Max-Planck-Institut für Mathematik
Bonn

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Preprint Series 2020 (2)

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MPIM 20-2

On normalizers of maximal tori in classical Lie groups

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Abstract. The normalizer $N_G(H_G)$ of a maximal torus H_G in a semisimple complex Lie group G does not in general allow a presentation as a semidirect product of H_G and the corresponding Weyl group W_G. Meanwhile, splitting holds for classical groups corresponding to the root systems A_ℓ, B_ℓ, D_ℓ. For the remaining classical groups corresponding to the root systems C_ℓ there still exists an embedding of the Tits extension of W_G into normalizer $N_G(H_G)$. We provide an explicit unified construction of the lifts of the Weyl groups into normalizers of maximal tori for classical Lie groups corresponding to the root systems A_ℓ, B_ℓ, D_ℓ using embeddings into general linear Lie groups. For symplectic series of classical Lie groups, we provide an explanation of the impossibility of embedding the Weyl group into the symplectic group. Also explicit formulas for the adjoint action of the lifts of the Weyl groups on $\mathfrak{g} = \mathrm{Lie}(G)$ are given. Finally some examples of the groups closely associated with classical Lie groups are considered.

1 Introduction

Normalizer $N_G(H_G)$ of a maximal torus H_G in a semisimple complex Lie group G allows a presentation as an extension of the corresponding Weyl group W_G by H_G

$$1 \longrightarrow H_G \longrightarrow N_G(H_G) \xrightarrow{p} W_G \longrightarrow 1. \tag{1.1}$$

This extension does not split in general [CWW], [AH]. For the Weyl groups there is a well-known presentation via generators and relations. To obtain the corresponding description of $N_G(H_G)$ one should pick a section of the projection p. A universal solution to this problem was given by Demazure [D] and Tits [T2] in terms of the Tits extension W_G^T of the Weyl group W_G (for recent discussions of the Tits groups see e.g. [N], [DW], [AH]). One way to understand the nature of the Tits extensions is via consideration of the maximal split real form $G(\mathbb{R}) \subset G(\mathbb{C})$ of a complex semisimple Lie group $G(\mathbb{C})$ [GLO].

In this note we consider the special class of classical Lie groups corresponding to the root systems A_ℓ, B_ℓ, C_ℓ and D_ℓ and provide an explicit description of the group $N_G(H_G)$ and lifts of the Weyl group which differs from the one proposed in [D], [T2]. We restrict ourselves to the Lie groups of classical type due to the fact that by simple reasoning, the exact sequence (1.1) is split for all classical groups except symplectic ones.

It is well-known that classical Lie groups may be defined as fixed point subgroups of general Lie groups under appropriate involutions. In particular this allows us to express generators of Weyl groups of the classical Lie groups via generators of the Weyl groups of general linear Lie groups. The explicit lift of the latter Weyl group into the general linear Lie groups is compatible with the action of the involution of the Lie algebra root data and then provides a lift of the Weyl groups for the corresponding classical Lie groups. This way we obtain sections of p in (1.1) for all classical Lie groups. In the case of orthogonal Lie groups corresponding to series B_ℓ and D_ℓ, this provides an embedding of the Weyl groups into the corresponding Lie groups while for symplectic groups we obtain an embedding of the Tits extension. In any case this provides a unified construction of the lifts of the Weyl groups of all classical Lie groups.

The fact that for C_ℓ-series of classical Lie groups the considered construction provides an embedding of the extension of the Weyl group by 2-torsion elements of a maximal torus, called the Tits group but not an embedding of the Weyl group itself looks rather surprising in this context. We propose an explanation for this phenomenon based on non-commutativity of quaternions. The main argument is based on the known fact that while general linear and orthogonal Lie groups are naturally associated with matrix groups over complex and real numbers, symplectic groups allow description in terms of matrix groups over quaternions. This leads to a modification of the standard notion of maximal torus by taking into account the non-commutative nature of quaternions. As a consequence, the notions of normalizer of maximal torus and the Weyl group are also modified. The standard Weyl groups of symplectic Lie groups appear as subgroups of thus defined Weyl groups. The Tits extension of the symplectic Weyl group then arises via construction of the universal cover of $\mathrm{Aut}(\mathbb{H}) = SO_3$ by the three-dimensional spinor group Spin_3. Concretely this might be traced back to the fact that that additional quaternionic unit $\jmath$ squares to minus one.

It is worth mentioning that the standard algorithm of the Gelfand-Zetlin construction of bases in finite-dimensional irreducible representations fails in the case of symplectic Lie groups. This fact obviously reverberates with the absence of the splitting of (1.1) in the symplectic case. We believe that this is not an accidental coincidence and the issue may be clarified using quaternionic geometry.

Let us note in this respect that the description of classical Lie groups in terms of matrix algebras over division algebras might be generalized to other, non-classical Lie groups (see e.g. [B] and references therein). We expect that the (im)possibility of embedding the Weyl groups into the corresponding Lie groups may be elucidated in all these cases, generalizing our considerations for symplectic Lie groups and quaternionic matrix algebras.

In this note we also consider the problem of the construction of a section of p in (1.1) for unimodular linear groups $SL_{\ell+1}$, special orthogonal groups $SO_{\ell+1}$, pinor/spinor groups $\mathrm{Pin}_{\ell+1}$ and $\mathrm{Spin}_{\ell+1}$. Although these groups are not classical Lie groups in the strict sense, they are related to classical Lie groups either via central extensions or via taking unimodular subgroups. In the case of central extension the resulting section is apparently given by a central extension of the corresponding Weyl group. Moreover the case of unimodular subgroups is also covered by central extension of the Weyl group. We provide an explicit description of maximal torus normalizers in terms of generators and relations in all these cases.

The plan of the paper is as follows. In Section 2 we recall required facts on semisimple Lie algebras and groups including normalizers of maximal tori (see also Appendix 10 for a detailed discussion of classical Lie groups). In Section 3 we recall constructions of classical Lie groups as fixed point subgroups of appropriate involutions of general Lie groups after E.Cartan. In Section 4 the structure of normalizers of maximal tori of general linear Lie groups is considered in detail including explicit lifts of the Weyl groups. In Section 5 we present a construction of the lifts of the Weyl groups of the classical Lie groups in terms of those for general Lie groups (see Theorem 5.1). In Section 6 we further clarify the explicit formulas of the previous Section 5 by establishing connections between maximal tori normalizers, the Weyl groups and their lifts for general linear groups and its classical subgroups. In Section 7 the underlying reasons for the special properties of symplectic groups are considered. In Section 8 we extend our analysis to the groups $SL_{\ell+1}$, $SO_{\ell+1}$, $\mathrm{Pin}_{\ell+1}$ and $\mathrm{Spin}_{\ell+1}$. In all these cases we construct explicit sections of p in (1.1) realized by central extensions of the corresponding Weyl groups. Finally in Section 9 we compute the adjoint action of the lifts of the Weyl groups on Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ for all classical Lie groups. Various technical details of the construction are provided in Appendix.

Acknowledgments: The research of the second and of the third authors was partially supported by the RSF grant 16-11-10075. The work of the third author was partially supported by the EPSRC grant EP/L000865/1. The third author is also thankful to the Max Planck Institute for Mathematics in Bonn, where his work on the project was started.

2 Preliminaries on semisimple Lie algebras and groups

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra of $\mathfrak{g}$, so that $\dim(\mathfrak{h}) = \mathrm{rank}(\mathfrak{g}) = \ell$. Let $\Pi = \{\alpha_i,\, i \in I\} \subset \mathfrak{h}^*$ be the set of simple roots of $\mathfrak{g}$, indexed by the set I of vertexes of the Dynkin diagram Γ. Let $\Pi^\vee = \{\alpha_i^\vee,\, i \in I\} \subset \mathfrak{h}$ be the set of corresponding coroots of $\mathfrak{g}$. Let Φ be the set of roots and let $\Phi^\vee$ be the set of co-roots of $\mathfrak{g}$. Then let $(\Pi, \Phi; \Pi^\vee, \Phi^\vee)$ be the root system associated with $\mathfrak{g}$, supplied with a non-degenerate $\mathbb{Z}$-valued pairing

$$\langle\, ,\, \rangle : \quad \Phi \times \Phi^\vee \longrightarrow \mathbb{Z}\,. \tag{2.1}$$

Introduce the weight lattice,

$$\Lambda_W \,=\, \left\{\gamma \in \mathfrak{h}^* \,:\, \langle \gamma,\, \alpha^\vee\rangle \in \mathbb{Z}\,, \forall \alpha^\vee \in \Phi^\vee\right\} \subset \mathfrak{h}^*\,, \tag{2.2}$$

and define the fundamental weights ϖ_i, $i \in I$ by

$$\langle \varpi_i,\, \alpha_j^\vee\rangle \,=\, \delta_{ij}\,. \tag{2.3}$$

Fundamental weights provide a basis of the weight lattice Λ_W and we have

$$\alpha_j \,=\, \sum_{i=1}^{\ell} a_{ij}\varpi_i\,, \qquad a_{ij} \,=\, \langle \alpha_j,\, \alpha_i^\vee\rangle\,, \tag{2.4}$$

where $A = \|a_{ij}\|$ is the Cartan matrix of $\mathfrak{g}$. The root lattice $\Lambda_R \subset \mathfrak{h}^*$ is generated by simple roots $\alpha_i \in \Pi$, $i \in I$ and appears to be a sublattice of the weight lattice Λ_W, so that $\Lambda_R \subset \Lambda_W$. The quotient group Λ_W/Λ_R is a finite group of order

$$|\Lambda_W/\Lambda_R| \,=\, \det(A)\,. \tag{2.5}$$

The co-weight and co-root lattices $\Lambda_W^\vee$, $\Lambda_R^\vee \subset \mathfrak{h}$ in the Euclidean space $\big(\mathfrak{h} = \mathbb{C}^\ell;\, \langle\cdot,\cdot\rangle\big)$ are generated by the fundamental co-weights and simple co-roots, given by

$$\langle \alpha_i,\, \varpi_j^\vee\rangle = \delta_{ij}\,, \qquad \varpi_j^\vee \,=\, \sum_{i=1}^{\ell} c_{ji}\alpha_i^\vee\,, \qquad \alpha_j^\vee \,=\, \sum_{k=1}^{\ell} a_{jk}\,\varpi_k^\vee\,, \tag{2.6}$$

where $C = \|c_{ij}\| = (A^T)^{-1}$ is the inverse transposed Cartan matrix A.

The corresponding Weyl group $W(\Phi)$ is generated by simple root reflections,

$$s_i(\alpha_j) = \alpha_j - \langle \alpha_j, \alpha_i^\vee\rangle\alpha_i \,=\, \alpha_j - a_{ij}\alpha_i\,, \tag{2.7}$$

and its action in $\mathfrak{h}^*$ preserves the root system $\Phi \subset \mathfrak{h}^*$. Each symmetry of the Dynkin diagram $\Gamma = \Gamma(\Phi)$ induces an automorphism of Φ, and the group $\mathrm{Aut}(\Phi)$ of all automorphisms of the root system $\Phi \subset \mathfrak{h}^*$ contains $W(\Phi)$ as a normal subgroup. Moreover, the following holds:

$$\mathrm{Aut}(\Phi) = W(\Phi) \rtimes \mathrm{Out}(\Phi)\,, \tag{2.8}$$

with $\mathrm{Out}(\Phi)$ being a group of symmetries of $\Gamma(\Phi)$.

Let $\{h_i = h_{\alpha_i},\, e_i = e_{\alpha_i},\, f_i = e_{-\alpha_i},\, i \in I\}$ be the standard set of generators of $\mathfrak{g}$:

$$\begin{aligned} &[h_i, h_j] \,=\, 0\,, \\ &\big[h_i,\, e_j\big] \,=\, a_{ij}e_j\,, \qquad \big[h_i,\, f_j\big] \,=\, -a_{ij}f_j\,, \\ &\big[e_i, f_i\big] \,=\, h_i\,, \qquad \big[e_i, f_j\big] \,=\, 0\,, \quad i \neq j\,; \end{aligned} \tag{2.9}$$

$$\mathrm{ad}_{e_i}^{1-a_{ij}}(e_j) \,=\, \mathrm{ad}_{f_i}^{1-a_{ij}}(f_j) \,=\, 0. \tag{2.10}$$

In the following a slightly modified presentation of (2.9) will be useful. Namely, the Lie algebra $\mathfrak{g}$ can be generated by $\{\varpi_i^\vee\,,\, e_i\,,\, f_i\, :\, i \in I\}$ subjected to the following relations:

$$[\varpi_i^\vee,\, e_j] \,=\, e_j\delta_{ij}\,, \qquad [\varpi_i^\vee,\, f_j] \,=\, -f_j\delta_{ij}\,, \qquad [e_i,\, f_j] \,=\, \delta_{ij}\sum_{k=1}^{\ell} a_{jk}\,\varpi_k^\vee\,. \tag{2.11}$$

Now let G_c be a connected compact semisimple Lie group of rank ℓ, such that $\mathfrak{g}_c = \mathrm{Lie}(G_c)$ is the tangent Lie algebra at $1 \in G_c$. Let $G = G_c \otimes \mathbb{C}$ be the complexification of G_c and let $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{g}_c \otimes \mathbb{C}$ be its Lie algebra. Then the Lie algebra $\mathfrak{g}_c$ is generated by the following elements:

$$H_k \,=\, \imath h_k\,, \qquad J_k \,=\, f_k - e_k\,, \qquad P_k \,=\, \imath(e_k + f_k)\,, \qquad k \in I\,. \tag{2.12}$$

In particular, the following relations hold:

$$\big[J_k,\, \imath\varpi_j^\vee\big] \,=\, \delta_{kj} P_k\,, \quad \big[\imath\varpi_j^\vee,\, P_k\big] \,=\, \delta_{kj} J_k\,, \quad [P_k, J_k] \,=\, 2\sum_{i=1}^{\ell} a_{ki}(\imath\varpi_i^\vee)\,. \tag{2.13}$$

Let $H_G \subset G$ be its maximal torus, such that $\mathfrak{h} = \mathrm{Lie}(H_G)$, and let $X^*(H_G)$ be the group of (rational) characters $\chi : H_G \to S^1$. Then the differential $d\chi$ at $1 \in G$ of a character $\chi \in X^*(H_G)$ is a linear form on $\mathfrak{h}$; hence it provides an embedding $X^*(H_G) \subset \mathfrak{h}^*$ as a discrete subgroup (lattice), supplied with a scalar product $\langle\, ,\, \rangle$.

The dual $X_*(H_G) \subset \mathfrak{h}$ of the lattice $X^*(H)$ is isomorphic to $\mathfrak{h}_{\mathbb{Z}}$. Moreover, using the above notations of the (co)root and (co)weight lattices we have

$$\Lambda_R \,\subseteq\, X^*(H_G) \,\subseteq\, \Lambda_W\,, \qquad \Lambda_R^\vee \,\subseteq\, X_*(H_G) \simeq \mathfrak{h}_{\mathbb{Z}} \,\subseteq\, \Lambda_W^\vee\,. \tag{2.14}$$

The adjoint action of maximal torus H_G on the complex Lie algebra $\mathfrak{g}$ provides the Cartan decomposition:

$$\begin{gathered} \mathfrak{g} \,=\, \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\,, \qquad \mathfrak{g}_\alpha \,=\, \big\{X \in \mathfrak{g} : \quad \mathrm{ad}_h(X) \,=\, \alpha(h)X,\ \forall h \in \mathfrak{h}\big\}\,; \\ \mathfrak{g}_\alpha \,=\, \mathbb{C}e_\alpha\,, \qquad \mathfrak{g}_{-\alpha} \,=\, \mathbb{C}f_\alpha\,, \qquad \alpha \in \Phi_+\,. \end{gathered} \tag{2.15}$$

The root system Φ and the lattice $X^*(H_G)$ (or its dual lattice $X_*(H_G) \simeq \mathfrak{h}_{\mathbb{Z}}$) determine a unique (up to isomorphism) connected semisimple Lie group G. In particular, G is simply connected if and only if $X^*(H) \,\cong\, \Lambda_W$.

The center $\mathcal{Z}(G)$ of a complex connected Lie group G allows for the following presentation:

$$\mathcal{Z}(G) \,\simeq\, \Lambda_W^\vee / X_*(H_G) \,\simeq\, X^*(H_G)/\Lambda_R\,, \tag{2.16}$$

and the group $\mathcal{Z}(G) \rtimes \mathrm{Out}(\Phi)$ is isomorphic to the group of symmetries of the extended Dynkin diagram $\widetilde{\Gamma}(\Phi)$. More specifically, in the case of simply-connected G its center is isomorphic to the quotient group (2.5):

$$\mathcal{Z}(G) \,\simeq\, \Lambda_W/\Lambda_R\,. \tag{2.17}$$

In particular, for simply-connected complex Lie groups of types A_ℓ, B_ℓ, C_ℓ and D_ℓ we have

$$\begin{gathered} \mathcal{Z}(A_\ell) \simeq \mathbb{Z}/(\ell+1)\mathbb{Z}\,, \qquad \mathcal{Z}(B_\ell) \,\simeq\, \mathbb{Z}/2\mathbb{Z}\,, \qquad \mathcal{Z}(C_\ell) \,\simeq\, \mathbb{Z}/2\mathbb{Z}\,. \\ \mathcal{Z}(D_\ell) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\,, & \ell \in 2\mathbb{Z} \\ \mathbb{Z}/4\mathbb{Z}\,, & \ell \in 1 + 2\mathbb{Z} \end{cases}\,. \end{gathered} \tag{2.18}$$

Let $\mathrm{Aut}(G)$ be the group of all automorphisms of a connected complex semisimple Lie group G. Its connected component can be identified with the group $\mathrm{Int}(G)$ of inner automorphisms, which is isomorphic to the adjoint group:

$$\mathrm{Int}(G) \,\simeq\, G/\mathcal{Z}(G)\,. \tag{2.19}$$

The quotient group over the connected component is the group of outer automorphisms:

$$\mathrm{Out}(G) \,=\, \mathrm{Aut}(G)/\mathrm{Int}(G)\,. \tag{2.20}$$

In the case of simply-connected Lie group G, the following holds (see e.g. [L], Chapter V Theorem 4.5.B):

$$\mathrm{Out}(G) \,\simeq\, \mathrm{Out}(\Phi)\,. \tag{2.21}$$

2.1 Normalizers of maximal tori

Given a connected semisimple complex Lie group G with a (fixed) maximal torus $H_G \subset G$ of finite rank ℓ, let $N_G = N_G(H_G)$ be the normalizer of the maximal torus and let us write

$$1 \longrightarrow H_G \longrightarrow N_G \stackrel{p}{\longrightarrow} W_G \longrightarrow 1\,. \tag{2.22}$$

The quotient group $W_G := N_G/H_G$ is finite and is isomorphic to the reflection group $W(\Phi_G)$ associated with the corresponding root system Φ_G. The group $W_G = W(\Phi_G)$ allows a presentation by simple root reflections $\{s_i,\, i \in I\}$ from (2.7) as generators subjected to the following relations:

$$s_i^2 = 1, \tag{2.23}$$

$$\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}, \qquad i \neq j \in I, \tag{2.24}$$

where $m_{ij} = 2,3,4,6$ for $a_{ij}a_{ji} = 0,1,2,3$, respectively. Here $a_{ij} = \langle \alpha_j,\, \alpha_i^\vee \rangle$ is the Cartan matrix entry (2.4). Equivalently these relations may be written in the Coxeter form:

$$s_i^2 = 1, \qquad (s_i s_j)^{m_{ij}} = 1\,, \qquad i \neq j \in I\,. \tag{2.25}$$

The exact sequence (2.22) defines the canonical action of W_G on H_G so that the corresponding action on the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H_G)$ is provided by (2.7):

$$s_i(h_j) \,=\, h_j - \langle \alpha_i, \alpha_j^\vee \rangle h_i \,=\, h_j - a_{ji} h_i\,. \tag{2.26}$$

This action preserves the scalar product $\langle\,,\,\rangle$ in $\mathfrak{h}$, which allows to identify Weyl group $W_G = W(\Phi_G)$ with a subgroup in the orthogonal group $O\big(\mathfrak{h},\, \langle\,,\,\rangle\big)$.

Let us stress that the situation is bit different in the case of non-connected groups. Thus in the following we encounter an example of a non-connected group, the orthogonal group $O_{2\ell}$ having two connected components. In this case, the quotient $N_{O_{2\ell}}(H)/H$ is larger then the Weyl group $W_{O_{2\ell}}$ defined by the generators and relations (2.23), (2.24) and contains the outer automorphism of the root system of the simple connected Lie groups $SO_{2\ell}$.

The important fact is that the exact sequence (2.22) does not split in general i.e. N_G is not necessarily isomorphic to the semi-direct product $W_G \ltimes H_G$ (for various details see [D],[T2], [CWW], [AH]). Thus in general we may only pick a suitable section of the projection map

p in (2.22). One such universal section for an arbitrary reductive Lie group was constructed by Tits [T2]. Namely, [T2] says that for general G, there exists a larger subgroup $W_G^T \subset N_G$ containing W_G as a quotient and fitting into the following exact sequence:

$$1 \longrightarrow H_G^{(2)} \longrightarrow W_G^T \longrightarrow W_G \longrightarrow 1 \quad . \tag{2.27}$$

Here $H_G^{(2)} \simeq (\mathbb{Z}/2\mathbb{Z})^\ell$ is the 2-torsion subgroup of the maximal torus H_G in the reductive complex Lie group G. In case the group G is semisimple the Tits group W_G^T allows for the following explicit presentation by the following generators [AH] (see the notations of (2.12))[1]

$$\dot{s}_i = e^{\frac{\pi}{2} J_i} , \qquad i \in I , \tag{2.28}$$

and relations (compare with (2.23), (2.24)):

$$(\dot{s}_i)^2 = e^{\pi \imath h_i} , \qquad \underbrace{\dot{s}_i \dot{s}_j \cdots}_{m_{ij}} = \underbrace{\dot{s}_j \dot{s}_i \cdots}_{m_{ij}} , \quad i \neq j ,$$
$$\mathrm{Ad}_{\dot{s}_i}(h) = s_i(h) , \quad \forall h \in \mathfrak{h} . \tag{2.29}$$

Using this presentation we readily observe that the Tits group W_G^T is not only a subgroup of the normalizer N_G, but is a subgroup of the corresponding compact group $G_c \subset G$, and more precisely of the normalizer subgroup $N_{G_c}(H_c)$ of the maximal torus H_c in the compact group G_c.

The true meaning of this construction is rather elusive; in [GLO] we have proposed some underlying reasons for the existence of this Tits construction. Below we will follow another direction and consider only the case of classical Lie groups G. Recall that a complex classical Lie group is a simple reductive Lie group allowing embedding in the general Lie group as a fixed point subgroup of an involution. Equivalently the classical groups may be defined as a stabilizer subgroup of bilinear forms. Essentially they are exhausted by the following families of groups

$$GL_{\ell+1}(\mathbb{C}), \qquad O_{2\ell+1}(\mathbb{C}), \qquad \mathrm{Sp}_{2\ell}(\mathbb{C}), \qquad O_{2\ell}(\mathbb{C}), \tag{2.30}$$

corresponding to the series A_ℓ, B_ℓ, C_ℓ and D_ℓ of Dynkin diagrams. Let us stress that we distinguish the classical Lie groups *per se* from their various cousins like $SL_{\ell+1}(\mathbb{C})$, $PSL_{\ell+1}(\mathbb{C})$, $\mathrm{Spin}_{\ell+1}(\mathbb{C})$, the metaplectic group $\mathrm{Mp}_{2\ell}(\mathbb{C})$ et cet.

In the case of classical Lie groups the results of [D], [T2], [CWW], [AH] might be formulated as the following statement:

- *In the case of the general linear group $GL_{\ell+1}(\mathbb{C})$ there is a section to the exact sequences (2.22) and (2.27), so that $N_{GL_{\ell+1}(\mathbb{C})}$ (and the Tits subgroup $W^T_{GL_{\ell+1}}$) contains the Weyl group $W(A_\ell) = W_{GL_{\ell+1}}$ associated with the root system A_ℓ;*

[1] In the following expressions we write the group elements as exponential of the linear combinations of Lie algebra generators using the canonical exponential map $\exp : \mathrm{Lie}(G) \to G$. Note that map has a non-trivial kernel so that for instance in the case $G = GL_{\ell+1}(\mathbb{C})$ we have $e^{2\pi J_i} = e^{2\pi P_i} = e^{2\pi \imath h_i} = 1$ for each $i \in I$.

- *In the case of the orthogonal groups* $O_{\ell+1}(\mathbb{C})$ *there exists a section to the corresponding exact sequence* (2.27), *so that* $W^T_{O_{\ell+1}}$ *contains a subgroup isomorphic to* $W_{O_{\ell+1}}$*;*
- *In the case of the symplectic group* $\mathrm{Sp}_{2\ell}(\mathbb{C})$ *no section of the corresponding exact sequence* (2.27) *exists.*

3 Classical Lie groups via involutive automorphisms

The definition of classical Lie groups as simple Lie groups isomorphic to fixed point subgroups with respect to certain involutive automorphism in $GL_{\ell+1}(\mathbb{C})$ goes back to E. Cartan and relevant expositions of the subject can be found in [H] and [L]. Below we recall the basics of this construction.

Given the general linear group $GL_{\ell+1}(\mathbb{C})$ and a maximal torus $H_{\ell+1}$ identified with the subgroup of diagonal elements, let us describe its group of outer automorphisms $\mathrm{Out}(GL_{\ell+1})$. Given Dynkin diagram $\Gamma(A_\ell)$, let us choose a natural ordering of the set of its vertices $I=\{1,2,\ldots,\ell\}$, and consider the outer automorphism of the root system induced by the Dynkin diagram symmetry:

$$\iota:\quad I\longrightarrow I\,,\qquad i\longmapsto \ell+1-i\,. \tag{3.1}$$

It defines the automorphism of the set Π_{A_ℓ} of simple roots, and therefore can be extended to an (outer) automorphism of the root system $\Phi(A_\ell)$. Thus the extended automorphism ι obviously preserves the lattice $\mathfrak{h}_{\mathbb{Z}}$ and can be lifted to the following involutive automorphism of the Lie group $GL_{\ell+1}(\mathbb{C})$ (we keep the same notation ι):

$$\iota:\quad g\longmapsto (g^\tau)^{-1}\,,\qquad g\in GL_{\ell+1}(\mathbb{C})\,, \tag{3.2}$$

where τ is the reflection at the opposite diagonal:

$$(g^\tau)_{ij}=g_{\ell+2-j,\ell+2-i},\qquad g=\|g_{ij}\|\in GL_{\ell+1}(\mathbb{C}). \tag{3.3}$$

Note that the automorphism ι respects the maximal torus $H_{\ell+1}\subset GL_{\ell+1}(\mathbb{C})$ given by invertible diagonal matrices. Given the involutive automorphism $\iota\in\mathrm{Out}(GL_{\ell+1})$ we have a family of involutive automorphisms still respecting the maximal torus $H_{\ell+1}$. Such automorphisms are obtained by combining ι with arbitrary inner automorphisms $\mathrm{Ad}_{\mathcal{S}}\in\mathrm{Int}(GL_{\ell+1})$, where $\mathcal{S}\in N_{GL_{\ell+1}(\mathbb{C})}(H_{\ell+1})$ is subjected to the condition $\mathcal{S}\mathcal{S}^\iota\in\mathcal{Z}(GL_{\ell+1})$. Each such $\theta=\mathrm{Ad}_{\mathcal{S}}\circ\iota$ defines the corresponding fixed points subgroups

$$G_\theta\;=\;(GL_{\ell+1}(\mathbb{C}))^\theta\,. \tag{3.4}$$

Classification of the corresponding subgroups is due to E. Cartan (see [H] Chapter IX, and [L] Chapter VII for details). A crucial fact following from his theory is that the requirement of G being a simple reductive Lie group imposes strong restrictions on possible choice of inner automorphism $\mathrm{Ad}_{\mathcal{S}}$. The corresponding list of simple reductive groups is exhausted by the following cases

$$
\begin{aligned}
O_{2\ell+1} &= \left\{ g \in GL_{2\ell+1} : \quad \theta_B(g) = g \right\}, \\
\mathrm{Sp}_{2\ell} &= \left\{ g \in GL_{2\ell} : \quad \theta_C(g) = g \right\}, \\
O_{2\ell} &= \left\{ g \in GL_{2\ell} : \quad \theta_D(g) = g \right\},
\end{aligned} \tag{3.5}
$$

where θ_G are the involutive automorphisms of general linear group

$$
\theta_G : \quad g \longmapsto \mathcal{S}_G \, g^\iota \, \mathcal{S}_G^{-1} , \tag{3.6}
$$

with the diagonal matrices S_G given by

$$
\begin{aligned}
\mathcal{S}_B &= \mathrm{diag}(1, -1, \ldots, 1) \in GL_{2\ell+1} , \qquad \det(\mathcal{S}_B) = (-1)^\ell ; \\
\mathcal{S}_C &= \mathrm{diag}(1, -1, \ldots, 1, -1) \in GL_{2\ell} , \qquad \det(\mathcal{S}_C) = (-1)^\ell ; \\
\mathcal{S}_D &= \mathcal{S}_C \cdot \eta , \qquad \eta = \mathrm{diag}(\mathrm{Id}_\ell, -\mathrm{Id}_\ell) , \qquad \det(\mathcal{S}_D) = 1 .
\end{aligned} \tag{3.7}
$$

There is also the trivial case of $\theta_G = \mathrm{id}$, which corresponds to the general linear Lie group $GL_{\ell+1}(\mathbb{C})$ itself. In the following we will use the term "*classical group*" in the narrow sense by excluding the trivial case of general linear groups.

Let us note that the construction of classical Lie groups via involutive automorphims described above differs from a more traditional one where instead of the reflection at the opposite diagonal (3.3) the standard transposition is used. For instance elements of the orthogonal subgroup $O_{\ell+1} \subset GL_{\ell+1}(\mathbb{C})$ are defined by the condition $g = (g^t)^{-1}$ where

$$
(g^t)_{ij} = g_{j,i}, \qquad g = \|g_{ij}\| \in GL_{\ell+1}(\mathbb{C}). \tag{3.8}
$$

Although two constructions are equivalent (one may be obtained from another by adjoint action of a certain group element of the general linear group) the construction (3.5) is more convenient when dealing with symmetries of the Dynkin diagrams and abstract root data [DS]. Thus in the following we use (3.5) as our basic definition.

4 The normalizer of maximal torus in $GL_{\ell+1}$

In the case of general linear Lie groups $GL_{\ell+1}$ the exact sequence (2.22) is split by simple reasons. Indeed each element $g \in N_{\ell+1}$ induces an (inner) automorphism Ad_g of the Lie algebra $\mathfrak{gl}_{\ell+1}(\mathbb{C})$, preserving the Cartan subalgebra $\mathfrak{h}$ as well as the scalar product $\langle \, , \rangle$ in it. This defines a group homomorphism from $N_{\ell+1}$ into the orthogonal group $O\big(\mathfrak{h}_{\mathbb{C}}, \langle \, , \rangle\big)$, so that its image is the Weyl group:

$$
N_{\ell+1}/\mathcal{Z}(GL_{\ell+1}) \longrightarrow W_{GL_{\ell+1}} \subset O_{\ell+1}(\mathbb{C}) \subset GL_{\ell+1}(\mathbb{C}) . \tag{4.1}
$$

This yields a canonical embedding of the Weyl group into the orthogonal group, which gives rise to the homomorphism $W_{GL_{\ell+1}} \to GL_{\ell+1}(\mathbb{C})$.

This argument implies the following explicit presentation of the lifts of the simple root generators $s_i \in W(A_\ell) = W_{GL_{\ell+1}}$, $1 \leq i \leq \ell$ as elementary permutation matrices

$$s_i \longmapsto S_i = \begin{pmatrix} \mathrm{Id}_{i-1} & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & \mathrm{Id}_{\ell-i} \end{pmatrix} \in N_{\ell+1} , \qquad S_i^2 = 1 , \tag{4.2}$$

satisfying the standard Coxeter relations

$$(S_iS_j)^2 = 1 , \quad |i-j| > 1 , \qquad (S_iS_j)^3 = 1 , \quad |i-j| = 1 , \tag{4.3}$$

of the generators of permutation group $\mathfrak{S}_{\ell+1}$.

The lifts S_i of the generators s_i of the Weyl group $W_{GL_{\ell+1}}$ should be compared with the generators of the corresponding Tits group $W^T_{GL_{\ell+1}}$ defined as follows. For the reductive group $GL_{\ell+1}(\mathbb{C})$ the Tits group $W^T_{GL_{\ell+1}} \subset N_{\ell+1}$ is given by the extension

$$1 \longrightarrow H^{(2)}_{\ell+1}(\mathbb{C}) \longrightarrow W^T_{GL_{\ell+1}} \longrightarrow W_{GL_{\ell+1}} \longrightarrow 1 \ , \tag{4.4}$$

where $H^{(2)}_{\ell+1}$ is the 2-torsion subgroup of the maximal torus $H_{\ell+1}$. Let $\{e_{kk},\, 1 \leq k \leq (\ell+1)\}$ be the bases of the coweight lattice of $GL_{\ell+1}(\mathbb{C})$ corresponding to the diagonal matrices with only one non-zero element being a unit on the diagonal (it might be expressed through the fundamental coweights as follows $e_{kk} = p_k^\vee - p_{k-1}^\vee$, see (10.14)). Then $H^{(2)}_{\ell+1}$ is generated by the elements

$$T_k := e^{\pi\imath e_{kk}} , \quad 1 \leq k \leq (\ell+1) . \tag{4.5}$$

The group $W^T_{GL_{\ell+1}}$ is generated by (2.28) for the roots system A_ℓ

$$\dot{s}_i = e^{\frac{\pi}{2}J_i} = \begin{pmatrix} \mathrm{Id}_{i-1} & & \\ & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & \\ & & \mathrm{Id}_{\ell-i} \end{pmatrix} \in N_{\ell+1} , \quad (\dot{s}_i)^2 = T_iT_{i+1}^{-1} , \tag{4.6}$$

together with an additional central element given by the product of all T_k, $k = 1, \ldots, \ell+1$. Here we write down the matrix form of generators $\dot{s}_i$ using the standard faithful representation of $GL_{\ell+1}(\mathbb{C})$. Comparing the matrix forms we arrive at the following relations:

$$S_i = T_i\dot{s}_i = \dot{s}_iT_{i+1} , \qquad 1 \leq i \leq \ell . \tag{4.7}$$

This leads to another presentation of the Tits group.

Lemma 4.1 *The elements*

$$S_i , \quad 1 \leq i \leq \ell \qquad \text{and} \qquad T_k , \quad 1 \leq k \leq (\ell+1) , \tag{4.8}$$

satisfying the relations

$$\begin{gathered} S_i^2 = T_k^2 = 1 , \qquad T_kT_n = T_nT_k , \\ (S_iS_j)^2 = 1 , \quad |i-j| > 1 , \qquad (S_iS_j)^3 = 1 , \quad |i-j| = 1 , \\ S_iT_k = T_{s_i(k)}S_i , \end{gathered} \tag{4.9}$$

provide a presentation of the Tits group $W^T_{GL_{\ell+1}}$.

Proof : The only non-obvious relation in the last line of (4.9) follows from explicit matrix computation:

$$T_iS_iT_i^{-1} \,=\, T_{i+1}S_iT_{i+1}^{-1} \,=\, S_iT_iT_{i+1} \,=\, T_iT_{i+1}S_i\,, \qquad 1\leq i\leq \ell\,. \tag{4.10}$$

□

In Section 3 we have introduced involutive automorphisms $\theta_G \in \mathrm{Aut}(GL_{\ell+1}(\mathbb{C}))$ (3.5), (3.6), (3.7) defining the classical Lie groups. The involutions θ_G do not act on the lift of $W_{GL_{\ell+1}}$ defined by the generators S_i but do act on its extension given by the Tits group. Explicitly we have

$$\begin{aligned} \theta_B\,:&\qquad \dot{s}_i \longmapsto \dot{s}_{2\ell+1-i}\,, \qquad T_i \longmapsto T_{2\ell+2-i}\,,\\ \theta_C\,:&\qquad \dot{s}_i \longmapsto \dot{s}_{2\ell-i}\,, \qquad T_i \longmapsto T_{2\ell+1-i}\,,\\ \theta_D\,:&\qquad \dot{s}_i \longmapsto \dot{s}_{2\ell-i}\,, \qquad T_i \longmapsto T_{2\ell+1-i}\,. \end{aligned} \tag{4.11}$$

Therefore, thus defined automorphism $\theta_G \in \mathrm{Aut}(W^T_{GL_{\ell+1}})$ preserves the normal subgroup $H^{(2)}_{\ell+1} = \langle T_k,\, 1\leq k\leq \ell+1\rangle$.

Let us describe the action of involutions θ_G in terms of another set of generators of $W^T_{GL_{\ell+1}}$; we introduce new elements given by

$$\overline{S}_i \,=\, T_iS_iT_i^{-1} \,=\, T_{i+1}S_iT_{i+1}^{-1} \,=\, \begin{pmatrix} \mathrm{Id}_{i-1} & & \\ & \begin{smallmatrix} 0 & -1\\ -1 & 0\end{smallmatrix} & \\ & & \mathrm{Id}_{\ell-i}\end{pmatrix}\,, \quad \overline{S}_i \,=\, (\overline{S}_i)^{-1}\,. \tag{4.12}$$

Lemma 4.2 *The elements* S_i, $\overline{S}_i \in W^T_{GL_{\ell+1}}$ *satisfy the following relations:*

$$\begin{aligned} S_i^2 \,=\, \overline{S}_i^2 \,=\, 1\,, \qquad S_i\overline{S}_i = \overline{S}_iS_i \,=\, T_iT_{i+1}\,, \qquad 1\leq i\leq\ell\,;\\ (S_iS_j)^2 \,=\, (S_i\overline{S}_j)^2 \,=\, (\overline{S}_i\overline{S}_j)^2 \,=\, 1\,, \qquad |i-j|>1\,,\\ (S_iS_j)^3 \,=\, (\overline{S}_i\overline{S}_j)^3 \,=\, 1\,, \quad S_iS_jS_i \,=\, \overline{S}_j\overline{S}_i\overline{S}_j\,, \qquad |i-j|=1\,. \end{aligned} \tag{4.13}$$

Proof : The relations follow from similar relations for the symmetric group (2.24), (2.23) and (2.25). The latter (braid) relation can be checked using

$$\overline{S}_i \,=\, \dot{s}_iT_i \,=\, T_{i+1}\dot{s}_i\,, \tag{4.14}$$

and the braid relation (2.29) in the Tits group. □

The action of the involutions θ_G (4.11) may be written as follows

$$\begin{aligned} \theta_B(S_i) \,=\, \overline{S}_{2\ell+1-i}\,, \qquad \theta_C(S_i) \,=\, \overline{S}_{2\ell-i}\,, \quad 1\leq i\leq\ell\,,\\ \theta_D(S_i) \,=\, \overline{S}_{2\ell-i}\,, \quad 1\leq i<\ell\,, \qquad \theta_D(S_\ell) \,=\, S_\ell\,. \end{aligned} \tag{4.15}$$

Note that the two sets $\{S_i\}$ and $\{\overline{S}_i\}$ of generators define two different embeddings of the Weyl group $W_{GL_{\ell+1}}$ into the Tits group $W^T_{GL_{\ell+1}}$ (and thus in the normalizer $N_{\ell+1}$) which are interchanged by involutions θ_G. Let us stress that for example it is possible to find an embedding $W_{GL_{\ell+1}} \subset GL_{\ell+1}(\mathbb{C})$ such that the image of $W_{GL_{\ell+1}}$ is invariant with respect to $\theta_{O_{\ell+1}}$. Indeed this fact is obvious in the case of the realization of the orthogonal subgroup $O_{\ell+1} \subset GL_{\ell+1}(\mathbb{C})$ based on (3.8) because for S_i defined by (4.2) we have $S_i \in O_{\ell+1}$. The same fact for the realization (3.5) follows from the equivalence of the two realizations. However the resulting matrix expressions for the lifts of s_i are more involved and require the use of complex numbers while in our realization the matrix entries are elements of the set $\{-1, 0, +1\}$. Thus in the following we will use the pair $\{S_i\}$, $\{\overline{S}_i\}$ of lifts related by (4.15).

5 Classical groups and their Weyl groups

For each complex classical Lie group $G = GL_{\ell+1}(\mathbb{C})^{\theta_G}$ the corresponding involution $\theta_G \in \mathrm{Aut}(GL_{\ell+1})$ respecting the embedding $H_{\ell+1} \subset GL_{\ell+1}(\mathbb{C})$ of the diagonal maximal torus $H_{\ell+1}$ naturally acts on the Lie algebra $\mathfrak{gl}_{\ell+1}(\mathbb{C})$, the corresponding root system $\Phi_{GL_{\ell+1}}$, normalizer $N_{\ell+1}(H_{\ell+1})$ of $H_{\ell+1}$ and the Weyl group $W_{GL_{\ell+1}} = N_{\ell+1}(H_{\ell+1})/H_{\ell+1}$. Both the maximal torus H_G of G, normalizer $N_G(H_G)$ and its quotient $W_G = N_G(H_G)/H_G$ can be expressed in terms of the corresponding objects of the underlying linear group $GL_{\ell+1}$. Let us give explicit expressions for the generators of the group of automorphisms $\mathrm{Aut}(\Phi_G)$ of the root systems and the corresponding Weyl groups in terms of the generators of the Weyl group of the corresponding general linear Lie group.

Proposition 5.1 *The following explicit description of the generators of the group* $\mathrm{Aut}(\Phi_G)$ *of automorphisms of the classical Lie algebra root systems* Φ_G *and the corresponding Weyl groups holds:*

- *For* $G = O_{2\ell+1}$ *one has* $\mathrm{Aut}(\Phi(B_\ell)) = W(\Phi(B_\ell))$, *and the simple root generators* $s_i^{B_\ell} \in W(\Phi(B_\ell))$ *can be expressed in terms of* $s_i^{A_{2\ell}} \in W(\Phi(A_{2\ell}))$ *as follows:*

$$\begin{aligned} s_1^{B_\ell} &= s_\ell^{A_{2\ell}} s_{\ell+1}^{A_{2\ell}} s_\ell^{A_{2\ell}} = s_{\ell+1}^{A_{2\ell}} s_\ell^{A_{2\ell}} s_{\ell+1}^{A_{2\ell}}, \\ s_k^{B_\ell} &= s_{\ell+1-k}^{A_{2\ell}} s_{\ell+k}^{A_{2\ell}} = s_{\ell+k}^{A_{2\ell}} s_{\ell+1-k}^{A_{2\ell}}, \quad 1 < k \leq \ell . \end{aligned} \tag{5.1}$$

- *For* $G = \mathrm{Sp}_{2\ell}$ *one has* $\mathrm{Aut}(\Phi(C_\ell)) = W(\Phi(C_\ell))$, *and the simple root generators* $s_i^{C_\ell} \in W(\Phi(C_\ell))$ *can be expressed in terms of* $s_i^{A_{2\ell-1}} \in W(\Phi(A_{2\ell-1}))$ *as follows:*

$$s_1^{C_\ell} = s_\ell^{A_{2\ell-1}}, \qquad s_k^{C_\ell} = s_{\ell+1-k}^{A_{2\ell-1}} s_{\ell-1+k}^{A_{2\ell-1}} = s_{\ell-1+k}^{A_{2\ell-1}} s_{\ell+1-k}^{A_{2\ell-1}}, \quad 1 < k \leq \ell . \tag{5.2}$$

- *For* $G = O_{2\ell}$ *one has* $\mathrm{Aut}(\Phi(D_\ell)) = W(\Phi(D_\ell)) \rtimes \mathrm{Out}(\Phi(D_\ell))$. *The simple root generators* $s_i^{D_\ell} \in W(\Phi(D_\ell))$ *together with the generator* $R \in \mathrm{Out}(\Phi(D_\ell)) = \mathbb{Z}/2\mathbb{Z}$ *can be expressed in terms of* $s_i^{A_{2\ell-1}} \in W(\Phi(A_{2\ell-1}))$ *as follows:*

$$\begin{aligned} s_1^{D_\ell} &= s_\ell^{A_{2\ell-1}} s_{\ell-1}^{A_{2\ell-1}} s_{\ell+1}^{A_{2\ell-1}} s_\ell^{A_{2\ell-1}} = s_\ell^{A_{2\ell-1}} s_{\ell+1}^{A_{2\ell-1}} s_{\ell-1}^{A_{2\ell-1}} s_\ell^{A_{2\ell-1}}, \\ s_k^{D_\ell} &= s_{\ell+1-k}^{A_{2\ell-1}} s_{\ell-1+k}^{A_{2\ell-1}} = s_{\ell-1+k}^{A_{2\ell-1}} s_{\ell+1-k}^{A_{2\ell-1}}, \quad 1 < k \leq \ell ; \\ R &= s_\ell^{A_{2\ell-1}} . \end{aligned} \tag{5.3}$$

Proof : We verify the assertion via case by case study in Lemmas 10.1, 10.4 and 10.7 in the following subsections. □

The Weyl group $W(A_\ell)$ allows embedding into the Lie group $GL_{\ell+1}$ as was discussed in Section 4. Therefore it is now natural to ask for a possibility to lift the presentations (5.1), (5.2), (5.3) of Weyl groups W_G into the corresponding classical groups $G = GL_{\ell+1}(\mathbb{C})^{\theta_G}$ via the pair of lifts of the Weyl group $W_{GL_{\ell+1}}$ into the general linear group $GL_{\ell+1}(\mathbb{C})$ considered previously. For the Tits extension $W^T_{GL_{\ell+1}}$ the following result is easily checked.

Proposition 5.2 *The explicit presentation for the Tits group* W^T_G *of the classical Lie group* $G = (GL_{\ell+1})^{\theta_G}$ *may be provided by replacement of all* s_i*'s by* $\dot{s}_i$*'s in* (5.1), (5.2) *and* (5.3).

Proof : We give the case by case verification in Lemmas 10.1, 10.2 and 10.3 by straightforward computation using standard faithful representations (3.5). □

For the Weyl groups we can not expect such a simple answer. Indeed we know that although for the groups $O_{2\ell}$ and $O_{2\ell+1}$, the corresponding Weyl groups allow embedding in the corresponding classical group this is not so for $\mathrm{Sp}_{2\ell}$. So for the case of the Weyl groups we have the following result.

Theorem 5.1 *Given a classical group* G *with its maximal torus* $H_G \subset G$, *let* S_i, $\overline{S}_i$, $1 \leq i \leq \ell$ *be the generators* (4.2), (4.12) *of the Tits group* $W^T_{GL_{\ell+1}}$ *and let* T_k, $1 \leq k \leq \ell+1$ *be the elements* (4.5). *Then the following holds:*

- *In the case* $G = O_{2\ell+1}$ *the generators* S^B_i *defined by*

$$\begin{aligned} S^B_1 &= S_{\ell+1}S_\ell S_{\ell+1} = S_\ell S_{\ell+1} S_\ell \,, \\ S^B_k &= S_{\ell+1-k}\overline{S}_{\ell+k} \,, \qquad 1 < k \leq \ell \,, \end{aligned} \tag{5.4}$$

  are θ_B*-invariant and generate a finite group isomorphic to* $W(B_\ell) = W_{O_{2\ell+1}}$.

- *In the case* $G = O_{2\ell}$ *the generators* S^D_i *defined by*

$$\begin{aligned} S^D_1 &= S_\ell S_{\ell-1}\overline{S}_{\ell+1}S_\ell \,, \\ S^D_k &= S_{\ell+1-k}\overline{S}_{\ell-1+k} \,, \qquad 1 < k \leq \ell \,, \end{aligned} \tag{5.5}$$

  are θ_D*-invariant and generate a finite group isomorphic to* $W(D_\ell) = W_{O_{2\ell}}$.

- *In the case* $G = \mathrm{Sp}_{2\ell}$ *the Tits group* $W^T_{Sp_{2\ell}}$ *is generated by* θ_C*-invariant generators*

$$\begin{aligned} S^C_1 &= T_\ell S_\ell \,, \\ S^C_k &= S_{\ell+1-k}\overline{S}_{\ell-1+k} \,, \qquad 1 < k \leq \ell \,, \end{aligned} \tag{5.6}$$

  while the group of θ_C*-invariant combinations of* S_i, $\overline{S}_i$, $1 \leq i \leq \ell$ *generated by*

$$\begin{aligned} \tilde{S}^C_1 &= S_\ell \overline{S}_\ell = T_\ell T^{-1}_{\ell+1} \,, \\ S^C_k &= S_{\ell+1-k}\overline{S}_{\ell-1+k} \,, \qquad 1 < k \leq \ell \,, \end{aligned} \tag{5.7}$$

  is isomorphic to a proper subgroup of the Weyl group $W(C_\ell)$.

Proof: We propose the detailed proof of the assertion via case by case study in Propositions 10.1, 10.2 and 10.3 below. □

6 Description of θ_G-invariants

The explicit expressions for the generators of the Weyl and Tits groups of the classical groups presented above implies some general relations between maximal torus normalizers, Weyl groups and Tits groups for general linear groups $GL_{\ell+1}(\mathbb{C})$ and their classical subgroups $G = (GL_{\ell+1}(\mathbb{C}))^{\theta_G}$. In this section we prove a set of such relations. First let us note the following property of the involutions θ_G defining classical Lie groups G.

Lemma 6.1 *The centralizer of the fixed point subset $H_{\ell+1}^{\theta_G}$ in $GL_{\ell+1}(\mathbb{C})$ is $H_{\ell+1}$.*

Proof We shall solve the set of equation for $g \in GL_{\ell+1}(\mathbb{C})$

$$gh = hg, \qquad h \in H_{\ell+1}^{\theta_G}. \tag{6.1}$$

Due to the obvious isomorphism $H_{\ell+1}^{\theta_G} = H_{\ell+1}^{\iota}$ we may use the basic automorphism ι (3.3) instead of θ_G in (6.1). Let $H_{\ell+1} \subset GL_{\ell+1}$ be the diagonal subgroup. Explicitly, elements $g \in C_{GL_{\ell+1}}(H_{\ell+1}^{\iota})$ of the centralizer $C_{GL_{\ell+1}}(H_{\ell+1}^{\iota})$ of invariant subtorus $H_{\ell+1}^{\iota} \subset H_{\ell+1}$ satisfy the conditions

$$x_i g_{ij} = g_{ij} x_j \,, \tag{6.2}$$

where

$$g = \|g_{ij}\|, \qquad h = \mathrm{diag}(x_1, x_2, \ldots, x_{\ell+1})\,, \tag{6.3}$$

with the additional condition $x_{\ell+2-i} = x_i^{-1}$ reflecting ι-invariance of h. Since g does not depend on x_i and there are no universal x-independent relations between entries of h (although they are subjected to quadratic relations $x_i x_{\ell+2-i} = 1$) we infer $g_{ij} = 0$, $i \neq j$ i.e. $g \in H_{\ell+1}$. □

As we already notice the definition of the Weyl group as a quotient of the normalizer $N_G(H_G)$ by H_G is not equivalent to the definition of the Weyl group via generators and relations (2.23), (2.24) for non-connected classical Lie groups. Precisely the group $N_G(H_G)/H_G$ for all classical Lie groups obtained as fixed point subgroups of the general linear group (we exclude the case of trivial involution) may be identified with the group of automorphims of the corresponding root system

$$N_G(H_G)/H_G \simeq \mathrm{Aut}(\Phi_G)\,. \tag{6.4}$$

This group fits into the following exact sequence

$$1 \longrightarrow W_G \longrightarrow \mathrm{Aut}(\Phi_G) \longrightarrow \mathrm{Out}(\Phi_G) \longrightarrow 1 \tag{6.5}$$

and we have

$$\mathrm{Aut}(B_\ell) \simeq W(B_\ell), \qquad \mathrm{Aut}(C_\ell) \simeq W(C_\ell), \qquad \mathrm{Aut}(D_\ell) \simeq W(D_\ell) \times (\mathbb{Z}/2\mathbb{Z}). \qquad (6.6)$$

Let us stress that the notation $\mathrm{Out}(\Phi_G)$ is natural for connected simple groups where indeed this group is the factor of all automorphims over inner automorphisms. In the case of non-connected groups the whole group $\mathrm{Aut}(\Phi_G)$ is realized by inner automorphims (see a detailed discussion of the group $O_{2\ell}$ in Section 10.4).

Proposition 6.1 *Let θ be the involution in $GL_{\ell+1}$ defining a classical Lie group G with maximal torus H_G and the Weyl group W_G. Then the following isomorphisms hold*

$$H_{\ell+1}^{\theta_G} = H_G, \qquad (6.7)$$

$$(N_{\ell+1}(H_{\ell+1}))^{\theta_G} = N_G(H_G), \qquad (6.8)$$

$$\left(W_{GL_{\ell+1}}\right)^{\theta_G} = \mathrm{Aut}(\Phi_G). \qquad (6.9)$$

Proof : Any two θ-fixed elements of $H_{\ell+1}$ define a pair of mutually commuting elements of G. Thus we have an embedding $H_{\ell+1}^{\theta} \subset H_G$. By Lemma 6.1 any element commuting with $H_{\ell+1}^{\theta}$ shall be in $H_{\ell+1}$ i.e. we have embedding $H_G \subset H_{\ell+1}$ and taking into account that the elements of H_G are θ_G-invariant, we obtain $H_G \simeq H_{\ell+1}^{\theta_G}$.

To establish (6.8), first note that elements $g \in (N_{\ell+1}(H_{\ell+1}))^{\theta_G}$ are θ-fixed elements of $GL_{\ell+1}$ stabilizing $H_{\ell+1}$ i.e. $ghg^{-1} \in H_{\ell+1}$ for any $h \in H_{\ell+1}$. It is clear that for any $h \in H_{\ell+1}^{\theta_G}$ we have $ghg^{-1} \in H_{\ell+1}^{\theta_G}$. This defines an embedding $(N_{\ell+1}(H_{\ell+1}))^{\theta_G} \subset N_G(H)$. In the following we use the fact that the involution θ_G on the commutative group $H_{\ell+1}$ defines the decomposition $H_{\ell+1} = H_+H_-$ into a product of commutative subgroups, so that for each element $h \in H_{\ell+1}$ there exist unique $h_\pm \in H_\pm$ such that

$$h = h_+h_- = h_-h_+, \qquad h_+^{\theta_G} = h_+, \qquad h_-^{\theta_G} = h_-^{-1}. \qquad (6.10)$$

Using (6.7) we can make the identification $H_+ \simeq H_G$.

Now we shall construct an embedding $N_G(H_G) \subset N_{\ell+1}(H_{\ell+1})^{\theta_G}$ i.e. prove that any θ-invariant element $g \in GL_{\ell+1}$ stabilizing H_G also transforms $h \in H_{\ell+1}$ into some other element $h' \in H_{\ell+1}$. This is trivial for elements of $H_+ \simeq H_G$ and thus we need to prove that gh_-g^{-1} is an element of $H_{\ell+1}$. The element gh_-g^{-1} commutes with all elements in $H_{\ell+1}^{\theta_G} \subset H_{\ell+1}$ due to the fact that H_+ and H_- mutually commute and g stabilizes H_+. Moreover for a given g all elements gh_-g^{-1} mutually commute. By Lemma 6.1 this entails that $gh_-g^{-1} \subset H_{\ell+1}$. Combining these two embeddings we obtain (6.8). To prove (6.9) let us compare two groups

$$N_{\ell+1}(H_{\ell+1})^{\theta}/H_{\ell+1}^{\theta}\,, \qquad (N_{\ell+1}(H_{\ell+1})/H_{\ell+1})^{\theta}\,. \qquad (6.11)$$

There is a natural map from the first group to the second one as each element in the l.h.s. coset defines an element of $g \in N_{\ell+1}(H_{\ell+1})$ modulo an element of $H_{\ell+1}^{\theta}$ and therefore modulo $H_{\ell+1}$. Note that elements of the second group in (6.11) are defined by the condition

$$g^{\theta} = g \cdot h, \qquad g \in N_{\ell+1}(H_{\ell+1}), \qquad h \in H_{\ell+1}, \qquad (6.12)$$

modulo right action of $H_{\ell+1}$ on g. Due to involutivity of θ_G we have

$$g = \left(g^\theta\right)^\theta = (gh)^\theta = ghh^\theta, \tag{6.13}$$

and thus

$$h^\theta = h^{-1}. \tag{6.14}$$

Changing representative $g \to \tilde{g} = g\tilde{h}$, $\tilde{h} \in H_{\ell+1}$ for g in the coset $N_{\ell+1}(H_{\ell+1})/H_{\ell+1}$ we may also modify (6.12) as follows

$$\tilde{g}^\theta = g \cdot h\tilde{h}^\theta = \tilde{g} \cdot h\tilde{h}^\theta \tilde{h}^{-1}. \tag{6.15}$$

Thus one can get rid of h in (6.12) if we manage to solve the equation

$$\tilde{h}^\theta = h\tilde{h}, \tag{6.16}$$

for any h. Using the decomposition

$$\tilde{h}_- = \tilde{h}_+\tilde{h}_-\,, \tag{6.17}$$

and taking into account that $h \in H_-$ we can easily check that the equation (6.16) can always be solved. Hence we conclude that two groups (6.11) are isomorphic and thus the last statement (6.9) follows from the fact that the Weyl groups are given by the quotients

$$W^{\theta_G}_{GL_{\ell+1}} = (N_{\ell+1}(H_{\ell+1})/H_{\ell+1})^\theta = N_{\ell+1}(H_{\ell+1})^\theta/H^\theta_{\ell+1} = N_G(H_G)/H_G = \mathrm{Aut}(\Phi_G).$$

□

Proposition 6.2 *Given a classical complex Lie group $G = (GL_{\ell+1}(\mathbb{C}))^{\theta_G}$, the following holds:*

$$\left(W^T_{GL_{\ell+1}}\right)^{\theta_G} = W^T_G\,. \tag{6.18}$$

The explicit presentation for (6.18) *may be provided by replacement of all s_i's by $\dot{s}_i$'s in* (5.1), (5.2) *and* (5.3).

Proof : To prove the assertion we use the following fact (see [GLO] Proposition 3.1). Let $G(\mathbb{R})$ be the totally split real form of $G(\mathbb{C})$. Then

$$\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R}))) = W^T_{G(\mathbb{C})}(H(\mathbb{C}))\,. \tag{6.19}$$

The connected component of the trivial element of $W^T_{G(\mathbb{R})} \subset N_{G(\mathbb{R})}(H(\mathbb{R}))$ is isomorphic to $\mathbb{R}^{\mathrm{rank}(G)}_{>0}$ and we have the split exact sequence

$$1 \longrightarrow \mathbb{R}^{\mathrm{rank}(G)}_{>0} \longrightarrow N_{G(\mathbb{R})}(H_G(\mathbb{R})) \longrightarrow W^T_{G(\mathbb{C})}(H_G(\mathbb{C})) \longrightarrow 1\,. \tag{6.20}$$

Thus we have

$$\Big(W^T_{GL_{\ell+1}(\mathbb{C})}(H_{\ell+1}(\mathbb{C}))\Big)^{\theta_G} = \big(N_{\ell+1}(\mathbb{R})(H_{\ell+1})/\mathbb{R}^{\ell+1}_{>0}\big)^{\theta_G} \,, \tag{6.21}$$

while on the other hand

$$W^T_{G(\mathbb{C})}(H_G(\mathbb{C})) \,=\, (N_{\ell+1}(\mathbb{R})(H_{\ell+1}))^{\theta_G} \,/ \big(\mathbb{R}^{\ell+1}_{>0}\big)^{\theta_G} \,. \tag{6.22}$$

There is an obvious map

$$(N_{\ell+1}(\mathbb{R})(H_{\ell+1}))^{\theta_G} \,/ \big(\mathbb{R}^{\ell+1}_{>0}\big)^{\theta_G} \,\longrightarrow\, \big(N_{\ell+1}(\mathbb{R})(H_{\ell+1})/\mathbb{R}^{\ell+1}_{>0}\big)^{\theta_G} \,, \tag{6.23}$$

and we have to show that it is actually an isomorphism. The obstruction to the isomorphism (6.23) is given by elements in $N_{\ell+1}(\mathbb{R})(H_{\ell+1})$, invariant with respect to θ_G only up to multiplication by an element $\Lambda \in \mathbb{R}^{\ell+1}_{>0}$ and we shall consider such elements up to the right action of $H_{\ell+1}$. As θ_G has order two we have the equation

$$\Lambda \cdot \Lambda^{\theta_G} = 1 \,. \tag{6.24}$$

Now consider another representative $\tilde{g} := gQ$ in the same $H_{\ell+1}$-coset such that

$$\tilde{g}^{\theta_G} = g^{\theta_G} Q^{\theta_G} = g\Lambda Q^{\theta_G} \,. \tag{6.25}$$

Thus we can choose a θ_G-invariant representative if we can solve the equation

$$Q = \Lambda Q^{\theta_G} \,. \tag{6.26}$$

and consistency of this equation follows from (6.24). Explicitly we have

$$q_{\ell+2-i} = \lambda_i q_i, \qquad Q = \mathrm{diag}(q_1, q_2, \cdots, q_{\ell+1}), \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{\ell+1}) \,, \tag{6.27}$$

where $Q, \Lambda \in \mathbb{R}^{\ell+1}_{>0}$. A direct check shows that such equations are always solvable and thus we have the isomorphism

$$(N_{\ell+1}(\mathbb{R})/(H_{\ell+1}))^{\theta_G} \,/\, \big(\mathbb{R}^{\ell+1}_{>0}\big)^{\theta_G} \simeq \big(N_{\ell+1}(\mathbb{R})(H_{\ell+1})/\mathbb{R}^{\ell+1}_{>0}\big)^{\theta_G} \,. \tag{6.28}$$

This completes the proof of (6.18). Note that the calculations above actually prove that the first cohomology group of $\mathbb{Z}/2\mathbb{Z}$ generated by θ_G acting in $\mathbb{R}^{\ell+1}_{>0}$ is trivial.

The explicit presentation for the generators of W^T_G is obtained in Lemmas 10.2, 10.5 and 10.9 in the following subsections. □

7 Why $\mathrm{Sp}_{2\ell}$ is different

According to Theorem 5.1 all classical Lie groups G except of symplectic type allow a lift of the Weyl group W_G into the corresponding Lie group G. In other words, the exact sequence (2.22) splits, so that the map p has a section. The splitting in these cases allows a simple explanation: the corresponding Weyl group action in $\mathfrak{h}_{\mathbb{R}}$ preserves the standard inner product, and therefore the Weyl group may be identified with a subgroup of $O(\mathfrak{h})$, which immediately implies the splitting of the sequence (1.1) in the cases of $GL_{\ell+1}$ and $O_{\ell+1}$. In contrast, for symplectic groups the exact sequence (2.22) does not split and as a section of p we encounter a non-trivial extension $W^T_{\mathrm{Sp}_{2\ell+2}}$ of the Weyl group $W_{\mathrm{Sp}_{2\ell+2}} = W(C_{\ell+1})$ introduced by Tits. Obviously the peculiarity of the symplectic series of classical Lie groups begs for explanation. In this section we propose an explanation of this phenomena relying on the properties of quaternion numbers $\mathbb{H}$, the unique non-commutative associative normed division algebra over $\mathbb{R}$.

Recall that in the case of the classical Lie groups along with the standard approach to the classification of the complex Lie algebras via root data, we may use another (Cartan's) approach based on the analysis of subalgebras of the general linear algebras fixed by certain involutions. A closely related approach is based on the classification of *normed division associative algebras*. The list of normed division associative algebras A is exhausted by the algebras of real, complex and quaternion numbers; to these algebras we associate three series of general linear groups:

$$GL_{\ell+1}(\mathbb{R}), \qquad GL_{\ell+1}(\mathbb{C}), \qquad GL_{\ell+1}(\mathbb{H}). \tag{7.1}$$

The corresponding maximal compact subgroups

$$O_{\ell+1}(\mathbb{R}), \qquad U_{\ell+1}, \qquad U\mathrm{Sp}_{\ell+1}, \tag{7.2}$$

may be defined as stabilizer subgroups of the standard quadratic form over the corresponding division algebra A:

$$(x,x) = \sum_{i=1}^{n} x_i^\dagger x_i = \sum_{i=1}^{n} \|x_i\|^2, \qquad x_i \in A. \tag{7.3}$$

Further complexification of the compact groups provides a complete list of the classical complex Lie groups:

$$\begin{gathered} O_{\ell+1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = O_{\ell+1}(\mathbb{C}), \qquad U_{\ell+1} \otimes_{\mathbb{R}} \mathbb{C} = GL_{\ell+1}(\mathbb{C}), \\ U\mathrm{Sp}_{\ell+1} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Sp}_{2\ell+2}(\mathbb{C}). \end{gathered} \tag{7.4}$$

Actually the point of view based on matrix groups over non-commutative fields implies some adjustment of standard definitions in the structure theory of Lie groups. An obvious instance is the notion of the maximal torus: for matrix groups over non-commutative fields, it is more natural to consider diagonal subgroups of general linear groups over A rather than maximal commutative subgroup. Namely, given a normed division algebra A, let A^* be its multiplicative group of invertible elements; then $H_A = (A^*)^{\ell+1}$ will be the diagonal subgroup of $GL_{\ell+1}(A)$. In the special case of symplectic groups the underlying normed division algebra $A = \mathbb{H}$ is non-commutative and the corresponding diagonal subgroup of $GL_{\ell+1}(\mathbb{H})$,

$$H_{\mathbb{H}} = \underbrace{(\mathbb{H}^* \times \cdots \times \mathbb{H}^*)}_{\ell+1} \subset GL_{\ell+1}(\mathbb{H}), \tag{7.5}$$

is a non-commutative Lie group. The modification of the notion of maximal torus implies the following modification of the definition of the Weyl group.

Definition 7.1 *Define the Weyl group $W_{GL_{\ell+1}}(A)$ of $GL_{\ell+1}(A)$ to be the group of inner automorphisms of the diagonal subgroup $H_A \subset GL_{\ell+1}(A)$.*

Note that while in the commutative case the group of inner automorphisms of the diagonal subgroup $H_{\ell+1} \subset GL_{\ell+1}$ is isomorphic to the quotient group $N_{\ell+1}(H_{\ell+1})/H_{\ell+1}$, for non-commutative A this is not true.

Lemma 7.1 *The group $W_{GL_{\ell+1}}(\mathbb{H})$ allows the following description:*

$$W_{GL_{\ell+1}}(\mathbb{H}) \,=\, \mathfrak{S}_{\ell+1} \ltimes (SO_3)^{\ell+1} \,. \tag{7.6}$$

Proof: The group $GL_{\ell+1}(\mathbb{H})$ acts on its diagonal subgroup $H_{\mathbb{H}}$ by conjugation, so let us find the normalizer subgroup $N_{\mathbb{H}} = N_{\ell+1}(\mathbb{H})(H_{\mathbb{H}})$ which preserves $H_{\mathbb{H}}$. The diagonal group $H_{\mathbb{H}}$ is generated by one-parametric subgroups

$$H_{\mathbb{H}}^{(k)} \,=\, \left\{ \begin{pmatrix} \mathrm{Id}_{k-1} & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & \mathrm{Id}_{\ell-k} \end{pmatrix} , \quad d \in \mathbb{H}^* \right\} \subset H_{\mathbb{H}} \,, \qquad 1 \leq k \leq \ell+1 \,. \tag{7.7}$$

Let us pick a diagonal element $D_k = \mathrm{diag}(\mathrm{Id}_{k-1}\,,\, d_k,\, \mathrm{Id}_{\ell-k}) \in H_{\mathbb{H}}^{(k)}$. Then $M \in GL_{\ell+1}(\mathbb{H})$ belongs to $N_{\mathbb{H}}$ if and only if it satisfies the following equations for each $1 \leq k \leq \ell+1$:

$$MD_k \,=\, D'M \,, \quad \text{for some} \quad D' = \mathrm{diag}(d_1', \ldots , d_{\ell+1}') \in H_{\mathbb{H}} \,. \tag{7.8}$$

Explicitly using the matrix notation $M = \|g_{ij}\|$, the above equation reads

$$g_{ij} \,=\, d_i' g_{ij} \,, \quad j \neq k \,; \qquad g_{ik} d_k \,=\, d_i' g_{ik} \,, \quad 1 \leq i,\, j\,, k \leq \ell+1 \,. \tag{7.9}$$

Now for each $1 \leq i \leq \ell+1$ the former relation in (7.9) implies either $g_{ij} = 0$, $\forall j \neq k$, or $d_i' = 1$. In the case $g_{ij} = 0$, $\forall j \neq k$ we necessarily obtain from the latter relation in (7.9) that $g_{ik} \neq 0$, otherwise M has the i-th zero row and $M \notin GL_{\ell+1}(\mathbb{H})$. In the case $d_i' = 1$ it follows from the latter relation in (7.9) that $g_{ik} d_k = g_{ik}$, which entails $g_{ik} = 0$, since we assume $d_k \neq 1$ so that $D_k \neq \mathrm{Id}_{\ell+1}$. Taking into account that g_{ij} are subjected to (7.9) for each $1 \leq k \leq \ell+1$, we deduce from the above that there is only one non-zero element in the i-th row of M.

Since the matrix M is invertible its rows are linearly independent, so for different i the non-zero entries g_{ij} have different j's. This defines the subgroup of monomial matrices $\mathfrak{M}_{\ell+1}(\mathbb{H}) \subset GL_{\ell+1}(\mathbb{H})$ consisting of matrices with only one non-zero element in each column and each row, and yields $N_{\mathbb{H}} = \mathfrak{M}_{\ell+1}(\mathbb{H})$. Clearly, the quotient group $N_{\mathbb{H}}/H_{\mathbb{H}}$ is isomorphic to the permutation group:

$$\mathfrak{M}_{\ell+1}(\mathbb{H})/H_{\mathbb{H}} \,=\, \mathfrak{S}_{\ell+1} \,, \tag{7.10}$$

so that Definition 7.1 reads

$$W_{GL_{\ell+1}}(\mathbb{H}) \,=\, \mathfrak{S}_{\ell+1} \ltimes \mathrm{Int}(\mathbb{H})^{\ell+1} \,. \tag{7.11}$$

where $\mathrm{Int}(\mathbb{H}) = \mathbb{H}^*/\mathbb{R}^*$ is the group of inner automorphisms of $\mathbb{H}$ (note that by Skolem-Noether theorem all automorphisms of $\mathbb{H}$ are inner). The norm homomorphism induces the following exact sequence:

$$1 \longrightarrow SU_2 \longrightarrow \mathbb{H}^* \stackrel{\mathrm{Nm}}{\longrightarrow} \mathbb{R}_{>0} \longrightarrow 1 \quad , \qquad \mathrm{Nm}(q) = q\bar{q} \, , \qquad \mathrm{Nm}(\mathbb{H}) = \mathbb{R}_{>0} \, . \tag{7.12}$$

Taking into account $\mathbb{R}^* = \mathcal{Z}(\mathbb{H}) \cap \mathbb{H}^* = \mathbb{R}_{>0} \times \mu_2$ with $\mu_2 = \{\pm 1\}$ being a group of roots of 1 in $\mathbb{R}$, we obtain

$$\mathrm{Aut}(\mathbb{H}) = \mathbb{H}^*/\mathbb{R}^* = SU(2)/\mu_2 \simeq SO_3 \, . \tag{7.13}$$

Thus $\mathrm{Aut}(\mathbb{H})$ is identified with the adjoint group SO_3 of the group SU_2 of unit quaternions. This complete the proof of (7.6). □

Let $\mathrm{Aut}_{\mathbb{C}}(\mathbb{H}) \subset \mathrm{Aut}(\mathbb{H})$ be the subgroup of automorphisms preserving the subalgebra $\mathbb{C} = (\mathbb{R} \oplus \mathbb{R}\imath) \subset \mathbb{H}$.

Lemma 7.2 *The following holds:*

$$\mathrm{Aut}_{\mathbb{C}}(\mathbb{H}) = (\mathbb{C}^* \sqcup \mathbb{C}^* \jmath)/\mathbb{R}^* \, , \tag{7.14}$$

and the following central extension splits

$$1 \longrightarrow \mathbb{C}^*/\mathbb{R}^* \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathbb{H}) \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1 \, . \tag{7.15}$$

Proof : Consider the faithful two-dimensional representation $\mathbb{H} \to \mathrm{Mat}_2(\mathbb{C})$, $q \mapsto \hat{q}$:

$$\hat{1} = \mathrm{Id}_2 \, , \qquad \hat{\imath} = \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix} , \qquad \hat{\jmath} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \hat{k} = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix} . \tag{7.16}$$

The norm homomorphism (7.12) is given by the matrix determinant:

$$q = q_0 + \imath q_1 + \jmath q_2 + k q_3 \in \mathbb{H} \, , \qquad |q|^2 = q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 \, , \qquad \hat{q} = \begin{pmatrix} q_0 + \imath q_1 & q_2 + \imath q_3 \\ -(q_2 - \imath q_3) & q_0 - \imath q_1 \end{pmatrix} , \qquad |q|^2 = \det \hat{q} \in \mathbb{R}_{>0} \, . \tag{7.17}$$

Then elements z of the subalgebra $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\imath \subset \mathbb{H}$ are identified with the diagonal matrices

$$z \longmapsto \hat{z} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} , \qquad |z|^2 = z\bar{z} = \det \hat{z} \, , \tag{7.18}$$

while the general quaternion has a representation for $a, b \in \mathbb{C}$

$$q \longmapsto \hat{q} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} , \qquad \det \hat{q} = |a|^2 + |b|^2 \, . \tag{7.19}$$

Now a quaternion q belongs to centralizer of subalgebra $\mathbb{C} \subset \mathbb{H}$ if and only if $\hat{q}\hat{z} = \hat{z}'\,\hat{q}$ for any $z \in \mathbb{C}$, which reads:

$$\begin{pmatrix} az & b\bar{z} \\ -\bar{b}z & \bar{a}\bar{z} \end{pmatrix} = \begin{pmatrix} z'a & z'b \\ -\bar{z}'\bar{b} & \bar{z}'\bar{a} \end{pmatrix}. \tag{7.20}$$

Since $z \in \mathbb{C}$ is arbitrary, either $z = z'$ and $b = 0$, or $z = \bar{z}'$ and $a = 0$. Obviously, the case $b = 0$ corresponds to $q \in \mathbb{C}^* \subset \mathbb{H}^*$, and in case $a = 0$ we have $q \in \mathbb{C}^*\jmath \subset \mathbb{H}^*$. Taking into account that $q \in \mathbb{R}^*$ implies (7.14). The second statement (7.15) follows from the fact that $\hat{\jmath}$ acts by complex conjugation:

$$\jmath\, z\, \jmath^{-1} = \bar{z}\,, \tag{7.21}$$

and its image in the quotient group $(\mathbb{C}^* \sqcup \mathbb{C}^*\jmath)/\mathbb{R}^*$ is precisely the generator of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ given by the complex conjugation. □

Recall that the standard Weyl group of the symplectic Lie group $\mathrm{Sp}_{2\ell+2}$ is given by

$$W_{\mathrm{Sp}_{2\ell+2}} = W(C_{\ell+1}) = \mathfrak{S}_{\ell+1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\ell+1}\,. \tag{7.22}$$

Here $\mathbb{Z}/2\mathbb{Z}$ in r.h.s. may be identified with the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and thus allowing via splitting of (7.15) an embedding

$$W_{\mathrm{Sp}_{2\ell+2}} \subset W_{GL_{\ell+1}}(\mathbb{H}). \tag{7.23}$$

The quaternionic analog $W_{GL_{\ell+1}}(\mathbb{H})$ of the standard Weyl group has obvious obstruction for the embedding into the general Lie group $GL_{\ell+1}(\mathbb{H})$. Indeed this lift implies in particular the transition from the adjoint action of quaternions on itself to its left action. On the other hand we have the following standard exact sequence

$$1 \longrightarrow \mathbb{R}^* \longrightarrow \mathbb{H}^* \stackrel{\pi}{\longrightarrow} \mathrm{Aut}(\mathbb{H}) \longrightarrow 1\ , \tag{7.24}$$

or taking into account the identifications $SU(2) = \mathbb{H}^*/\mathrm{Nm}(\mathbb{H})$ (for the notations see (7.12)) and $\mathrm{Aut}(\mathbb{H}) = SO_3$

$$1 \longrightarrow \mu_2 \longrightarrow SU_2 \stackrel{\pi}{\longrightarrow} SO_3(\mathbb{R}) \longrightarrow 1\ . \tag{7.25}$$

The generator of μ_2 above might be identified with the square $\jmath^2$ of the quaternionic unity. Indeed, the group $\mathrm{Aut}(\mathbb{H}) \simeq SO_3$ acts in the subspace $\mathbb{R}^3$ of purely imaginary quaternions

$$\hat{x} = \imath x_1 + \jmath x_2 + k x_3\,, \tag{7.26}$$

via standard three-dimensional rotations. Then the section of projection π may be identified with the lift of the orthogonal rotations to the conjugation action of the unit norm quaternions $q \in SU_2$:

$$\mathrm{Ad}_q : \quad \hat{x} \longrightarrow q\hat{x}q^{-1}\,. \tag{7.27}$$

Let us pick a rotation given by the diagonal element

$$g \,=\, \mathrm{diag}(-1,+1,-1) \,\in\, SO_3\,. \tag{7.28}$$

Solving the equation $\widehat{gx} \,=\, q\widehat{x}q^{-1}$ for (7.28), that is

$$-\imath x_1 \,+\, \jmath x_2 \,-\, kx_3 \,=\, q\big(\imath x_1 \,+\, \jmath x_2 \,+\, kx_3\big)\bar{q}\,, \tag{7.29}$$

we derive that $q = \jmath$ and thus

$$\hat{g} \,=\, \jmath\,, \qquad \hat{g}^2 = -1\,. \tag{7.30}$$

The fact that the square is equal to minus a unit element implies that indeed we have a central extension with generator which may be identified with $\jmath^2$.

Now taking into account (7.11) we arrive at the following result.

Proposition 7.1 *The quaternionic Weyl group* $W_{GL_{\ell+1}}(\mathbb{H})$ *allows the extension* $\widetilde{W}_{GL_{\ell+1}}(\mathbb{H})$

$$1 \longrightarrow (\mu_2)^{\ell+1} \longrightarrow \widetilde{W}_{GL_{\ell+1}}(\mathbb{H}) \longrightarrow W_{GL_{\ell+1}}(\mathbb{H}) \longrightarrow 1\,. \tag{7.31}$$

The extended group $\widetilde{W}_{GL_{\ell+1}}(\mathbb{H})$ *allows a natural embedding*

$$\widetilde{W}_{GL_{\ell+1}}(\mathbb{H}) \subset GL_{\ell+1}(\mathbb{H})\,. \tag{7.32}$$

The Tits group extension

$$1 \longrightarrow (\mu_2)^{\ell+1} \longrightarrow W^T_{\mathrm{Sp}_{2\ell+2}} \longrightarrow W_{\mathrm{Sp}_{2\ell+2}} \longrightarrow 1\,, \tag{7.33}$$

is then obtained by restriction of (7.31) *to the subgroup* $W_{\mathrm{Sp}_{2\ell+2}} \subset W_{GL_{\ell+1}}(\mathbb{H})$.

Thus the underlying reason for the appearance of the Tits groups in the case of the symplectic Lie groups may be traced back to the extensions (7.24), (7.25). The non-triviality of this extension is closely related with the basic μ_2-extension of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ characterizing quaternions. Actually the cohomology class measuring non-triviality of this extension is directly related with the basic invariant characterizing the real central simple division algebras. Indeed such algebras are characterized by the invariant taking values in the 2-torsion of the Brauer group (see e.g. [S], [PR]), which coincides with the whole Brauer group in case of $\mathbb{R}$:

$$\mathrm{Br}(\mathbb{R}) \,=\, H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}),\mathbb{C}^*) \simeq H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}),\mu_2) \simeq \mu_2\,. \tag{7.34}$$

The corresponding cohomology group describes the extensions of the form

$$1 \longrightarrow \mathbb{C}^* \longrightarrow A \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1\,, \tag{7.35}$$

and the case of the semidirect product $\mathbb{C} \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ with the natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ via complex conjugation corresponds to the trivial cocycle. The only other non-trivial case corresponds to the algebra $A = \mathbb{H}$ of quaternions so that the lift of the generator $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ may be identified with the quaternionic unit $\jmath$, $\jmath^2 = -1$. This provides a link between the non-trivial extension (7.25) and the construction of quaternions via the non-trivial extension (7.35) (for more conceptual explanations of the relation between universal μ_2-extension of the orthogonal groups and the second cohomology of the Galois group of the base field see [BD] and references therein). In particular this directly relates the fact that $\jmath^2 = -1$ in the algebra of quaternions with the unavoidable appearance of the Tits extension of the Weyl group for classical symplectic Lie groups.

8 Lie groups closely related with classical Lie groups

In this section we consider several examples of constructions of suitable sections in the extensions (1.1) for the Lie groups closely related to classical Lie groups. Precisely we will treat the cases of the unimodular subgroup $SL_{\ell+1} \subset GL_{\ell+1}$ of the general linear group, of the groups $\mathrm{Pin}_{2\ell+1}$, $\mathrm{Pin}_{2\ell}$, $\mathrm{Spin}_{2\ell+1}$ and $\mathrm{Spin}_{2\ell}$. In all cases we provide an explicit construction of sections of (1.1) realized as central extensions of the corresponding Weyl groups. Let us stress that these sections differ from the Tits lifts (which are not central extensions of the corresponding Weyl groups in general). Although the possibility to define sections via central extensions is obvious for Pin groups (they are central extensions of the corresponding classical orthogonal groups) it is a bit less obvious for unimodular and spinor Lie groups.

8.1 Construction of a section for $G = SL_{\ell+1}$

We will freely use the notations for root data of type A_ℓ defined in the Appendix. Recall that the unimodular subgroup $SL_{\ell+1} \subset GL_{\ell+1}$ is defined as a kernel of the determinant map thus fitting the following exact sequence:

$$1 \longrightarrow SL_{\ell+1}(\mathbb{C}) \stackrel{\varphi}{\longrightarrow} GL_{\ell+1}(\mathbb{C}) \stackrel{\det}{\longrightarrow} \mathbb{C}^* \longrightarrow 1 \quad . \tag{8.1}$$

The group center of the unimodular group is identified with the cyclic group: $\mathcal{Z}(SL_{\ell+1}(\mathbb{C})) = \mathbb{Z}/(\ell+1)\mathbb{Z}$ generated by $\zeta = e^{2\pi\imath\varpi_\ell^\vee}$. The image $\varphi(\zeta)$ belongs to the center $\mathcal{Z}(GL_{\ell+1}) = \mathbb{C}^*$ and is given by:

$$\varphi(\zeta) \,=\, e^{\frac{2\pi\imath}{\ell+1}\,p_{\ell+1}^\vee} \,\in\, \mathcal{Z}(GL_{\ell+1}(\mathbb{C}))\,. \tag{8.2}$$

In Section 4 we constructed a lift of the Weyl group $W_{GL_{\ell+1}} = W(A_\ell)$ corresponding to the root system A_ℓ to the group $GL_{\ell+1}$. As the Weyl groups for $SL_{\ell+1}$ and $GL_{\ell+1}$ coincide to construct a section of (1.1) for $SL_{\ell+1}(\mathbb{C})$, we shall invert the homomorphims φ on the subgroup $W_{GL_{\ell+1}} \subset GL_{\ell+1}$. Formally the inverse of φ may be written as follows

$$\varphi^{-1}(g) \,=\, g\cdot\left[\det{}^{(-1)}(\det g)\right]^{-1}, \qquad g \in GL_{\ell+1}(\mathbb{C})\,, \tag{8.3}$$

where $\det^{(-1)}$ is a multi-valued inverse of the determinant map det in (8.1), which can be defined by

$$\det{}^{(-1)}:\quad \mathbb{C}^* \longrightarrow \mathcal{Z}(GL_{\ell+1})\,, \qquad \det{}^{(-1)}(z) \,=\, e^{\frac{1}{\ell+1}\,p_{\ell+1}^\vee\,\log(z)}\,, \quad z\in\mathbb{C}^*\,. \tag{8.4}$$

To make this map single-valued we shall choose a branch of the log-function. Note that all generators S_i of $W_{GL_{\ell+1}}(\mathbb{C})$ defined in (4.2) satisfy the relation $\det S_i = -1$. Thus we pick $\log(-1) = \imath\pi$ and introduce the following lift of generators S_i in $SL_{\ell+1}(\mathbb{C})$:

$$\sigma_i \,=\, e^{\frac{\pi\imath}{\ell+1}\,p_{\ell+1}^\vee}\,S_i\,, \qquad 1 \le i \le \ell\,. \tag{8.5}$$

Proposition 8.1 *The group $\widetilde{W}(A_\ell)$ generated by the elements (8.5) provides a central extension*

$$1 \longrightarrow \mathcal{Z}(SL_{\ell+1}) \longrightarrow \widetilde{W}(A_\ell) \longrightarrow W(A_\ell) \longrightarrow 1 \quad , \tag{8.6}$$

of the Weyl group $W(A_\ell)$ with the defining relations

$$\begin{aligned} \sigma_k^2 \,&=\, \zeta\,, \quad \text{for each} \quad 1 \leq k \leq \ell\,; \\ \sigma_k\sigma_j \,&=\, \sigma_j\sigma_k\,, \quad \text{if} \quad m_{kj} = 2, \\ \text{and} \qquad \sigma_k\sigma_j\sigma_k \,&=\, \sigma_j\sigma_k\sigma_j\,, \quad \text{if} \quad m_{kj} = 3\,, \end{aligned} \tag{8.7}$$

where $\zeta = e^{2\pi\imath\omega_\ell^\vee}$ is the generator of the center $\mathcal{Z}(SL_{\ell+1})$. The group $\tilde{W}(A_\ell)$ allows an embedding in $N_{SL_{\ell+1}}(H_{SL_{\ell+1}})$ compatible with the exact sequence

$$1 \longrightarrow H_{SL_{\ell+1}} \longrightarrow N_{SL_{\ell+1}}(H_{SL_{\ell+1}}) \longrightarrow W(A_\ell) \longrightarrow 1\,, \tag{8.8}$$

and the embedding $\mathcal{Z}(SL_{\ell+1}) \subset H_{SL_{\ell+1}}$.

Proof . The element $e^{\frac{\imath\pi}{\ell+1}p_{\ell+1}^\vee}$ is in the center of $GL_{\ell+1}(\mathbb{C})$ and its square is equal to ζ. The relations (8.7) then follow from the relations (4.2) and (4.3) for the generators of the Weyl group $W(A_\ell)$. □

8.2 Construction of a section for $SO_{\ell+1}$

The problem of lifting the Weyl groups $W(B_\ell)$ and $W(D_\ell)$ into special orthogonal subgroups is similar to that for $SL_{\ell+1} \subset GL_{\ell+1}$. In (5.4), (5.5) we introduced a lift of Weyl groups $W(B_\ell)$ and $W(D_\ell)$ to the corresponding orthogonal groups. It is natural to split this construction into two parts depending on the parity of the rank.

In case of the odd orthogonal group the following sequence splits,

$$1 \longrightarrow SO_{2\ell+1} \stackrel{\varphi}{\longrightarrow} O_{2\ell+1} \stackrel{\det}{\longrightarrow} \mu_2 \longrightarrow 1 \quad , \tag{8.9}$$

so that

$$O_{2\ell+1} \simeq \mathcal{Z}(O_{2\ell+1}) \times SO_{2\ell+1}\,. \tag{8.10}$$

Here the generator z of the center $\mathcal{Z}(O_{2\ell+1}) = \mathbb{Z}/2\mathbb{Z}$ is given by (see Section 10.2 for notations):

$$z \,=\, T_0^B T_1^B \cdots T_\ell^B \,=\, T_0^B e^{\pi\imath\varpi_1^\vee} = -\mathrm{Id}_{2\ell+1} \in \mathcal{Z}(O_{2\ell+1})\,, \tag{8.11}$$

The Weyl group generators S_i^B (5.4) satisfy

$$\det(S_1^B) \,=\, -1\,, \qquad \det(S_k^B) \,=\, 1\,, \quad 1 < k \leq \ell\,, \tag{8.12}$$

so that $S_1^B \in (O_{2\ell+1} \setminus SO_{2\ell+1})$ and $S_k^B \in SO_{2\ell+1}$, $1 < k \leq \ell$.

Lemma 8.1 *The elements $\sigma_i \in SO_{2\ell+1}$ defined by*

$$\sigma_1 \,=\, zS_1^B\,, \qquad \sigma_k \,=\, S_k^B\,, \quad 1 < k \leq \ell\,, \tag{8.13}$$

generate the group isomorphic to the Weyl group $W(B_\ell)$.

Proof : Follows from the fact that $z^2 = 1$ in $SO_{2\ell+1}$. □

Note that here we follow the analogous construction for the case of the unimodular group $SL_{\ell+1} \subset GL_{\ell+1}$ by using (8.11) as a lift of the generator of μ_2 in (8.9).

In the case of the even orthogonal group generators S_i^D (5.5) of $W(D_\ell)$, they already belong to $SO_{2\ell}$ due to $\det(S_i^D) = 1$, and therefore provide an embedding $W(D_\ell) \subset N_{SO_{2\ell}}$.

8.3 Construction of a section for $\mathrm{Pin}_{\ell+1}$

The group $\mathrm{Pin}(V)$ is a central extension of $O(V)$, which can be defined as follows. Let V be real vector space supplied with the standard quadratic form

$$\|v\|^2 \,=\, \sum_{i=1}^{\dim(V)} v_i^2, \qquad v = (v_1, \ldots, v_{\dim(V)}) \in V\,. \tag{8.14}$$

Let $\mathcal{C}(V)$ be the corresponding Clifford algebra, defined as a quotient of the tensor algebra $T(V)$ as follows:

$$\mathcal{C}(V) \,=\, T(V)\Big/\big(v \cdot v \,-\, \|v\|^2 \cdot 1\big)\,. \tag{8.15}$$

The standard $\mathbb{Z}$-grading on the tensor algebra induces a $\mathbb{Z}/2\mathbb{Z}$-grading $\alpha : \mathcal{C}(V) \to \mathcal{C}(V)$ of the Clifford algebra, thus splitting it into the direct sum:

$$\mathcal{C}(V) \,=\, \mathcal{C}(V)^+ \oplus \mathcal{C}(V)^-\,. \tag{8.16}$$

Let $\top : \mathcal{C}(V) \to \mathcal{C}(V)$ be the transposition anti-automorphism of the Clifford algebra, defined by $(v_1 \cdot v_2)^\top = v_2 \cdot v_1$ for any $v_1,\, v_2 \in V \subset \mathcal{C}(V)$. Define the conjugation anti-automorphism $x \to \bar{x} = \alpha(x)^\top$ on $\mathcal{C}(V)$. The spinor norm is then given by:

$$\mathrm{Nm} : \quad \mathcal{C}(V) \longrightarrow \mathbb{R}^*\,, \qquad x \longmapsto \mathrm{Nm}(x) \,=\, x\,\bar{x}\,, \tag{8.17}$$

so that for all $u,\, v \in V$ we have

$$\begin{gathered} (u,v) \,=\, \frac{1}{2}\Big(\mathrm{Nm}(u) \,+\, \mathrm{Nm}(v) \,-\, \mathrm{Nm}(u+v)\Big) \,=\, \frac{u \cdot v \,+\, v \cdot u}{2}\,, \\ \mathrm{Nm}(v) \,=\, -\, v \cdot v \,=\, -\|v\|^2 \,=\, -(v,v)\,. \end{gathered} \tag{8.18}$$

Definition 8.1 *(see [ABS]) The Clifford group $\Gamma(V)$ is the subgroup of the group of invertible elements of $\mathcal{C}(V)$ that respects the linear subspace $V \subset \mathcal{C}(V)$ i.e.*

$$\Gamma(V) \,=\, \big\{x \in (\mathcal{C}(V) \setminus \{0\}) \,|\, xV\alpha(x)^{-1} \subseteq V\big\}\,. \tag{8.19}$$

The action of $\Gamma(V)$ on V defined above respects the norm (8.18) and thus allows a homomorphism to $O(V)$ for all $u \in V \subset \mathcal{C}(V)$:

$$u \longmapsto s_u(v) := u \cdot v \cdot \alpha(u)^{-1} = v - 2\frac{(u,v)}{(u,u)}u\,, \qquad \forall\, v \in V\,. \tag{8.20}$$

Moreover, since the orthogonal group is generated by reflections with respect to elements $u \in V$, the group $\Gamma(V)$ maps onto $O(V)$. Clearly, $\mathbb{R}^* \subset \Gamma(V)$ acts trivially and this leads to the following exact sequence:

$$1 \longrightarrow \mathbb{R}^* \longrightarrow \Gamma(V) \stackrel{\pi}{\longrightarrow} O(V) \longrightarrow 1 \quad . \tag{8.21}$$

Restriction to the subgroup of elements with norm in $\mu_2 = \{\pm 1\}$ results in taking a quotient over the subgroup $\mathrm{Nm}(\mathbb{R}^*) = \mathbb{R}_{>0}$, which yields the central extension $\mathrm{Pin}(V)$ of $O(V)$:

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Pin}(V) \stackrel{\pi}{\longrightarrow} O(V) \longrightarrow 1 \quad , \tag{8.22}$$

where $\mu_2 = \{\pm 1\}$ is a subgroup of the center $\mathcal{Z}(\mathrm{Pin}(V))$ of the pinor group $\mathrm{Pin}(V)$.

Taking into account (8.20), we may define a properly normalized lift $\hat{u}$ of a simple reflection s_u with respect to a vector $u \in V$ into $\mathrm{Pin}(V)$ as follows

$$\hat{u} := \frac{u}{\sqrt{|\mathrm{Nm}(u)|}}\,, \quad \hat{u}^2 = 1\,, \quad \mathrm{Nm}(\hat{u}) = -1\,, \qquad \forall u \in V \subset \mathcal{C}(V)\,. \tag{8.23}$$

Lemma 8.2 *Let* $\{\epsilon_1, \ldots, \epsilon_{\dim(V)}\} \subset V$ *be an orthonormal basis. Let* T_k, $1 \leq k \leq \dim(V)$ *be reflections at* $\epsilon_k \in V$ *and let* S_i, $1 \leq i \leq (\dim(V) - 1)$ *be reflections at simple root* $\epsilon_i - \epsilon_{i+1}$. *Then the following elements of* $\mathrm{Pin}(V)$ *represent liftings of the reflections* S_i, $T_k \in O(V)$*:*

$$\begin{aligned} \mathcal{T}_k &= \hat{\epsilon}_k\,, \qquad (\mathcal{T}_k)^2 = 1, \qquad 1 \leq k \leq \dim(V)\,, \\ \mathcal{S}_i &= \frac{\hat{\epsilon}_i - \hat{\epsilon}_{i+1}}{\sqrt{2}}\,, \qquad (\mathcal{S}_i)^2 = 1\,, \qquad 1 \leq i \leq \ell\,. \end{aligned} \tag{8.24}$$

Proof : One shall check that the action of the lifts $\mathcal{T}_i$ and $\mathcal{S}_i$ on V according to (8.20) coincides with the action of T_i and S_i. Thus for $\mathcal{T}_i$ we have the following:

$$\pi(\mathcal{T}_k)(v) = \mathcal{T}_k \cdot v \cdot \alpha(\mathcal{T}_k)^{-1} = \hat{\epsilon}_k \cdot v \cdot (-\hat{\epsilon}_k)$$
$$= \hat{\epsilon}_k \cdot (x_1\hat{\epsilon}_1 + \ldots + x_k\hat{\epsilon}_k + \ldots + x_{\dim(V)}\hat{\epsilon}_{\dim(V)}) \cdot (-\hat{\epsilon}_k) = T_k v\,.$$

Similarly the action of $\mathcal{S}_i$ can be verified. □

Lemma 8.3 *The elements* (8.24) *satisfy the following relations (compare with* (4.9)*):*

$$\begin{aligned} \mathcal{T}_i\mathcal{T}_j &= -\mathcal{T}_j\mathcal{T}_i\,, \quad i \neq j\,; \qquad \mathcal{S}_i\mathcal{S}_j = -\mathcal{S}_j\mathcal{S}_i\,, \quad |i-j| > 1\,, \\ \mathcal{S}_i\mathcal{S}_j\mathcal{S}_i &= \mathcal{S}_j\mathcal{S}_i\mathcal{S}_j\,, \qquad |i-j| = 1\,, \\ \mathcal{T}_i\mathcal{S}_i &= -\mathcal{S}_i\mathcal{T}_{i+1}\,, \end{aligned} \tag{8.25}$$

and therefore provide a central extension of the Tits group $W^T_{GL(V)}$ *(and of its subgroup* $W_{GL(V)}$ *generated by* $\mathcal{S}_i$, $1 \leq i \leq \dim(V) - 1$*) by* $\mu_2 \subseteq \mathcal{Z}(\mathrm{Pin}(V))$*.*

Proof : The relations can be easily checked by straightforward computation. For example let us verify the 3-move braid relation: one has

$$\begin{aligned} \mathcal{S}_i\mathcal{S}_{i+1} &= \frac{\hat{\epsilon}_i\hat{\epsilon}_{i+1} - \hat{\epsilon}_i\hat{\epsilon}_{i+2} + \hat{\epsilon}_{i+1}\hat{\epsilon}_{i+2} - 1}{2}, \\ (\mathcal{S}_i\mathcal{S}_{i+1})^2 &= \frac{-\hat{\epsilon}_i\hat{\epsilon}_{i+1} + \hat{\epsilon}_i\hat{\epsilon}_{i+2} - \hat{\epsilon}_{i+1}\hat{\epsilon}_{i+2} - 1}{2}. \end{aligned} \tag{8.26}$$

This implies the Coxeter relation $(\mathcal{S}_i\mathcal{S}_j)^3 = 1$ for $|i-j| = 1$, which is equivalent to the 3-move braid relation due to $\mathcal{S}_i^{-1} = \mathcal{S}_i$. □

Now let us construct a lift of the generators of the Weyl group $W_{O(V)}$ into $\mathrm{Pin}(V)$. We use a version of the embedding that is based on the expressions given in Theorem 5.1 for the lift of $W_{O(V)}$ to $O(V)$. Note that the conjugated generators $\overline{S}_i^A$ are expressed thorough S_i^A and T_i^A according to (4.12) and the lift of the generators S_i^A and T_i^A is provided by Lemma 8.3. We will consider the cases of $W(B_\ell)$ and $W(D_\ell)$ separately.

In the odd orthogonal group case we have $O(V) = O_{2\ell+1}$. By (5.4), Weyl group $W(B_\ell)$ can be imbedded into the θ_B-invariant subgroup $(W^T_{GL_{2\ell+1}})^{\theta_B}$ via

$$S_1^B = S_\ell S_{\ell+1} S_\ell = S_{\ell+1} S_\ell S_{\ell+1}, \qquad S_k^B = S_{\ell+1-k}\overline{S}_{\ell+k}, \quad 1 < k \le \ell. \tag{8.27}$$

Using (4.12) and (8.24) we obtain the following:

$$\mathcal{S}_1^B = \mathcal{S}_\ell \mathcal{S}_{\ell+1} \mathcal{S}_\ell, \qquad \mathcal{S}_k^B = \mathcal{S}_{\ell+1-k}\mathcal{T}_{\ell+k}\mathcal{S}_{\ell+k}\mathcal{T}_{\ell+k}, \quad 1 < k \le \ell. \tag{8.28}$$

In the odd orthogonal group case we have $O(V) = O_{2\ell}$. By (5.5), Weyl group $W(D_\ell)$ can be imbedded into the θ_D-invariant subgroup $(W^T_{GL_{2\ell}})^{\theta_D}$ via

$$S_1^D = S_\ell S_{\ell-1}\overline{S}_{\ell+1} S_\ell, \qquad S_k^D = S_{\ell+1-k}\overline{S}_{\ell-1+k}, \quad 1 < k \le \ell. \tag{8.29}$$

Therefore, using (4.12) and (8.24) for the lift to $\mathrm{Pin}_{2\ell}$ we obtain

$$\begin{aligned} \mathcal{S}_1^D &= \mathcal{S}_\ell\mathcal{S}_{\ell-1}\mathcal{T}_{\ell+1}\mathcal{S}_{\ell+1}\mathcal{T}_{\ell+1}\mathcal{S}_\ell, \\ \mathcal{S}_k^D &= \mathcal{S}_{\ell+1-k}\mathcal{T}_{\ell-1+k}\mathcal{S}_{\ell-1+k}\mathcal{T}_{\ell-1+k}, \quad 1 < k \le \ell. \end{aligned} \tag{8.30}$$

The subgroups in $\mathrm{Pin}_{2\ell+1}$ and $\mathrm{Pin}_{2\ell}$ generated by the elements $\mathcal{S}_i^B$ and $\mathcal{S}_i^D$, respectively, appears to be central group extensions of the corresponding Weyl groups.

Proposition 8.2 *The elements of pinor group* $\mathrm{Pin}(V)$ *introduced in* (8.28), (8.30) *satisfy the following relations for any* $i, j \in I$*:*

$$\begin{aligned} (\mathcal{S}_1^B)^2 &= 1, \qquad (\mathcal{S}_k^B)^2 = -1, \quad 1 < k \le \ell, \\ \mathcal{S}_i^B\mathcal{S}_j^B &= \mathcal{S}_j^B\mathcal{S}_i^B, \qquad a_{ij} = 0, \\ \mathcal{S}_i^B\mathcal{S}_j^B\mathcal{S}_i^B &= \mathcal{S}_j^B\mathcal{S}_i^B\mathcal{S}_j^B, \qquad a_{ij}a_{ji} = 1, \\ \mathcal{S}_i^B\mathcal{S}_j^B\mathcal{S}_i^B\mathcal{S}_j^B &= \mathcal{S}_j^B\mathcal{S}_i^B\mathcal{S}_j^B\mathcal{S}_i^B, \qquad a_{ij}a_{ji} = 2, \end{aligned} \tag{8.31}$$

and

$$
\begin{aligned}
(\mathcal{S}_i^D)^2 \;=\; -1\,, \quad \mathcal{S}_1^D\mathcal{S}_2^D \;=\; -\mathcal{S}_2^D\mathcal{S}_1^D\,, \quad \mathcal{S}_1^D\mathcal{S}_3^D\mathcal{S}_1^D \;=\; -\mathcal{S}_3^D\mathcal{S}_1^D\mathcal{S}_3^D\,,\\
\mathcal{S}_i^D\mathcal{S}_j^D \;=\; \mathcal{S}_j^D\mathcal{S}_i^D\,, \quad a_{ij}=0\,, \qquad i,j\neq(1,2)\,,\\
\mathcal{S}_i^D\mathcal{S}_j^D\mathcal{S}_i^D \;=\; \mathcal{S}_j^D\mathcal{S}_i^D\mathcal{S}_j^D\,, \qquad a_{ij}a_{ji}=1\,, \quad i,j\neq(1,3)\,.
\end{aligned}
\tag{8.32}
$$

This results in a central extension of the Weyl group $W_{O(V)}$ *by* $\mu_2\subseteq\mathcal{Z}(\mathrm{Pin}(V))$*:*

$$
1\longrightarrow\mu_2\longrightarrow\widetilde{W}_{O(V)}\stackrel{\pi}{\longrightarrow}W_{O(V)}\longrightarrow 1\quad. \tag{8.33}
$$

Proof : Relations in the first lines of (8.31), (8.32) follow from $\hat{\epsilon}_i^2=1$ and $\hat{\epsilon}_i\cdot\hat{\epsilon}_j\;=\;-\hat{\epsilon}_j\cdot\hat{\epsilon}_i$ for $i\neq j$. The other relations may be verified by writing explicitly from (8.24):

$$
\begin{aligned}
\mathcal{S}_1^B \;&=\; \frac{\hat{\epsilon}_\ell-\hat{\epsilon}_{\ell+2}}{\sqrt{2}}\,, \qquad \mathcal{T}_i\mathcal{S}_i\mathcal{T}_i^{-1} \;=\; \frac{\hat{\epsilon}_i+\hat{\epsilon}_{i+1}}{\sqrt{2}}\,,\\
\mathcal{S}_1^D \;&=\; \frac{\hat{\epsilon}_{\ell-1}\hat{\epsilon}_\ell+\hat{\epsilon}_{\ell-1}\hat{\epsilon}_{\ell+2}+\hat{\epsilon}_\ell\hat{\epsilon}_{\ell+1}-\hat{\epsilon}_{\ell+1}\hat{\epsilon}_{\ell+2}}{2}\,.
\end{aligned}
\tag{8.34}
$$

In particular, one can verify in a straightforward way that

$$
\mathcal{S}_1^D\mathcal{S}_3^D\mathcal{S}_1^D \;=\; \frac{\hat{\epsilon}_{\ell-2}\hat{\epsilon}_\ell+\hat{\epsilon}_\ell\hat{\epsilon}_{\ell+1}+\hat{\epsilon}_{\ell+1}\hat{\epsilon}_{\ell+3}-\hat{\epsilon}_{\ell-2}\hat{\epsilon}_{\ell+3}}{2} \;=\; -\mathcal{S}_3^D\mathcal{S}_1^D\mathcal{S}_3^D\,. \tag{8.35}
$$

The other 3-move braid relations from the last lines of (8.31), (8.32) follow by (8.25). $\square$

8.4 Construction of a section for $\mathrm{Spin}_{\ell+1}$

In this section we describe a lift of the Weyl group $W_{O(V)}$ into the spinor group $\mathrm{Spin}(V)$ combining the results of previous Sections 8.2 and 8.3 with the constructions of Section 5. The construction will be compatible with the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & \mu_2 & = & \mu_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathrm{Spin}(V) & \stackrel{\varphi}{\longrightarrow} & \mathrm{Pin}(V) & \stackrel{\det}{\longrightarrow} & \mu_2 & \longrightarrow & 1\\
 & & \downarrow\pi & & \downarrow\pi & & \| & & \\
1 & \longrightarrow & \mathrm{SO}(V) & \stackrel{\varphi}{\longrightarrow} & O(V) & \stackrel{\det}{\longrightarrow} & \mu_2 & \longrightarrow & 1
\end{array}
\quad. \tag{8.36}
$$

Note that the elements of $\mathrm{Spin}(V)\subset\mathrm{Pin}(V)$ are singled out by the condition that they are represented by a product of even number of elements $\hat{u}\in\mathrm{Pin}(V)$, or equivalently, $\mathrm{Spin}(V)=\mathrm{Pin}(V)\cap\mathcal{C}(V)^+$. Moreover, we have an epimorphism $\mathrm{Spin}(V)\to SO(V)$. As we have already managed to construct the inverse image under $\varphi: SO(V)\to O(V)$ of the generators of the corresponding Weyl groups we can use (8.23) to lift generators (8.13) and generators S_i^D of $W(D_\ell)$ into the corresponding spinor groups. We again consider the cases of odd and even orthogonal groups separately.

Lemma 8.4 *Introduce the following elements* $\widetilde{\mathcal{S}}_i^B \in \mathrm{Spin}_{2\ell+1}$

$$\widetilde{\mathcal{S}}_1^B = \hat{z}\,\mathcal{S}_1^B\,, \qquad \widetilde{\mathcal{S}}_k^B = \mathcal{S}_k^B\,, \quad 1<k\leq \ell\,, \tag{8.37}$$

where $\mathcal{S}_i^B$ *are given by* (8.28) *and*

$$\hat{z} = \hat{\epsilon}_1\cdot \ldots \cdot \hat{\epsilon}_{2\ell+1} \in \mathrm{Pin}_{2\ell+1}\,, \tag{8.38}$$

is a lift of the central element $z = T_0^B T_1^B \cdots T_\ell^B \in \mathcal{Z}(SO_{2\ell+1})$*. Then* (8.37) *satisfy the following relations:*

$$\begin{gathered}(\widetilde{\mathcal{S}}_i^B)^2 = (-1)^\ell\,, \qquad (\widetilde{\mathcal{S}}_k^B)^2 = -1\,, \quad 1<k\leq\ell\,,\\ \widetilde{\mathcal{S}}_i^B\widetilde{\mathcal{S}}_j^B = \widetilde{\mathcal{S}}_j^B\widetilde{\mathcal{S}}_i^B\,, \quad a_{ij}=0\,,\\ \widetilde{\mathcal{S}}_i^B\widetilde{\mathcal{S}}_j^B\widetilde{\mathcal{S}}_i^B = \widetilde{\mathcal{S}}_i^B\widetilde{\mathcal{S}}_i^B\widetilde{\mathcal{S}}_j^B\,, \quad a_{ij}=-1\,,\\ \widetilde{\mathcal{S}}_1^B\widetilde{\mathcal{S}}_2^B\widetilde{\mathcal{S}}_1^B\widetilde{\mathcal{S}}_2^B = \widetilde{\mathcal{S}}_2^B\widetilde{\mathcal{S}}_1^B\widetilde{\mathcal{S}}_2^B\widetilde{\mathcal{S}}_1^B\,.\end{gathered} \tag{8.39}$$

Therefore, (8.37) *generate a central extension of Weyl group* $W(B_\ell)$ *by* $\mathcal{Z}(\mathrm{Spin}_{2\ell+1}) = \mu_2$*:*

$$1 \longrightarrow \mathcal{Z}(\mathrm{Spin}_{2\ell+1}) \longrightarrow \widetilde{W}(B_\ell) \longrightarrow W(B_\ell) \longrightarrow 1 \quad . \tag{8.40}$$

Proof : By (8.25) one has $\hat{z}\mathcal{S}_i^B = \mathcal{S}_i^B\hat{z}$, $\forall i\in I$, therefore using (8.31) one finds out,

$$(\widetilde{\mathcal{S}}_1^B)^2 = \hat{z}^2(\mathcal{S}_1^B)^2 = \hat{z}^2 = (-1)^\ell\,. \tag{8.41}$$

The same argument implies the remaining relations in (8.39). □

The even orthogonal group case is already covered by Proposition 8.2 as the elements $\mathcal{S}_i^D$ entering (8.32) are already in $\mathrm{Spin}_{2\ell}$.

9 Adjoint action of the Tits groups W_G^T

In this section, we compute the action of the elements $S_i^G \in W_G^T$ in the corresponding Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$ for classical Lie groups G, including $G = GL_{\ell+1}$. Given a classical group G, the calculation of the adjoint action can be done using the appropriate faithful representation. We provide an explicit description of the action using the description of the groups W_G^T from Sections 4 and 5 above. Note that the resulting formulas easily follow from the explicit expressions for $\mathrm{Ad}_{\dot{s}_i}$, $i\in I$ obtained in [GLO].

Proposition 9.1 *The adjoint action of the Tits group* W_G^T *on the Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$ *via homomorphism* (2.29) *is given by*

$$\dot{s}_i\, e_i\, \dot{s}_i^{-1} = -f_i, \qquad \dot{s}_i\, f_i\, \dot{s}_i^{-1} = -e_i\,, \tag{9.1}$$

$$\dot{s}_i\, e_j\, \dot{s}_i^{-1} = e_j, \qquad \dot{s}_i\, f_j\, \dot{s}_i^{-1} = f_j, \qquad a_{ij}=0\,, \tag{9.2}$$

$$\dot{s}_i\, e_j\, \dot{s}_i^{-1} \;=\; \frac{1}{|a_{ij}|!}\,\underbrace{[e_i,[\ldots[e_i}_{|a_{ij}|},\, e_j]\ldots]]\,,$$

$$\dot{s}_i\, f_j\, \dot{s}_i^{-1} \;=\; \frac{(-1)^{|a_{ij}|}}{|a_{ij}|!}\,\underbrace{[f_i,[\ldots[f_i}_{|a_{ij}|},\, f_j]\ldots]], \qquad i \neq j\,. \tag{9.3}$$

Let us emphasize that we keep the ordering $I = \{1, \ldots, \ell\}$ of the corresponding set of vertices of the Dynkin diagram for each classical G, and use the explicit realization of the Cartan-Weyl generators $\phi(e_i)$, $\phi(f_i)$, $i \in I$ via the standard faithful representation ϕ.

Proposition 9.2 (The case A_ℓ) *The adjoint action of the elements $S_i \in W^T_{GL_{\ell+1}}$ (4.2) for $i \in \{1, \ldots, \ell\}$ is given by*

$$\mathrm{Ad}_{S_i}(e_j) \;=\; \begin{cases} f_j\,, & i = j \\ e_j\,, & |i-j| > 1 \\ \mathrm{ad}_{e_i}(e_j)\,, & i-j = -1 \\ -\mathrm{ad}_{e_i}(e_j)\,, & i-j = 1 \end{cases}\,, \tag{9.4}$$

and

$$\mathrm{Ad}_{S_i}(f_j) \;=\; \begin{cases} e_j\,, & i = j \\ f_j\,, & |i-j| > 1 \\ -\mathrm{ad}_{f_i}(f_j)\,, & i-j = -1 \\ \mathrm{ad}_{f_i}(f_j)\,, & i-j = 1 \end{cases}\,. \tag{9.5}$$

Proposition 9.3 (The case B_ℓ) *The adjoint action of the elements $S_i^B \in W^T_{SO_{2\ell+1}}$ (5.4) for $i \in \{1, \ldots, \ell\}$ is given by*

$$\mathrm{Ad}_{S_i^B}(e_j) \;=\; \begin{cases} f_j\,, & i = j \\ e_j\,, & |i-j| > 1 \\ -\dfrac{(-1)^{\delta_{i,1}}}{|a_{ij}|!}\mathrm{ad}_{e_i}^{-a_{ij}}(e_j)\,, & i-j = -1 \\ \mathrm{ad}_{e_i}(e_j)\,, & i-j = 1 \end{cases}\,, \tag{9.6}$$

and

$$\mathrm{Ad}_{S_i^B}(f_j) \;=\; \begin{cases} e_j\,, & i = j \\ f_j\,, & |i-j| > 1 \\ \dfrac{1}{|a_{ij}|!}\mathrm{ad}_{f_i}^{-a_{ij}}(f_j)\,, & i-j = -1 \\ -\mathrm{ad}_{f_i}(f_j)\,, & i-j = 1 \end{cases}\,. \tag{9.7}$$

Proposition 9.4 (The case C_ℓ) *The adjoint action of the elements $S_i^C \in W^T_{\mathrm{Sp}_{2\ell}}$ (5.7) for $i \in \{1, \ldots, \ell\}$ is given by*

$$\mathrm{Ad}_{S_i^C}(e_j) = \begin{cases} (-1)^{\delta_{i,1}} f_j\,, & i = j \\ e_j\,, & |i-j| > 1 \\ -\mathrm{ad}_{e_i}(e_j)\,, & i - j = -1 \\ \dfrac{1}{|a_{ij}|!}\mathrm{ad}_{e_i}^{-a_{ij}}(e_j)\,, & i - j = 1 \end{cases}, \tag{9.8}$$

and

$$\mathrm{Ad}_{S_i^C}(f_j) = \begin{cases} (-1)^{\delta_{i,1}} e_j\,, & i = j \\ f_j\,, & |i-j| > 1 \\ \mathrm{ad}_{f_i}^{-a_{ij}}(f_j)\,, & i - j = -1 \\ \dfrac{(-1)^{|a_{ij}|}}{|a_{ij}|!}\mathrm{ad}_{f_i}^{-a_{ij}}(f_j)\,, & i - j = 1 \end{cases}. \tag{9.9}$$

Proposition 9.5 (The case D_ℓ) *The adjoint action of the elements $S_i^D \in W^T_{O_{2\ell}}$ (5.5) for $i \in \{1, 2; 3, \ldots, \ell\}$ is given by*

$$\mathrm{Ad}_{S_i^D}(e_j) = \begin{cases} f_j\,, & i = j \\ -e_j\,, & a_{ij} = 0\,, \quad \iota(i) = j \\ e_j\,, & a_{ij} = 0\,, \quad \iota(i) \neq j \\ -\mathrm{ad}_{e_i}(e_j)\,, & a_{ij} = -1\,, \quad i < j \\ \mathrm{ad}_{e_i}(e_j)\,, & a_{ij} = -1\,, \quad i > j \end{cases} \tag{9.10}$$

and

$$\mathrm{Ad}_{S_i^D}(f_j) = \begin{cases} e_j\,, & i = j \\ -f_j\,, & \iota(i) = j\,, \quad \iota(i) = j \\ f_j\,, & a_{ij} = 0\,, \quad \iota(i) \neq j \\ \mathrm{ad}_{f_i}(f_j)\,, & a_{ij} = -1\,, \quad i < j \\ -\mathrm{ad}_{f_i}(f_j)\,, & a_{ij} = -1\,, \quad i > j \end{cases}. \tag{9.11}$$

10 Appendix: General linear and classical Lie groups

10.1 General linear group $GL_{\ell+1}(\mathbb{C})$

Let $\mathfrak{gl}_{\ell+1}(\mathbb{C}) = \mathfrak{gl}(V)$ be the Lie algebra induced by endomorphisms $\phi : V \to V$ of the complex vector space $V \simeq \mathbb{C}^{\ell+1}$ via the commutator Lie bracket:

$$[\phi_1,\, \phi_2] = \phi_1 \circ \phi_2 - \phi_2 \circ \phi_1\,, \tag{10.1}$$

where $\circ$ is a composition of linear maps $V \to V$. Let $\mathfrak{h} \subset \mathfrak{gl}_{\ell+1}(\mathbb{C})$ be its maximal commutative subalgebra. Choosing a basis $\{\varepsilon_1, \ldots, \varepsilon_{\ell+1}\} \subset V$ of vector space V we identify $\mathfrak{gl}_{\ell+1}(\mathbb{C})$ with the Lie algebra of matrices with basis e_{ij}, $1 \leq i, j \leq \ell + 1$ defined by the following endomorphisms:

$$e_{ij} : \quad V \longrightarrow V \,, \qquad \varepsilon_k \longmapsto \delta_{ik}\varepsilon_j \,, \quad 1 \leq k \leq \ell+1 \,. \tag{10.2}$$

The corresponding Lie brackets is given by

$$\big[e_{ij},\, e_{kl}\big] \,=\, \delta_{jk}e_{il} \,-\, \delta_{il}e_{kj} \,. \tag{10.3}$$

The Cartan subalgebra $\mathfrak{h}_{\ell+1} \subset \mathfrak{gl}_{\ell+1}(\mathbb{C})$ is identified with a subalgebra of diagonal matrices.

Fix an orthonormal basis $\{\epsilon_1, \ldots, \epsilon_{\ell+1}\} \subset \mathfrak{h}^*$ in the Euclidean space $\big(\mathfrak{h}^* \simeq \mathbb{C}^{\ell+1};\, \langle\,,\,\rangle\big)$ and identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the standard scalar product $\langle\,,\rangle$ on $\mathbb{C}^{\ell+1}$. Thus we can consider the diagonal entries $\epsilon_k := e_{kk}$ as functions on $\mathfrak{h}$, so that they provide a (complex) linear coordinate system on $\mathfrak{h}$. Then any linear functional $\lambda \in \mathfrak{h}^*$ has the form

$$\lambda \,=\, \lambda_1\epsilon_1 \,+\, \ldots \,+\, \lambda_{\ell+1}\epsilon_{\ell+1} \,. \tag{10.4}$$

The general linear group $GL_{\ell+1}(\mathbb{C})$ is isomorphic to the group of invertible rank $\ell + 1$ matrices, and maximal torus $H_{\ell+1} \subset GL_{\ell+1}(\mathbb{C})$ is identified with the subgroup of diagonal matrices. The diagonal entries of $t_k = t_k(g)$, $g \in H_{\ell+1}$ provide coordinates on the maximal torus $H_{\ell+1}$ and generate the group of (rational) characters

$$X^*(H_{\ell+1}) \,=\, \mathrm{Hom}(H_{\ell+1},\, \mathbb{C}^*) \simeq \mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}^{\ell+1} \,. \tag{10.5}$$

Namely, any $\lambda \in \mathfrak{h}^*$ gives rise to a homomorphism

$$\begin{gathered} e^{\lambda} : \quad H_{\ell+1} \longrightarrow \mathbb{C}^* \,, \qquad e^h \longmapsto e^{2\pi\lambda(h)} \,=\, e^{2\pi\langle\lambda,\, h\rangle} \,, \quad h \in \mathfrak{h} \,; \\ e^{\lambda} \,=\, t_1^{\lambda_1} t_2^{\lambda_2} \cdot \ldots \cdot t_{\ell+1}^{\lambda_{\ell+1}} \,. \end{gathered} \tag{10.6}$$

The elements $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$ consisting of $\lambda \in \mathfrak{h}^*$ with integral components $(\lambda_1, \ldots, \lambda_{\ell+1}) \in \mathbb{Z}^{\ell+1}$ are called the weights; they span the weight lattice:

$$\Lambda_W \,=\, \mathbb{Z}\epsilon_1 \oplus \ldots \oplus \mathbb{Z}\epsilon_{\ell+1} \,\simeq\, \mathfrak{h}^*_{\mathbb{Z}} \,. \tag{10.7}$$

The adjoint action of $H_{\ell+1}$ in Lie algebra $\mathfrak{gl}_{\ell+1}(\mathbb{C})$ provides the Cartan decomposition of $\mathfrak{gl}_{\ell+1}(\mathbb{C})$ with respect to $H_{\ell+1}$

$$\begin{gathered} \mathfrak{gl}_{\ell+1}(\mathbb{C}) \,=\, \mathfrak{h} \,\oplus \bigoplus_{\alpha_{ij}\in\Phi(A_\ell)} \big(\mathfrak{gl}_{\ell+1}\big)_{ij} \,, \\ (\mathfrak{gl}_{\ell+1})_{ij} \,=\, \big\{X \in \mathfrak{gl}_{\ell+1} : \quad \mathrm{ad}_h(X) \,=\, \alpha_{ij}(h)X,\ \forall h \in \mathfrak{h}\big\} \,=\, \mathbb{C}e_{ij} \,, \quad i \neq j \,, \end{gathered} \tag{10.8}$$

where the corresponding root system $\Phi(A_\ell)$ of type A_ℓ reads

$$\Phi(A_\ell) \,=\, \big\{\alpha_{ij} \,=\, \epsilon_i - \epsilon_j \,, \quad i \neq j\big\} \,\subset\, \mathfrak{h}^*_{\mathbb{Z}} \,. \tag{10.9}$$

The simple root system $\Pi_{A_\ell} \subset \Phi_+(A_\ell)$ is given by

$$\begin{aligned} \Pi_{A_\ell} \,&=\, \big\{\alpha_i \,=\, \epsilon_i - \epsilon_{i+1}\big\} \,\subset\, \Phi_+(A_\ell)\,, \\ \Phi_+(A_\ell) \,&=\, \big\{\alpha_{ij} \,=\, \epsilon_i - \epsilon_j\,, \quad i<j\big\} \,\subset\, \mathfrak{h}^*_{\mathbb{Z}}\,, \end{aligned} \tag{10.10}$$

and the root lattice is

$$\Lambda_R \,=\, \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_\ell \,\subset\, \Lambda_W \simeq \mathfrak{h}^*_{\mathbb{Z}}\,. \tag{10.11}$$

Let $X_*(H_{\ell+1}) = \mathrm{Hom}(\mathbb{C}^*, H_{\ell+1})$ be the group of co-characters (i.e. one-parametric subgroups) in the maximal torus:

$$\chi^\vee_a \,:\quad \mathbb{C}^* \longrightarrow H_{\ell+1}\,, \qquad t \longmapsto \mathrm{diag}\{t^{a_1}, \ldots, t^{a_{\ell+1}}\}\,. \tag{10.12}$$

The group of co-characters $X_*(H_{\ell+1}) \simeq \mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}^{\ell+1}$ is dual to $X^*(H_{\ell+1})$ via

$$e^\lambda\big(\chi^\vee_a(t)\big) \,=\, t^{a_1\lambda_1+\ldots+a_{\ell+1}\lambda_{\ell+1}} \,=\, t^{\langle a^\vee,\,\lambda\rangle}\,. \tag{10.13}$$

The dual basis in $\mathfrak{h}_{\mathbb{Z}}$ is referred to as *fundamental co-weights*:

$$p^\vee_k \,=\, e_{11} + \ldots + e_{kk}\,, \qquad \langle p^\vee_k,\, \alpha_i\rangle \,=\, \delta_{ki}\,, \qquad 1 \le k, i \le \ell+1\,, \tag{10.14}$$

that span the co-weight lattice:

$$\Lambda^\vee_W \,=\, \big\{h \in \mathfrak{h} \,:\, \lambda(h) \in \mathbb{Z}\,, \quad \forall \lambda \in \Lambda_W\big\} \,=\, \mathbb{Z}p^\vee_1 \oplus \ldots \oplus \mathbb{Z}p^\vee_{\ell+1}\,. \tag{10.15}$$

The lattice $\Lambda^\vee_W$ can be identified with the kernel of the map $\exp : \mathfrak{h} \to H_{\ell+1}$, so that $H_{\ell+1} = \mathfrak{h}/\Lambda^\vee_W$.

The determinant map $\det : GL_{\ell+1}(\mathbb{C}) \to \mathbb{C}^*$ may be defined as a unique homomorphim such that

$$\det \,:\quad e^{\sum\limits_{i=1}^{\ell+1} t_i e_{ii}} \longrightarrow e^{\sum\limits_{i=1}^{\ell+1} t_i}\,. \tag{10.16}$$

We have the following exact sequence

$$1 \longrightarrow SL_{\ell+1}(\mathbb{C}) \stackrel{\pi}{\longrightarrow} GL_{\ell+1}(\mathbb{C}) \stackrel{\det}{\longrightarrow} \mathbb{C}^* \longrightarrow 1 \quad , \tag{10.17}$$

where $SL_{\ell+1}(\mathbb{C})$ is the unimodular subgroup. Its Lie algebra $\mathfrak{sl}_{\ell+1}$ has the standard faithful representation given by the following homomorphism of associative algebras:

$$\begin{gathered} \phi \,:\quad \mathcal{U}\big(\mathfrak{sl}_{\ell+1}(\mathbb{C})\big) \longrightarrow \mathrm{End}(\mathbb{C}^{\ell+1})\,, \\ \phi(X_{\alpha_i}) \,=\, e_{i,i+1}\,, \qquad \phi(X_{-\alpha_i}) \,=\, -e_{i+1,i}\,, \qquad \phi(h_{\alpha_i}) \,=\, e_{i,i} - e_{i+1,i+1}\,, \end{gathered} \tag{10.18}$$

where e_{ij} are the matrix units (10.2). The Lie algebra $\mathfrak{sl}_{\ell+1}$ may be identified with the Lie subalgebra of matrices with zero trace in $\mathfrak{gl}_{\ell+1}$, so that

$$\mathfrak{gl}_{\ell+1} \,=\, \mathfrak{sl}_{\ell+1} \oplus \mathbb{C}\,. \tag{10.19}$$

The Lie algebras $\mathfrak{sl}_{\ell+1}$ and $\mathfrak{gl}_{\ell+1}$ share the Dynkin diagram Γ_{A_ℓ} with the set of vertices I of size ℓ, the reduced root system $\Phi(A_\ell)$ and the Weyl group W_{A_ℓ}. However, in the case of $\mathfrak{sl}_{\ell+1}$ its rank is equal to the rank of the root lattice, contrary to the $\mathfrak{gl}_{\ell+1}$ case. More precisely, let $\{\varepsilon_1,\dots,\varepsilon_{\ell+1}\}$ be an orthonormal basis in $\mathbb{C}^{\ell+1}$. Then the fundamental weights p_k of $\mathfrak{gl}_{\ell+1}$ are linear forms defined by $p_k(h_{\alpha_i})=\delta_{ki}$ (see (10.7), (10.14)):

$$p_k \,=\, \epsilon_1+\ldots+\epsilon_k\,,\qquad 1\leq k\leq \ell+1\,. \tag{10.20}$$

Define a collection of vectors in $\mathbb{R}^{\ell+1}$,

$$\epsilon_k^{A_\ell}\,=\,\epsilon_k-\frac{\epsilon_1+\ldots+\epsilon_{\ell+1}}{\ell+1}\,;\qquad 1\leq k\leq \ell+1\,, \tag{10.21}$$

which span a codimension one Euclidean subspace, due to $\epsilon_1^{A_\ell}+\ldots+\epsilon_{\ell+1}^{A_\ell}\,=\,0$. Then the simple roots and fundamental weights of $\mathfrak{sl}_{\ell+1}$ can be written as follows:

$$\alpha_k^{A_\ell}\,=\,\epsilon_k^{A_\ell}-\epsilon_{k+1}^{A_\ell}\,,\qquad \varpi_k^{A_\ell}\,=\,p_k-\frac{k}{\ell+1}\,p_{\ell+1}\,=\,\epsilon_1^{A_\ell}+\ldots+\epsilon_k^{A_\ell}\,. \tag{10.22}$$

The Cartan matrix $A\,=\,\|a_{ij}^{A_\ell}\|$ with $a_{ij}=\langle\alpha_j^{A_\ell},(\alpha_i^\vee)^{A_\ell}\rangle$, and its inverse $A^{-1}\,=\,\|c_{ij}^{A_\ell}\|$ take the form:

$$A\,=\,\begin{pmatrix} 2 & -1 & 0 & \dots & 0\\ -1 & 2 & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & -1 & 0\\ \vdots & \ddots & -1 & 2 & -1\\ 0 & \dots & 0 & -1 & 2\end{pmatrix}\,,\qquad A^{-1}\,=\,\frac{1}{\ell+1}\begin{pmatrix} \ell & \ell-1 & \ell-2 & \dots & 2 & 1\\ \ell-1 & 2(\ell-1) & 2(\ell-2) & \dots & 2\cdot 2 & 2\\ \ell-2 & 2(\ell-2) & 3(\ell-2) & \dots & 3\cdot 2 & 3\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 2 & 2\cdot 2 & 3\cdot 2 & \dots & (\ell-1)2 & \ell-1\\ 1 & 2 & 3 & \dots & \ell-1 & \ell\end{pmatrix}\,. \tag{10.23}$$

In these terms one obtains the following expressions for the coroots and coweights:

$$\begin{gathered} h_i^{A_\ell}\,=\,(\alpha_i^{A_\ell})^\vee\,=\,\sum_{j\in I}\langle\alpha_j^{A_\ell},\,(\alpha_i^{A_\ell})^\vee\rangle(\varpi_j^{A_\ell})^\vee\,=\,\sum_{j\in I}a_{ij}^{A_\ell}(\varpi_j^{A_\ell})^\vee\,,\\ (\varpi_i^{A_\ell})^\vee\,=\,\sum_{j\in I}c_{ij}^{A_\ell}(\alpha_j^{A_\ell})^\vee\,,\qquad i\in I\,. \end{gathered} \tag{10.24}$$

10.2 The case of type B_ℓ root system

The Lie algebra $\mathfrak{so}_{2\ell+1}$ can be identified with the θ_B-fixed subalgebra:

$$\begin{gathered} \mathfrak{so}_{2\ell+1}\,=\,\big\{X\in\mathfrak{gl}_{2\ell+1}(\mathbb{C})\,:\quad X\,=\,\theta_B(X)\big\}\,,\qquad \theta_B(X)\,=\,-S_BX^\tau S_B^{-1}\,,\\ S_B\,=\,\mathrm{diag}\big(1,-1,\,\ldots,\,1\big)\,\in\,GL_{2\ell+1}\,, \end{gathered} \tag{10.25}$$

where τ is the transposition along the opposite diagonal (3.3). This provides the faithful representation,

$$\phi\,:\quad \mathfrak{so}_{2\ell+1}\,\longrightarrow\,\mathfrak{gl}_{2\ell+1}(\mathbb{C})\,, \tag{10.26}$$

given by the following presentation of the Chevalley-Weyl generators $h_k = \alpha_k^\vee$, e_k, f_k, $k \in I$ of $\mathfrak{so}_{2\ell+1}$ in terms of the generators of $\mathfrak{gl}_{2\ell+1}(\mathbb{C})$:

$$\begin{aligned} \phi(e_1) &= \sqrt{2}(e_{\ell,\ell+1} + e_{\ell+1,\ell+2}) \,, \qquad \phi(f_1) = \sqrt{2}(e_{\ell+1,\ell} + e_{\ell+2,\ell+1}) \,; \\ \phi(e_k) &= e_{\ell+1-k,\ell+2-k} + e_{\ell+k,\ell+1+k} \,, \quad \phi(f_k) = e_{\ell+2-k,\ell+1-k} + e_{\ell+1+k,\ell+k} \,, \\ & 1 < k \leq \ell \,; \\ \phi(h_1) &= 2(e_{\ell\ell} - e_{\ell+2,\ell+2}) \,, \\ \phi(h_k) &= (e_{\ell+1-k,\ell+1-k} + e_{\ell+k,\ell+k}) - (e_{\ell+2-k,\ell+2-k} + e_{\ell+1+k,\ell+1+k}) \,, \\ & 1 < k \leq \ell \,. \end{aligned} \tag{10.27}$$

The representation (10.27) implies the following presentation of the type B_ℓ root system in terms of the root system of type $A_{2\ell}$

$$\alpha_i^{B_\ell} = \alpha_{\ell+1-i}^{A_{2\ell}} + \alpha_{\ell+i}^{A_{2\ell}} \,, \qquad i \in I \,. \tag{10.28}$$

Explicitly, the symmetry of the Dynkin diagram of type $A_{2\ell}$ is given by

$$\begin{aligned} \iota \,: \quad i &\longmapsto 2\ell + 1 - i \,, \qquad i \in \{1, 2, \ldots, 2\ell\} \,, \\ \alpha_i^{A_{2\ell}} &\longmapsto \alpha_{2\ell+1-i}^{A_{2\ell}} \,. \end{aligned} \tag{10.29}$$

Then (10.28) reads

$$\alpha_i^{B_\ell} = \alpha_{\ell+1-i}^{A_{2\ell}} + \iota\big(\alpha_{\ell+1-i}^{A_{2\ell}}\big) \,. \tag{10.30}$$

Introduce the Euclidean space $\mathbb{R}^{2\ell+1}$ with a standard orthonormal basis $\big\{\epsilon_1, \ldots, \epsilon_{2\ell+1}\big\}$, acted on by the involution ι as follows:

$$\iota \,: \quad \mathbb{R}^{2\ell+1} \longrightarrow \mathbb{R}^{2\ell+1} \,, \qquad \epsilon_i \longmapsto -\epsilon_{2\ell+2-i} \,. \tag{10.31}$$

Consider the ι-fixed Euclidean subspace $\mathbb{R}^\ell \subset \mathbb{R}^{2\ell+1}$ spanned by

$$\epsilon_k^{B_\ell} = \epsilon_{\ell+1-k} + \iota\big(\epsilon_{\ell+1-k}\big) = \epsilon_{\ell+1-k} - \epsilon_{\ell+1+k} \,, \qquad 1 \leq k \leq \ell \,, \tag{10.32}$$

then the root data of type B_ℓ is given by

$$\begin{gathered} \begin{cases} \alpha_1^{B_\ell} = \epsilon_1^{B_\ell} \\ \alpha_k^{B_\ell} = \epsilon_k^{B_\ell} - \epsilon_{k-1}^{B_\ell} \,, \\ 1 < k \leq \ell \,; \end{cases} \qquad \begin{cases} \varpi_1^{B_\ell} = \dfrac{\epsilon_1^{B_\ell} + \ldots + \epsilon_\ell^{B_\ell}}{2} \\ \varpi_k^{B_\ell} = \epsilon_k^{B_\ell} + \ldots + \epsilon_\ell^{B_\ell} \,, \\ 1 < k \leq \ell \,; \end{cases} \\ \Phi(B_\ell) = \big\{\pm\epsilon_i^{B_\ell} \pm \epsilon_j^{B_\ell} \,, \quad \pm\epsilon_i^{B_\ell}\big\} \,. \end{gathered} \tag{10.33}$$

The simple co-roots and fundamental co-weights are determined via

$$\langle \alpha_i^{B_\ell}, (\varpi_j^{B_\ell})^\vee \rangle = \langle (\alpha_i^{B_\ell})^\vee, \varpi_j^{B_\ell} \rangle = \delta_{ij} \,, \tag{10.34}$$

and have the following form:

$$\begin{cases} \left(\alpha_1^{B_\ell}\right)^\vee \,=\, 2\epsilon_1^{B_\ell} \\ \left(\alpha_k^{B_\ell}\right)^\vee \,=\, \epsilon_k^{B_\ell} - \epsilon_{k-1}^{B_\ell}\,, \\ 1 < k \le \ell\,; \end{cases} \qquad \begin{cases} \left(\varpi_i^{B_\ell}\right)^\vee \,=\, \epsilon_i^{B_\ell} + \ldots + \epsilon_\ell^{B_\ell}\,, \\ i \in I \end{cases} \tag{10.35}$$

$$\Phi^\vee(B_\ell) \,=\, \left\{\pm\epsilon_i^{B_\ell} \pm \epsilon_j^{B_\ell}\,, \quad \pm 2\epsilon_i^{B_\ell}\right\}.$$

The Cartan matrix $A = \|a_{ij}^{B_\ell}\| = \|\langle\alpha_j^{B_\ell},\, \left(\alpha_i^{B_\ell}\right)^\vee\rangle\|$ and its inverse $A^{-1} = \|c_{ij}\|$ are given by

$$A \,=\, \begin{pmatrix} 2 & -2 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \ldots & 0 & -1 & 2 \end{pmatrix}, \qquad A^{-1} \,=\, \begin{pmatrix} \frac{\ell}{2} & \ell-1 & \ldots & 3 & 2 & 1 \\ \frac{\ell-1}{2} & \ell-1 & \ldots & 3 & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ \frac{3}{2} & 3 & \ldots & 3 & 2 & 1 \\ 1 & 2 & \ldots & 2 & 2 & 1 \\ \frac{1}{2} & 1 & \ldots & 1 & 1 & 1 \end{pmatrix},$$

$$h_i^{B_\ell} \,=\, \left(\alpha_i^{B_\ell}\right)^\vee \,=\, \sum_{j\in I}\langle\alpha_j^{B_\ell},\, \left(\alpha_i^{B_\ell}\right)^\vee\rangle\left(\varpi_j^{B_\ell}\right)^\vee \,=\, \sum_{j\in I} a_{ij}^{B_\ell}\left(\varpi_j^{B_\ell}\right)^\vee, \tag{10.36}$$

$$\left(\varpi_i^{B_\ell}\right)^\vee \,=\, \sum_{j\in I} c_{ij}\left(\alpha_j^{B_\ell}\right)^\vee, \qquad i \in I\,.$$

Lemma 10.1 *The Weyl group $W(B_\ell)$ is isomorphic to a subgroup in $W(A_{2\ell})$ via*

$$\begin{aligned} s_1^{B_\ell} \,&=\, s_\ell^{A_{2\ell}} s_{\ell+1}^{A_{2\ell}} s_\ell^{A_{2\ell}} \,=\, s_{\ell+1}^{A_{2\ell}} s_\ell^{A_{2\ell}} s_{\ell+1}^{A_{2\ell}}\,, \\ s_k^{B_\ell} \,&=\, s_{\ell+1-k}^{A_{2\ell}} s_{\ell+k}^{A_{2\ell}} \,=\, s_{\ell+k}^{A_{2\ell}} s_{\ell+1-k}^{A_{2\ell}}\,, \quad 1 < k \le \ell\,. \end{aligned} \tag{10.37}$$

Proof: For $1 < k \le \ell$ given $\lambda = \lambda_1\epsilon_1^{B_\ell} + \ldots + \lambda_\ell\epsilon_\ell^{B_\ell}$ by straightforward computation one has

$$\begin{aligned} s_k^{B_\ell}(\lambda) \,&=\, \lambda \,-\, \langle\lambda,\, (\alpha_k^{B_\ell})^\vee\rangle_{B_\ell}\alpha_k^{B_\ell} \,=\, \lambda \,-\, \langle\lambda,\, \epsilon_k^{B_\ell} - \epsilon_{k-1}^{B_\ell}\rangle_{B_\ell}(\epsilon_k^{B_\ell} - \epsilon_{k-1}^{B_\ell}) \\ &=\, \lambda \,-\, (\lambda_k - \lambda_{k-1})(\epsilon_k^{B_\ell} - \epsilon_{k-1}^{B_\ell}) \\ &=\, \lambda_1\epsilon_1^{B_\ell} + \ldots + \lambda_k\epsilon_{k-1}^{B_\ell} + \lambda_{k-1}\epsilon_k^{B_\ell} + \ldots + \lambda_\ell\epsilon_\ell^{B_\ell} \\ &=\, \lambda_1\epsilon_\ell^{A_{2\ell}} + \ldots + \lambda_k\epsilon_{\ell+2-k}^{A_{2\ell}} + \lambda_{k-1}\epsilon_{\ell+1-k}^{A_{2\ell}} + \ldots + \lambda_\ell\epsilon_1^{A_{2\ell}} \\ &\quad - \lambda_1\epsilon_{\ell+2}^{A_{2\ell}} - \ldots - \lambda_k\epsilon_{\ell+k}^{A_{2\ell}} - \lambda_{k-1}\epsilon_{\ell+1+k}^{A_{2\ell}} - \ldots - \lambda_\ell\epsilon_{2\ell+1}^{A_{2\ell}} \\ &=\, \lambda \,-\, \langle\lambda,\, (\alpha_{\ell+1-k}^{A_{2\ell}})^\vee\rangle_{A_{2\ell}}\alpha_{\ell+1-k}^{A_{2\ell}} \,-\, \langle\lambda,\, (\alpha_{\ell+k}^{A_{2\ell}})^\vee\rangle_{A_{2\ell}}\alpha_{\ell+k}^{A_{2\ell}} \\ &=\, s_{\ell+k}^{A_{2\ell}} s_{\ell+1-k}^{A_{2\ell}}(\lambda)\,. \end{aligned} \tag{10.38}$$

Similarly, for $k = 1$ one has

$$\begin{aligned} s_1^{B_\ell}(\lambda) \,&=\, \lambda \,-\, \langle\lambda,\, (\alpha_1^{B_\ell})^\vee\rangle_{B_\ell}\alpha_1^{B_\ell} \,=\, \lambda \,-\, \langle\lambda,\, 2\epsilon_1^{B_\ell}\rangle_{B_\ell}\epsilon_1^{B_\ell} \\ &=\, -\lambda_1\epsilon_1^{B_\ell} + \lambda_2\epsilon_2^{B_\ell} + \ldots + \lambda_\ell\epsilon_\ell^{B_\ell} \\ &=\, \lambda_\ell\epsilon_1^{A_{2\ell}} + \ldots + \lambda_2\epsilon_{\ell-1}^{A_{2\ell}} - \lambda_1\epsilon_\ell^{A_{2\ell}} \,+\, \lambda_1\epsilon_{\ell+2}^{A_{2\ell}} - \lambda_2\epsilon_{\ell+3}^{A_{2\ell}} - \ldots - \lambda_\ell\epsilon_{2\ell+1}^{A_{2\ell}} \\ &=\, s_\ell^{A_{2\ell}} s_{\ell+1}^{A_{2\ell}} s_\ell^{A_{2\ell}}(\lambda)\,, \end{aligned} \tag{10.39}$$

since $s_\ell^{A_{2\ell}} s_{\ell+1}^{A_{2\ell}} s_\ell^{A_{2\ell}}$ simply swaps $\epsilon_\ell^{A_{2\ell}} \leftrightarrow \epsilon_{\ell+2}^{A_{2\ell}}$. □

The Lie group $O_{2\ell+1}$ may be presented as θ_B-invariant subgroup of $GL_{2\ell+1}$:

$$O_{2\ell+1} \,=\, \big\{ g \in GL_{2\ell+1} : \quad g \,=\, g^{\theta_B} \big\} \,. \tag{10.40}$$

Let us describe the corresponding Tits group $W^T_{O_{2\ell+1}}$ in terms of generators of $W^T_{GL_{2\ell+1}}$.

Lemma 10.2 *Let $\dot{s}_i^B = e^{\frac{\pi}{2} J_i}$, $i \in I$ be the Tits generators* (2.28)*. Then the following hold:*

1. *The Tits group $W^T_{O_{\ell+1}}$ is isomorphic to a subgroup of $W^T_{GL_{2\ell+1}}$ via*

$$\dot{s}_1^B \,=\, \dot{s}_\ell^{A_{2\ell}} \dot{s}_{\ell+1}^{A_{2\ell}} \dot{s}_\ell^{A_{2\ell}} \,, \qquad \dot{s}_k^B \,=\, \dot{s}_{\ell+1-k}^{A_{2\ell}} \dot{s}_{\ell+k}^{A_{2\ell}} \,, \quad 1 < k \leq \ell \,. \tag{10.41}$$

2. *The elements* (10.41) *belong to the θ_B-fixed subgroup $(W^T_{GL_{2\ell+1}})^{\theta_B}$.*

3. *Presentation* (10.41) *matches with* (10.37) *due to*

$$\mathrm{Ad}_{\dot{s}_i^B}|_{\mathfrak{h}} \,=\, s_i^B \,. \tag{10.42}$$

Proof : (1) Since $\dot{s}_i^B \in SO_{2\ell+1}$, $i \in I$, one might verify (10.41) using the standard faithful representation $\phi: SO_{2\ell+1} \to GL_{2\ell+1}$. For $i = 1$, we have

$$\phi(\dot{s}_1^B) \,=\, \phi(e^{\frac{\pi}{2} J_1}) \,=\, \begin{pmatrix} \mathrm{Id}_{\ell-1} & & & & \\ & 0 & 0 & 1 & \\ & 0 & -1 & 0 & \\ & 1 & 0 & 0 & \\ & & & & \mathrm{Id}_{\ell-1} \end{pmatrix} \,=\, \phi(\dot{s}_\ell^{A_{2\ell}} \dot{s}_{\ell+1}^{A_{2\ell}} \dot{s}_\ell^{A_{2\ell}}) \,, \tag{10.43}$$

and similarly, for $1 < k \leq \ell$ we obtain

$$\phi(\dot{s}_k^B) \,=\, \phi(e^{\frac{\pi}{2} J_k}) \,=\, \begin{pmatrix} \mathrm{Id}_{\ell-k} & & & & & & \\ & 0 & -1 & & & & \\ & 1 & 0 & & & & \\ & & & \mathrm{Id}_{2k-3} & & & \\ & & & & 0 & -1 & \\ & & & & 1 & 0 & \\ & & & & & & \mathrm{Id}_{\ell-k} \end{pmatrix} \,=\, \phi(\dot{s}_{\ell+1-k}^{A_{2\ell}} \dot{s}_{\ell+k}^{A_{2\ell}}) \,. \tag{10.44}$$

(2) From (4.11) one reads

$$\theta_B : \qquad \dot{s}_i^{A_{2\ell}} \longmapsto \dot{s}_{2\ell+1-i}^{A_{2\ell}} \,, \qquad 1 \leq i \leq 2\ell+1 \,, \tag{10.45}$$

and using the Tits relations (2.29) one infers that (10.41) are θ_B-invariant.

(3) Follows by (2.29) and (10.37). □

By analogy with the general linear group case, there is another way to lift simple root generators $s_i^B \in W(B_\ell)$ into the Tits group $W^T_{O_{2\ell+1}}$. Recall that $W^T_{GL_{2\ell+1}}$ contains the following elements:

$$\begin{gathered} T_k \,:=\, e^{\pi \imath e_{kk}} \,, \qquad 1 \leq k \leq (2\ell+1) \,; \\ S_i \,=\, T_i \dot{s}_i \,=\, \dot{s}_i T_{i+1} \,, \qquad \overline{S}_i \,=\, T_{i+1} \dot{s}_i \,=\, \dot{s}_i T_i \,, \qquad 1 \leq i \leq 2\ell \,. \end{gathered} \tag{10.46}$$

Lemma 10.3 *The following elements belong to the fixed point subgroup* $(W^T_{GL_{2\ell+1}})^{\theta_B}$*:*

$$S_1^B = S_{\ell+1} S_\ell S_{\ell+1}\,, \qquad S_k^B = S_{\ell+1-k}\overline{S}_{\ell+k}\,, \quad 1 < k \le \ell\,;$$
$$T_i^B = T_{\ell+1-i}T_{\ell+1+i} = e^{\pi\imath\epsilon_i^B}\,, \quad 1\le i\le \ell \qquad \text{and} \qquad T_0^B := T_{\ell+1}\,, \tag{10.47}$$

where T_0^B *has* $\det(T_0^B) = -1$ *and generates the center* $\mathcal{Z}\big(W^T_{O_{2\ell+1}}\big) = \mu_2$.

Proof : The explicit action of θ_B on generators (10.46) gives (4.15):

$$\theta_B(S_i) = \overline{S}_{2\ell+1-i}\,, \qquad \theta_B(\overline{S}_i) = S_{2\ell+1-i}\,, \qquad \theta_B(T_k) = T_{2\ell+2-k}\,. \tag{10.48}$$

Using (4.13) this implies that (10.47) are invariant under θ_B. □

Corollary 10.1 *The elements* (10.41) *and* (10.47) *can be identified via*

$$S_1^B = T_0^B\,\dot{s}_1^B\,, \qquad S_k^B = T_k^B\,\dot{s}_k^B\,, \quad 1<k\le\ell\,. \tag{10.49}$$

Proof : By straightforward computation using faithful representation (10.40) one finds

$$\phi(S_1^B) = \begin{pmatrix} \mathrm{Id}_{\ell-1} & & \\ & \begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix} & \\ & & \mathrm{Id}_{\ell-1}\end{pmatrix}, \qquad \phi(S_k^B) = \begin{pmatrix} \mathrm{Id}_{\ell-k} & & & & \\ & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & & & \\ & & \mathrm{Id}_{2k-3} & & \\ & & & \begin{smallmatrix}0&-1\\-1&0\end{smallmatrix} & \\ & & & & \mathrm{Id}_{\ell-k}\end{pmatrix}. \tag{10.50}$$

Identifying this with (10.43), (10.44) one deduces (10.49). □

Proposition 10.1 *The elements* $S_i^B \in W^T_{O_{2\ell+1}}$, $i\in I$ *defined in* (10.47) *satisfy the relations* (2.23)*,* (2.24) *of type* B_ℓ*:*

$$(S_1^B)^2 = \ldots = (S_\ell^B)^2 = 1\,,$$
$$S_i^B S_j^B = S_j^B S_i^B\,, \quad i,j\in I \quad \text{such that} \quad a_{ij}=0\,;$$
$$S_i^B S_j^B S_i^B = S_j^B S_i^B S_j^B\,, \qquad i,j\in I \quad \text{such that} \quad a_{ij}=-1\,, \tag{10.51}$$
$$S_1^B S_2^B S_1^B S_2^B = S_2^B S_1^B S_2^B S_1^B\,.$$

Proof : One might prove (10.51) b applying faithful representation (10.40). Alternatively, since the relations (10.51) are exactly the relations (2.24), they can be derived from (2.29) using substitution (10.49). For the first line of (10.51) we have:

$$(S_1^B)^2 = T_0^B\dot{s}_1^B T_0^B\dot{s}_1^B = (T_0^B)^2(\dot{s}_1^B)^2 = e^{\pi\imath h_1} = e^{2\pi\imath(e_{\ell\ell}-e_{\ell+2,\ell+2})} = 1\,, \tag{10.52}$$

and since $\dot{s}_k^B T_k^B = T_{k-1}^B\dot{s}_k^B$ we derive

$$(S_k^B)^2 = T_k^B\dot{s}_k^B T_k^B\dot{s}_k^B = T_k^B T_{k-1}^B(\dot{s}_k^B)^2 = (T_k^B T_{k-1}^B)^2 = 1\,. \tag{10.53}$$

For $i,j\in I$ such that $a_{ij}=0$, since $\dot{s}_i^B T_j^B = T_j^B \dot{s}_i^B$ we have

$$S_i^B S_j^B \;=\; T_i^B \dot{s}_i^B T_j^B \dot{s}_j^B \;=\; T_j^B \dot{s}_j^B T_i^B \dot{s}_i^B \;=\; S_j^B S_i^B\,, \tag{10.54}$$

and similarly, since $\dot{s}_1^B T_2^B = T_2^B \dot{s}_1^B$, we obtain

$$\begin{aligned} S_1^B S_2^B S_1^B S_2^B \;&=\; T_0^B \dot{s}_1^B T_2^B \dot{s}_2^B T_0^B \dot{s}_1^B T_2^B \dot{s}_2^B \;=\; (T_0^B T_2^B)^2 \dot{s}_1^B \dot{s}_2^B \dot{s}_1^B \dot{s}_2^B \\ &=\; \dot{s}_2^B \dot{s}_1^B \dot{s}_2^B \dot{s}_1^B \;=\; S_2^B S_1^B S_2^B S_1^B\,. \end{aligned} \tag{10.55}$$

The 3-move braid relation for $i,\, j=i+1$ we have:

$$\begin{aligned} S_i^B S_{i+1}^B S_i^B \;&=\; T_i^B \dot{s}_i^B T_{i+1}^B \dot{s}_{i+1}^B T_i^B \dot{s}_i^B \;=\; T_i^B T_{i+1}^B \dot{s}_i^B T_{i+1}^B \dot{s}_{i+1}^B \dot{s}_i^B \\ =\; T_i^B (T_{i+1}^B)^2 \dot{s}_i^B \dot{s}_{i+1}^B \dot{s}_i^B \;&=\; T_i^B (T_{i+1}^B)^2 \dot{s}_{i+1}^B \dot{s}_i^B \dot{s}_{i+1}^B \;=\; T_i^B T_{i+1}^B \dot{s}_{i+1}^B T_i^B \dot{s}_i^B \dot{s}_{i+1}^B \\ &=\; T_{i+1}^B \dot{s}_{i+1}^B T_i^B T_{i+1}^B \dot{s}_i^B \dot{s}_{i+1}^B \;=\; S_{i+1}^B S_i^B S_{i+1}^B\,, \end{aligned} \tag{10.56}$$

since $T_{i+1} \dot{s}_{i+1}^B \;=\; \dot{s}_{i+1}^B T_i^B$. $\square$

10.3 The case of type C_ℓ root system

The Lie algebra $\mathfrak{sp}_{2\ell}$ is identified with the θ_C-fixed subalgebra of $\mathfrak{gl}_{2\ell}$:

$$\begin{aligned} \mathfrak{sp}_{2\ell} \;=\; \bigl\{X\in\mathfrak{gl}_{2\ell}:\quad X \;=\; \theta_C(X)\bigr\} \;\subseteq\; \mathfrak{gl}_{2\ell}\,, \qquad \theta_C(X) \;=\; -S_C X^\tau S_C^{-1}\,, \\ S_C \;=\; \mathrm{diag}\bigl(1,\,-1,\,\ldots,\,1,\,-1\bigr)\,, \end{aligned} \tag{10.57}$$

where τ is the matrix transposition with respect to the opposite diagonal (3.3). This provides the standard faithful representation:

$$\phi:\quad \mathfrak{sp}_{2\ell} \longrightarrow \mathfrak{gl}_{2\ell}\,, \tag{10.58}$$

given by the following presentation of the Chevalley-Weyl generators $h_i,\, e_i,\, f_i,\, i\in I$:

$$\begin{aligned} &\phi(h_1) \;=\; e_{\ell\ell} \,-\, e_{\ell+1,\ell+1}\,, \\ &\phi(h_k) \;=\; (e_{\ell+1-k,\ell+1-k} + e_{\ell+k-1,\ell+k-1}) \,-\, (e_{\ell+2-k,\ell+2-k} + e_{\ell+k,\ell+k})\,, \\ &\qquad 1<k\le\ell\,; \\ &\phi(e_1) \;=\; e_{\ell,\ell+1}\,, \quad \phi(e_k) \;=\; e_{\ell+1-k,\,\ell+2-k} + e_{\ell+k-1,\ell+k}\,, \quad 1<k\le\ell\,, \\ &\phi(f_1) \;=\; e_{\ell+1,\ell}\,, \quad \phi(f_k) \;=\; e_{\ell+2-k,\,\ell+1-k} + e_{\ell+k,\ell+k-1}\,, \quad 1<k\le\ell\,. \end{aligned} \tag{10.59}$$

The representation (10.59) yields the following presentation of the type C_ℓ root system:

$$\alpha_1^{C_\ell} \;=\; 2\alpha_\ell^{A_{2\ell-1}}\,, \qquad \alpha_k^{C_\ell} \;=\; \alpha_{\ell+1-k}^{A_{2\ell-1}} \,+\, \alpha_{\ell-1+k}^{A_{2\ell-1}}\,, \qquad 1<k\le\ell\,. \tag{10.60}$$

Consider the root system of type $A_{2\ell-1}$ endowed with the automorphism ι of its Dynkin diagram:

$$\begin{aligned} \iota:\quad i\longmapsto 2\ell-i\,, \qquad i\in\{1,2,\ldots,2\ell-1\}\,; \\ \alpha_i^{A_{2\ell}} \longmapsto \alpha_{2\ell-i}^{A_{2\ell-1}}\,, \end{aligned} \tag{10.61}$$

so that (10.60) reads

$$\alpha_1^{C_\ell} \,=\, \alpha_\ell^{A_{2\ell-1}} + \iota\big(\alpha_\ell^{A_{2\ell-1}}\big)\,, \quad \alpha_k^{C_\ell} \,=\, \alpha_{\ell+1-k}^{A_{2\ell-1}} \,+\, \iota\big(\alpha_{\ell+1-k}^{A_{2\ell-1}}\big)\,, \quad 1<k\leq\ell\,. \tag{10.62}$$

Introduce the Euclidean vector space $\mathbb{R}^{2\ell}$ with orthonormal basis $\big\{\epsilon_1,\ldots,\epsilon_{2\ell}\big\}$, supplied with an action of involution

$$\iota:\quad \mathbb{R}^{2\ell}\longrightarrow\mathbb{R}^{2\ell}\,,\qquad \epsilon_i\longmapsto-\epsilon_{2\ell+1-i}\,,\quad 1\leq i\leq 2\ell\,. \tag{10.63}$$

Consider the ι-fixed Euclidean subspace $\mathbb{R}^\ell\subset\mathbb{R}^{2\ell}$, spanned by

$$\epsilon_i^{C_\ell} \,=\, \epsilon_{\ell+1-i} \,+\, \iota(\epsilon_{\ell+1-i}) \,=\, \epsilon_{\ell+1-i} \,-\, \epsilon_{\ell+i}\,,\qquad 1\leq i\leq\ell\,. \tag{10.64}$$

Then the root data of type C_ℓ reads

$$\begin{gathered}
\begin{cases} \alpha_1^{C_\ell} \,=\, 2\epsilon_1^{C_\ell} \\ \alpha_k^{C_\ell} \,=\, \epsilon_k^{C_\ell}-\epsilon_{k-1}^{C_\ell}\,, \\ 1<k\leq\ell \end{cases} \qquad \begin{cases} \varpi_i^{C_\ell} \,=\, \epsilon_i^{C_\ell}+\ldots+\epsilon_\ell^{C_\ell}\,, \\ i\in I\,; \end{cases} \\
\langle\alpha_i^{C_\ell},\,(\varpi_j^{C_\ell})^\vee\rangle \,=\, \langle(\alpha_i^{C_\ell})^\vee,\,\varpi_j^{C_\ell}\rangle \,=\, \delta_{ij}\,, \\
\begin{cases} (\alpha_1^{C_\ell})^\vee \,=\, \epsilon_1^{C_\ell} \\ (\alpha_k^{C_\ell})^\vee \,=\, \epsilon_k^{C_\ell}-\epsilon_{k-1}^{C_\ell}\,, \\ 1<k\leq\ell\,; \end{cases} \qquad \begin{cases} (\varpi_1^{C_\ell})^\vee \,=\, \dfrac{\epsilon_1^{C_\ell}+\ldots+\epsilon_\ell^{C_\ell}}{2} \\ (\varpi_k^{C_\ell})^\vee \,=\, \epsilon_k^{C_\ell}+\ldots+\epsilon_\ell^{C_\ell}\,, \\ 1<k\leq\ell\,. \end{cases}
\end{gathered} \tag{10.65}$$

The Cartan matrix $A\,=\,\|\langle\alpha_j^{C_\ell},\,(\alpha_i^{C_\ell})^\vee\rangle\|$ and its inverse are given by

$$A\,=\,\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -2 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \ldots & 0 & -1 & 2 \end{pmatrix}\,,\qquad A^{-1}\,=\,\begin{pmatrix} \frac{\ell}{2} & \frac{\ell-1}{2} & \ldots & \frac{3}{2} & 1 & \frac{1}{2} \\ \ell-1 & \ell-1 & \ldots & 3 & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 3 & 3 & \ldots & 3 & 2 & 1 \\ 2 & 2 & \ldots & 2 & 2 & 1 \\ 1 & 1 & \ldots & 1 & 1 & 1 \end{pmatrix}\,,$$

$$h_i^{C_\ell} \,=\, (\alpha_i^{C_\ell})^\vee \,=\, \sum_{j\in I}\langle\alpha_j^{C_\ell},\,(\alpha_i^{C_\ell})^\vee\rangle\varpi_j^\vee \,=\, \sum_{j\in I}a_{ij}^{C_\ell}(\varpi_j^{C_\ell})^\vee\,,\qquad \varpi_i^\vee \,=\, \sum_{j\in I}c_{ij}^{C_\ell}(\alpha_j^{C_\ell})^\vee\,,\qquad i\in I\,.$$

Lemma 10.4 *The Weyl group $W(C_\ell)$ is isomorphic to a subgroup of $W(A_{2\ell-1})$ via*

$$s_1^{C_\ell} \,=\, s_\ell^{A_{2\ell-1}}\,,\qquad s_k^{C_\ell} \,=\, s_{\ell+1-k}^{A_{2\ell-1}}s_{\ell-1+k}^{A_{2\ell-1}} \,=\, s_{\ell-1+k}^{A_{2\ell-1}}s_{\ell+1-k}^{A_{2\ell-1}}\,,\quad 1<k\leq\ell\,. \tag{10.67}$$

Proof: For $k=1$ given $\lambda=\lambda_1\epsilon_1^{C_\ell}+\ldots+\lambda_\ell\epsilon_\ell^{C_\ell}$ one has

$$\begin{aligned}
s_1^{C_\ell}(\lambda) \,&=\, \lambda \,-\, \langle\lambda,\,(\alpha_1^{C_\ell})^\vee\rangle_{C_\ell}\alpha_1^{C_\ell} \,=\, \lambda \,-\, \langle\lambda,\,\epsilon_1^{C_\ell}\rangle_{C_\ell}2\epsilon_1^{C_\ell} \\
&=\, -\lambda_1\epsilon_1^{C_\ell}+\lambda_2\epsilon_2^{C_\ell}+\ldots+\lambda_\ell\epsilon_\ell^{C_\ell} \\
&=\, \lambda_\ell\epsilon_1^{A_{2\ell-1}}+\ldots+\lambda_2\epsilon_{\ell-1}^{A_{2\ell}}-\lambda_1\epsilon_\ell^{A_{2\ell-1}} \,+\, \lambda_1\epsilon_{\ell+1}^{A_{2\ell-1}}-\lambda_2\epsilon_{\ell+2}^{A_{2\ell-1}}-\ldots-\lambda_\ell\epsilon_{2\ell}^{A_{2\ell-1}} \\
&=\, s_\ell^{A_{2\ell-1}}(\lambda)\,.
\end{aligned} \tag{10.68}$$

For $1 < k \le \ell$ the computation reproduces the one from Lemma 10.1. □

The symplectic Lie group $\mathrm{Sp}_{2\ell}$ may be identified with the θ_C-fixed subgroup of $GL_{2\ell}$:

$$\mathrm{Sp}_{2\ell} = \left\{ g \in GL_{2\ell} : \quad g = g^{\theta_C} \right\} \subset GL_{2\ell} . \tag{10.69}$$

Let us describe the Tits group $W^T_{\mathrm{Sp}_{2\ell}}$, which is the extension of the Weyl group $W(C_\ell)$ (10.67), and identify it with the fixed point subgroup in $W^T_{GL_{2\ell}}$.

Lemma 10.5 *Let* $\dot{s}^C_i = e^{\frac{\pi}{2}J_i}$, $i \in I$ *be the Tits generators* (2.28). *Then the following hold:*

1. *The Tits group* $W^T_{\mathrm{Sp}_{2\ell}}$ *is isomorphic to a subgroup of* $W^T_{GL_{2\ell}}$ *via*

$$\dot{s}^C_1 = \dot{s}^{A_{2\ell-1}}_\ell , \qquad \dot{s}^C_k = \dot{s}^{A_{2\ell-1}}_{\ell+1-k} \dot{s}^{A_{2\ell-1}}_{\ell-1+k} , \quad 1 < k \le \ell . \tag{10.70}$$

2. *The elements* (10.70) *belong to the* θ_C*-fixed subgroup* $(W^T_{GL_{2\ell}})^{\theta_C}$.

3. *Presentation* (10.70) *matches with* (10.67) *due to*

$$\mathrm{Ad}_{\dot{s}^C_i}|_{\mathfrak{h}} = s^C_i , \qquad i \in I . \tag{10.71}$$

Proof : (1) Since $\dot{s}^C_i \in \mathrm{Sp}_{2\ell}$, $i \in I$, one might verify (10.70) using the standard faithful representation $\phi : \mathrm{Sp}_{2\ell} \to GL_{2\ell}$. For $i = 1$, we have

$$\phi(\dot{s}^C_1) = \phi(e^{\frac{\pi}{2}J_1}) = \begin{pmatrix} \mathrm{Id}_{\ell-1} & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & \mathrm{Id}_{\ell-1} \end{pmatrix} = \phi(\dot{s}^{A_{2\ell-1}}_\ell , \tag{10.72}$$

and similarly, for $1 < k \le \ell$ we obtain

$$\phi(\dot{s}^C_k) = \phi(e^{\frac{\pi}{2}J_k}) = \begin{pmatrix} \mathrm{Id}_{\ell-k} & & & & & & \\ & 0 & -1 & & & & \\ & 1 & 0 & & & & \\ & & & \mathrm{Id}_{2k-4} & & & \\ & & & & 0 & -1 & \\ & & & & 1 & 0 & \\ & & & & & & \mathrm{Id}_{\ell-k} \end{pmatrix} = \phi(\dot{s}^{A_{2\ell-1}}_{\ell+1-k} \dot{s}^{A_{2\ell-1}}_{\ell-1+k}) . \tag{10.73}$$

(2) From (4.11) one reads

$$\theta_C : \qquad \dot{s}^{A_{2\ell-1}}_i \longmapsto \dot{s}^{A_{2\ell-1}}_{2\ell-i} , \qquad 1 \le i \le 2\ell , \tag{10.74}$$

so that using (2.29) the expressions (10.70) are θ_C-invariant.

(3) Follows from (2.29) and (10.67). □

By analogy with thegeneral linear group case, there is another way to lift simple root generators $s^C_i \in W(C_\ell)$ into the Tits group $W^T_{\mathrm{Sp}_{2\ell}}$. Recall that $W^T_{GL_{2\ell}}$ contains the following elements:

$$\begin{gathered} T_k := e^{\pi \imath e_{kk}} , \qquad 1 \le k \le (2\ell) ; \\ S_i = T_i \dot{s}_i = \dot{s}_i T_{i+1} , \qquad \overline{S}_i = T_{i+1} \dot{s}_i = \dot{s}_i T_i , \qquad 1 \le i \le 2\ell . \end{gathered} \tag{10.75}$$

Lemma 10.6 *The following elements belong to the fixed point subgroup* $(W^T_{GL_{2\ell}})^{\theta_C}$*:*

$$S_1^C = T_\ell S_\ell = T_{\ell+1}\overline{S}_\ell\,, \qquad S_k^C = S_{\ell+1-k}\overline{S}_{\ell-1+k}\,, \quad 1<k\leq\ell\,;$$
$$T_i^C = T_{\ell+1-i}T_{\ell+i} = e^{\pi\imath\epsilon_i^C}\,, \qquad 1\leq i\leq\ell\,. \tag{10.76}$$

Proof : The explicit action of θ_C on generators (10.75) gives (4.15):

$$\theta_C(S_i) = \overline{S}_{2\ell-i}\,, \qquad \theta_C(\overline{S}_i) = S_{2\ell-i}\,, \qquad \theta_C(T_k) = T_{2\ell+1-k}\,. \tag{10.77}$$

This implies that (10.76) are invariant under θ_C. □

Corollary 10.2 *The elements* (10.70) *and* (10.76) *can be identified via*

$$S_1^C = \dot{s}_1^C\,, \qquad S_k^C = T_k^C\dot{s}_k^C\,, \quad 1<k\leq\ell\,. \tag{10.78}$$

Proof : By straightforward computation using faithful representation (10.69) one finds

$$\phi(S_1^C) = \begin{pmatrix} \mathrm{Id}_{\ell-1} & & \\ & \begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix} & \\ & & \mathrm{Id}_{\ell-1}\end{pmatrix} = \phi(\dot{s}_\ell^{A_{2\ell-1}})\,,$$
$$\phi(S_k^C) = \begin{pmatrix} \mathrm{Id}_{\ell-k} & & & & \\ & \begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix} & & & \\ & & \mathrm{Id}_{2k-4} & & \\ & & & \begin{smallmatrix}0 & -1\\ -1 & 0\end{smallmatrix} & \\ & & & & \mathrm{Id}_{\ell-k}\end{pmatrix}. \tag{10.79}$$

Identifying this with (10.72), (10.73) one deduces (10.78). □

Proposition 10.2 *The elements* S_i^C*,* $i\in I$ *of the Tits group* $W^T_{\mathrm{Sp}_{2\ell}}$ *satisfy the following relations:*

$$(S_1^C)^2 = T_1^C = e^{\pi\imath h_1^C}\,, \qquad (S_2^C)^2 = \ldots = (S_\ell^C)^2 = 1\,,$$
$$S_i^C S_j^C = S_j^C S_i^C\,, \quad i,j\in I \quad \textit{such that} \quad a_{ij}=0\,,\,;$$
$$S_i^C S_j^C S_i^C = S_j^C S_i^C S_j^C\,, \qquad i,j\in I \quad \textit{such that} \quad a_{ij}=-1\,, \tag{10.80}$$
$$(S_1^C S_2^C)^4 = (S_2^C S_1^C)^4 = 1\,,$$

and they generate the whole Tits group $W^T_{\mathrm{Sp}_{2\ell}}$*.*

Proof : One might verify (10.80) using the standard fathful representation (10.69). Alternatively, similarly to the proof of Proposition 10.1 the relations (10.80) can be deduced from (2.29) and (10.78). □

10.4 The case of type D_ℓ root system

The Lie algebra $\mathfrak{so}_{2\ell}$ is identified with a subalgebra of $\mathfrak{gl}_{2\ell}$ as follows:

$$\begin{aligned} \mathfrak{so}_{2\ell} \,=\, \big\{X \in \mathfrak{gl}_{2\ell} : \quad X \,=\, \theta_D(X)\big\} \subseteq \mathfrak{gl}_{2\ell}\,, \qquad \theta_D(X) \,=\, -S_D X^\tau S_D^{-1}\,, \\ S_D \,=\, \mathrm{diag}\big(1,\, -1,\, \ldots,\, (-1)^{\ell-1};\, (-1)^{\ell-1},\, (-1)^{\ell},\, \ldots,\, 1\big)\,, \end{aligned} \tag{10.81}$$

where τ is the matrix transposition with respect to the reverse diagonal (3.3). This provides the standard faithful representation:

$$\phi : \quad \mathfrak{so}_{2\ell} \longrightarrow \mathfrak{gl}_{2\ell}\,, \tag{10.82}$$

which implies the following presentation of the Chevalley-Weyl generators h_i, e_i, f_i, $i \in I$:

$$\begin{aligned} \phi(h_1) \,&=\, (e_{\ell-1,\ell-1} + e_{\ell\ell}) \,-\, (e_{\ell+1,\ell+1} + e_{\ell+2,\ell+2})\,, \\ \phi(h_k) \,&=\, (e_{\ell+1-k,\ell+1-k} + e_{\ell+k-1,\ell+k-1}) \,-\, (e_{\ell+2-k,\ell+2-k} + e_{\ell+k,\ell+k})\,, \\ &\qquad 1 < k \leq \ell\,, \\ \phi(e_1) \,&=\, e_{\ell-1,\ell+1} + e_{\ell,\ell+2}\,, \\ \phi(e_k) \,&=\, e_{\ell+1-k,\,\ell+2-k} + e_{\ell+k-1,\ell+k}\,, \quad 1 < k \leq \ell\,, \\ \phi(f_1) \,&=\, e_{\ell+1,\ell-1} + e_{\ell+2,\ell}\,, \\ \phi(f_k) \,&=\, e_{\ell+2-k,\,\ell+1-k} + e_{\ell+k,\ell+k-1}\,, \quad 1 < k \leq \ell\,. \end{aligned} \tag{10.83}$$

The representation (10.83) yields the following presentation of the type D_ℓ root system:

$$\begin{aligned} \alpha_1^{D_\ell} \,&=\, \alpha_{\ell-1}^{A_{2\ell-1}} \,+\, 2\alpha_\ell^{A_{2\ell-1}} \,+\, \alpha_{\ell+1}^{A_{2\ell-1}}\,, \\ \alpha_k^{D_\ell} \,&=\, \alpha_{\ell+1-k}^{A_{2\ell-1}} \,+\, \alpha_{\ell-1+k}^{A_{2\ell-1}}\,, \qquad 1 < k \leq \ell\,. \end{aligned} \tag{10.84}$$

Consider the root system of type $A_{2\ell-1}$ endowed with the automorphism ι of its Dynkin diagram:

$$\begin{aligned} \iota : \quad i \longmapsto 2\ell - i\,, \qquad i \in \{1, 2, \ldots, 2\ell-1\}\,; \\ \alpha_i^{A_{2\ell}} \longmapsto \alpha_{2\ell-i}^{A_{2\ell-1}}\,. \end{aligned} \tag{10.85}$$

so that (10.84) reads

$$\begin{aligned} \alpha_1^{D_\ell} \,&=\, \alpha_{\ell-1}^{A_{2\ell-1}} + \alpha_\ell^{A_{2\ell-1}} \,+\, \iota\big(\alpha_{\ell-1}^{A_{2\ell-1}} + \alpha_\ell^{A_{2\ell-1}}\big)\,, \\ \alpha_k^{D_\ell} \,&=\, \alpha_{\ell+1-k}^{A_{2\ell-1}} \,+\, \iota\big(\alpha_{\ell+1-k}^{A_{2\ell-1}}\big)\,, \qquad 1 < k \leq \ell\,. \end{aligned} \tag{10.86}$$

Introduce the Euclidean vector space $\mathbb{R}^{2\ell}$ with orthonormal basis $\big\{\epsilon_1, \ldots, \epsilon_{2\ell}\big\}$, supplied with an action of involution:

$$\iota : \quad \mathbb{R}^{2\ell} \longrightarrow \mathbb{R}^{2\ell}\,, \qquad \epsilon_i \longmapsto -\epsilon_{2\ell+1-i}\,, \quad 1 \leq i \leq (2\ell)\,. \tag{10.87}$$

Consider the ι-fixed Euclidean subspace $\mathbb{R}^\ell \subset \mathbb{R}^{2\ell}$, spanned by

$$\epsilon_i^{D_\ell} \,=\, \epsilon_{\ell+1-i}^{A_{2\ell-1}} + \iota\big(\epsilon_{\ell+1-i}^{A_{2\ell-1}}\big) \,=\, \epsilon_{\ell+1-i}^{A_{2\ell-1}} - \epsilon_{\ell+i}^{A_{2\ell-1}}\,, \qquad 1 \leq i \leq \ell\,. \tag{10.88}$$

Then the root data of type D_ℓ reads

$$\begin{cases} \alpha_1^{D_\ell} \,=\, \epsilon_2^{D_\ell} + \epsilon_1^{D_\ell} \\ \\ \alpha_k^{D_\ell} \,=\, \epsilon_k^{D_\ell} - \epsilon_{k-1}^{D_\ell}\,, \\ 1 < k \leq \ell \end{cases} \qquad \begin{cases} \varpi_1^{D_\ell} \,=\, \dfrac{\epsilon_1^{D_\ell} + \epsilon_2^{D_\ell} + \ldots + \epsilon_\ell^{D_\ell}}{2} \\ \varpi_2^{D_\ell} \,=\, \dfrac{-\epsilon_1^{D_\ell} + \epsilon_2^{D_\ell} + \ldots + \epsilon_\ell^{D_\ell}}{2} \\ \varpi_k^{D_\ell} \,=\, \epsilon_k^{D_\ell} + \ldots + \epsilon_\ell^{D_\ell}\,, \quad 2 < k \leq \ell\,; \end{cases} \tag{10.89}$$

$$\langle\alpha_i^{D_\ell},\, (\varpi_j^{D_\ell})^\vee\rangle \,=\, \langle(\alpha_i^{D_\ell})^\vee,\, \varpi_j^{D_\ell}\rangle \,=\, \delta_{ij}\,,$$

and the Cartan matrix $A \,=\, \|\langle\alpha_j^{D_\ell},\, (\alpha_i^{D_\ell})^\vee\rangle\|$ and its inverse are given by

$$A \,=\, \begin{pmatrix} 2 & 0 & -1 & 0 & \ldots & 0 \\ 0 & 2 & -1 & 0 & \ddots & \vdots \\ -1 & -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & -1 & 2 & -1 \\ 0 & \ldots & \ldots & 0 & -1 & 2 \end{pmatrix}, \quad A^{-1} \,=\, \|c_{ij}^{D_\ell}\| \,=\, \begin{pmatrix} \frac{\ell}{4} & \frac{\ell-2}{4} & \frac{\ell-2}{2} & \ldots & \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{\ell-2}{4} & \frac{\ell}{4} & \frac{\ell-2}{2} & \ldots & \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{\ell-2}{2} & \frac{\ell-2}{2} & \ell-2 & \ldots & 3 & 2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{3}{2} & \frac{3}{2} & 3 & \ldots & 3 & 2 & 1 \\ 1 & 1 & 2 & \ldots & 2 & 2 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & \ldots & 1 & 1 & 1 \end{pmatrix},$$

$$h_i^{D_\ell} \,=\, (\alpha_i^{D_\ell})^\vee \,=\, \sum_{j\in I}\langle\alpha_j^{D_\ell},\, (\alpha_i^{D_\ell})^\vee\rangle\varpi_j^\vee \,=\, \sum_{j\in I} a_{ij}^{D_\ell}(\varpi_j^{D_\ell})^\vee\,, \tag{10.90}$$

$$\varpi_i^\vee \,=\, \sum_{j\in I} c_{ij}^{D_\ell}(\alpha_j^{D_\ell})^\vee\,, \qquad i \in I\,.$$

Lemma 10.7 *The Weyl group $W(D_\ell)$ is isomorphic to a subgroup of $W(A_{2\ell-1})$ via*

$$\begin{aligned} s_1^{D_\ell} \,&=\, s_\ell^{A_{2\ell-1}} s_{\ell-1}^{A_{2\ell-1}} s_{\ell+1}^{A_{2\ell-1}} s_\ell^{A_{2\ell-1}} \,=\, s_\ell^{A_{2\ell-1}} s_{\ell+1}^{A_{2\ell-1}} s_{\ell-1}^{A_{2\ell-1}} s_\ell^{A_{2\ell-1}}\,, \\ s_k^{D_\ell} \,&=\, s_{\ell+1-k}^{A_{2\ell-1}} s_{\ell-1+k}^{A_{2\ell-1}} \,=\, s_{\ell-1+k}^{A_{2\ell-1}} s_{\ell+1-k}^{A_{2\ell-1}}\,, \quad 1 < k \leq \ell\,. \end{aligned} \tag{10.91}$$

In particular, one has $s_1^{D_\ell} \,=\, s_\ell^{A_{2\ell-1}} s_2^{D_\ell} s_\ell^{A_{2\ell-1}}$, so that $\mathrm{Out}(\Phi(D_\ell)) = \langle s_\ell^{A_{2\ell-1}}\rangle \simeq \mathbb{Z}/2\mathbb{Z}$.

Proof : For $1 < k \leq \ell$ the proof literarily follows similar statement of Lemma 5.1. For $k = 1$ given $\lambda = \lambda_1\epsilon_1^{D_\ell} + \ldots + \lambda_\ell\epsilon_\ell^{D_\ell}$ one has

$$\begin{aligned} s_1^{D_\ell}(\lambda) \,&=\, \lambda \,-\, \langle\lambda,\, (\alpha_1^{D_\ell})^\vee\rangle_{D_\ell}\alpha_1^{D_\ell} \,=\, \lambda \,-\, \langle\lambda,\, \epsilon_1^{D_\ell} + \epsilon_2^{D_\ell}\rangle_{D_\ell}(\epsilon_1^{D_\ell} + \epsilon_2^{D_\ell}) \\ &=\, -\lambda_2\epsilon_1^{D_\ell} - \lambda_1\epsilon_2^{D_\ell} + \lambda_3\epsilon_3^{D_\ell} + \ldots + \lambda_\ell\epsilon_\ell^{D_\ell} \\ &=\, \lambda_\ell\epsilon_1^{A_{2\ell-1}} + \ldots + \lambda_3\epsilon_{\ell-2}^{A_{2\ell}} - \lambda_1\epsilon_{\ell-1}^{A_{2\ell-1}} - \lambda_2\epsilon_\ell^{A_{2\ell-1}} \\ &\quad + \lambda_2\epsilon_{\ell+1}^{A_{2\ell-1}} + \lambda_1\epsilon_{\ell+2}^{A_{2\ell-1}} - \lambda_3\epsilon_{\ell+3}^{A_{2\ell-1}} - \ldots - \lambda_\ell\epsilon_{2\ell}^{A_{2\ell-1}} \\ &=\, s_\ell^{A_{2\ell-1}} s_{\ell-1}^{A_{2\ell-1}} s_{\ell+1}^{A_{2\ell-1}} s_\ell^{A_{2\ell-1}}(\lambda)\,, \end{aligned} \tag{10.92}$$

since $s_\ell^{A_{2\ell-1}}s_{\ell-1}^{A_{2\ell-1}}s_{\ell+1}^{A_{2\ell-1}}s_\ell^{A_{2\ell-1}}$ simply swaps $\epsilon_\ell^{A_{2\ell-1}} \leftrightarrow \epsilon_{\ell+2}^{A_{2\ell-1}}$ and $\epsilon_{\ell-1}^{A_{2\ell-1}} \leftrightarrow \epsilon_{\ell+1}^{A_{2\ell-1}}$. $\square$

The orthogonal group $O_{2\ell}$ may be identified with the fixed point subgroup of the general linear group as follows:

$$O_{2\ell} \,=\, \big\{g \in GL_{2\ell} : \quad g \,=\, g^{\theta_D}\big\} \,\subset\, GL_{2\ell}\,. \tag{10.93}$$

The group $O_{2\ell}$ is not simply connected and there is the following exact sequence:

$$1 \longrightarrow SO_{2\ell} \longrightarrow O_{2\ell} \underset{\overset{\longleftarrow}{z}}{\overset{\det}{\longrightarrow}} \{\pm 1\} \longrightarrow 1 \quad , \tag{10.94}$$

with the non-trivial action of $\{\pm 1\}$ on $SO_{2\ell}$.

Lemma 10.8 *The even orthogonal group allows for the following decomposition:*

$$\begin{gathered} O_{2\ell} \,=\, SO_{2\ell} \sqcup S_\ell \cdot SO_{2\ell}\,, \qquad S_\ell \,=\, \begin{pmatrix} \mathrm{Id}_{\ell-1} & & & \\ & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & & \mathrm{Id}_{\ell-1} \end{pmatrix}, \\ \mathrm{Ad}_{S_\ell} : \quad \alpha_1^{D_\ell} \longleftrightarrow \alpha_2^{D_\ell}\,. \end{gathered} \tag{10.95}$$

so that Ad_{S_ℓ} *represents the symmetry of the Dynkin diagram of type* D_ℓ*. Moreover,* $O_{2\ell}$ *has a structure of a semidirect product:* $O_{2\ell} \,=\, SO_{2\ell} \rtimes \mathrm{Out}(SO_{2\ell})$.

Proof : We have

$$(H_{GL_{2\ell}})^{\theta_D} \,=\, H_{O_{2\ell}} \,=\, H_{\mathrm{Sp}_{2\ell}} \,=\, (H_{GL_{2\ell}})^{\theta_C}\,, \tag{10.96}$$

hence $H_{O_{2\ell}} = H_{\mathrm{SO}_{2\ell}}$. It is convenient to pick $z(-1) = M_\ell \in O_{2\ell}$ as a representative of section $z : \{\pm 1\} \to O_{2\ell}$, since it represents the outer automorphism of $SO_{2\ell}$. $\square$

Lemma 10.9 *Let* $\dot{s}_i^D = e^{\frac{\pi}{2}J_i}$, $i \in I$ *be the Tits generators* (2.28)*. Then the following hold:*

1. *The Tits group* $W^T_{\mathrm{SO}_{2\ell}}$ *is isomorphic to a subgroup of* $W^T_{GL_{2\ell}}$ *via*

$$\begin{aligned} \dot{s}_1^D \,&=\, \dot{s}_{\ell-1}^{A_{2\ell-1}}\dot{s}_\ell^{A_{2\ell-1}}(\dot{s}_{\ell+1}^{A_{2\ell-1}})^{-1}(\dot{s}_{\ell-1}^{A_{2\ell-1}})^{-1}\,, \\ \dot{s}_k^D \,&=\, \dot{s}_{\ell+1-k}^{A_{2\ell-1}}\dot{s}_{\ell-1+k}^{A_{2\ell-1}}\,, \quad 1 < k \leq \ell\,. \end{aligned} \tag{10.97}$$

2. *The elements* (10.97) *belong to the* θ_D*-fixed subgroup* $(W^T_{GL_{2\ell}})^{\theta_D}$.

3. *Presentation* (10.97) *matches with* (10.91) *due to*

$$\mathrm{Ad}_{\dot{s}_i^D}|_{\mathfrak{h}} \,=\, s_i^D\,, \qquad i \in I\,. \tag{10.98}$$

4. The Tits group $W^T_{O_{2\ell}}$ fits into the following exact sequence:

$$1 \longrightarrow W^T_{SO_{2\ell}} \longrightarrow W^T_{O_{2\ell}} \longrightarrow \mathrm{Out}(\Phi(D_\ell)) \longrightarrow 1 \quad , \tag{10.99}$$

that actually splits, so that $W^T_{O_{2\ell}} = W^T_{SO_{2\ell}} \rtimes \mathrm{Out}(\Phi(D_\ell))$.

Proof : (1) Since $\dot{s}^D_i \in SO_{2\ell}$, $i \in I$, one might verify (10.97) using the standard faithful representation $\phi:\ SO_{2\ell} \to GL_{2\ell}$. For $i = 1$, we have

$$\phi(\dot{s}^D_1) \;=\; \phi(e^{\frac{\pi}{2}J_1}) \;=\; \begin{pmatrix} \mathrm{Id}_{\ell-2} & & \\ & \begin{smallmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix} & \\ & & \mathrm{Id}_{\ell-2} \end{pmatrix} \;=\; \phi(\dot{s}^{A_{2\ell-1}}_\ell)\,, \tag{10.100}$$

and similarly, for $1 < k \le \ell$ we obtain

$$\phi(\dot{s}^D_k) \;=\; \phi(e^{\frac{\pi}{2}J_k}) \;=\; \begin{pmatrix} \mathrm{Id}_{\ell-k} & & & & \\ & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & & \\ & & \mathrm{Id}_{2k-4} & & \\ & & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & \\ & & & & \mathrm{Id}_{\ell-k} \end{pmatrix} \;=\; \phi(\dot{s}^{A_{2\ell-1}}_{\ell+1-k}\dot{s}^{A_{2\ell-1}}_{\ell-1+k})\,. \tag{10.101}$$

(2) From (4.11) one reads

$$\theta_D\ : \qquad \dot{s}^{A_{2\ell-1}}_i \longmapsto \dot{s}^{A_{2\ell-1}}_{2\ell-i}\,, \qquad 1 \le i \le 2\ell\,, \tag{10.102}$$

so that (10.97) are θ_D-invariant.

(3) Follows from (2.29) and (10.91).

(4) The element $\dot{R} = \dot{s}_\ell$ represents $R \in \mathrm{Out}(D_\ell)$ from (10.91); it is not a reflection at any root of $\Phi(D_\ell)$, that is $\mathrm{Ad}_{\dot{s}_\ell}|_{\mathfrak{h}} = s_\ell \notin W(D_\ell)$ due to (10.91). Moreover, by (2.21) $\mathrm{Out}(\Phi(D_\ell)) = \mathrm{Out}(SO_{2\ell})$ one obtains (10.99). □

By analogy with the general linear group case, there is another way to lift simple root generators $s^D_i \in W(D_\ell)$ into the Tits group $W^T_{\mathrm{SO}_{2\ell}}$. Recall that $W^T_{GL_{2\ell}}$ contains the following elements:

$$\begin{aligned} T_k \;&:=\; e^{\pi \imath e_{kk}}\,, \qquad 1 \le k \le 2\ell\,; \\ S_i \;=\; T_i\dot{s}_i \;=\; \dot{s}_iT_{i+1}\,, \qquad \overline{S}_i \;&=\; T_{i+1}\dot{s}_i \;=\; \dot{s}_iT_i\,, \qquad 1 \le i \le 2\ell\,. \end{aligned} \tag{10.103}$$

Lemma 10.10 *The following elements belong to the fixed point subgroup $(W^T_{GL_{2\ell}})^{\theta_D}$:*

$$\begin{aligned} S^D_1 \;&=\; S_\ell S_{\ell-1}\overline{S}_{\ell+1}S_\ell\,, \\ S^D_k \;&=\; S_{\ell+1-k}\overline{S}_{\ell-1+k}\,, \qquad 1 < k \le \ell\,; \\ T^D_i \;&=\; T_{\ell+1-i}T_{\ell+i}\,, \qquad 1 \le i \le \ell\,, \end{aligned} \tag{10.104}$$

Proof : The explicit action of θ_C on generators (10.103) gives (4.15):

$$\theta_D(S_i) \;=\; \overline{S}_{2\ell-i}\,, \qquad \theta_D(\overline{S}_i) \;=\; S_{2\ell-i}\,, \qquad \theta_D(T_k) \;=\; T_{2\ell+1-k}\,. \tag{10.105}$$

This implies that (10.104) are invariant under θ_D. □

Corollary 10.3 *The elements* (10.97) *and* (10.104) *can be identified via*

$$S_1^D \,=\, T_2^D \dot{s}_1^D\,, \qquad S_k^D \,=\, T_k^D \dot{s}_k^D\,, \quad 1 < k \leq \ell\,. \tag{10.106}$$

Proof : By straightforward computation using faithful representation (10.93) one finds

$$\begin{aligned}\phi(S_1^D) \,&=\, \begin{pmatrix} \mathrm{Id}_{\ell-2} & & \\ & \begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{smallmatrix} & \\ & & \mathrm{Id}_{\ell-2} \end{pmatrix}\,, \\ \phi(S_k^D) \,&=\, \begin{pmatrix} \mathrm{Id}_{\ell-k} & & & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & & & \\ & & \mathrm{Id}_{2k-4} & & \\ & & & \begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix} & \\ & & & & \mathrm{Id}_{\ell-k} \end{pmatrix}\,.\end{aligned} \tag{10.107}$$

Identifying this with (10.72), (10.73) one deduces (10.78). □

Proposition 10.3 *The elements* S_i^D*,* $i \in I$ *from* (10.104) *satisfy the relations* (2.23)*,* (2.24) *of type* D_ℓ*:*

$$\begin{aligned} (S_1^D)^2 \,=\, (S_2^D)^2 \,=\, \ldots \,=\, (S_\ell^D)^2 \,=\, 1\,, \\ S_i^D S_j^D \,=\, S_j^D S_i^D\,, \quad i,j \in I \quad \textit{such that} \quad a_{ij} = 0\,,\,; \\ S_i^D S_j^D S_i^D \,=\, S_j^D S_i^D S_j^D\,, \qquad i,j \in I \quad \textit{such that} \quad a_{ij} = -1\,. \end{aligned} \tag{10.108}$$

Proof : One might verify (10.108) using the standard faithful representation (10.93). Alternatively, similarly to our proof of Proposition 10.1 the relations (10.108) can be deduced from (2.29) and (10.106). □

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