. In our circumstances, we apply Browder's theorem in conjunction with Wall's finite dimensionality criterion as follows. Refering to the notation and the set–up of the introduction, suppose we have constructed an infinite dimensinal (as it happens in most homotopy theoretic constructions) space X with $\pi_1(X) = \Gamma$, such that the finite group G operates freely on the regular covering space $\tilde{X}$ with $\pi_1(\tilde{X}) = \pi$. First, we give a finite dimensionality criterion for $\tilde{X}$ to be G–homotopy equivalent to a finite dimensional free G–complex $\tilde{Y}$ (in terms of restrictions to suitable subgroups of G). Next, we pose the hypothesis that $\tilde{X}$ is a Poincaré complex, so that $\tilde{Y}$ becomes a Poincaré complex as well. Now $\tilde{Y}/G = Y$ is seen to satisfy Poincaré duality, since $\dim \tilde{Y} < \infty$. This shows that X satisfies Poincaré duality, hence, by Browder's theorem X is finitely dominated and step (a) of the Introduction is carried through. (See [A3] also.)

For simplicity of exposition, and without loss of generality as far as applications to compact manifolds are concerned, we assume that π is a finitely presented discrete group, and we work in the category of CW complexes of finite type and cellular maps. Let G be a finite group, and let $\Gamma = \Gamma(G)$ be a discrete group satisfying the exact sequence $\eta(G)$:

$\underline{\eta(G)}$: $$1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

Let $X = X(G)$ be a connected space with $\pi_1(X) = \Gamma$. Denote by $\tilde{X}$ the universal covering space of X. For each subgroup $H \subseteq G$, we consider the corresponding exact sequence $\eta(H)$:

$\underline{\eta(H)}$: $$1 \longrightarrow \pi \longrightarrow \Gamma(H) \longrightarrow H \longrightarrow 1$$

We set $X(H) = \tilde{X}/\Gamma(H)$ which is a covering space of X and $\pi_1(X(H)) = \Gamma(H)$. Thus $X(1)$ is a free G–space which is of interest to us, and $X = X(G) = X(1)/G$; $X(1)$ is homotopy equivalent to the given manifold W in consideration. Our first result in this direction is of local–to–global nature. Namely, we show that the problem of finite dimensionality of a free G–space (up to G–homotopy) can be decided by restricting to elementary abelian subgroups of G.

1.1. Theorem. (a) Let X be a CW complex with $\pi_1(X) = \Gamma$ as above. Then X is homotopy equivalent to a finite dimensional complex if and only if X(A) is homotopy equivalent to a finite dimensional complex for all elementary abelian p–subgroups of G and all primes p.

(b) Further, suppose that X(1) satisfies Poincaré duality. Then X is finitely dominated if and only if X(A) is finitely dominated for all elementary abelian p–subgroups of G and all p.

Before giving the proof of 1.1, we state the following conjecture and a supporting theorem.

1.2. Conjecture. Let G be a nontrivial group. (a) Let X be a CW complex with $\pi_1(X) = \Gamma$ as above. Then X is homotopy equivalent to a finite dimensional complex if and only if X(C) is homotopy equivalent to a finite dimensional complex for all cyclic subgroups $C \subseteq G$ of prime order.

(b) Suppose further that X(1) is a Poincaré complex. Then X is finitely dominated if and only if X(C) is finitely dominated for all $C \subseteq G$ of prime order.

We have proved this conjecture for the cases where Γ is a finite group, or more generally where for some n sufficiently large, $\mathrm{Ker}(d : C_n \longrightarrow C_{n-1})$ is a finitely presented $\mathbb{Z}\Gamma$–module, where $C_* = C_*(\tilde{X})$ = cellular chains of the universal covering space (Assadi [A3]).

We need some auxiliary algebraic concepts first. Recall that an $R\Gamma$–module M is called $(R\Gamma,R\pi)$–projective, if there exists an $f \in \mathrm{Hom}_{R\pi}(M,M)$ such that $\sum_{g \in G} gf(g^{-1}x) = x$ for all $x \in M$. This is a generalization of the concept of "weakly projective" (cf. [Cartan–Eilenberg]).

1.3. Lemma ([Rim] Proposition 2.2) An $R\Gamma$–module M is $(R\Gamma,R\pi)$–projective if and only if the Tate cohomology $\hat{H}(G;\mathrm{Hom}_{R\pi}(M,M))$ is trivial.

1.4. Lemma. An $R\Gamma$–module M is $R\Gamma$–projective if and only if M is $(R\Gamma,R\pi)$–projective and $R\pi$–projective.

Proof of Theorem 1.1. Note that if X is finitely dominated, then so are all finite covering spaces of X. Therefore, the non–trivial direction is to pass from elementary abelian p–groups to the group G itself. Assume that for such a $A \subseteq G$, $X(A)$ is homotopy equivalent to a finite dimensional complex. In particular, this holds for the covering space $X(1)$ with $\pi_1(X(1)) = \pi$. Consider the cellular chains $C_* = C_*(\tilde{X})$ of the universal cover $\tilde{X}$, which is a free finitely generated $\Gamma(G)$–complex, and let $d : C_* \longrightarrow C_*$ be its boundary homomorphism. Choose m large enough so that (C_*,d_*) is exact in all dimensions $n \geq m$. By Wall's Theorem above $M \equiv \mathrm{Ker}(d_n : C_n \longrightarrow C_{n-1})$ is $\mathbb{Z}\pi$–projective, and by our standing hypotheses, it is finitely generated. Further, for every p–elementary abelian group $A \subseteq G$, $\mathrm{Ker}\, d_n$ is $\mathbb{Z}\Gamma(A)$–projective, since $X(A)$ is homotopy equivalent to a finite dimensional complex. By Lemmas 1.3 and 1.4, $\hat{H}(A;\mathrm{Hom}_{\mathbb{Z}\pi}(M,M)) = 0$. By Chouinard's Theorem (cf. [Jackowski] Theorem 3.1 for a topological proof, or [Chouinard]), the G–module $\mathrm{Hom}_{\mathbb{Z}\pi}(M,M)$ is cohomologically trivial in the sense of Tate–Nakayama (see [Rim]). Since this module is also $\mathbb{Z}$–free, it is $\mathbb{Z}G$–projective. By Lemma 1.4, M is $(\mathbb{Z}\Gamma,\mathbb{Z}\pi)$–projective. Hence M is $\mathbb{Z}\Gamma$–projective. The proof of (b) follows from Browder's theorem ([Browder] Corollary 2), once we observe that if a finite group G acts freely on a finite dimensional Poincaré duality complex, then the orbit space also satisfies Poincaré duality. (See e.g. Gottlieb, Proc. AMS 76 (1979) 148–150 or Quinn. Bull AMS 78 (1972) 262–267.)

■

To indicate how the above theorem may apply to prove the above conjecture 1.2, we consider the special case where W is a closed Poincaré complex of dimension four with a finite fundamental group.

1.5. Theorem. Let $X(1)$ be a (possibly) infinite dimensional free G–space, where G is any finite group. Let $\pi_1(X(1)) = \pi$ be a finite group, and assume that non–equivariantly $X(1)$ is homotopy equivalent to a finite 4–dimensional Poincaré complex. Assume that $\tilde{X}$ is the universal covering space on which Γ acts freely, and as before, $1 \longrightarrow \pi \longrightarrow \Gamma \xrightarrow{\theta} G \longrightarrow 1$ is exact. Then the following are necessary and sufficient for X to be finitely dominated. For each $C \subseteq \Gamma$, $|C| = \text{prime}$ and $C \cap \pi = 1$:

(I) The spectral sequence of the Borel construction $E_C \times_C \tilde{X} \longrightarrow BC$ does not collapse.

(II) $\dim_{\mathbb{F}_p} H^1(C;H^2(\tilde{X})) \geq 2$ when $|C| = p$.

Outline of the proof: Suppose X is finitely dominated. Then one verifies directly that some differentials in the spectral sequence of the indicated fibration must be non–trivial in order that the total cohomology $H^*(X)$ be finitely generated. Thus (I) follows. As for (II), again a direct computation with chain complexes shows that $H^2(\tilde{X}) \cong I \oplus I \oplus P$ where I is the augmentation ideal of $\mathbb{Z}C$ and P is $\mathbb{Z}C$–projective. To prove that these conditions are sufficient, we apply Theorem 1.1 above to reduce the problem to the case of a p–elementary abelian group. The strategy is to reduce the problem to the case $G = \mathbb{Z}_p$. First, we note that we need to prove a finite dimensionality statement in view of the proof of Theorem 1.1. Secondly, observe that the reduction from G to $\mathbb{Z}_p$ is a special case of Conjecture 1.2 above, which we formulate as follows.

1.6. Lemma. In the situation of 1.5, X(1) is G–homotopy equivalent to a finite dimensional free G–complex, if and only if for each prime order subgroup $C \subseteq \Gamma$, $\tilde{X}/C$ is homotopy equivalent to a finite dimensional complex.

We postpone the outline of proof of this Lemma, and proceed to prove 1.5. First, notice that for any $C \subset \pi$, $|C|$ = prime, conditions (I) and (II) of the theorem are satisfied, since X(1) is homotopy equivalent (non–equivariantly) to a finite dimensional complex. Therefore, by Lemma 1.6, we are reduced to the case $G = \mathbb{Z}_p$ and $\pi_1(X(1)) = 1$, and we need to show that (I) and (II) imply the desired finite dimensionality result. Here, we use the notion of "free equivalence" of [A1]. Namely, finite dimensionality (up to equivariant homotopy) is preserved under "free equivalence" of G–spaces and G–complexes (cf. [A4] also).
This translates into:

1.7. Lemma. Let $\pi_1(X(1)) = 0$, in the above notation. Let X' be a free G–complex obtained from X(1) by adding free orbits of G–cells of dimension 3 and 4 so that $\pi_i(X') = 0$ for $i \leq 3$. Then X'/G is homotopy equivalent to a finite dimensional complex if and only if X is homotopy equivalent to a finite–dimensional complex. (In view of 1.6 above, we may take $G = \mathbb{Z}_p$ here and in 1.8 below for simplicity, although this restriction is not necessary.)

This is a special case of a more general result in [A2], and we leave out the proof. Next, we reduce the problem to cohomology computations, taking advantage of the fact that $H^*(\mathbb{Z}_p)$ is periodic in positive dimensions.

1.8. Lemma. Let X' be as above. Then X'/G is homotopy equivalent to a finite dimensional complex if and only if $H_G^i(X') = 0$ for $i \geq 5$, and this happens if and only if $H_G^i(X(1)) = 0$ for $i \geq 5$.

The proof of this lemma is computational, using the spectral sequence $H^i(BG, H^j(X')) \Rightarrow \mathscr{G}r\,(H_G^{i+j}(X'))$.

We further compute that in the spectral sequence of $E_G \times_G X(1) \longrightarrow BG$, if the differentials $d_3^{j,2}$ do not identically vanish, then $E_\infty^{j,0} = 0$ for all $j \geq 3$. Further, if $d_3(\zeta) = 0$ in $E_3^{i+3,0}$, then $\zeta \in \text{Image}\,(d_3 : E_3^{i-3,4} \longrightarrow E_3^{i,2})$. This implies, of course, that $E_r^{i,2} = 0$ for $i \geq 2$ and $r \geq 4$. The proof of the latter statement is based on the periodicity of $H^*(G)$ and the multiplicative properties of the spectral sequence. Another computational point is that if d_3 is not identically zero on $E_3^{j,2}$, then $H^{2i}(G, H^2(X(1))) = 0$ for $i > 0$. Putting all these together, it follows that $E_\infty^{i,j} = 0$ when $i+j > 4$, and the theorem follows from 1.8. ■

It remains to indicate the proof of Lemma 1.6.

Outline of the proof of Lemma 1.6. Consider the free Γ–space $\tilde{X}$, and assume that $\tilde{X}/C$ is homotopy equivalent to a finite dimensional complex. Let $M = \text{Ker}(d_n : C_n \longrightarrow C_{n-1})$, where $C_* = C_*(\tilde{X})$ as before, for some sufficiently large n. Using Theorem 1.0 above ([Wall II]), the hypotheses imply that M is $\mathbb{Z}C$–projective for all prime order cyclic subgroups $C \subset \Gamma$. We want to show that M is $\mathbb{Z}\Gamma$–projective, and this will prove 1.6 (using Theorem 1.0 again). $\mathbb{Z}\Gamma$–projectivity of M follows, in principle, from the projectivity criterion of [A1] (see also [A3]). We make a few comments in this direction. Let $k = \overline{\mathbb{F}}_p$, and $A \subset \Gamma$ be a p–elementary abelian subgroup. We need the following:

1.9. Lemma. The kA–free complex $C_*(\tilde{X}) \otimes k$ is chain homotopy equivalent to a finite dimensional free kA–complex if and only if $H^*(A;k)$ acts nilpotently on $H^*_A(\tilde{X};k)$.

This result is contained in [A2] (see Assadi [A3] for a summary and further results). The idea is as follows. In [A2], we associate certain homogeneous affine varieties $V_A(C_*(\tilde{X}) \otimes k)$ and $V^r_A(C_*(\tilde{X}) \otimes k)$ which are algebro–geometric invariants of $C_*(\tilde{X}) \otimes k$. The variety $V_A(C_*(\tilde{X}) \otimes k)$ is constructed from the support of the $H^*(A;k)$–module $H^*_A(\tilde{X};k)$. When $H^*(A;k)$ acts nilpotently on $H^*_A(\tilde{X};k)$, it follows that $V_A(C_*(\tilde{X}) \otimes k) = 0$. Furthermore, $C_*(\tilde{X}) \otimes k$ is shown to be chain homotopy equivalent to a free finite dimensional kA–chain complex if and only if $V^r_A(C_*(\tilde{X}) \otimes k) = 0$. On the other hand, according to ([A2] Theorem 1.4) for connected kA–complexes with finitely generated cohomology, $V_A(C_*) \cong V^r_A(C_*)$. These statements together imply Lemma 1.9.

■

As we have seen in Theorem 1.1, we need to consider only prime order subgroups of p–elementary abelian groups $A \subseteq \Gamma$, and show that M is kA–projective, or equivalently, $H^*(A;k)$ acts nilpotently on $H^*_A(\tilde{X};k)$, using 1.9. The proof of the projectivity criterion of [A1] can be modified in this set–up to show that:

1.10. Lemma: $H^*(A;k)$ acts nilpotently on $H^*_A(\tilde{X};k)$ if and only if for each cyclic subgroup $C \subseteq A$, $H^*(C;k)$ acts nilpotently on $H^*_C(\tilde{X};k)$.

This statement, of course, implies Lemma 1.6, using 1.9 again for each C. The proof of Lemma 1.10 is based: (a) The fact that $H^*_C(\tilde{X};k)$ is equipped with an "Steenrod algebra" operation, and (b) the notion of support varieties. The details are similar to the proof of ([A1] Theorem 2.1).

1.11 Corollary. In the situation of Theorem 1.5. if the necessary conditions are satisfied, then there exists a well–defined obstruction $\theta(X) \in \tilde{K}_0(\mathbb{Z}\Gamma)$ such that $\theta(X) = 0$ if and only if X is G–homotopy equivalent to a finite Poincaré complex with a free G–action.

This corollary follows from the general theory of [Wall I] once we have shown that X is finitely dominated via Theorem 1.5. Here, one should remark that $\theta(X)$ can be determined in terms of the G–module $H^2(\tilde{X})$ directly. For example, when $G = \mathbb{Z}_p$, one computes that $H^2(X) \cong I \oplus I \oplus P$, where $I \subset \mathbb{Z}G$ is the augmentation ideal and P is a projective module. Then $\theta(X)$ is the class $[P] \in \tilde{K}_0(\mathbb{Z}G)$ if $\tilde{X}$ is simply–connected. If $\theta(X) = 0$, then one has a finite Poincaré complex, and one can apply M. Freedman's topological surgery in dimension four to discuss the surgery obstruction. It is possible to determine the precise obstructions in this case by studying the intersectin form of $\tilde{X}$. This analysis is carried out for a special class of finite groups in a somewhat different context by I. Hambleton and M. Kreck. Theorem 1.5 holds in higher dimensions as well, although the statement should be suitably modified. These matters will be considered in a future paper.

Finally, we make some remarks about the validity of Conjecture 1.2 under the additional hypothesis that for some sufficiently large n, $\mathrm{Ker}(d : C_n \longrightarrow C_{n-1})$ is a finitely presented $\mathbb{Z}\Gamma$–module (using the previous notation etc.). As we have seen in the proof of Theorem 1.1, the basic step for finite dimensionality up to Γ–equivariant homotopy (of free Γ–chain complexes) is the projectivity of the Γ–module $M \equiv \mathrm{Ker}(d : C_n \longrightarrow C_{n-1})$ for some sufficiently large n. We will mention the relevant algebraic fact below (Lemma 1.12) which together with the projectivity criterion of [A4] Theorem 2.1 prove Conjecture 1.2 (a) in this case (cf. the proof of 1.1 (a) above). The proof of 1.2 (b) proceeds as in Theorem 1.1 (b), replacing elementary abelian subgroups by prime order subgroups in that argument. The following lemma is quite useful in other circumstances as well (see [A3]).

<u>1.12. Lemma</u>. In the above situation, suppose that M is a finitely presented $R\Gamma$–module which is $R\pi$–projective. Then $\mathrm{Hom}_{R\pi}(M,M)$ is RG–projective if and only if $\mathrm{Hom}_R(R \otimes_\pi M, R \otimes_\pi M)$ is RG–projective. (In particular, either condition implies that M is $R\Gamma$–projective.)

<u>Sketch of proof</u>. Suppose $\mathrm{Hom}_R(R \otimes_\pi M, R \otimes_\pi M)$ is RG–projective. It follows that $R \otimes_\pi M$ is also RG–projective. On the other hand the only non–vanishing term in $\mathrm{Tor}_*^{R\pi}(N,M)$ is $\mathrm{Tor}_0^{R\pi}(N,M) \cong N \otimes_\pi M$ for any $R\pi$–module N, since M is $R\pi$–projective. These two facts, together with an argument using a Grothendieck–type spectral sequence: $\mathrm{Tor}_i^{RG}(\mathrm{Tor}_j^{R\pi}(M,-),-) \Rightarrow \mathrm{Tor}_{i+j}^{R\Gamma}(M,-)$ imply that M is $R\Gamma$–flat. Since M is also finitely presented, it follows that M is $R\Gamma$–projective (see e.g. Bourbaki's Commutative Algebra, Ch.I, p.64, Ex. 15). Thus, M is $(R\Gamma,R\pi)$–projective which implies that

$\mathrm{Hom}_{R\pi}(M,M)$ is RG–projective. Conversely, if $\mathrm{Hom}_{R\pi}(M,M)$ is RG–projective, then M is RΓ–projective (being (RΓ,Rπ)–projective and Rπ–projective). It follows easily that $R \otimes_\pi M$ is RG–projective. Consequently $\mathrm{Hom}_R(R \otimes_\pi M, R \otimes_\pi M)$ is also RG–projective.

■

SECTION TWO. THE ALGEBRAIC WHITEHEAD TRANSFER. Let $(\eta): 1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ be an extension of groups and let u be a section (not a homomorphism necessarily). Here G is a finite group of order g and π and Γ are discrete groups. Let $\underline{A}$ be the category whose objects consists of pairs (M,B), where M is a finitely generated $\mathbb{Z}\Gamma$–projective module which is free over $\mathbb{Z}\pi$ and B is a finite $\mathbb{Z}\pi$–basis for M. Let $(M_1,B_1) \sim (M_2,B_2)$ if there exists a $\mathbb{Z}\Gamma$–isomorphism $f: M_1 \longrightarrow M_2$ such that f is π–simple with respect to B_1 and B_2. The set of equivalence classes $\underline{A}' = \underline{A}/\sim$ has a monoid structure under direct sum of modules and disjoint union of bases, and $(0,\phi)$ is the neutral element. Let $\underline{R}$ be the submonoid generated by $(\mathbb{Z}\Gamma, u(G))$. Then $Wh_1^T(\pi \longrightarrow \Gamma)$ is defined to be the quotient monoid $\underline{A}'/\underline{R}$. As in [AV] (Proposition 1.1) it follows that $Wh_1^T(\pi \longrightarrow \Gamma)$ is an abelian group. The forgetful functor $(M,B) \longrightarrow M$ induces a homomorphism $\beta: Wh_1^T(\pi \longrightarrow \Gamma) \longrightarrow \tilde{K}_0(\mathbb{Z}\Gamma)$. On the other hand, given $t \in Wh_1(\pi)$, we define $\alpha(t)$ to be the equivalence class of (M,B) where $M = (\mathbb{Z}\Gamma)^k$ and B is obtained from twisting the standrad basis $u(G)^k$ by t, i.e., $id: (M,u(G)^k) \longrightarrow (M,B)$ has π–torsion t. It follows that the sequence $Wh_1(\pi) \xrightarrow{\alpha} Wh_1^T(\pi \longrightarrow \Gamma) \xrightarrow{Tr} \tilde{K}_0(\mathbb{Z}\Gamma)$ is exact. In [AV], this sequence is extended to a five term exact sequence involving the transfers in Wh_1 and $\tilde{K}_0$ where $\Gamma = \pi \times G$ (cf. [AV] Proposition 1.2).

2.1 Proposition. The following sequence is exact.

$$Wh_1(\Gamma) \xrightarrow{Tr} Wh_1(\pi) \xrightarrow{\beta} Wh_1^T(\pi \longrightarrow \Gamma) \xrightarrow{\alpha} \tilde{K}_0(\mathbb{Z}\Gamma) \xrightarrow{tr} \tilde{K}_0(\mathbb{Z}\pi) .$$

Several other properties of Wh_1^T extended from the product case $\Gamma = \pi \times G$ to the present case. Let $\underline{P}_*$ denote the category of bounded finitely generated projective $\mathbb{Z}\Gamma$–complexes. Let $\underline{P}_*^h$ be the category of $\mathbb{Z}\Gamma$–complexes which have the chain homotopy type of a complex in $\underline{P}_*$. Denote by $\underline{A}_*$ the category of pairs (C_*,B_*) where C_* is a complex in $\underline{P}_*$ and $(C_i,B_i) \in \underline{A}$ for all i. For $(C_*,B_*) \in \underline{A}_*$, we define $\chi(C_*,B_*) = \sum_n (-1)^n [C_n,B_n] \in Wh_1^T(\pi \longrightarrow \Gamma)$. If C_* is acyclic, then its torsion $\tau(C_*,B_*) \in Wh_1(\pi)$ is defined.

2.2 Proposition. (a) For $(C_*,B_*) \in \underline{A}_*$, $\chi(C_*,B_*) = \beta(\tau(C_*,B_*))$ if C_* is acyclic.

(b) Let (C_*,B_*) and (C'_*,B'_*) be objects in $\underline{A}_*$ and let $f: C_* \longrightarrow C'_*$ be a $\mathbb{Z}\Gamma$–chain homotopy equivalence. Then $\chi(C'_*,B'_*) = \chi(C_*,B_*) + \beta(\tau(f))$, where $\tau(f)$ is the π–Whitehead torsion of f.

2.3 Proposition and Definition. Let f be a $\mathbb{Z}\pi$–chain homotopy equivalence from a finitely $\mathbb{Z}\pi$–based $\mathbb{Z}\Gamma$–complex D_* to a chain complex $C_* \in \underline{P}^h_*$. Let g be a $\mathbb{Z}\Gamma$–chain homotopy equivalence from C_* to a chain complex C'_* with a $\mathbb{Z}\pi$–basis B'_* such that $(C'_*,B'_*) \in \underline{A}_*$. Then the element $\chi(C'_*,B'_*) - \beta(\tau(g \cdot f)) \in Wh_1^T(\pi \longrightarrow \Gamma)$ does not depend on the choice of (C'_*,B'_*) and g. This element is denoted by $\chi(f)$.

A topological application of this element is based on the following:

2.4 Proposition. Let $D_* \in \underline{D}_*$, $C_* \in \underline{P}^h_*$, and $f: D_* \longrightarrow C_*$ be a $\mathbb{Z}\pi$–homotopy equivalence. Then $\chi(f) = 0$ in $Wh_1^T(\pi \longrightarrow \Gamma)$ if and only if there exists a finitely $\mathbb{Z}\pi$–based projective $\mathbb{Z}\Gamma$–chain complex C'_* and a $\mathbb{Z}\Gamma$–chain homotopy equivalence $g: C_* \longrightarrow C'_*$ such that $g \cdot f$ is π–simple.

2.5 Theorem. Let X be a connected CW complex (of finite type) with $\pi_1(X) = \Gamma$, and let Y be a connected finite CW subcomplex of X with $\pi_1(Y) = \pi_1(X)$. Let $\overline{X}$ and $\overline{Y}$ be the covering space of X and Y with $\pi_1(\overline{X}) = \pi = \pi_1(\overline{Y})$, i.e., $X = \overline{X}/G$ and $Y = \overline{Y}/G$. Let $\overline{\overline{X}}$ be a connected finite CW complex with the commutative diagram:

$$\begin{array}{ccc} \bar{Y} & \longrightarrow & \bar{\bar{X}} \\ \downarrow & & \downarrow \alpha \\ Y & \longrightarrow & X \end{array}$$

such that $\alpha : \bar{\bar{X}} \longrightarrow X$ induces a homotopy equivalence from $\bar{\bar{X}}$ to $\bar{X}$. And suppose that the inclusion of the n–skeleton $(X^{(n)},Y)$ to (X,Y) is a finite domination for some sufficiently large n. Then $\chi(\alpha_* : C_*(\bar{\bar{X}},\bar{Y};\mathbb{Z}\pi) \longrightarrow C_*(X,Y;\mathbb{Z}\Gamma)) = 0$ in $Wh_1^T(\pi \longrightarrow \Gamma)$ if and only if there exists a finite complex $Z \supset Y$ and a homotopy equivalence $g : X \longrightarrow Z$ (rel. Y) such that $g \cdot \alpha : \bar{\bar{X}} \longrightarrow X \longrightarrow Z$ induces a simple homotopy equivalence $\bar{\bar{X}} \longrightarrow \bar{Z}$, where $\bar{Z}$ is the covering space of Z with $\pi_1(\bar{Z}) = \pi$, i.e., $Z = \bar{Z}/G$. Here α_* denotes the composition of the $\mathbb{Z}\pi$–chain homotopy equivalence: $C_*(\bar{\bar{X}},\bar{Y};\mathbb{Z}\pi) \longrightarrow C_*(\bar{X},\bar{Y};\mathbb{Z}\pi)$ and $\mathbb{Z}\Gamma$–chain isomrophism: $C_*(\bar{X},\bar{Y};\mathbb{Z}\pi) \longrightarrow C_*(X,Y;\mathbb{Z}\Gamma)$.

Let $\underline{D}_*$ be the category whose objects are finitely $\mathbb{Z}\pi$–based $\mathbb{Z}\Gamma$–chain complexes D_* such that $D_* \otimes_{\mathbb{Z}} \mathbb{Z}_q$ is $\mathbb{Z}_q\Gamma$–chain homotopic to the trivial complex 0. We wish to use the well–defined element $\chi(-)$ above to define an invariant of $D_* \in \underline{D}_*$. Let R_* be a projective $\mathbb{Z}G$–resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$–module such that $R_0 = \mathbb{Z}G$. Then it turns out that the standard $\mathbb{Z}\pi$–chain map $f_s : D_* \longrightarrow D_* \otimes_{\mathbb{Z}} R_*$ is a $\mathbb{Z}\pi$–chain homotopy equivalence, (f_s is given by $f_s(x) = x \otimes 1 \in D_i \otimes R_0$ for $x \in D_i$, where 1 is the unit of $\mathbb{Z}G = R_0$). Furthermore, $D_* \otimes_{\mathbb{Z}} R_*$ is an object in $\underline{P}_*^h$, and there is an object $(C'_*,B'_*) \in \underline{A}_*$ and a $\mathbb{Z}\Gamma$–chain homotopy equivalence $g : D_* \otimes_{\mathbb{Z}} R_* \longrightarrow C'_*$. Hence we define $\gamma(D_*)$ to be $\chi(f_s) = \chi([C'_*,B'_*]) - \beta(\tau(g \cdot f_s))$. If $\Gamma = \pi \times G$ and G acts trivially on D_*, then $\gamma(D_*)$ depends only on the Reidemeister torsion of D_* in $Wh_1(\pi;\mathbb{Z}_q)$, but this may not hold in general.

<u>SECTION THREE. THE TOPOLOGICAL WHITEHEAD TRANSFER</u>. As pointed out in the Introduction, it is possible to formulate the algebraic construction of Section 2 in terms of spaces, in accordance with the current emphasis on "spaces" rather than "groups". Thus,

we replace the Whitehead groups by Whitehead spaces following Hatcher's higher simple homotopy theory [H]. However, there are some technical points which must be dealt with. For example "the naturally suggested transfer functor" in Hatcher's theory is not a homotopy functor. In [W] Waldhausen introduces a "homotopification" procedure for functors, and this resolves the above–mentioned difficulty. One advantage of this approach is that the 5–term exact sequence of the type introduced in Proposition 2.1 turns out as the lower portion of the homotopy exact sequence of "the transfer fibration" between Whitehead spaces. Another point which should be remarked is that the following approach applies to more general extensions $1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$; e.g., G could be the fundamental group of an aspherical manifold. While it would be more natural and appropriate to present this material in the semisimplicial language, we continue the discussion in the topological context.

To every extension $(\eta): 1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$, where G is a finite group, one associates the fibration $G \longrightarrow B\pi \longrightarrow B\Gamma$. This is a special case of a compact ANR fibration:

Definition: Let $\mathscr{C}$ be a subcategory of the category of topological spaces. A triple $\eta = (E \xrightarrow{\pi} B)$ is called a compact ANR fibration in $\mathscr{C}$ with the following properties:

(i) π is a proper map;
(ii) $\pi : E \longrightarrow B$ is a Hurewicz fibration;
(iii) all the fibres of π are compact ANR.

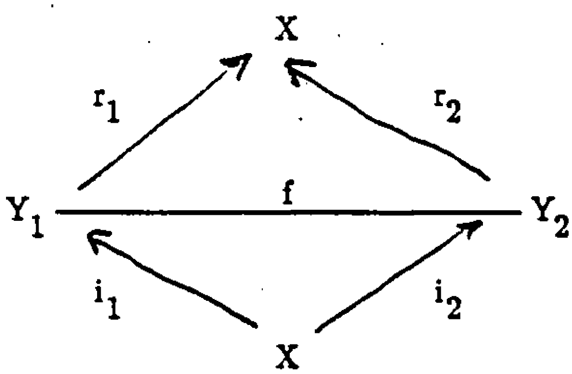
In a combinatorial category, e.g. that of simplicial complexes, we assume that the fibres are finite simplicial complexes.

Examples.

(1) Let K be a finite simplicial complex. The product fibration $B \times K \longrightarrow B$ is a compact ANR fibration.
(2) Any fibre bundle with compact fibres is a compact ANR fibration. For instance, covering spaces with finite groups of deck transformations are compact ANR fibrations.
(3) Let π be a "geometric group", i.e. the fundamental group of a closed aspherical manifold (e.g. closed hyperbolic manifolds), and let $1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow \pi' \longrightarrow 1$ be an extension of groups. Then the fibration $B\pi \longrightarrow B\Gamma \longrightarrow B\pi'$ has compact fibres. In many cases (e.g. if $\Gamma = \pi \times \pi'$) the obstructions for converting this fibration to a

compact ANR fibration vanish. We will see below in such cases, one can define the transfer and consequently the group $Wh_0(\pi \longrightarrow \Gamma)$. If π' is an infinite group, then the procedure of Section 2 does not apply here.

For each topological space X, Hatcher has defined a category $Wh^{PL}(X)$ such that $\pi_0(BWh^{PL}(X)) \cong Wh_1(\pi_1(X))$, where $BWh^{PL}(X)$ is the classifying space of $Wh^{PL}(X)$. We refer the reader to [H] and [W] (in particular § 5) for details and justifications of what follows. We are interested in a modified version of Hatcher's construction. Let $\mathcal{T}op$ be the category of topological spaces and $\mathcal{C}at$ the category of categories. Define a functor $\mathcal{H}^{sp}: \mathcal{T}op \longrightarrow \mathcal{C}at$ as follows. The objects of $\mathcal{H}^{sp}(X)$ consist of diagrams $X \underset{r}{\overset{i}{\rightleftarrows}} Y$, where $X,Y \in \mathcal{T}op$, and i is an inclusion while r is a deformation retraction, and (Y,X) is a relatively finite CW complex. Further, we assume that all maps are cell–like, i.e., the inverse images of points are contractible. A morphism f between $X \underset{i_1}{\overset{r_1}{\rightleftarrows}} Y_1$ and $X \underset{i_2}{\overset{r_2}{\rightleftarrows}} Y_2$ consists of a strictly commutative diagram of cell–like maps:



We call $\mathcal{H}^{sp}(X)$ the special Hatcher–Whitehead category of X and $B\mathcal{H}^{sp}(X)$ the special Hatcher–Whitehead space of X. In the combinatorial version of this category, if the reader prefers, we have simplicial complexes and simplicial maps such that the inverse image of every simplex is contractible.

Let $\eta = (E \xrightarrow{\pi} B)$ be a compact ANR fibration. Given $Y \underset{r}{\overset{i}{\rightleftarrows}} B$ in $\mathcal{H}^{sp}(B)$ representing an object x, we define $Pretr(x)$ by the pull–back $r^*\eta = (E' \xrightarrow{\pi} Y)$. Then one has the natural inclusion $i' : E \longrightarrow E'$ and the retraction $r' : \Delta' \longrightarrow E$ covering $r : Y \longrightarrow B$. Moreover, r' is cell–like since r is cell–like and F is a compact ANR. In addition, (E',E) is a relatively finite CW complex. Similar comments apply to the combinatorial case. Thus, we have a candidate for the transfer, namely $(Y \underset{r}{\overset{i}{\rightleftarrows}} B) \longmapsto (E \underset{r'}{\overset{i'}{\rightleftarrows}} E)$ which defines a functor $\mathrm{Pretr} : \mathscr{H}^{sp}(B) \longrightarrow \mathscr{H}^{sp}(E)$.

The technical problems which arise here are:

(a) the classifying space $B\mathscr{H}^{sp}(X)$ is not the Whitehead space of X;

(b) the functor $X \longmapsto B\mathscr{H}^{sp}(X)$ is not a homotopy functor.

According to Waldhausen [W] (pp. 55–58) these difficulties are overcome by using his "homotopification of functors". First, let us remark that if $\mathscr{X}$ is a simplicial object in the category of topological spaces, so that for each $n \geq 0$ X_n is a topological space and the faces and degeneracies $X_{n+1} \xrightarrow{d_i} X_n$ and $X_n \xrightarrow{s_i} X_{n+1}$, $0 \leq i \leq n$ are homotopy equivalences, then the geometric realization $|\mathscr{X}|$ is homotopy equivalent to each X_n. We will use this remark in the following construction.

Let Δ^{∞} be the standard semisimplicial object for which $(\Delta^{\infty})_n$ is the standard simplex Δ^n and the boundaries and degeneracies are the usual maps $\partial_i : \Delta^n \longrightarrow \Delta^{n-1}$ and $s_i : \Delta^{n-1} \longrightarrow \Delta^n$. To each topological space X, we associate the simplicial object $\Delta^{\infty}(X)$ such that $\Delta^{\infty}(X)_n = \Delta^n \times X$ and the boundaries and degeneracies are $\partial_i \times id_X$ and $s_i \times id_X$, $0 \leq i \leq n$. Now if F is a functor on the category of topological spaces, we define the simplicial object $F\Delta^{\infty}(X)$ to have $(F\Delta^{\infty}(X))_n = F(\Delta^n \times X)$ and $F(\partial_i \times id_X)$ and $F(s_i \times id_X)$ for its boundaries and degeneracies. We denote by $hF(X)$ the geometric realization of this simplicial object. The functor $X \longrightarrow hF(X)$ is a homotopy functor, and if F itself is a homotopy functor, then $hF(X)$ and $F(X)$ are homotopy equivalent [W]. Let us call hF the homotopification of F.

3.1 Proposition. The homotopification of $\phi : X \longmapsto B\mathscr{H}^{sp}(X)$ is the functor $X \longmapsto Wh^{PL}(X)$ (as defined by Hatcher) up to homotopy.

Proof. Let $\mathscr{H}^{H}$ be Hatcher's Whitehead category. Then the forgetful functor which forgets the retraction $r : Y \longrightarrow X$ in $\mathscr{H}^{sp}(X)$ yields a functor $\varphi : \mathscr{H}^{sp}(X) \longrightarrow \mathscr{H}^{H}(X)$. This defines $Br : B\mathscr{H}^{sp}(X) \longrightarrow Wh^{PL}(X)$ and one has the commutative diagram:

$$\begin{array}{ccc} B\mathscr{H}^{sp}(X) & \xrightarrow{\quad\varphi\quad} & Wh^{PL}(X) \\ \downarrow & & \downarrow \\ hB\mathscr{H}^{sp}(X) & \xrightarrow{\quad hB\varphi\quad} & hWh^{PL}(X) \end{array}$$

in which α and β are induced by "inclusions", and the bottom row is obtained from the top row by the homotopification procedure above. The map β is a homotopy equivalence, because Wh^{PL} is a homotopy functor and in $Wh^{PL}\Delta^{\infty}(X)$ all the boundaries and degeneracies are homotopy equivalences. Thus the above remark applies to the geometric realization $hWh^{PL}\Delta^{\infty}(X)$.

3.2 Corollary. The map $\varphi : B\mathscr{H}^{sp}(X) \longrightarrow Wh^{PL}(X)$ factors through $hB\mathscr{H}^{sp}(X)$.

Now we apply the above discussion to the functor $\text{Pretr} : \mathscr{H}^{sp}(B) \longrightarrow \mathscr{H}^{sp}(E)$ defined above. Apply the homotopification to this functor, to get "hPretr", which we call "the transfer" and denote it by "Tr" or "Tr(η)" if reference to $\eta = (E \xrightarrow{\pi} B)$ is needed. By Proposition 3.1, we have defined $Tr : Wh^{PL}(B) \longrightarrow Wh^{PL}(E)$. Delooping Tr, we get $Tr : Wh(B) \longrightarrow Wh(E)$, where Wh is the delooping of Wh^{PL}. Let $Tr(\eta)$ be the fibre of this natural transformation ([Q] section one), so that we have the fibration $Tr(\eta) \longrightarrow Wh(B) \longrightarrow Wh(E)$.

A particularly interesting situation arises from the following. Let $(e) : 1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ be an extension, where G is a finite group. Then we have the compact ANR–fibration (η) given by the finite covering $G \longrightarrow B\pi \longrightarrow B\Gamma$. Such extensions (e) form the objects of a category $\mathscr{E}$ whose morphisms are homomorphisms of such exact sequences. Thus, one has two naturally defined functors, namely, $(e) \longmapsto Wh_1^T(\pi \longrightarrow \Gamma)$ and $(e) \longmapsto \pi_0(Tr(\eta))$, defined on the category extensions $\mathscr{E}$. Judging from the long exact sequence of the fibration $Tr(\eta) \longrightarrow Wh(B) \longrightarrow Wh(E)$

in homotopy, and comparing it with the 5–term exact sequence of 2.2, it is natural to conjecture the following:

3.3 Conjecture. There is a natural isomorphism between the functors $\pi_0(\mathrm{Tr}(\eta))$ and $\mathrm{Wh}_1^T(\pi \longrightarrow \Gamma)$.

This will imply, of course that there is a long exact sequence of higher Whitehead groups extending the five–term sequence of Proposition 1.2. So far this conjecture has been verified only in special cases, and we plan to take up this subject in another paper.

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"On Finite Domination and Simple Homotopy Type of Nonsimply–Connected G–spaces"

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Introduction. Suppose W is a compact manifold with $\pi_1(W) = \pi$ and G is a finite group, and we wish to construct a free G–action on W with certain desired properties. In the relative version, we may be given, in addition, a submanifold $V \subset W$ which has a G–action $\varphi : G \times V \longrightarrow V$ and we require that the G–action $\psi : G \times W \longrightarrow W$ restricts to φ on V, i.e. $\psi | G \times V = \varphi$. This is the extension problem, considered in [AV], for example. (See [Wr] for a survey and further examples and applications.) In [AV] and [Wr], the extra condition is that G should act trivially on homology, and $\pi_1(W/G) = \pi \times G$.

One systematic approach to construct such group actions is the following. Using some homotopy theoretic tools, one constructs a space X (with $\pi_1(X) = \Gamma$, where Γ fits into an exact sequence $1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$) such that the regular covering space of X

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with $\pi_1(\tilde{X}) = \pi$, is homotopy equivalent to the given W. Thus G acts freely on a space $\tilde{X}$ homotopy equivalent to W and the problem reduces to the following:

(a) Find a finitely dominated space Y which is homotopy equivalent to X.
(b) Find a finite complex K homotopy equivalent to Y and such that the induced covering $\tilde{K}$, with $\pi_1(\tilde{K}) = \pi$, is π–simple homotopy equivalent to W.

Once K is found, the problem is reduced to surgery theory; that is, to find an appropriate normal invariant and to show that the surgery obstruction vanishes for a suitable choice of the normal invariant. Of course, it could happen that at some stage there is an obstruction and one does not succeed to carry out this procedure. The question arises then, as to how to measure such obstructions, and how to express them in terms of the topological or other invariants of W.

In this paper, we consider problems (a) and (b) above in a fairly general setting. Namely, in Section 1, we discuss the problem of finite domination of nonsimply–connected free G–spaces for any finite group G, and we show that the question reduces to the case of $G = \mathbb{Z}/p\mathbb{Z}$, where p is a prime. This reduction is a significant step in computations, and we illustrate this by an application.

Next, we address the problem (b) above in Section 2, and formulate the appropriate obstruction group $Wh_1^T(\pi \longrightarrow \Gamma)$ from an algebraic point of view, following our earlier treatment in [AV] for the case $\Gamma = \pi \times G$ and $Wh_1^T(\pi \subset \pi \times G)$. As in [AV], this abelian group is defined as the Grothendieck group of a certain category of projective modules. Wh_1^T is closely related to the functor Wh_1 and $\tilde{K}_0$ via a five–term exact sequence involving transfer homomorphisms. It is fair to say that Wh_1^T plays the same role in the nonsimply–connected cases that $\tilde{K}_0$ does in the simply–connected ones (at least in construction and classification problems of group actions).

This generalization of Wh_1^T as a functor of extensions $1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$

(rather than pairs of groups $\pi \subset \pi \times G$) suggests a further generalization. Namley, to define Wh_1^T on the space level, as it has been the current trend ever since Hatcher's "Whitehead spaces" [H]. The recent developments in geometric topology, in particular surgery theory, have put more emphasis on spaces rather than their corresponding algebraic objects, and this has proved quite fruitful. Moreover, the topological construction has a wider domain of definition than the algebraic one, and naturally it is expected to have more topological applications. Thus we have included a discussion of this point of view in Section 3, based on Hatcher's and Waldhausen's theories [H] [W] with suitable modifications. While the semi-simplicial language is more natural in this context, we have chosen to informally discuss the matters in the topological category leaving the details and applications for a future opportunity.

SECTION ONE. FINITE DIMENSIONALITY AND FINITE DOMINATION. Recall that a topological space is called finitely dominated if there exists a finite CW complex K and a map $f : K \longrightarrow X$ which has a right homotopy inverse $r : X \longrightarrow K$. A CW complex X is said to be of finite type if every finite dimensional skeleton of X is a finite complex, i.e. X has finitely many cells in each dimension. It is easy to see that a finitely dominated complex is homotopy equivalent to a finite dimensional CW complex with finitely generated total homology. An algebraic criterion for finite dimensionality of complexes can be formulated with the help of the following result of [Wall II]:

1.0. Theorem ([Wall II] p. 137, Theorem 6). A projective positive chain complex C_* is chain hmotopy equivalent to an n–dimensional complex if and only if $H_i(C_*) = 0$ for $i > n$ and $\mathrm{Im}\,(d : C_{n+1} \longrightarrow C_n)$ is a projective module.

With the help of ([Wall II] Theorem 2) as well as related results of [Wall I], one can translate the above mentioned finite dimensionality criterion of [Wall II] into the following: A CW complex X with bounded $H_*(X;\mathbb{Z}\pi_1(X))$ is homotopy equivalent to a finite dimensional complex if and only if for $C_* \equiv C_*(X;\mathbb{Z}\pi_1(X))$ (= cellular chain complex of the universal covering space $\tilde{X}$ of X) and some sufficiently large n, $\mathrm{Ker}(d : C_{n+1} \longrightarrow C_n)$ is a projective $\mathbb{Z}\pi_1(X)$–module. Clearly if the latter condition holds for some large n, then it holds for all $m \geq n$.

The passage from finite dimensionality to finite domination for spaces is technical in general. However, for applications to manifolds etc., the results of [Bieri–Eckmann] and especially, Browder's theorem ([Browder] Corollary 2) are quite useful. Namely, if X is a Poincaré space with finitely presented fundamental group, then X is finitely dominated. In our circumstances, we apply Browder's theorem in conjunction with Wall's finite dimensionality criterion as follows. Refering to the notation and the set–up of the introduction, suppose we have constructed an infinite dimensinal (as it happens in most homotopy theoretic constructions) space X with $\pi_1(X) = \Gamma$, such that the finite group G operates freely on the regular covering space $\tilde{X}$ with $\pi_1(\tilde{X}) = \pi$. First, we give a finite dimensionality criterion for $\tilde{X}$ to be G–homotopy equivalent to a finite dimensional free G–complex $\tilde{Y}$ (in terms of restrictions to suitable subgroups of G). Next, we pose the hypothesis that $\tilde{X}$ is a Poincaré complex, so that $\tilde{Y}$ becomes a Poincaré complex as well. Now $\tilde{Y}/G = Y$ is seen to satisfy Poincaré duality, since $\dim \tilde{Y} < \infty$. This shows that X satisfies Poincaré duality, hence, by Browder's theorem X is finitely dominated and step (a) of the Introduction is carried through. (See [A3] also.)

For simplicity of exposition, and without loss of generality as far as applications to compact manifolds are concerned, we assume that π is a finitely presented discrete group, and we work in the category of CW complexes of finite type and cellular maps. Let G be a finite group, and let $\Gamma = \Gamma(G)$ be a discrete group satisfying the exact sequence $\eta(G)$:

$\underline{\eta(G)}$: $$1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

Let $X = X(G)$ be a connected space with $\pi_1(X) = \Gamma$. Denote by $\tilde{X}$ the universal covering space of X. For each subgroup $H \subseteq G$, we consider the corresponding exact sequence $\eta(H)$:

$\underline{\eta(H)}$: $$1 \longrightarrow \pi \longrightarrow \Gamma(H) \longrightarrow H \longrightarrow 1$$

We set $X(H) = \tilde{X}/\Gamma(H)$ which is a covering space of X and $\pi_1(X(H)) = \Gamma(H)$. Thus $X(1)$ is a free G–space which is of interest to us, and $X = X(G) = X(1)/G$; $X(1)$ is homotopy equivalent to the given manifold W in consideration. Our first result in this direction is of local–to–global nature. Namely, we show that the problem of finite dimensionality of a free G–space (up to G–homotopy) can be decided by restricting to elementary abelian subgroups of G .

1.1. Theorem. (a) Let X be a CW complex with $\pi_1(X) = \Gamma$ as above. Then X is homotopy equivalent to a finite dimensional complex if and only if X(A) is homotopy equivalent to a finite dimensional complex for all elementary abelian p–subgroups of G and all primes p .

(b) Further, suppose that X(1) satisfies Poincaré duality. Then X is finitely dominated if and only if X(A) is finitely dominated for all elementary abelian p–subgroups of G and all p .

Before giving the proof of 1.1, we state the following conjecture and a supporting theorem.

1.2. Conjecture. Let G be a nontrivial group. (a) Let X be a CW complex with $\pi_1(X) = \Gamma$ as above. Then X is homotopy equivalent to a finite dimensional complex if and only if X(C) is homotopy equivalent to a finite dimensional complex for all cyclic subgroups $C \subseteq G$ of prime order.

(b) Suppose further that X(1) is a Poincaré complex. Then X is finitely dominated if and only if X(C) is finitely dominated for all $C \subseteq G$ of prime order.

We have proved this conjecture for the cases where Γ is a finite group, or more generally where for some n sufficiently large, $\mathrm{Ker}(d : C_n \longrightarrow C_{n-1})$ is a finitely presented $\mathbb{Z}\Gamma$–module, where $C_* = C_*(\tilde{X})$ = cellular chains of the universal covering space (Assadi [A3]).

We need some auxiliary algebraic concepts first. Recall that an $R\Gamma$–module M is called $(R\Gamma,R\pi)$–projective, if there exists an $f \in \mathrm{Hom}_{R\pi}(M,M)$ such that $\sum_{g\in G} gf(g^{-1}x) = x$ for all $x \in M$. This is a generalization of the concept of "weakly projective" (cf. [Cartan–Eilenberg]).

1.3. Lemma ([Rim] Proposition 2.2) An $R\Gamma$–module M is $(R\Gamma,R\pi)$–projective if and only if the Tate cohomology $\hat{H}(G;\mathrm{Hom}_{R\pi}(M,M))$ is trivial.

<u>1.4. Lemma</u>. An $R\Gamma$–module M is $R\Gamma$–projective if and only if M is $(R\Gamma, R\pi)$–projective and $R\pi$–projective.

<u>Proof of Theorem 1.1.</u> Note that if X is finitely dominated, then so are all finite covering spaces of X. Therefore, the non–trivial direction is to pass from elementary abelian p–groups to the group G itself. Assume that for such a $A \subseteq G$, X(A) is homotopy equivalent to a finite dimensional complex. In particular, this holds for the covering space X(1) with $\pi_1(X(1)) = \pi$. Consider the cellular chains $C_* = C_*(\tilde{X})$ of the universal cover $\tilde{X}$, which is a free finitely generated $\Gamma(G)$–complex, and let $d : C_* \longrightarrow C_*$ be its boundary homomorphism. Choose m large enough so that (C_*, d_*) is exact in all dimensions $n \geq m$. By Wall's Theorem above $M \equiv \mathrm{Ker}(d_n : C_n \longrightarrow C_{n-1})$ is $\mathbb{Z}\pi$–projective, and by our standing hypotheses, it is finitely generated. Further, for every p–elementary abelian group $A \subseteq G$, Ker d_n is $\mathbb{Z}\Gamma(A)$–projective, since X(A) is homotopy equivalent to a finite dimensional complex. By Lemmas 1.3 and 1.4, $\hat{H}(A; \mathrm{Hom}_{\mathbb{Z}\pi}(M,M)) = 0$. By Chouinard's Theorem (cf. [Jackowski] Theorem 3.1 for a topological proof, or [Chouinard]), the G–module $\mathrm{Hom}_{\mathbb{Z}\pi}(M,M)$ is cohomologically trivial in the sense of Tate–Nakayama (see [Rim]). Since this module is also $\mathbb{Z}$–free, it is $\mathbb{Z}G$–projective. By Lemma 1.4, M is $(\mathbb{Z}\Gamma, \mathbb{Z}\pi)$–projective. Hence M is $\mathbb{Z}\Gamma$–projective. The proof of (b) follows from Browder's theorem ([Browder] Corollary 2), once we observe that if a finite group G acts freely on a finite dimensional Poincaré duality complex, then the orbit space also satisfies Poincaré duality. (See e.g. Gottlieb, Proc. AMS 76 (1979) 148–150 or Quinn. Bull AMS 78 (1972) 262–267.)

■

To indicate how the above theorem may apply to prove the above conjecture 1.2, we consider the special case where W is a closed Poincaré complex of dimension four with a finite fundamental group.

<u>1.5. Theorem</u>. Let X(1) be a (possibly) infinite dimensional free G–space, where G is any finite group. Let $\pi_1(X(1)) = \pi$ be a finite group, and assume that non–equivariantly X(1) is homotopy equivalent to a finite 4–dimensional Poincaré complex. Assume that $\tilde{X}$ is the universal covering space on which Γ acts freely, and as before, $1 \longrightarrow \pi \longrightarrow \Gamma \xrightarrow{\theta} G \longrightarrow 1$ is exact. Then the following are necessary and sufficient for X to be finitely dominated. For each $C \subseteq \Gamma$, $|C| = \text{prime}$ and $C \cap \pi = 1$:

(I) The spectral sequence of the Borel construction $E_C \times_C \tilde{X} \longrightarrow BC$ does not collapse.

(II) $\dim_{\mathbb{F}_p} H^1(C;H^2(\tilde{X})) \geq 2$ when $|C| = p$.

<u>Outline of the proof</u>: Suppose X is finitely dominated. Then one verifies directly that some differentials in the spectral sequence of the indicated fibration must be non–trivial in order that the total cohomology $H^*(X)$ be finitely generated. Thus (I) follows. As for (II), again a direct computation with chain complexes shows that $H^2(\tilde{X}) \cong I \oplus I \oplus P$ where I is the augmentation ideal of $\mathbb{Z}C$ and P is $\mathbb{Z}C$–projective. To prove that these conditions are sufficient, we apply Theorem 1.1 above to reduce the problem to the case of a p–elementary abelian group. The strategy is to reduce the problem to the case $G = \mathbb{Z}_p$. First, we note that we need to prove a finite dimensionality statement in view of the proof of Theorem 1.1. Secondly, observe that the reduction from G to $\mathbb{Z}_p$ is a special case of Conjecture 1.2 above, which we formulate as follows.

<u>1.6. Lemma</u>. In the situation of 1.5, X(1) is G–homotopy equivalent to a finite dimensional free G–complex, if and only if for each prime order subgroup $C \subseteq \Gamma$, $\tilde{X}/C$ is homotopy equivalent to a finite dimensional complex.

We postpone the outline of proof of this Lemma, and proceed to prove 1.5. First, notice that for any $C \subset \pi$, $|C|$ = prime, conditions (I) and (II) of the theorem are satisfied, since X(1) is homotopy equivalent (non–equivariantly) to a finite dimensional complex. Therefore, by Lemma 1.6, we are reduced to the case $G = \mathbb{Z}_p$ and $\pi_1(X(1)) = 1$, and we need to show that (I) and (II) imply the desired finite dimensionality result. Here, we use the notion of "free equivalence" of [A1]. Namely, finite dimensionality (up to equivariant homotopy) is preserved under "free equivalence" of G–spaces and G–complexes (cf. [A4] also).
This translates into:

<u>1.7. Lemma</u>. Let $\pi_1(X(1)) = 0$, in the above notation. Let X' be a free G–complex obtained from X(1) by adding free orbits of G–cells of dimension 3 and 4 so that $\pi_i(X') = 0$ for $i \leq 3$. Then X'/G is homotopy equivalent to a finite dimensional complex if and only if X is homotopy equivalent to a finite–dimensional complex. (In view of 1.6 above, we may take $G = \mathbb{Z}_p$ here and in 1.8 below for simplicity, although this restriction

is not necessary.)

This is a special case of a more general result in [A2], and we leave out the proof. Next, we reduce the problem to cohomology computations, taking advantage of the fact that $H^*(\mathbb{Z}_p)$ is periodic in positive dimensions.

1.8. Lemma. Let X' be as above. Then X'/G is homotopy equivalent to a finite dimensional complex if and only if $H^i_G(X') = 0$ for $i \geq 5$, and this happens if and only if $H^i_G(X(1)) = 0$ for $i \geq 5$.

The proof of this lemma is computational, using the spectral sequence $H^i(BG,H^j(X')) \Rightarrow \mathcal{G}r\,(H^{i+j}_G(X'))$.

We further compute that in the spectral sequence of $E_G \times_G X(1) \longrightarrow BG$, if the differentials $d_3^{j,2}$ do not identically vanish, then $E_\infty^{j,0} = 0$ for all $j \geq 3$. Further, if $d_3(\zeta) = 0$ in $E_3^{i+3,0}$, then $\zeta \in \text{Image}\,(d_3 : E_3^{i-3,4} \longrightarrow E_3^{i,2})$. This implies, of course, that $E_r^{i,2} = 0$ for $i \geq 2$ and $r \geq 4$. The proof of the latter statement is based on the periodicity of $H^*(G)$ and the multiplicative properties of the spectral sequence. Another computational point is that if d_3 is not identically zero on $E_3^{j,2}$, then $H^{2i}(G,H^2(X(1))) = 0$ for $i > 0$. Putting all these together, it follows that $E_\infty^{i,j} = 0$ when $i+j > 4$, and the theorem follows from 1.8. ∎

It remains to indicate the proof of Lemma 1.6.

Outline of the proof of Lemma 1.6. Consider the free Γ–space $\tilde{X}$, and assume that $\tilde{X}/C$ is homotopy equivalent to a finite dimensional complex. Let $M = \text{Ker}(d_n : C_n \longrightarrow C_{n-1})$, where $C_* = C_*(\tilde{X})$ as before, for some sufficiently large n. Using Theorem 1.0 above ([Wall II]), the hypotheses imply that M is $\mathbb{Z}C$–projective for all prime order cyclic subgroups $C \subset \Gamma$. We want to show that M is $\mathbb{Z}\Gamma$–projective, and this will prove 1.6 (using Theorem 1.0 again). $\mathbb{Z}\Gamma$–projectivity of M follows, in principle, from the projectivity criterion of [A1] (see also [A3]). We make a few comments in this direction. Let $k = \bar{\mathbb{F}}_p$, and $A \subset \Gamma$ be a p–elementary abelian subgroup. We need the following:

1.9. Lemma. The kA–free complex $C_*(\tilde{X}) \otimes k$ is chain homotopy equivalent to a finite dimensional free kA–complex if and only if $H^*(A;k)$ acts nilpotently on $H^*_A(\tilde{X};k)$.

This result is contained in [A2] (see Assadi [A3] for a summary and further results). The idea is as follows. In [A2], we associate certain homogeneous affine varieties $V_A(C_*(\tilde{X}) \otimes k)$ and $V^\Gamma_A(C_*(\tilde{X}) \otimes k)$ which are algebro–geometric invariants of $C_*(\tilde{X}) \otimes k$. The variety $V_A(C_*(\tilde{X}) \otimes k)$ is constructed from the support of the $H^*(A;k)$–module $H^*_A(\tilde{X};k)$. When $H^*(A;k)$ acts nilpotently on $H^*_A(\tilde{X};k)$, it follows that $V_A(C_*(\tilde{X}) \otimes k) = 0$. Furthermore, $C_*(\tilde{X}) \otimes k$ is shown to be chain homotopy equivalent to a free finite dimensional kA–chain complex if and only if $V^\Gamma_A(C_*(\tilde{X}) \otimes k) = 0$. On the other hand, according to ([A2] Theorem 1.4) for connected kA–complexes with finitely generated cohomology, $V_A(C_*) \cong V^\Gamma_A(C_*)$. These statements together imply Lemma 1.9.

■

As we have seen in Theorem 1.1, we need to consider only prime order subgroups of p–elementary abelian groups $A \subseteq \Gamma$, and show that M is kA–projective, or equivalently, $H^*(A;k)$ acts nilpotently on $H^*_A(\tilde{X};k)$, using 1.9. The proof of the projectivity criterion of [A1] can be modified in this set–up to show that:

1.10. Lemma: $H^*(A;k)$ acts nilpotently on $H^*_A(\tilde{X};k)$ if and only if for each cyclic subgroup $C \subseteq A$, $H^*(C;k)$ acts nilpotently on $H^*_C(\tilde{X};k)$.

This statement, of course, implies Lemma 1.6, using 1.9 again for each C. The proof of Lemma 1.10 is based: (a) The fact that $H^*_C(\tilde{X};k)$ is equipped with an "Steenrod algebra" operation, and (b) the notion of support varieties. The details are similar to the proof of ([A1] Theorem 2.1).

1.11 Corollary. In the situation of Theorem 1.5. if the necessary conditions are satisfied, then there exists a well–defined obstruction $\theta(X) \in \tilde{K}_0(\mathbb{Z}\Gamma)$ such that $\theta(X) = 0$ if and only if X is G–homotopy equivalent to a finite Poincaré complex with a free G–action.

This corollary follows from the general theory of [Wall I] once we have shown that X is finitely dominated via Theorem 1.5. Here, one should remark that $\theta(X)$ can be determined in terms of the G–module $H^2(\tilde{X})$ directly. For example, when $G = \mathbb{Z}_p$, one computes that $H^2(X) \cong I \oplus I \otimes P$, where $I \subset \mathbb{Z}G$ is the augmentation ideal and P is a projective module. Then $\theta(X)$ is the class $[P] \in \tilde{K}_0(\mathbb{Z}G)$ if $\tilde{X}$ is simply–connected. If $\theta(X) = 0$, then one has a finite Poincaré complex, and one can apply M. Freedman's topological surgery in dimension four to discuss the surgery obstruction. It is possible to determine the precise obstructions in this case by studying the intersectin form of $\tilde{X}$. This analysis is carried out for a special class of finite groups in a somewhat different context by I. Hambleton and M. Kreck. Theorem 1.5 holds in higher dimensions as well, although the statement should be suitably modified. These matters will be considered in a future paper.

Finally, we make some remarks about the validity of Conjecture 1.2 under the additional hypothesis that for some sufficiently large n, $\mathrm{Ker}(d : C_n \longrightarrow C_{n-1})$ is a finitely presented $\mathbb{Z}\Gamma$–module (using the previous notation etc.). As we have seen in the proof of Theorem 1.1, the basic step for finite dimensionality up to Γ–equivariant homotopy (of free Γ–chain complexes) is the projectivity of the Γ–module $M \equiv \mathrm{Ker}(d : C_n \longrightarrow C_{n-1})$ for some sufficiently large n. We will mention the relevant algebraic fact below (Lemma 1.12) which together with the projectivity criterion of [A4] Theorem 2.1 prove Conjecture 1.2 (a) in this case (cf. the proof of 1.1 (a) above). The proof of 1.2 (b) proceeds as in Theorem 1.1 (b), replacing elementary abelian subgroups by prime order subgroups in that argument. The following lemma is quite useful in other circumstances as well (see [A3]).

1.12. Lemma. In the above situation, suppose that M is a finitely presented $R\Gamma$–module which is $R\pi$–projective. Then $\mathrm{Hom}_{R\pi}(M,M)$ is RG–projective if and only if $\mathrm{Hom}_R(R \otimes_\pi M, R \otimes_\pi M)$ is RG–projective. (In particular, either condition implies that M is $R\Gamma$–projective.)

Sketch of proof. Suppose $\mathrm{Hom}_R(R \otimes_\pi M, R \otimes_\pi M)$ is RG–projective. It follows that $R \otimes_\pi M$ is also RG–projective. On the other hand the only non–vanishing term in $\mathrm{Tor}_*^{R\pi}(N,M)$ is $\mathrm{Tor}_0^{R\pi}(N,M) \cong N \otimes_\pi M$ for any $R\pi$–module N, since M is $R\pi$–projective. These two facts, together with an argument using a Grothendieck–type spectral sequence: $\mathrm{Tor}_i^{RG}(\mathrm{Tor}_j^{R\pi}(M,-),-) \Rightarrow \mathrm{Tor}_{i+j}^{R\Gamma}(M,-)$ imply that M is $R\Gamma$–flat. Since M is also finitely presented, it follows that M is $R\Gamma$–projective (see e.g. Bourbaki's Commutative Algebra, Ch.I, p.64, Ex. 15). Thus, M is $(R\Gamma,R\pi)$–projective which implies that

$\mathrm{Hom}_{R\pi}(M,M)$ is RG–projective. Conversely, if $\mathrm{Hom}_{R\pi}(M,M)$ is RG–projective, then M is RΓ–projective (being (RΓ,Rπ)–projective and Rπ–projective). It follows easily that $R \otimes_\pi M$ is RG–projective. Consequently $\mathrm{Hom}_R(R \otimes_\pi M, R \otimes_\pi M)$ is also RG–projective.

■

<u>SECTION TWO. THE ALGEBRAIC WHITEHEAD TRANSFER</u>. Let $(\eta) : 1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ be an extension of groups and let u be a section (not a homomorphism necessarily). Here G is a finite group of order g and π and Γ are discrete groups. Let $\underline{A}$ be the category whose objects consists of pairs (M,B) , where M is a finitely generated $\mathbb{Z}\Gamma$–projective module which is free over $\mathbb{Z}\pi$ and B is a finite $\mathbb{Z}\pi$–basis for M . Let $(M_1,B_1) \sim (M_2,B_2)$ if there exists a $\mathbb{Z}\Gamma$–isomorphism $f : M_1 \longrightarrow M_2$ such that f is π–simple with respect to B_1 and B_2. The set of equivalence classes $\underline{A}' = \underline{A}/\sim$ has a monoid structure under direct sum of modules and disjoint union of bases, and $(0,\phi)$ is the neutral element. Let $\underline{R}$ be the submonoid generated by $(\mathbb{Z}\Gamma,u(G))$. Then $Wh_1^T(\pi \longrightarrow \Gamma)$ is defined to be the quotient monoid $\underline{A}'/\underline{R}$. As in [AV] (Proposition 1.1) it follows that $Wh_1^T(\pi \longrightarrow \Gamma)$ is an abelian group. The forgetful functor $(M,B) \longrightarrow M$ induces a homomorphism $\beta : Wh_1^T(\pi \longrightarrow \Gamma) \longrightarrow \tilde{K}_0(\mathbb{Z}\Gamma)$. On the other hand, given $t \in Wh_1(\pi)$, we define $\alpha(t)$ to be the equivalence class of (M,B) where $M = (\mathbb{Z}\Gamma)^k$ and B is obtained from twisting the standrad basis $u(G)^k$ by t , i.e., $id : (M,u(G)^k) \longrightarrow (M,B)$ has π–torsion t . It follows that the sequence $Wh_1(\pi) \xrightarrow{\alpha} Wh_1^T(\pi \longrightarrow \Gamma) \xrightarrow{Tr} \tilde{K}_0(\mathbb{Z}\Gamma)$ is exact. In [AV], this sequence is extended to a five term exact sequence involving the transfers in Wh_1 and $\tilde{K}_0$ where $\Gamma = \pi \times G$ (cf. [AV] Proposition 1.2).

<u>2.1 Proposition</u>. The following sequence is exact.

$$Wh_1(\Gamma) \xrightarrow{Tr} Wh_1(\pi) \xrightarrow{\beta} Wh_1^T(\pi \longrightarrow \Gamma) \xrightarrow{\alpha} \tilde{K}_0(\mathbb{Z}\Gamma) \xrightarrow{tr} \tilde{K}_0(\mathbb{Z}\pi) .$$

Several other properties of Wh_1^T extended from the product case $\Gamma = \pi \times G$ to the present case. Let $\underline{P}_*$ denote the category of bounded finitely generated projective $\mathbb{Z}\Gamma$–complexes. Let $\underline{P}_*^h$ be the category of $\mathbb{Z}\Gamma$–complexes which have the chain homotopy type of a complex in $\underline{P}_*$. Denote by $\underline{A}_*$ the category of pairs (C_*,B_*) where C_* is a complex in $\underline{P}_*$ and $(C_i,B_i) \in \underline{A}$ for all i. For $(C_*,B_*) \in \underline{A}_*$, we define $\chi(C_*,B_*) = \sum_n (-1)^n [C_n,B_n] \in Wh_1^T(\pi \longrightarrow \Gamma)$. If C_* is acyclic, then its torsion $\tau(C_*,B_*) \in Wh_1(\pi)$ is defined.

2.2 Proposition. (a) For $(C_*,B_*) \in \underline{A}_*$, $\chi(C_*,B_*) = \beta(\tau(C_*,B_*))$ if C_* is acyclic.

(b) Let (C_*,B_*) and (C'_*,B'_*) be objects in $\underline{A}_*$ and let $f: C_* \longrightarrow C'_*$ be a $\mathbb{Z}\Gamma$–chain homotopy equivalence. Then $\chi(C'_*,B'_*) = \chi(C_*,B_*) + \beta(\tau(f))$, where $\tau(f)$ is the π–Whitehead torsion of f.

2.3 Proposition and Definition. Let f be a $\mathbb{Z}\pi$–chain homotopy equivalence from a finitely $\mathbb{Z}\pi$–based $\mathbb{Z}\Gamma$–complex D_* to a chain complex $C_* \in \underline{P}_*^h$. Let g be a $\mathbb{Z}\Gamma$–chain homotopy equivalence from C_* to a chain complex C'_* with a $\mathbb{Z}\pi$–basis B'_* such that $(C'_*,B'_*) \in \underline{A}_*$. Then the element $\chi(C'_*,B'_*) - \beta(\tau(g \cdot f)) \in Wh_1^T(\pi \longrightarrow \Gamma)$ does not depend on the choice of (C'_*,B'_*) and g. This element is denoted by $\chi(f)$.

A topological application of this element is based on the following:

2.4 Proposition. Let $D_* \in \underline{D}_*$, $C_* \in \underline{P}_*^h$, and $f: D_* \longrightarrow C_*$ be a $\mathbb{Z}\pi$–homotopy equivalence. Then $\chi(f) = 0$ in $Wh_1^T(\pi \longrightarrow \Gamma)$ if and only if there exists a finitely $\mathbb{Z}\pi$–based projective $\mathbb{Z}\Gamma$–chain complex C'_* and a $\mathbb{Z}\Gamma$–chain homotopy equivalence $g: C_* \longrightarrow C'_*$ such that $g \cdot f$ is π–simple.

2.5 Theorem. Let X be a connected CW complex (of finite type) with $\pi_1(X) = \Gamma$, and let Y be a connected finite CW subcomplex of X with $\pi_1(Y) = \pi_1(X)$. Let $\bar{X}$ and $\bar{Y}$ be the covering space of X and Y with $\pi_1(\bar{X}) = \pi = \pi_1(\bar{Y})$, i.e., $X = \bar{X}/G$ and $Y = \bar{Y}/G$. Let $\bar{\bar{X}}$ be a connected finite CW complex with the commutative diagram:

$$\begin{array}{ccc} \bar{Y} & \hookrightarrow & \bar{\bar{X}} \\ \big\downarrow & & \big\downarrow \alpha \\ Y & \hookrightarrow & X \end{array}$$

such that $\alpha : \bar{\bar{X}} \longrightarrow X$ induces a homotopy equivalence from $\bar{\bar{X}}$ to $\bar{X}$. And suppose that the inclusion of the n–skeleton $(X^{(n)},Y)$ to (X,Y) is a finite domination for some sufficiently large n. Then $\chi(\alpha_* : C_*(\bar{\bar{X}},\bar{Y};\mathbb{Z}\pi) \longrightarrow C_*(X,Y;\mathbb{Z}\Gamma)) = 0$ in $Wh_1^T(\pi \longrightarrow \Gamma)$ if and only if there exists a finite complex $Z \supset Y$ and a homotopy equivalence $g : X \longrightarrow Z$ (rel. Y) such that $g \cdot \alpha : \bar{\bar{X}} \longrightarrow X \longrightarrow Z$ induces a simple homotopy equivalence $\bar{\bar{X}} \longrightarrow \bar{Z}$, where $\bar{Z}$ is the covering space of Z with $\pi_1(\bar{Z}) = \pi$, i.e., $Z = \bar{Z}/G$. Here α_* denotes the composition of the $\mathbb{Z}\pi$–chain homotopy equivalence: $C_*(\bar{\bar{X}},\bar{Y};\mathbb{Z}\pi) \longrightarrow C_*(\bar{X},\bar{Y};\mathbb{Z}\pi)$ and $\mathbb{Z}\Gamma$–chain isomrophism: $C_*(\bar{X},\bar{Y};\mathbb{Z}\pi) \longrightarrow C_*(X,Y;\mathbb{Z}\Gamma)$.

Let $\underline{D}_*$ be the category whose objects are finitely $\mathbb{Z}\pi$–based $\mathbb{Z}\Gamma$–chain complexes D_* such that $D_* \otimes_{\mathbb{Z}} \mathbb{Z}_q$ is $\mathbb{Z}_q\Gamma$–chain homotopic to the trivial complex 0. We wish to use the well–defined element $\chi(-)$ above to define an invariant of $D_* \in \underline{D}_*$. Let R_* be a projective $\mathbb{Z}G$–resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$–module such that $R_0 = \mathbb{Z}G$. Then it turns out that the standard $\mathbb{Z}\pi$–chain map $f_s : D_* \longrightarrow D_* \otimes_{\mathbb{Z}} R_*$ is a $\mathbb{Z}\pi$–chain homotopy equivalence, (f_s is given by $f_s(x) = x \otimes 1 \in D_i \otimes R_0$ for $x \in D_i$, where 1 is the unit of $\mathbb{Z}G = R_0$). Furthermore, $D_* \otimes_{\mathbb{Z}} R_*$ is an object in $\underline{P}_*^h$, and there is an object $(C'_*,B'_*) \in \underline{A}_*$ and a $\mathbb{Z}\Gamma$–chain homotopy equivalence $g : D_* \otimes_{\mathbb{Z}} R_* \longrightarrow C'_*$. Hence we define $\gamma(D_*)$ to be $\chi(f_s) = \chi([C'_*,B'_*]) - \beta(\tau(g \cdot f_s))$. If $\Gamma = \pi \times G$ and G acts trivially on D_*, then $\gamma(D_*)$ depends only on the Reidemeister torsion of D_* in $Wh_1(\pi;\mathbb{Z}_q)$, but this may not hold in general.

<u>SECTION THREE. THE TOPOLOGICAL WHITEHEAD TRANSFER</u>. As pointed out in the Introduction, it is possible to formulate the algebraic construction of Section 2 in terms of spaces, in accordance with the current emphasis on "spaces" rather than "groups". Thus,

we replace the Whitehead groups by Whitehead spaces following Hatcher's higher simple homotopy theory [H]. However, there are some technical points which must be dealt with. For example "the naturally suggested transfer functor" in Hatcher's theory is not a homotopy functor. In [W] Waldhausen introduces a "homotopification" procedure for functors, and this resolves the above–mentioned difficulty. One advantage of this approach is that the 5–term exact sequence of the type introduced in Proposition 2.1 turns out as the lower portion of the homotopy exact sequence of "the transfer fibration" between Whitehead spaces. Another point which should be remarked is that the following approach applies to more general extensions $1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$; e.g., G could be the fundamental group of an aspherical manifold. While it would be more natural and appropriate to present this material in the semisimplicial language, we continue the discussion in the topological context.

To every extension $(\eta) : 1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$, where G is a finite group, one associates the fibration $G \longrightarrow B\pi \longrightarrow B\Gamma$. This is a special case of a compact ANR fibration:

Definition: Let $\mathscr{C}$ be a subcategory of the category of topological spaces. A triple $\eta = (E \xrightarrow{\pi} B)$ is called a compact ANR fibration in $\mathscr{C}$ with the following properties:

(i) π is a proper map;
(ii) $\pi : E \longrightarrow B$ is a Hurewicz fibration;
(iii) all the fibres of π are compact ANR.

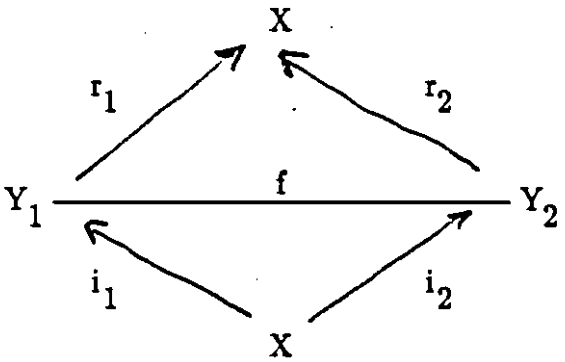
In a combinatorial category, e.g. that of simplicial complexes, we assume that the fibres are finite simplicial complexes.

Examples.

(1) Let K be a finite simplicial complex. The product fibration $B \times K \longrightarrow B$ is a compact ANR fibration.
(2) Any fibre bundle with compact fibres is a compact ANR fibration. For instance, covering spaces with finite groups of deck transformations are compact ANR fibrations.
(3) Let π be a "geometric group", i.e. the fundamental group of a closed aspherical manifold (e.g. closed hyperbolic manifolds), and let $1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow \pi' \longrightarrow 1$ be an extension of groups. Then the fibration $B\pi \longrightarrow B\Gamma \longrightarrow B\pi'$ has compact fibres. In many cases (e.g. if $\Gamma = \pi \times \pi'$) the obstructions for converting this fibration to a

compact ANR fibration vanish. We will see below in such cases, one can define the transfer and consequently the group $Wh_0(\pi \longrightarrow \Gamma)$. If π' is an infinite group, then the procedure of Section 2 does not apply here.

For each topological space X, Hatcher has defined a category $Wh^{PL}(X)$ such that $\pi_0(BWh^{PL}(X)) \cong Wh_1(\pi_1(X))$, where $BWh^{PL}(X)$ is the classifying space of $Wh^{PL}(X)$. We refer the reader to [H] and [W] (in particular § 5) for details and justifications of what follows. We are interested in a modified version of Hatcher's construction. Let $\mathcal{T}op$ be the category of topological spaces and $\mathcal{C}at$ the category of categories. Define a functor $\mathcal{H}^{sp} : \mathcal{T}op \longrightarrow \mathcal{C}at$ as follows. The objects of $\mathcal{H}^{sp}(X)$ consist of diagrams $X \underset{r}{\overset{i}{\rightleftarrows}} Y$, where $X,Y \in \mathcal{T}op$, and i is an inclusion while r is a deformation retraction, and (Y,X) is a relatively finite CW complex. Further, we assume that all maps are cell–like, i.e., the inverse images of points are contractible. A morphism f between $X \underset{i_1}{\overset{r_1}{\rightleftarrows}} Y_1$ and $X \underset{i_2}{\overset{r_2}{\rightleftarrows}} Y_2$ consists of a strictly commutative diagram of cell–like maps:



We call $\mathcal{H}^{sp}(X)$ the special Hatcher–Whitehead category of X and $B\,\mathcal{H}^{sp}(X)$ the special Hatcher–Whitehead space of X. In the combinatorial version of this category, if the reader prefers, we have simplicial complexes and simplicial maps such that the inverse image of every simplex is contractible.

Let $\eta = (E \xrightarrow{\pi} B)$ be a compact ANR fibration. Given $Y \underset{r}{\overset{i}{\rightleftarrows}} B$ in $\mathcal{H}^{sp}(B)$ representing an object x, we define $Pretr(x)$ by the pull–back $r^*\eta = (E' \xrightarrow{\pi} Y)$. Then one has the natural inclusion $i' : E \longrightarrow E'$ and the retraction $r' : \Delta' \longrightarrow E$ covering $r : Y \longrightarrow B$. Moreover, r' is cell–like since r is cell–like and F is a compact ANR. In addition, (E',E) is a relatively finite CW complex. Similar comments apply to the combinatorial case. Thus, we have a candidate for the transfer, namely $(Y \underset{r}{\overset{i}{\rightleftarrows}} B) \longmapsto (E \underset{r'}{\overset{i'}{\rightleftarrows}} E)$ which defines a functor $\mathrm{Pretr} : \mathscr{H}^{sp}(B) \longrightarrow \mathscr{H}^{sp}(E)$.

The technical problems which arise here are:

(a) the classifying space $B\mathscr{H}^{sp}(X)$ is not the Whitehead space of X;

(b) the functor $X \longmapsto B\mathscr{H}^{sp}(X)$ is not a homotopy functor.

According to Waldhausen [W] (pp. 55–58) these difficulties are overcome by using his "homotopification of functors". First, let us remark that if $\mathscr{X}$ is a simplicial object in the category of topological spaces, so that for each $n \geq 0$ X_n is a topological space and the faces and degeneracies $X_{n+1} \xrightarrow{d_i} X_n$ and $X_n \xrightarrow{s_i} X_{n+1}$, $0 \leq i \leq n$ are homotopy equivalences, then the geometric realization $|\mathscr{X}|$ is homotopy equivalent to each X_n. We will use this remark in the following construction.

Let Δ^∞ be the standard semisimplicial object for which $(\Delta^\infty)_n$ is the standard simplex Δ^n and the boundaries and degeneracies are the usual maps $\partial_i : \Delta^n \longrightarrow \Delta^{n-1}$ and $s_i : \Delta^{n-1} \longrightarrow \Delta^n$. To each topological space X, we associate the simplicial object $\Delta^\infty(X)$ such that $\Delta^\infty(X)_n = \Delta^n \times X$ and the boundaries and degeneracies are $\partial_i \times \mathrm{id}_X$ and $s_i \times \mathrm{id}_X$, $0 \leq i \leq n$. Now if F is a functor on the category of topological spaces, we define the simplicial object $F\Delta^\infty(X)$ to have $(F\Delta^\infty(X))_n = F(\Delta^n \times X)$ and $F(\partial_i \times \mathrm{id}_X)$ and $F(s_i \times \mathrm{id}_X)$ for its boundaries and degeneracies. We denote by $hF(X)$ the geometric realization of this simplicial object. The functor $X \longrightarrow hF(X)$ is a homotopy functor, and if F itself is a homotopy functor, then $hF(X)$ and $F(X)$ are homotopy equivalent [W]. Let us call hF the homotopification of F.

<u>3.1 Proposition</u>. The homotopification of $\psi : X \longmapsto B\mathscr{H}^{sp}(X)$ is the functor $X \longmapsto Wh^{PL}(X)$ (as defined by Hatcher) up to homotopy.

Proof. Let $\mathcal{H}^{H}$ be Hatcher's Whitehead category. Then the forgetful functor which forgets the retraction $r : Y \longrightarrow X$ in $\mathcal{H}^{sp}(X)$ yields a functor $\varphi : \mathcal{H}^{sp}(X) \longrightarrow \mathcal{H}^{H}(X)$. This defines $Br : B\mathcal{H}^{sp}(X) \longrightarrow Wh^{PL}(X)$ and one has the commutative diagram:

$$\begin{array}{ccc} B\mathcal{H}^{sp}(X) & \xrightarrow{\quad\varphi\quad} & Wh^{PL}(X) \\ \downarrow & & \downarrow \\ hB\mathcal{H}^{sp}(X) & \xrightarrow{\quad hB\varphi\quad} & hWh^{PL}(X) \end{array}$$

in which α and β are induced by "inclusions", and the bottom row is obtained from the top row by the homotopification procedure above. The map β is a homotopy equivalence, because Wh^{PL} is a homotopy functor and in $Wh^{PL}\Delta^{\infty}(X)$ all the boundaries and degeneracies are homotopy equivalences. Thus the above remark applies to the geometric realization $hWh^{PL}\Delta^{\infty}(X)$.

3.2 Corollary. The map $\varphi : B\mathcal{H}^{sp}(X) \longrightarrow Wh^{PL}(X)$ factors through $hB\mathcal{H}^{sp}(X)$.

Now we apply the above discussion to the functor $Pretr : \mathcal{H}^{sp}(B) \longrightarrow \mathcal{H}^{sp}(E)$ defined above. Apply the homotopification to this functor, to get "hPretr", which we call "the transfer" and denote it by "Tr" or "Tr(η)" if reference to $\eta = (E \xrightarrow{\pi} B)$ is needed. By Proposition 3.1, we have defined $Tr : Wh^{PL}(B) \longrightarrow Wh^{PL}(E)$. Delooping Tr, we get $Tr : Wh(B) \longrightarrow Wh(E)$, where Wh is the delooping of Wh^{PL}. Let $Tr(\eta)$ be the fibre of this natural transformation ([Q] section one), so that we have the fibration $Tr(\eta) \longrightarrow Wh(B) \longrightarrow Wh(E)$.

A particularly interesting situation arises from the following. Let $(e) : 1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ be an extension, where G is a finite group. Then we have the compact ANR–fibration (η) given by the finite covering $G \longrightarrow B\pi \longrightarrow B\Gamma$. Such extensions (e) form the objects of a category $\mathcal{E}$ whose morphisms are homomorphisms of such exact sequences. Thus, one has two naturally defined functors, namely, $(e) \longmapsto Wh_1^T(\pi \longrightarrow \Gamma)$ and $(e) \longmapsto \pi_0(Tr(\eta))$, defined on the category extensions $\mathcal{E}$. Judging from the long exact sequence of the fibration $Tr(\eta) \longrightarrow Wh(B) \longrightarrow Wh(E)$

in homotopy, and comparing it with the 5–term exact sequence of 2.2, it is natural to conjecture the following:

3.3 Conjecture. There is a natural isomorphism between the functors $\pi_0(\mathrm{Tr}(\eta))$ and $\mathrm{Wh}_1^T(\pi \longrightarrow \Gamma)$.

This will imply, of course that there is a long exact sequence of higher Whitehead groups extending the five–term sequence of Proposition 1.2. So far this conjecture has been verified only in special cases, and we plan to take up this subject in another paper.

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