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RELATIVISTIC TODA CHAIN AT ROOT OF UNITY III. RELATIVISTIC TODA CHAIN HIERARCHY

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ABSTRACT. The hierarchy of the classical nonlinear integrable equations associated with relativistic Toda chain model is considered. It is formulated for the N-powers of the quantum operators of the corresponding quantum integrable models. Following the ideas of the paper [11] it is shown how one can obtain such a system from 2D Toda lattice system. The reduction procedure is described explicitly. The soliton solutions for the relativistic Toda chain are constructed using results of [12] in terms of the rational tau-functions. The vanishing properties of these tau-functions are investigated.

INTRODUCTION

This paper is a supplement to the papers [1, 2]. It contains proofs of several statements used in these papers and related to the classical relativistic Toda chain (RTCh) model. The papers [1, 2] deal with the quantum integrable model based on the Weyl algebra at the roots of unity

$$\mathbf{u}_m \cdot \mathbf{w}_m = \omega\, \mathbf{w}_m \cdot \mathbf{u}_m\,, \quad \omega = \mathrm{e}^{2\pi i/N}\,. \tag{1}$$

Namely, the simplest possible model which is the RTCh model at the root of unity was considered there. The parameter ω being different from unity describes the deviation of the quantum model from the corresponding classical counterpart.

In the quantum integrable models of this type, in contrast to the models where the quantization parameter is in general position, there exists a non-standard classical limit. In usual, in order to get a classical model from the quantum one, the quantization parameter is sent

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to unity and the Poisson bracket appears in this classical limit as normalized commutator. In this limit the quantum observables becomes classical ones, namely, $\mathbb{C}$-valued functions depending on the evolution parameters (times or coordinates). In the quantum integrable models based on the Weyl algebra at the root of unity there is another possibility. One may consider the N-th powers of the quantum dynamical variables

$$u_m \equiv \mathbf{u}_m^N \,, \quad w_m \equiv \mathbf{w}_m^N \,. \tag{2}$$

Writing the deformation parameter in the form

$$\omega \;=\; \mathrm{e}^{2\,\pi\, i/N} \cdot \mathrm{e}^{\varepsilon/N^2} \tag{3}$$

and considering the limit $\varepsilon \to 0$ [3] one may obtain from the commutation relations (1) the Poisson bracket

$$\{u_m, w_n\} = \delta_{nm} u_m w_n \,. \tag{4}$$

It is clear that logarithms of the dynamical variables form the canonical symplectic form

$$\Omega = \sum d \ln\, u_m \wedge d \ln\, w_m \,. \tag{5}$$

The important feature of this unusual classical limit is that the quantum variables do not disappear and their N-th powers being the parameters of the unitary representations of the Weyl algebra form the classical integrable system. The quantum analysis of such a combined systems becomes more transparent. In particular, in the paper [2] a modified $\mathbf{Q}$-operator was constructed for the quantum RTCh model at the roots of unity using several result from the corresponding classical integrable system.

The aim of this paper is to prove some of these results from the point of view of the standard theory of integrable nonlinear differential equations. In this approach the integrable equation appears usually as a representative of some family of equations sometimes called hierarchy of integrable equations. In some cases, these hierarchies are certain reductions of a most general system of equations. The subject of the present paper is the RTCh equation and the associated hierarchy. It is known that this model can be obtained as a reduction of the 2D Toda lattice (2DTL) system [11, 12] and its integrable discrete analogs.

Relativistic generalizations of the classical integrable equations attracted much attention after seminal work by S. Ruijsenaars [10] and have been investigated in several papers (see, for example, [7, 8, 9]). The bilinear formalism for one of most simple example of these relativistic generalizations, namely, for RTCh equation was presented in [12]. In the latter paper the M-soliton solution to this equation were constructed in terms of Casorati determinants. We show that these solutions coincide identically with the standard soliton tau-functions constructed in KP or 2DTL theories [15, 16, 14] providing that parameters of solitons lies on the rational curve, which is related to Baxter's curve appeared in the quantum integrable models based on the Weyl algebra at the root of unity.

The paper is organized as follows. In Section 1 we consider RTCh model in finite volume (with lattice size equal to M) and using L-operator approach construct RTCh hierarchy in the framework of slightly modified AKNS construction. In Section 2 we obtain the RTCh hierarchy as certain reduction from 2DTL theory following ideas of the paper [11]. In last Section the soliton solutions to the first equation of RTCh hierarchy are considered and identified with those used in papers [1, 2]. The Appendix contains a short review of Casoratian technique, which helped authors of the paper [12] to find soliton solutions of RTCh equation.

1. RTCh hierarchy in r-matrix formalism

In order to establish relation with the papers [1, 2] we construct in this Section the RTCh hierarchy using language of L-operators and r-matrix formalism. We will show that RTCh hierarchy can be obtained as slightly modified AKNS system. In particular, we will explain the meaning of the constant shift in the corresponding trigonometric classical r-matrix mentioned in the paper [8].

The L-operator for RTCh in the form of 2×2 matrix

$$L_m = \frac{1}{\sqrt{u_m w_m}} \begin{pmatrix} z^2 + \kappa u_m w_m & z\, u_m \\ z\, w_m & 0 \end{pmatrix} \tag{6}$$

appeared first in [8]. In (6) z is a spectral parameter, $\kappa \in \mathbb{C}$ is the complex parameter and u_m, w_m, $m \in \mathbb{Z}$ satisfy the local Poisson bracket (4). This L-operator[1] is associated with lattice site m. In this section we will consider the lattice of the finite volume, so $m = 1, \dots, M$.

[1]We used here a gauge-equivalent dependence of the L-operator (6) on the spectral parameter in contrast to [1, 2] and also changed its normalization.

It is easy to show [8] that the Poisson bracket (4) can be rewritten in r-matrix form [5, 8]

$$\{L_m(z) \overset{\otimes}{,} L_n(y)\} = \delta_{nm}[\tilde{r}(z,y), L_n(z) \otimes L_m(y)] \ , \tag{7}$$

where $\tilde{r}(z,y) = r(z,y) + s$ and

$$r(z,y) = \begin{pmatrix} \alpha(z,y) & 0 & 0 & 0 \\ 0 & 0 & \beta(z,y) & 0 \\ 0 & \beta(z,y) & 0 & 0 \\ 0 & 0 & 0 & \alpha(z,y) \end{pmatrix}, \tag{8}$$

$$\alpha(z,y) = \frac{1}{2}\frac{z^2+y^2}{z^2-y^2} \ , \quad \beta(z,y) = \frac{zy}{z^2-y^2} \ ,$$

is the standard trigonometric classical r-matrix for $\widetilde{\mathfrak{sl}}_2$ and

$$s = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

is a constant shift. One may show that after quantization this constant shift is related to twisting of XXZ quantum R-matrix in the commutation relation of the corresponding quantum L-operators (see [4, 1]). In the quantum case the twisting of the quantum R-matrix is necessary in order to use minimal finite-dimensional representations of the Weyl algebra at the root of unity. On the classical level such a shift of the classical r-matrix will simplify the corresponding classical integrable hierarchy by removing higher non-linear terms (see discussion around (24)).

In order to introduce the evolution of L-operator (6) with respect to continuous times one have to consider the monodromy matrix

$$\widehat{T} = L_1 \ L_2 \ \ldots \ L_M = \begin{pmatrix} A(z^2) & z\,B(z^2) \\ z\,C(z^2) & D(z^2) \end{pmatrix}, \tag{10}$$

where $A(z^2)$, $B(z^2)$, $C(z^2)$ and $D(z^2)$ are polynomials with respect to z^2 of the orders M, $M-1$, $M-1$ and $M-2$ respectively. The commutation relation (7) shows that the coefficients

T_k, $k = 0, \ldots, M$ of the trace of the monodromy matrix (10)

$$T(z^2) = A(z^2) + D(z^2) = \sum_{k=0}^{M} T_k\, z^{2k} \tag{11}$$

are in involution with respect to this Poisson bracket

$$\{T_k, T_n\} = 0$$

and can be taken as Hamiltonians which correspond to evolution of the L-operator (6) with respect to times t_k

$$\frac{\partial L_m}{\partial t_k} = \{L_m, T_k\}\ . \tag{12}$$

Our goal is to rewrite the equation (12) in the zero curvature form. To do this we will use a method described in the book [5]. We have

$$\{L_m(z), T(y)\} = V_{m-1}(z,y)\, L_m(z) - L_m(z)\, V_m(z,y)\ , \tag{13}$$

where

$$\begin{aligned} V_m(z,y) \;&=\; \mathrm{tr}_2[(\mathrm{id}\otimes L_1(y))\cdots(\mathrm{id}\otimes L_m(y))\,\tilde{r}(z,y)\;\times \\ &\quad\times\;(\mathrm{id}\otimes L_{m+1}(y))\cdots(\mathrm{id}\otimes L_M(y))] \end{aligned} \tag{14}$$

and $\mathrm{tr}_2(A\otimes B) \equiv \mathrm{tr}(B)\cdot A$. Moreover, since the singular part of the r-matrix (8) is proportional to the permutation operator the following equality holds

$$\lim_{z^2\to y^2}(z^2-y^2)[V_{m-1}(z,y)\, L_m(z) - L_m(z)\, V_m(z,y)] = 0$$

which shows that r.h.s. of (13) is polynomial with respect to z and y

$$V_m(z,y) = \sum_{k=0}^{M} \tilde{Q}_m^{(k)}(z) y^{2k}\ , \tag{15}$$

where

$$\tilde{Q}_m^{(k)}(z) = \sum_{p=0}^{k} a_{kp}(T) Q_m^{(p)}(z)\ , \tag{16}$$

$$
\begin{aligned}
Q^{(n)}_m(z) &= \begin{pmatrix} z^{2n}/2 & 0 \\ 0 & -z^{2n}/2 \end{pmatrix} + \sum_{k=1}^{n} z^{2(n-k)} \begin{pmatrix} h_{2k}/2 & z^{-1}e_{2k-1} \\ z^{-1}f_{2k-1} & -h_{2k}/2 \end{pmatrix} \\
&+ \varepsilon \begin{pmatrix} h_{2n}/2 & 0 \\ 0 & -h_{2n}/2 \end{pmatrix} , \quad \varepsilon = -\frac{1}{2}
\end{aligned} \tag{17}
$$

and the entries of the matrix polynomials $Q^{(n)}_m(z)$ h_{2k}, e_{2k-1} and f_{2k-1} are differential polynomials of the functions u_m and w_{m+1} only. The coefficients $a_{kp}(T)$ in decomposition (16) are polynomial functions of the integrals of motions T_k.

We can see that the structure of 2×2 matrix polynomials (17) looks very similar to those in the AKNS construction. Let us introduce new time variables x_k such that evolution of L-operator (6) is given by the zero curvature equation

$$
\frac{\partial L_m(z)}{\partial x_n} + Q^{(n)}_{m-1}(z)\, L_m(z) - L_m(z)\, Q^{(n)}_m(z) = 0 \,. \tag{18}
$$

In (18) we can choose integer n to be any non-negative integer and call the obtained hierarchy of equations as RTCh hierarchy. Let us write explicitly several first equations of this hierarchy (see the Section 3, formula (51), where linear combination of the systems (19) and (20) is identified with RTCh equation):

$$
-\frac{\partial \ln u_m}{\partial x_0} = \frac{\partial \ln w_m}{\partial x_0} = 1 \tag{19}
$$

$$
\begin{aligned}
\frac{\partial \ln u_m}{\partial x_1} &= w_m(\kappa u_m + u_{m-1}) \,, \\
-\frac{\partial \ln w_m}{\partial x_1} &= u_m(\kappa w_m + w_{m+1}) \,,
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
-\frac{\partial \ln u_m}{\partial x_2} &= u_m w_m w_{m+1}(\kappa u_m + u_{m-1}) + \\
&+ u_{m-1}w_{m-1}w_m(\kappa u_{m-1} + u_{m-2}) + w_m^2(\kappa u_m + u_{m-1})^2 \,, \\
\frac{\partial \ln w_m}{\partial x_2} &= u_{m-1}u_m w_m(\kappa w_m + w_{m+1}) + \\
&+ u_m u_{m+1} w_{m+1}(\kappa w_{m+1} + w_{m+2}) + u_m^2(\kappa w_m + w_{m+1})^2 \,.
\end{aligned} \tag{21}
$$

The compatibility condition between the systems (20) and (21) falls into quite simple form:

$$
\begin{aligned}
\frac{\partial u_m}{\partial x_2} &= -\frac{\partial^2 u_m}{\partial x_1^2} - 2u_m w_{m+1}\frac{\partial u_m}{\partial x_1} \,, \\
\frac{\partial w_{m+1}}{\partial x_2} &= \frac{\partial^2 w_{m+1}}{\partial x_1^2} - 2u_m w_{m+1}\frac{\partial w_{m+1}}{\partial x_1} \,.
\end{aligned} \tag{22}
$$

As we already mentioned, the structure of the matrices $Q_m^{(n)}$ (17) shows that the whole RTCh hierarchy can be obtained as a version of AKNS hierarchy [6, 11] with small modification related to the shift matrix s (9). In (17) we decompose the matrices $Q_m^{(n)}$ into two pieces. The first one correspond to the r-matrix (8) and to standard AKNS construction, while the second one to the constant shift matrix s (9). If the parameter ε in (17) is free then the first equation in the corresponding AKNS construction, written as the zero-curvature condition for matrices (17),

$$\frac{\partial Q^{(1)}}{\partial x_2} - \frac{\partial Q^{(2)}}{\partial x_1} + [Q^{(2)}, Q^{(1)}] = 0 \,, \tag{23}$$

has the form (we renamed $u_m \equiv e$ and $w_{m+1} \equiv f$ for simplicity)

$$\begin{aligned} \frac{\partial e}{\partial x_2} &= -\frac{\partial^2 e}{\partial x_1^2} + 4\varepsilon e f \frac{\partial e}{\partial x_1} + 2(2\varepsilon+1)\left(e^2 \frac{\partial f}{\partial x_1} + (\varepsilon+1) e^3 f^2 \right) \,, \\ \frac{\partial f}{\partial x_2} &= \frac{\partial^2 f}{\partial x_1^2} + 4\varepsilon e f \frac{\partial f}{\partial x_1} + 2(2\varepsilon+1)\left(f^2 \frac{\partial e}{\partial x_1} - (\varepsilon+1) e^2 f^3 \right) \,. \end{aligned} \tag{24}$$

It is seen from (24) that these nonlinear equations are simplified essentially at the choice of the parameter $\varepsilon = -1/2$ by removing the higher nonlinear terms. The same phenomena occurs with all higher hierarchy equations. We would like to repeat once more that such a simplification is a classical analog of the possibility to use the minimal representation of the Weyl algebra in the corresponding quantum model.

Due to locality property ($Q_m^{(n)}$ depends only on u_m and w_{m+1}) the AKNS construction do not describe discrete time evolution (18). As well as in the usual AKNS hierarchy the dependence on the discrete time or the discrete evolution (18) can be reconstructed after using the Bäcklund-Schlezinger transformation [13]. The L-operator (6) plays the role of matrix which generates this transformation. It is possible to develop this AKNS construction in full details using the methods described in the book [6], but instead we will describe in the next section the RTCh hierarchy as a reduction of 2DTL.

2. RTCh as reduction of 2DTL

In this section we recall the basic facts on 2DTL model following the fundamental paper [14]. Then we consider a certain reduction of this system which leads to the RTCh hierarchy.

2.1. **2D Toda lattice.** In [14] 2DTL was defined in terms of $\mathbb{Z}\times\mathbb{Z}$ matrices $\mathcal{L}$ and $\overline{\mathcal{L}}$ of the following form

$$\mathcal{L}=\sum_{j\le 1}\mathrm{diag}[b_j(m)]\Lambda^j\ ,\quad \overline{\mathcal{L}}=\sum_{j\ge -1}\mathrm{diag}[c_j(m)]\Lambda^j\ , \tag{25}$$

where $b_1(m)=1$, $c_{-1}(m)\neq 0$ for any $m\in\mathbb{Z}$ and $\Lambda_{m,k}\equiv\delta_{m+1,k}$ is a shift matrix. In components, these infinite matrices can be written as follows

$$\begin{aligned}\mathcal{L}_{m,k} &= \delta_{m+1,k}+\sum_{j=0}^{\infty}b_{-j}(m)\delta_{m-j,k}\ ,\\ \overline{\mathcal{L}}_{m,k} &= c_{-1}(m)\delta_{m-1,k}+\sum_{j=0}^{\infty}c_j(m)\delta_{m+j,k}\ .\end{aligned} \tag{26}$$

For arbitrary infinite matrix $A=\sum_{j\in\mathbb{Z}}\mathrm{diag}[a_j(m)]\Lambda^j$ we define its projections on the upper-diagonal (including diagonal) A_+ and the strictly lower-diagonal (excluding diagonal) A_- matrices by the formulas

$$A_+=\sum_{j\ge 0}\mathrm{diag}[a_j(m)]\Lambda^j\ ,\quad A_-=\sum_{j<0}\mathrm{diag}[a_j(m)]\Lambda^j\ . \tag{27}$$

2DTL is a system of differential equations for the functions $b_{-j}(m)$, $c_j(m)$, $j\ge 0$ and $c_{-1}(m)$ with respect to two copies of the continuous parameters (times) $x=\{x_1,x_2,\dots\}$ and $y=\{y_1,y_2,\dots\}$ of the form

$$\begin{aligned}\partial_{x_s}\mathcal{L} &= [\mathcal{B}_s,\mathcal{L}]\ , &\qquad \partial_{x_s}\overline{\mathcal{L}} &= [\mathcal{B}_s,\overline{\mathcal{L}}]\ ,\\ \partial_{y_s}\mathcal{L} &= [\mathcal{C}_s,\mathcal{L}]\ , &\qquad \partial_{y_s}\overline{\mathcal{L}} &= [\mathcal{C}_s,\overline{\mathcal{L}}]\ ,\end{aligned} \tag{28}$$

where $\mathcal{B}_s=(\mathcal{L}^s)_+$ and $\mathcal{C}_s=\left(\overline{\mathcal{L}}^s\right)_-$.

The system (28) can be obtained as the compatibility conditions of the spectral problems

$$\mathcal{L}\,\mathcal{W}(x,y)=\mathcal{W}(x,y)\Lambda\ ,\quad \overline{\mathcal{L}}\,\overline{\mathcal{W}}(x,y)=\overline{\mathcal{W}}(x,y)\Lambda^{-1} \tag{29}$$

and the linear problems

$$\begin{aligned}\partial_{x_s}\mathcal{W}(x,y) &= \mathcal{B}_s\mathcal{W}(x,y)\ , &\qquad \partial_{x_s}\overline{\mathcal{W}}(x,y) &= \mathcal{B}_s\overline{\mathcal{W}}(x,y)\ ,\\ \partial_{y_s}\mathcal{W}(x,y) &= \mathcal{C}_s\mathcal{W}(x,y)\ , &\qquad \partial_{y_s}\overline{\mathcal{W}}(x,y) &= \mathcal{C}_s\overline{\mathcal{W}}(x,y)\ ,\end{aligned} \tag{30}$$

where $\mathcal{W}(x,y)$ and $\overline{\mathcal{W}}(x,y)$ are Baker-Akhiezer functions. Writing them in the form

$$\begin{aligned}
\mathcal{W}(x,y) &= \sum_{j=0}^{\infty} \mathrm{diag}[\psi_j(m;x,y)]\Lambda^{-j} \cdot \exp\left(\xi(x,\Lambda)\right) \ , \\
\overline{\mathcal{W}}(x,y) &= \sum_{j=0}^{\infty} \mathrm{diag}[\overline{\psi}_j(m;x,y)]\Lambda^{j} \cdot \exp\left(\xi(y,\Lambda^{-1})\right) \ , \\
\mathcal{W}(x,y)^{-1} &= \exp\left(-\xi(x,\Lambda)\right) \cdot \sum_{j=0}^{\infty} \Lambda^{-j} \ \mathrm{diag}[\psi_j^*(m+1;x,y)] \ , \\
\overline{\mathcal{W}}(x,y)^{-1} &= \exp\left(-\xi(y,\Lambda^{-1})\right) \cdot \sum_{j=0}^{\infty} \Lambda^{j} \ \mathrm{diag}[\overline{\psi}_j^*(m+1;x,y)] \ ,
\end{aligned}$$

with $\psi_0(m;x,y) = \psi_0^*(m;x,y) = 1$ and $\overline{\psi}_0(m;x,y) \neq 0$ for $m \in \mathbb{Z}$ and using a bilinear formalism one can prove [14] that the scalar generating series

$$\begin{aligned}
\Psi(m;x,y;\lambda) &= \lambda^m \exp\xi(x,\lambda) \sum_{j=0}^{\infty} \psi_j(m;x,y)\lambda^{-j} \ , \\
\Psi^*(m;x,y;\lambda) &= \lambda^{-m} \exp\xi(-x,\lambda) \sum_{j=0}^{\infty} \psi_j^*(m;x,y)\lambda^{-j} \ , \\
\overline{\Psi}(m;x,y;\lambda) &= \lambda^m \exp\xi(y,\lambda^{-1}) \sum_{j=0}^{\infty} \overline{\psi}_j(m;x,y)\lambda^{j} \ , \\
\overline{\Psi}^*(m;x,y;\lambda) &= \lambda^{-m} \exp\xi(-y,\lambda^{-1}) \sum_{j=0}^{\infty} \overline{\psi}_j^*(m;x,y)\lambda^{j}
\end{aligned} \tag{31}$$

can be expressed in terms of family of tau-function $\tau_n(x,y)$, $n \in \mathbb{Z}$

$$\begin{aligned}
\Psi(m;x,y;\lambda) &= \frac{\tau_m(x-\varepsilon(\lambda^{-1}),y)}{\tau_m(x,y)} \ , \\
\Psi^*(m;x,y;\lambda) &= \frac{\tau_m(x+\varepsilon(\lambda^{-1}),y)}{\tau_m(x,y)} \ , \\
\overline{\Psi}(m;x,y;\lambda) &= \frac{\tau_{m+1}(x,y-\varepsilon(\lambda))}{\tau_m(x,y)} \ , \\
\overline{\Psi}^*(m;x,y;\lambda) &= \frac{\tau_{m-1}(x,y+\varepsilon(\lambda))}{\tau_m(x,y)} \ .
\end{aligned} \tag{32}$$

In (31) the function $\xi(x,\lambda)$ is

$$\xi(x,\lambda) = \sum_{s=1}^{\infty} x_s\lambda^s$$

and in (32) the notation $\varepsilon(\lambda)$ means the set $\{\lambda, \lambda^2/2, \lambda^3/3, \dots\}$.

The formulas (32) and the spectral problems (30) allow to express all the entries of the matrices $\mathcal{L}$ and $\overline{\mathcal{L}}$ as certain combinations of tau-function $\tau_n(x,y)$. In particular, the first coefficient $c_{-1}(n)$ in the decomposition of the matrix $\overline{\mathcal{L}}$ has the form

$$c_{-1}(m;x,y)=\frac{\tau_{m+1}(x,y)\tau_{m-1}(x,y)}{\tau_m^2(x,y)}\,. \tag{33}$$

2.2. **RTCh hierarchy as reduction from 2DTL.** Consider the following substitution for the entries of the matrices $\mathcal{L}$ and $\overline{\mathcal{L}}$ ($j\geq 0$) [11]:

$$\begin{aligned} b_{-j}(m) &= (-)^{j+1}\prod_{i=1}^{j} u_{m-i}\prod_{i=0}^{j} w_{m-i}(\kappa u_{m-j}+u_{m-j-1})\,,\\ c_j(m) &= (\kappa)^{-j-1}\prod_{i=0}^{j-1} u_{m+i}^{-1}\prod_{i=0}^{j} w_{m+i}^{-1}(\kappa u_{m+j}^{-1}+u_{m+j+1}^{-1})\,,\\ c_{-1}(m) &= \frac{u_{m-1}}{u_m}\,. \end{aligned} \tag{34}$$

In (34) we introduced the functions $u_m(x,y)$ and $w_m(x,y)$ which depend on the continuous times x_s and y_s and $\kappa\in\mathbb{C}$ is a parameter, the same as in the previous Section.

Comparing the last formula in (34) with (33) we can choose the following parametrization of the function $u_m(x,y)$ in terms of tau-function

$$u_m(x,y)=\frac{\tau_m(x,y)}{\tau_{m+1}(x,y)}\,. \tag{35}$$

We consider the similar ansatz for the variable $w_m(x,y)$

$$w_m(x,y)=\frac{\theta_{m+1}(x,y)}{\theta_m(x,y)}\,. \tag{36}$$

The formulas (35), (36) and (34) allows to get simple expressions relations for the matrices $\mathcal{L}$ and $\overline{\mathcal{L}}$ in terms of RTCh dynamical variables u_m and w_m. We have

Proposition 1.

$$\begin{aligned} \mathcal{L} &= \frac{\Theta}{\mathcal{T}}\left[W\,\frac{1}{1+\Lambda^{-1}}\,U\,(\Lambda-\kappa)\right]\frac{\mathcal{T}}{\Theta}\,,\\ \overline{\mathcal{L}} &= \frac{\Theta}{\mathcal{T}}\left[\frac{1}{1-\kappa^{-1}\Lambda}\,U^{-1}\,(1+\Lambda^{-1})\,W^{-1}\right]\frac{\mathcal{T}}{\Theta}\,, \end{aligned} \tag{37}$$

where U, W, $\mathcal{T}$ and Θ are diagonal matrices

$$U^{\pm1}=\mathrm{diag}[u_m^{\pm1}]\,,\quad W^{\pm1}=\mathrm{diag}[w_m^{\pm1}]\,,\quad \mathcal{T}^{\pm1}=\mathrm{diag}[\tau_m^{\pm1}]\,,\quad \Theta^{\pm1}=\mathrm{diag}[\theta_m^{\pm1}] \tag{38}$$

and the rational functions $\frac{1}{1-x}$ in formulas (37) is always understood as the series

$$\frac{1}{1-x}=\sum_{j=0}^{\infty}x^j\,.$$

Corollary 1. *The matrices $\mathcal{L}$ and $\overline{\mathcal{L}}$ which correspond to RTCh reduction satisfy the relation*

$$\mathcal{L}\cdot\overline{\mathcal{L}} = \overline{\mathcal{L}}\cdot\mathcal{L} = -\kappa \ . \tag{39}$$

Proof. Substituting formulas (34) into (26) we obtain

$$\begin{aligned} 1-\Lambda^{-1}\mathcal{L} &= (1+\kappa)\frac{\Theta}{\mathcal{T}_-}\,\frac{\Lambda^{-1}}{1+\Lambda^{-1}}\,\frac{\widetilde{\Theta}}{\mathcal{T}_+}\ , \\ \overline{\mathcal{L}}\Lambda+\kappa = &= (1+\kappa)\frac{\Theta}{\mathcal{T}}\,\frac{\kappa}{\kappa-\Lambda}\,\frac{\widetilde{\Theta}}{\mathcal{T}}\ , \end{aligned} \tag{40}$$

where we introduced the diagonal matrices

$$\mathcal{T}_\pm = \mathrm{diag}[\tau_{n\pm1}]\ ,\quad \widetilde{\Theta} = \mathrm{diag}[\tilde{\theta}_m]\ ,\quad \tilde{\theta}_m = \frac{\tau_{m+1}\tau_{m-1}+\kappa\tau_m^2}{(1+\kappa)\theta_m}\ . \tag{41}$$

Excluding from formulas (40) the diagonal matrix $\widetilde{\Theta}$ we can find first the relation (39) and using this relation in (40) we prove the statement of the Proposition. □

At the first sight the formulas (37) does not depend just on variables u_n and w_n, since they includes explicitly the diagonal matrices $\mathcal{T}$ and Θ. But this is not true since, we have obvious formulas

$$\begin{aligned} \frac{\Theta}{\mathcal{T}}\,\Lambda^j\,\frac{\mathcal{T}}{\Theta} &= \mathrm{diag}\left[\prod_{i=0}^{j-1} u_{m+i}^{-1}w_{m+i}^{-1}\right]\ ,\quad j>0\ , \\ \frac{\Theta}{\mathcal{T}}\,\Lambda^j\,\frac{\mathcal{T}}{\Theta} &= \mathrm{diag}\left[\prod_{i=1}^{-j} u_{m-i}w_{m-i}\right]\ ,\quad j<0\ . \end{aligned} \tag{42}$$

Using formulas (37) we can obtain explicit formulas for the elements of the matrices $\mathcal{B}_s$ and $\mathcal{C}_s$ which describe the evolutions of the matrices $\mathcal{L}$ and $\overline{\mathcal{L}}$ by the formulas (28). This will give the evolution equations of the functions $u_m(x,y)$ and $w_m(x,y)$ with respect to continuous times x_s and y_s. Form of these equations is given by the following

Proposition 2. *The substitution (34) describes the reduction of 2DTL to the couple of the RTCh hierarchies with respect to the sets of the times x_s and y_s. These hierarchies are given by the formulas*

$$\begin{aligned} \frac{\partial \ln u_m}{\partial x_s} &= -\sum_{\substack{j_k\geq 0\\ j_1+\dots+j_s=s}}\ \prod_{i=0}^{s-1}\mathsf{U}_{j_k}\left(m+i-\sum_{r=1}^{i} j_r\right)\ , \\ \frac{\partial \ln w_m}{\partial x_s} &= \sum_{\substack{j_k\geq 0\\ j_1+\dots+j_s=s}}\ \prod_{i=0}^{s-1}\mathsf{W}_{j_k}\left(m-i+\sum_{r=1}^{i} j_r\right)\ , \end{aligned} \tag{43}$$

and

$$(44)\qquad \begin{aligned} (-\kappa)^s\frac{\partial \ln u_m^{-1}}{\partial y_s} &= \sum_{\substack{j_k\geq 0\\ j_1+\ldots+j_s=s}} \prod_{i=0}^{s-1} \overline{\mathsf{U}}_{j_k}\left(m-i+\sum_{r=1}^{i} j_r\right)\ , \\ (-\kappa)^s\frac{\partial \ln w_m^{-1}}{\partial y_s} &= -\sum_{\substack{j_k\geq 0\\ j_1+\ldots+j_s=s}} \prod_{i=0}^{s-1} \overline{\mathsf{W}}_{j_k}\left(m+i-\sum_{r=1}^{i} j_r\right)\ , \end{aligned}$$

where

$$(45)\qquad \begin{aligned} \mathsf{U}_j(m) &= (-)^j w_m\left(\kappa(1-\delta_{j,0})u_{m-j+1}+u_{m-j}\right)\ , \\ \mathsf{W}_j(m) &= (-)^j u_m\left(\kappa(1-\delta_{j,0})w_{m+j-1}+w_{m+j}\right)\ , \end{aligned}$$

and

$$(46)\qquad \begin{aligned} \overline{\mathsf{U}}_j(m) &= (-)^j w_m^{-1}\left(\kappa(1-\delta_{j,0})u^{-1}_{m+j-1}+u^{-1}_{m+j}\right)\ , \\ \overline{\mathsf{W}}_j(m) &= (-)^j u_m^{-1}\left(\kappa(1-\delta_{j,0})w^{-1}_{m-j+1}+w^{-1}_{m-j}\right)\ . \end{aligned}$$

The meaning of the Proposition 2 can be understood as follows. Take the equations from the system (28), which describe the evolution of the operator $\mathcal{L}$ and $\overline{\mathcal{L}}$ with respect to the time x_s. This set contains infinite number of equations, which describe the evolution of the entries $b_{-j}(m)$ and $c_j(m)$ with respect to this time. The statement of the Proposition 2 means that the whole set of these equations reduces after substitution (34) just to pair of equations given by the formulas (43). The same is true for the evolutions with respect to times y_s. One may verify that first two pair of equations from (43) which correspond to $s=1,2$ coincide precisely with equations (20) and (21) obtained from L-operator approach. A general proof that the hierarchy (18) obtained in L-operator approach coincide with hierarchy (43) for any s should be a repetition of the analogous proof for the equivalence of such a presentations in case of standard AKNS hierarchy. We do not go into these details here.

Explicit formulas (43) and (44) yield the following

Corollary 2. *The systems (43) and (44) transforms to each other after the simultaneous transformations*

$$(47)\qquad x_s \longleftrightarrow -(-\kappa)^{-s}y_s\ , \quad u_n \longleftrightarrow w_n^{-1}\ .$$

The first equation in (43) goes into the second equation of (44) while the second of (43) to first of (44).

The transformation (47) allows to forget about dependence of our dynamical variables on times y_s and consider only the set of equations (43).

This symmetry of the reduced 2DTL is related to certain Bäcklund transformations of 2DTL investigated in [17] and used in the paper [12] in order to obtain the bilinear formulation of RTCh equation. Ended, formulas (40) contains 'new' tau-function $\tilde{\theta}_m$ given by (41) as a certain rational combination of τ_m and θ_m. It is actually unclear from our presentation whether the functions θ_m and $\tilde{\theta}_m$ are also the tau-functions. The last equation of the substitution (34) and (33) allows to identify only τ_m with some tau-function of 2DTL.

Nevertheless, this assertion is true and one can use the following argument to see that the function θ_m is also a tau-function. It is clear that the symmetry transformation (47) moves the substitution (34) into another one:

$$\begin{aligned} b_{-j}(m) &= (-)^{j+1}\prod_{i=1}^{j} w_{m-i}^{-1}\prod_{i=0}^{j} u_{m-i}^{-1}(\kappa w_{m-j}^{-1}+w_{m-j-1}^{-1})\ , \\ c_j(m) &= (\kappa)^{-j-1}\prod_{i=0}^{j-1} w_{m+i}\prod_{i=0}^{j} u_{m+i}(\kappa w_{m+j}+w_{m+j+1})\ , \\ c_{-1}(m) &= \frac{w_m}{w_{m-1}}\ , \end{aligned} \tag{48}$$

where now the ratio of functions w_m should be identified with a ratio of tau-functions according to the formulas (32). The ansatz (36) allows to identify the function θ_m with some another tau-function of 2DTL. This tau-function should be different since otherwise the substitutions (34) and (48) becomes trivial (the product $u_m w_m = 1$ for any $m \in \mathbb{Z}$ in this case).

On the other hand if one uses the substitution (48) in order to get expressions for the matrices $\mathcal{L}$ and $\overline{\mathcal{L}}$ similar to (40) a new rational combination of tau-functions

$$\tilde{\tau}_m = \frac{\theta_{m+1}\theta_{m-1}+\kappa\theta_m^2}{(1+\kappa)\tau_m} \tag{49}$$

will appear. Let us denote all these tau-functions as a set $\tau_m^{(k)}$

$$\tilde{\theta}_m \equiv \tau_m^{(1)}\ , \quad \tau_m \equiv \tau_m^{(0)}\ , \quad \theta_m \equiv \tau_m^{(-1)}\ , \quad \tilde{\tau}_m \equiv \tau^{(-2)}\ .$$

We can extend the value of the index k to be any integer number by repeating the transformations given by the formulas (41) and (49). It was shown in [12] that the whole set of these tau-functions satisfy a reduced discrete 2DTL (the transformations (41) and (49) describe the evolution with respect to discrete parameter k) and all of them are some tau-functions of original 2DTL hierarchy.

We will see in the next Section that in the soliton sectors the tau-functions $\tau_m^{(k)}$ differs by the proportional change of the soliton amplitudes (see (63)). In terms of the set $\tau_m^{(k)}$ the transformation (47) can be written in the form

$$\tau_m^{(k)} \longleftrightarrow \tau_m^{(1-k)}\,, \quad k \in \mathbb{Z}\,. \tag{50}$$

3. Soliton solutions of the RTCh equation

In this Section we will give the solitonic solution of the RTCh equation obtained in [12] in terms of rational tau-functions. We will identify them with tau-functions used in the papers [1, 2] as well as with soliton tau-functions appeared in KP [15, 16] and in 2DTL [14] theories. In order to do this we have to transform the RTCh equation (20) into bilinear form analogous to one used in the [12]. The authors of this paper used so called Casoratian technique to find a solitonic solutions to the RTCh equation in form of the Casorati determinants. For the readers convenience we will describe part of this technique relevant for our paper in the Appendix.

We start with rewriting the equations (20) into bilinear form. Actually, in order to use results on the reduction described in the previous section, we have to consider certain linear combinations of equations from the RTCh hierarchy. Instead of (20) we consider a linear combination of (19) and (20)

$$\begin{aligned} \frac{\partial \ln u_m}{\partial x_1} &= w_m(\kappa u_m + u_{m-1}) - (\kappa+1)\,, \\ -\frac{\partial \ln w_m}{\partial x_1} &= u_m(\kappa w_m + w_{m+1}) - (\kappa+1)\,. \end{aligned} \tag{51}$$

such that the system (51) has a constant ($u_m = w_m^{-1} = \mathrm{const}$) solution.

Excluding from (51) w_m and replacing $u_m = \mathrm{e}^{q_m((\sqrt{1+\kappa})x_1)}$ we obtain ($q'_m = \partial q_m/\partial x_1$)

$$q''_m = \frac{(\sqrt{1+\kappa}+q'_{m-1})(\sqrt{1+\kappa}+q'_m)}{1+\kappa\,\mathrm{e}^{q_m-q_{m-1}}} - \frac{(\sqrt{1+\kappa}+q'_m)(\sqrt{1+\kappa}+q'_{m+1})}{1+\kappa\,\mathrm{e}^{q_{m+1}-q_m}}\,. \tag{52}$$

The latter is the relativistic Toda chain equation. In the non-relativistic limit $\kappa\to\infty$ we obtain from (52) the standard Toda chain equation

$$q''_m = \mathrm{e}^{q_{m-1}-q_m} - \mathrm{e}^{q_m-q_{m+1}}\,.$$

Proposition 3. [12] *The substitution*

$$u_m(x_1) = \frac{\tau_m(x_1)}{\tau_{m+1}(x_1)}\,, \qquad w_m(x_1) = \frac{\theta_{m+1}(x_1)}{\theta_m(x_1)}\,, \tag{53}$$

transform the equations (51) into bilinear form

$$\begin{aligned} D_1^2(\tau_m \circ \tau_m) &= 2(1+\kappa)(\theta_{m+1}\tilde{\theta}_{m-1} - \tau_m^2)\,, \\ D_1(\tau_m \circ \tau_{m-1}) &= (1+\kappa)(\tau_m\tau_{m-1} - \theta_m\tilde{\theta}_{m-1})\,, \\ (1+\kappa)\theta_m\tilde{\theta}_m &= \tau_{m+1}\tau_{m-1} + \kappa\tau_m^2\,. \end{aligned} \tag{54}$$

In (54) the Hirota operator $D(f \circ g) = f'g - fg'$ have been used.

We can also find a bilinear relation between tau-functions τ_m and θ_m

$$D_1(\theta_{m-1} \circ \tau_m) = (1+\kappa)(\theta_m\tau_{m-1} - \theta_{m-1}\tau_m) \tag{55}$$

which is a consequence of the RTCh equation and Bäcklund transformation $\tau_m \to \theta_m$ discussed at the end of the previous section. In the Appendix we will show how this bilinear equation can be solved using Casoratian technique.

Let Δ^* and Δ are two independent complex parameters which belong to the rational curve

$$\kappa = \Delta^*\Delta - \Delta^* - \Delta\,. \tag{56}$$

In the paper [1] we used the following uniformization of the curve (56)

$$\begin{aligned} \Delta_\phi^* &= \mathrm{e}^{-i\phi}\left(\sqrt{\cos^2\phi + \kappa} + \cos\phi\right)\,, \\ \Delta_\phi &= \mathrm{e}^{i\phi}\left(\sqrt{\cos^2\phi + \kappa} + \cos\phi\right) \end{aligned} \tag{57}$$

which is convenient for restricting the model to the chain of the finite size with periodic boundary conditions. For real κ and ϕ the parameters Δ^* and Δ are complex conjugated.

Define functions

$$\begin{aligned} \varphi_m^{(k)}(x_1;\phi,f) &= \frac{(1-\Delta_\phi^*)^m(1-{\Delta_\phi}^{-1})^k\mathrm{e}^{(1-\Delta_\phi^*)x_1}}{(1+\kappa)^{m/2}} - \\ &- f\,\frac{(1-\Delta_\phi)^m(1-{\Delta_\phi^*}^{-1})^k\mathrm{e}^{(1-\Delta_\phi)x_1}}{(1+\kappa)^{m/2}} \end{aligned} \tag{58}$$

which depend on discrete time m, first continuous time x_1 and some additional integer parameter k. Define also the M-soliton tau-function by the determinant[2]

$$\tau_m^{(\nu)}\left(x_1;\,\{\phi_k,f_k\}|_{k=1}^M\right)=\det\left|\varphi_{m+i-1}^{(m+\nu)}(x_1;\phi_j,f_j)\right|_{i,j=1}^M \tag{59}$$

One of the statements of the paper [12] is the following

Proposition 4. [12] *The tau-functions which solve the equations (54) are given by the following specialization of the function(59)*

$$\begin{aligned}
\tau_m\left(x_1;\,\{\phi_k,f_k\}|_{k=1}^M\right) &= \tau_m^{(0)}\left(x_1;\,\{\phi_k,f_k\}|_{k=1}^M\right)\;,\\
\theta_m\left(x_1;\,\{\phi_k,f_k\}|_{k=1}^M\right) &= \tau_m^{(-1)}\left(x_1;\,\{\phi_k,f_k\}|_{k=1}^M\right)\;,\\
\tilde\theta_m\left(x_1;\,\{\phi_k,f_k\}|_{k=1}^M\right) &= \tau_m^{(1)}\left(x_1;\,\{\phi_k,f_k\}|_{k=1}^M\right)\;.
\end{aligned} \tag{60}$$

The tau-functions given by the Proposition 4 coincide with those used in the papers [1, 2] up to certain normalization factor. Let us change this normalization in order to fix precise relation to formulas of these papers and also to demonstrate the connection of these tau-functions with standard solitonic tau-functions known from the 2DTL theory [14]. We obtain ($\Delta_i^*\equiv\Delta_{\phi_i}^*$ and $\Delta_i\equiv\Delta_{\phi_i}$)

$$\tau_m^{(\nu)}=\prod_{i<j}(\Delta_i^*-\Delta_j^*)\prod_i\frac{(-\Delta_i)^{\nu-m}}{(1-\Delta_i)^{\nu}}\mathrm{e}^{x_1\sum(1-\Delta_i^*)}\mathcal{T}_m^{(\nu)}\;, \tag{61}$$

where

$$\mathcal{T}_m^{(\nu)}=\prod_{i<j}(\Delta_i^*-\Delta_j^*)^{-1}\det\left|(1-\Delta_j^*)^{i-1}-X_m^{(\nu)}(x_1;\phi_j,f_j)(1-\Delta_j)^{i-1}\frac{\Delta_j^*}{\Delta_j}\right|_{i,j=1}^M \tag{62}$$

and

$$X_m^{(\nu)}(x_1;\phi,f)=f\mathrm{e}^{(\Delta_\phi^*-\Delta_\phi)x_1}\left(\frac{\Delta_\phi}{\Delta_\phi^*}\right)^{m+1}(c(\phi))^{-\nu}\;,\quad c(\phi)=\frac{\Delta_\phi^*}{\Delta_\phi}\,\frac{(1-\Delta_\phi)}{(1-\Delta_\phi^*)}\;. \tag{63}$$

It is clear that factor before $\mathcal{T}_m^{(\nu)}$ in formula (61) cannot spoil the form of the equations (54) so can be dropped. Now the tau-functions $\mathcal{T}_m\equiv\mathcal{T}_m^{(0)}$ and $\Theta_m\equiv\mathcal{T}_m^{(-1)}$ coincide identically with those used in papers [1, 2]. The only difference is the shift in amplitude (63), but this is because we used $u_m=\tau_m/\tau_{m+1}$ instead of $u_m=\tau_{m-1}/\tau_m$ in [1, 2].

[2]The superscript $^{(\nu)}$ counts the number of Bäcklund transformations (55), not the number of solitons as in the papers [1, 2].

The formula (62) may be rewritten in the form analogous to the standard presentation of solitonic tau-functions appeared in KP [15] or 2DTL theories [14] with all higher continuous times but x_1 are freezed:

$$\mathcal{T}_n^{(\nu)} = \det \left| \delta_{ij} - X_m^{(\nu)}(x_1; \phi_j, f_j) \prod_{k \neq j} s_{j,k} \frac{\Delta_i^* - \Delta_i}{\Delta_i^* - \Delta_j} \right|_{i,j=1}^{M} \tag{64}$$

where

$$s_{j,k} = s(\phi, \phi') \ , \quad s(\phi, \phi') = \frac{\Delta_\phi^*}{\Delta_\phi} \, \frac{\Delta_\phi - \Delta_{\phi'}^*}{\Delta_\phi^* - \Delta_{\phi'}^*} \tag{65}$$

In order to come from formula (62) to (64) one have to use explicitly the inverse Vandermonde matrix and this is a reason of getting product of factors $s_{j,k}$ in soliton amplitudes. Actually, the determinant presentation (64) is equivalent to the recurrent relation which relates the tau-functions with different number of solitons:

$$\mathcal{T}_m^{(\nu)}(\{\phi_k, f_k\}_{k=1}^{M}) = \mathcal{T}_m^{(\nu)}(\{\phi_k, f_k\}_{k=1}^{M-1}) - X_m^{(\nu)}(\phi_M, f_M) \prod_{k=1}^{M-1} s_{M,k} \mathcal{T}_m^{(\nu)}(\{\phi_k, d_{k,n} f_k\}_{k=1}^{M-1}) \ ,$$

where the phase shift is given by cross-ratio

$$d_{i,j} = \frac{(\Delta_i^* - \Delta_j^*)(\Delta_i - \Delta_j)}{(\Delta_i^* - \Delta_j)(\Delta_i - \Delta_j^*)} \ . \tag{66}$$

Specializing the M-soliton tau-function $\mathcal{T}_m^{(0)}(x_1; \{\phi_k, f_k\}|_{k=1}^{M})$ to the zero value of continuous time $x_1 = 0$ and setting the amplitudes $f_k = 1$ we obtain that it becomes proportional to

$$\mathcal{T}_m^{(0)} \left(0; \{\phi_k, 1\}|_{k=1}^{M}\right) \sim \det \left| (-)^{j-1} \left(\Delta_i^{*\ j+m-1} - \Delta_i^{\ j+m-1} \right) \right|_{i,j=1}^{M} \ . \tag{67}$$

This proves the following

Proposition 5. *The rational solitonic tau function $\mathcal{T}_m^{(0)}(0; \{\phi_k, 1\}|_{k=1}^{M})$ vanishes for values of the discrete time $m = -M+1, -M+2, \ldots, 0$ irrespective the values of the spectral parameters ϕ_k.*

This property of the rational tau-function was crucial in [2] in order to construct modified **Q**-operators.

4. Discussion

So far the discrete variable m in the previous sections runs over all integer numbers. It is clear how to apply the results of the previous section to finite volume system or how to satisfy the periodic boundary condition with respect to discrete time m

$$u_m(x_n) = u_{m+M}(x_n)\ , \quad w_m(x_n) = w_{m+M}(x_n)\ . \tag{68}$$

One should specialize the spectral variable ϕ which uniformize the curve (56) to the finite set

$$\phi_k = \frac{2\pi i k}{M}\ , \quad k = 1, \dots, M-1\ . \tag{69}$$

The case $k = 0$ or $k = M$ are excluded from (69) because at this value of the parameter ϕ, $\Delta^*_\phi = \Delta_\phi$ and the solitons disappear. It is clear that in this case there is only finite number (M including a zero soliton case) rational tau-functions.

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Appendix: Casoratian technique

The goal of this Appendix is to demonstrate Casoratian technique [12] and prove the bilinear relation (55) with tau-function given by the Casorati determinant (59). Let us fix values of the discrete parameters m and ν. The functions (58) satisfy the following dispersion relations

$$\begin{aligned} \frac{\partial \varphi^{(m+\nu)}_{m+k}}{\partial x_1} &= \sqrt{1+\kappa}\,\varphi^{(m+\nu)}_{m+k+1}\ , \\ \sqrt{1+\kappa}\left(\varphi^{(m+\nu)}_{m+k} - \varphi^{(m+\nu-1)}_{m+k}\right) &= \varphi^{(m+\nu)}_{m+k+1}\ , \quad k \in \mathbb{Z}\ . \end{aligned} \tag{70}$$

Let $\overline{\varphi}_m^{(k)}$ means the column

$$\overline{\varphi}_m^{(\nu)} = \begin{pmatrix} \varphi_m^{(m+\nu)}(x_1;\phi_1,f_1) \\ \vdots \\ \varphi_m^{(m+\nu)}(x_1;\phi_M,f_M) \end{pmatrix} \tag{71}$$

and consider the following identity for $2M \times 2M$ determinant

$$\begin{vmatrix} \overline{\varphi}_m^{(\nu-1)} & \cdots & \overline{\varphi}_{m+M-3}^{(\nu-1)} & \overline{\varphi}_{m+M-1}^{(\nu-1)} & \overline{\varphi}_{m+M-2}^{(\nu)} & & 0 & & \overline{\varphi}_{m+M-2}^{(\nu-1)} \\ & 0 & & \overline{\varphi}_{m+M-1}^{(\nu-1)} & \overline{\varphi}_{m+M-2}^{(\nu)} & \overline{\varphi}_{m-1}^{(\nu-1)} & \cdots & \overline{\varphi}_{m+M-3}^{(\nu-1)} & \overline{\varphi}_{m+M-2}^{(\nu-1)} \end{vmatrix} = 0\ . \tag{72}$$

The identity is valid since this determinant can be transformed to

$$\begin{vmatrix} \overline{\varphi}_m^{(\nu-1)} & \cdots & \overline{\varphi}_{m+M-3}^{(\nu-1)} & 0 & 0 & -\overline{\varphi}_{m-1}^{(\nu-1)} & & 0 & & 0 \\ & 0 & & \overline{\varphi}_{m+M-1}^{(\nu-1)} & \overline{\varphi}_{m+M-2}^{(\nu)} & 0 & \overline{\varphi}_m^{(\nu-1)} & \cdots & \overline{\varphi}_{m+M-3}^{(\nu-1)} & \overline{\varphi}_{m+M-2}^{(\nu-1)} \end{vmatrix}$$

by adding and subtracting the columns and rows. Applying the Laplace expansion to the l.h.s. of identity (72) we obtain the bilinear relation for $M \times M$ determinants

$$\begin{aligned} &\left|\overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-1}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu)}\right| \cdot \left|\overline{\varphi}_{m-1}^{(\nu-1)}\ \overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu-1)}\right| - \\ -&\left|\overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu)}\right| \cdot \left|\overline{\varphi}_{m-1}^{(\nu-1)}\ \overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-1}^{(\nu-1)}\right| - \\ -&\left|\overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-1}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu-1)}\right| \cdot \left|\overline{\varphi}_{m-1}^{(\nu-1)}\ \overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu)}\right| = 0\ . \end{aligned}$$

Using now the dispersion relations (70) we can identify

$$\begin{aligned} \left|\overline{\varphi}_{m-1}^{(\nu-1)}\ \overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu-1)}\right| &= \tau_{m-1}^{(\nu-1)}\ , \\ \left|\overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-1}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu)}\right| &= \frac{1}{1+\kappa}\frac{\partial \tau_m^{(\nu)}}{\partial x_1} - \tau_m^{(\nu)}\ , \\ \left|\overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu)}\right| &= \frac{1}{\sqrt{1+\kappa}}\tau_m^{(\nu)}\ , \\ \left|\overline{\varphi}_{m-1}^{(\nu-1)}\ \overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-1}^{(\nu-1)}\right| &= \frac{1}{\sqrt{1+\kappa}}\frac{\partial \tau_{m-1}^{(\nu-1)}}{\partial x_1}\ , \\ \left|\overline{\varphi}_{m-1}^{(\nu-1)}\ \overline{\varphi}_m^{(\nu-1)} \cdots \overline{\varphi}_{m+M-3}^{(\nu-1)}\ \overline{\varphi}_{m+M-2}^{(\nu)}\right| &= \tau_{m-1}^{(\nu)} \end{aligned}$$

so the bilinear identity for determinants becomes the bilinear identity for tau-functions

$$\left(\frac{1}{1+\kappa}\frac{\partial \tau_m^{(\nu)}}{\partial x_1} - \tau_m^{(\nu)}\right) \cdot \tau_{m-1}^{(\nu-1)} - \frac{1}{\sqrt{1+\kappa}}\tau_m^{(\nu)} \cdot \frac{1}{\sqrt{1+\kappa}}\frac{\partial \tau_{m-1}^{(\nu-1)}}{\partial x_1} + \tau_m^{(\nu-1)}\tau_{m-1}^{(\nu)} = 0$$

which can be written in the form

$$D_1\left(\tau_{m-1}^{(\nu-1)}\circ\tau_m^{(\nu)}\right)=(1+\kappa)\left(\tau_m^{(\nu-1)}\tau_{m-1}^{(\nu)}-\tau_{m-1}^{(\nu-1)}\tau_m^{(\nu)}\right)\ .$$

Last equality coincide with (55) at $\nu=0$.

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