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YAMABE FLOWS AND EXTREMAL ENTROPY ON COMPLETE MANIFOLDS

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Abstract. We introduce curvature-normalized versions of the Yamabe flow on complete manifolds with negative scalar curvature, for which we show long time existence of solutions and convergence of these flows to complete Yamabe metrics. We apply them to study the extrema of the topological entropy in conformal classes and offer an entropy-rigidity theorem for convex-cocompact surfaces: extrema of the entropy are the metrics whose closed geodesics coincide with those of the unique hyperbolic metric conformally equivalent to the initial one. On convex-cocompact manifolds of higher dimension, we show that local extrema of the entropy have constant scalar curvature on their non-wandering set.

1. Introduction

Geometric flows have been extensively used during the last twenty years to deform a given Riemannian manifold into one with some more symmetries, such as constant curvatures.

In this article, we focus on the case of manifolds with negative scalar curvature and infinite volume. On these, we introduce Curvature-normalized versions of the Yamabe flow, which converge smoothly to a Yamabe metric as soon as the Riemannian curvature tensor of the original metric is uniformly bounded. We then use these flows to study the extrema of the topological entropy on convex-cocompact manifolds. A similar study was carried out for compact manifolds in [SST11]. Some technical computations are identical in the compact case and in the complete case: for these we will refer to [SST11], and here we will focus on the specifics of non-compact manifolds.

Let (M, g) be a complete Riemannian manifold whose scalar curvature R_g satisfies $R_{min} \leq R_g \leq R_{max} < 0$. We will say that a family of complete metrics $(g_t)_{t\in[0,T)}$ is an ***increasing Curvature-normalized Yamabe flow*** if it is a solution of the PDE

$$\frac{\partial g_t}{\partial t} = (R_{max} - R_{g_t})g_t \quad \text{with initial condition} \quad g_0 = g.$$

We will denote such a solution by CYF$^+$. Similarly, we will say a family of metrics $(g_t)_{t\in[0,T)}$ is a ***decreasing Curvature-normalized Yamabe flow*** if it is a solution of the PDE

$$\frac{\partial g_t}{\partial t} = (R_{min} - R_{g_t})g_t \quad \text{with initial condition} \quad g_0 = g.$$

It will be denoted by CYF$^-$. Recall that a metric with constant scalar curvature is called a ***Yamabe metric***. Section 2 is dedicated to the study of these flows.

The Yamabe flow was introduced by Hamilton in [Ham89], and its first properties on compact manifolds were published by Ye in [Ye94]. Long time existence and convergence of the Volume-normalized Yamabe flow have been established for most compact manifolds, see the work by Schwetlick-Struwe [SchS03] and by Brendle [Bre05], [Bre07] and references given there. On complete manifolds, our work only considers the case of negative scalar curvature, which is analytically

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simpler. Indeed, as we will see in Section 3, our approach to study the extrema of topological entropy is only relevant for negatively curved manifolds.

Theorem 1.1. *Let (M, g) be a complete Riemannian manifold whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$.*

(1) *The increasing Curvature-normalized Yamabe flow CYF$^+$ (with initial condition $g_0 = g$) has a solution defined for all $t \geq 0$, which we will denote by $(g_t^+)_{t\geq 0}$.*
Similarly, the CYF$^-$ has a solution defined for all $t \geq 0$, which we will denote by $(g_t^-)_{t\geq 0}$.
(2) *For all $t \geq 0$, the scalar curvature bounds are preserved along these flows:*
$$R_{min} \leq R_{g_t^+} \leq R_{max},$$
$$R_{min} \leq R_{g_t^-} \leq R_{max}.$$
(3) *For all $x \in M$, the application $t \mapsto g_t^+(x)$ is increasing and $t \mapsto g_t^-(x)$ is decreasing.*
(4) *For all $x \in M$, the flows' solutions g_t^+ and g_t^- are uniformly bounded:*
$$g_0 \leq g_t^+ \leq \left|\frac{R_{min}}{R_{max}}\right| g_0 \quad \text{and} \quad g_0 \geq g_t^- \geq \left|\frac{R_{max}}{R_{min}}\right| g_0.$$
(5) *Assume moreover that the Riemannian curvature tensor of g_0 is uniformly bounded, and let g_Y be the unique Yamabe metric in the conformal class of g_0 with scalar curvature $R_{g_Y} \equiv -1$. Then on all compact sets and for all $k \geq 0$, the CYF$^+$ converges exponentially fast in the $\mathcal{C}^k$ topology to*
$$g_{max} = \frac{g_Y}{|R_{max}|}.$$
Similarly, for all $k \geq 0$ the CYF$^-$ converges exponentially fast in the $\mathcal{C}^k$ topology on compact sets to
$$g_{min} = \frac{g_Y}{|R_{min}|}.$$

Remark 1. In general, CYF could have many solutions, as has been shown by Giesen and Topping in [GT09] for surfaces. However, it seems likely that there is a unique solution with uniformly bounded conformal factor with respect to the initial metric.

The monotonicity of these Curvature-normalized Yamabe flows implies, in particular, the following Schwarz Lemma, whose first proof is due to Yau in [Yau73].

Corollary 1.2 (Conformal Schwarz Lemma)**.** *Let (M, g) be a complete Riemannian manifold with bounded sectional curvatures whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$. Let g_Y be the Yamabe metric in the conformal class of g with constant scalar curvature -1. Then*
$$\frac{g_Y}{|R_{min}|} \leq g \leq \frac{g_Y}{|R_{max}|}.$$

We conclude our study of the Curvature-normalized Yamabe flows by showing that for a short time they preserve negative sectional curvatures (this is an issue since our manifolds are not compact).

Theorem 1.3. *Let (M, g) be a complete Riemannian n-manifold, $n \geq 3$, with pinched negative sectional curvatures K_g:*
$$-b^2 \leq K_g \leq -a^2 < 0.$$
Assume moreover that $\|\nabla_g R_g\|$ and $|\Delta_g R_g|$ are uniformly bounded. Let $(g_t^+)_{t\geq 0}$ and $(g_t^-)_{t\geq 0}$ be the solutions of the CYF$^+$ and the CYF$^-$ with initial metric g. Then for all $\epsilon > 0$, there exists $T_\epsilon > 0$ such that for all $t \in [0, T_\epsilon]$,
$$-b^2 - \epsilon \leq K_{g_t^+} \leq -a^2 + \epsilon \quad \text{and} \quad -b^2 - \epsilon \leq K_{g_t^-} \leq -a^2 + \epsilon.$$

We will actually prove in Theorem 2.9 a stronger statement, showing that $T_\epsilon \geq C\epsilon^\beta$ for some positive constant C and $\beta \in (0,1)$.

Extrema of the entropy have been shown to be very symmetric metrics in many situations. This study was initiated by Katok in [Kat82] showing that on compact surfaces, the hyperbolic metrics are the entropy minimizers. It was then shown by Hamenstaedt in [Ham90] that, if (M, g_S) is a locally symmetric compact Riemannian manifold such that $\sup K_{g_S} = -1$, then it is the unique minimum for the topological entropy among all metrics in M with sectional curvatures less than or equal to -1. This is called ***entropy-rigidity of symmetric spaces***. A stronger version of this result was shown by Besson, Courtois and Gallot in [BCG95], showing that when the volume is fixed, a compact locally symmetric manifold with negative sectional curvatures is the unique minimum of the topological entropy. An extension of this result to the case of non-compact locally symmetric manifolds with finite volume, as well as product of such spaces, was published by Connell and Farb in [ConFar03]. On the other hand, Ledrappier and Wang have shown in [LW09] that given a lower bound on the Ricci curvature of a compact Riemannian manifold, a hyperbolic metric strictly maximizes the topological entropy among all metrics. They have also extended their result to complex-hyperbolic and quaternionic-hyperbolic manifolds. On *compact* manifolds with negative sectional curvatures which do not admit any locally symmetric metric, not much else seems to be known about the extrema of the entropy. We have shown in [SST11] that in each conformal class, given bounds on the scalar curvature, the extrema of the entropy are exactly the Yamabe metrics.

When the manifold has infinite volume, even if it admits a locally symmetric metric, barycenter methods as well as Hamenstaedt's rank-rigidity methods may not work. This is usually the case when the limit set of the fundamental group of the manifold *is not the full boundary at infinity of the universal cover*. Even for a simple complete pair of pants (i.e. a complete metric on a thrice punctured $\mathbb{S}^2$ with infinite volume ends), no characterization of the extrema of the entropy was previously known. When the hyperbolic metrics on a pair of pants run over the Teichmueller space, results due to Patterson imply that the topological entropy goes over exactly the interval $(0,1)$ (see Theorem 4.2). However, in each conformal class, there is a unique hyperbolic metric. We will show in Section 4 that in each conformal class, the hyperbolic metric characterizes the extrema of the entropy. A full entropy-rigidity statement in the conformal class will be shown in this context. For higher dimensional manifolds, we get interesting properties of entropy extrema in Section 3.

Recall that a manifold with negative sectional curvature and bounded scalar curvature has a uniformly bounded Riemannian curvature tensor. A manifold with negative sectional curvatures is said to be convex-cocompact if there is a compact set containing all its closed geodesics. This terminology was introduced by Sullivan for hyperbolic manifolds in [Sul79]. In Section 3, we apply this flow to study extrema of the topological entropy on such manifolds. For any Riemannian manifold (M, g), we will denote by $\pi\Omega_g \subset M$ the closure of the set of closed geodesics for g, this notation will be explained in Section 3.

The key fact which allow us to relate our study of Curvature-normalized Yamabe flows to the topological entropy is the following, which will be shown in Proposition 3.3. Assume (M, g) is a convex-cocompact Riemannian manifold with negative scalar curvature, which satisfies $R_{min} \leq R_g \leq R_{max} < 0$. Then the topological entropy decreases along CYF$^+$ and increases along CYF$^-$ as long as the sectional curvatures remain negative on both flows. Moreover, this change is *strictly* monotonic along both flows when the scalar curvature is not extremal on some closed geodesics. This implies:

Theorem 1.4. *Let (M, g) be a convex-cocompact Riemannian manifold with negative sectional curvatures whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$ and such that $\|\nabla_g R_g\|$ and*

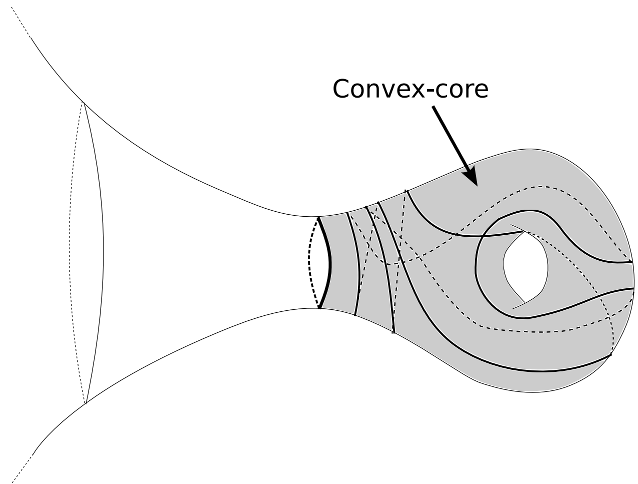

FIGURE 1. On convex-cocompact manifolds, closed geodesics are contained but not dense in the convex-core.

$|\Delta_g R_g|$ are uniformly bounded. Let $\mathcal{M}_{R_{min},R_{max}}$ be the class of all complete metrics $\tilde{g}$ with $R_{min} \leq R_{\tilde{g}} \leq R_{max} < 0$. We have the following.

(1) *If (M, g) is a local minimum for the entropy in $\mathcal{M}_{R_{min},R_{max}}$ then $R_g = R_{max}$ on the set of closed geodesics $\pi\Omega_g$.*
(2) *If (M, g) is a local maximum for the entropy $\mathcal{M}_{R_{min},R_{max}}$ then $R_g = R_{min}$ on $\pi\Omega_g$.*

If the Yamabe metric g_Y has negative sectional curvatures or if the flows $\text{CYF}^{\pm}$ preserve negative sectional curvature, we obtain stronger results:

Theorem 1.5. *Let (M, g) be a convex-cocompact Riemannian manifold with negative sectional curvatures whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$. Let g_Y be the Yamabe metric conformally equivalent to g with $R_{g_Y} \equiv -1$, and $(g_t^+)_{t\geq 0}$ and $(g_t^-)_{t\geq 0}$ be the solutions of CYF$^+$ and CYF$^-$ given by Theorem 1.1.*

(1) *If the sectional curvatures of g_Y are strictly negative, then the topological entropy of g satisfies*
$$\sqrt{|R_{max}|}h_{top}(g_Y) = h_{top}(g_{max}) \leq h_{top}(g) \leq h_{top}(g_{min}) = \sqrt{|R_{min}|}h_{top}(g_Y).$$
(2) *Assume that the sectional curvatures $K_{g_t^+}$ remain negative along the CYF$^+$.*
If $h_{top}(g) = \sqrt{|R_{max}|}h_{top}(g_Y)$ then on the set of closed geodesics $\pi\Omega_{g_Y}$, we have
$$g = g_{max} = \frac{g_Y}{|R_{max}|} \text{ and } R_g = R_{max};$$
(3) *Assume that the sectional curvatures $K_{g_t^-}$ remain negative along the CYF$^-$.*
If $h_{top}(g) = \sqrt{|R_{min}|}h_{top}(g_Y)$ then on the set of closed geodesics $\pi\Omega_{g_Y}$, we have
$$g = g_{min} = \frac{g_Y}{|R_{min}|} \text{ and } R_g = R_{min}.$$

In general it is very difficult to know whether the sectional curvatures remain negative along the flow. Therefore, for manifolds of dimension $n \geq 3$, Theorem 1.4 is much more interesting than Theorem 1.5. In Section 4, we deal with the case of surfaces, on which CYF automatically preserve negative sectional curvature. We get a full characterization of entropy extrema on convex-cocompact surfaces, which generalizes the results of Katok [Kat82] to the convex-cocompact setting.

Recall that the Yamabe metrics (with negative curvature) on surfaces are proportional to hyperbolic metrics. Our result is the following.

Theorem 1.6 (Conformal entropy-rigidity on convex-cocompact surfaces)**.** *Let* (S, g) *be a complete convex-cocompact Riemannian surface whose Gauss curvature satisfies* $K_{min} \leq K_g \leq K_{max} < 0$*. Let* g_H *be the unique hyperbolic metric conformally equivalent to* g*.*

(1) *The entropy of* g *satisfies*

$$\sqrt{|K_{max}|}h_{top}(g_H) \leq h_{top}(g) \leq \sqrt{|K_{min}|}h_{top}(g_H);$$

(2) $h_{top}(g) = \sqrt{|K_{max}|}h_{top}(g_H)$ *if and only if the closed geodesics of* g *and* g_H *coincide, with* $g = \frac{g_H}{|K_{max}|}$ *and* $K_g = K_{max}$ *on* $\pi\Omega_g = \pi\Omega_{g_H}$ *;*

(3) $h_{top}(g) = \sqrt{|K_{min}|}h_{top}(g_H)$ *if and only if the closed geodesics of* g *and* g_H *coincide, with* $g = \frac{g_H}{|K_{min}|}$ *and* $K_g = R_{max}$ *on* $\pi\Omega_g = \pi\Omega_{g_H}$*.*

On convex-cocompact surfaces with infinite volume, the closed geodesics are contained in the compact *convex-core*, also called *Nielsen core* (cf. Figure 1). However, they are not necessarily dense in this convex set, see for instance the study of the Hausdorff dimension of the closure of the set of closed geodesics made by Ledrappier and Lindenstrauss in [LL03]. Theorem 1.6 shows that the convex-cocompact surfaces with extremal entropy are exactly those whose closed geodesics coincide with the closed geodesics of hyperbolic metrics, and the metrics itself coincide (up to scaling) on these geodesics. Combined with Yau's Schwarz Lemma, this can be reformulated as follows.

Theorem 1.7 (Conformal entropy-rigidity of hyperbolic convex-cocompact surfaces)**.** *Let* (S, g_H) *be a convex-cocompact hyperbolic surface. Let* $\mathcal{M}_{(-\infty,-1]}$ *be the set of complete metrics on* S *with Gauss curvature in* $(-A, -1]$ *for some* $A > 0$ *and* $\mathcal{M}_{[-1,0)}$ *be the set of complete metrics on* S *with Gauss curvature in* $[-1, -\epsilon]$ *for some* $\epsilon > 0$*.*

(1) g_H *minimizes the topological entropy on* $\mathcal{M}_{(-\infty,-1]}$*. Moreover, any other metric* $g_m \in \mathcal{M}_{(-\infty,-1]}$ *is a minimizer for the entropy* if and only if *it has the same set of closed geodesics as* g_H*, and on this set, the metrics coincide.*

(2) g_H *maximizes the topological entropy on* $\mathcal{M}_{[-1,0)}$*. Moreover, any other metric* $g_M \in \mathcal{M}_{[-1,0)}$ *is a maximizer for the entropy* if and only *if has the same set of closed geodesics as* g_H*, and on this set, the metrics coincide.*

This is the analogous result for infinite volume surfaces of the results by Hamenstädt and by Ledrappier and Wang mentioned above, restricted to a conformal class. As explained previously, due to classical results of Patterson, it is not possible to get a global entropy-rigidity result gathering all conformal classes.

Our line of attack is inspired by the paper of Manning [Man04], which shows that on compact surfaces with negative curvature, Volume-normalized Ricci-Yamabe flow decreases topological entropy. Nevertheless, the proof we develop is quite different as Manning's arguments do not generalize to infinite volume surfaces.

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2. Curvature-normalized Yamabe Flows on complete manifolds

Let (M,g) be a Riemannian n-manifold, $n \geq 3$ such that $R_{min} \leq R_g \leq R_{max} < 0$. It has been shown by Aviles and Mc Owen in [AMc88] that there exists a Yamabe metric $g_{max} = v_{max} g$ conformally equivalent to g with constant scalar curvature $R_{g_{max}} \equiv R_{max}$. It follows from Corollary 1.2 of [Yau73] that, as soon as the Riemannian curvature tensor is bounded, this Yamabe metric is unique. We now introduce our Curvature-normalized Yamabe flows, which will converge to the Yamabe metric when the curvature is bounded.

Definition 2.1. *Let (M,g) be a Riemannian manifold such that $R_{min} \leq R_g \leq R_{max} < 0$. We will call* ***Curvature-normalized increasing Yamabe Flow*** *(in short, CYF^+) a family of Riemannian metrics $(g_t)_{t\geq 0}$ on M satisfying the following equations:*

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= (R_{max} - R_{g_t}) g_t \\ g_0 &= g \end{aligned}$$

Let us first show the existence of a global solution to this Yamabe flow.

Theorem 2.1. *Let (M,g) be a Riemannian manifold such that $R_{min} \leq R_g \leq R_{max} < 0$. Then there exists a solution $g_t = v_t g$ to the CYF^+, with $v_0 = 1$, defined for all time $t \geq 0$. Moreover, it satisfies for all $t \geq 0$,*

$$R_{min} \leq R_{g_t} \leq R_{max} \quad \text{and} \quad v_t \leq \left| \frac{R_{min}}{R_{max}} \right|.$$

Proof. Let (M,g) be a Riemannian manifold such that $R_{min} \leq R_g \leq_{max} < 0$. Let $(K_\alpha)_{\alpha\in\mathbb{N}}$ be an increasing family of compact subset of M, with smooth boundary, such that $M = \cup_{\alpha\in\mathbb{N}} K_\alpha$. We assume that whenever $\alpha < \beta$, then $K_\alpha \subset \overset{\circ}{K}_\beta$. For all $\alpha \in \mathbb{N}$, we set $U_\alpha := \overset{\circ}{K}_\alpha$. We will first find a solution for this Curvature-normalized Yamabe flow on compact sets with fixed boundary conditions. Let $\alpha \in \mathbb{N}$ be fixed for a while.

Lemma 2.2. *There exists $\epsilon_\alpha > 0$ and a unique family of metrics $(g_{\alpha,t})_{t\in[0,\epsilon_\alpha]}$ on M satisfying the following conditions: for all $t \in [0,\epsilon_\alpha]$,*

$$\begin{aligned} g_{\alpha,t} &= g && \text{on } M\backslash U_\alpha, \\ \frac{\partial g_{\alpha,t}}{\partial t} &= (R_{max} - R_{g_t}) g_{\alpha,t} && \text{on } U_\alpha, \\ g_{\alpha,0} &= g && \text{on } M. \end{aligned}$$

We will call this $(g_{\alpha,t})_{t\in[0,\epsilon_\alpha]}$ the CYF^+ on K_α with fixed boundary.

Proof. It has been shown by Yamabe in [Yam60] that, writing $g_{\alpha,t} = u_{\alpha,t}^{\frac{4}{n-2}} g$, we have

$$R_{g_{\alpha,t}} = u_{\alpha,t}^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \Delta_g u_t + R_g u_{\alpha,t} \right). \tag{1}$$

Therefore $\frac{\partial g_{\alpha,t}}{\partial t} = (R_{max} - R_{g_{\alpha,t}}) g_t$ is equivalent to the following equation:

$$\frac{\partial}{\partial t}\left(u_{\alpha,t}^N\right) = \frac{n+2}{4}\left(R_{max} u_{\alpha,t}^N - R_g u_{\alpha,t}\right) - \frac{(n+2)(n-1)}{n-2} \Delta_g u_{\alpha,t}, \tag{2}$$

where $N = \frac{n+2}{n-2} > 1$. This is a strictly parabolic equation as long as $u_{\alpha,t}$ stays bounded. Since we solve it on an open set with compact closure, with boundary condition $u_{\alpha,t} = 1$ on ∂U_α, it has a unique solution on $t \in [0,\epsilon_\alpha]$, for some $\epsilon_\alpha > 0$. □

It follows from Lemma 9 in [SST11] that for all $t \in [0, \epsilon_\alpha)$ and all $x \in U_\alpha$, the scalar curvature of $g_{\alpha,t}$ satisfies:

$$\frac{\partial R_{g_{\alpha,t}}}{\partial t} = -(n-1)\Delta_{g_{\alpha,t}} R_{g_{\alpha,t}} + R_{g_{\alpha,t}}(R_{g_{\alpha,t}} - R_{max}). \tag{3}$$

Using the Maximum Principle given in Proposition A.1, this implies that for all time $t \in [0, \epsilon_\alpha)$, we have

$$R_{min} \leq R_{g_{\alpha,t}} \leq R_{max}.$$

Now, it follows from Lemma 12 in [SST11] (using the relevant form of the Maximum Principle) that $u_{\alpha,t}$ is well defined and satisfies for all $t \in (0, \epsilon_\alpha)$,

$$1 \leq u_{\alpha,t}^{\frac{4}{n-2}} \leq \left| \frac{R_{min}}{R_{max}} \right|. \tag{4}$$

This implies the global existence of the solution $g_{\alpha,t}$ for all $t \geq 0$.

If $\alpha < \beta \in \mathbb{N}$, the Maximum principle given in Proposition A.2 implies that for all $t \geq 0$ and at all $x \in M$, we have $u_{\alpha,t} \leq u_{\beta,t}$. Let us define

$$u_t := \sup_{\alpha \in \mathbb{N}} u_{\alpha,t} \quad \text{and} \quad g_t := u_t^{\frac{4}{n-2}} g.$$

Lemma 2.3. *For all $k \in \mathbb{N}$, all $t \in [0, \infty)$ and every compact set $K \subset M$, the functions $(u_{\alpha,t})$ converge when $\alpha \to \infty$ to u_t uniformly in the $\mathcal{C}^k$ topology. Similarly, $\frac{\partial u_{\alpha,t}}{\partial t}$ converges to $\frac{\partial u_t}{\partial t}$. In particular, u_t is smooth, the metric $g_t = u_t^{\frac{4}{n-2}} g$ is a smooth solution to CYF$^+$ defined for all $t \geq 0$, and it satisfies*

$$R_{min} \leq R_{g_t} \leq R_{max}.$$

Proof. Let $\alpha < \beta \in \mathbb{N}$, and let $(g_{\alpha,t} = u_{\alpha,t}^{\frac{4}{n-2}})_{t \geq 0}$ and $(g_{\beta,t} = u_{\beta,t}^{\frac{4}{n-2}})_{t \geq 0}$ be the solutions of the CYF$^+$ with fixed boundary conditions on K_α and K_β respectively.

It follows from equation (2) that, for all $t \geq 0$:

$$\frac{\partial}{\partial t}\left(u_{\beta,t}^N - u_{\alpha,t}^N\right) = \frac{n+2}{4}\left(R_{max}(u_{\beta,t}^N - u_{\alpha,t}^N) - R_g(u_{\beta,t} - u_{\alpha,t})\right) - \frac{(n+2)(n-1)}{n-2}\Delta_g(u_{\beta,t} - u_{\alpha,t}). \tag{5}$$

Moreover, let us define

$$v_{\alpha,\beta,t} = \frac{u_{\beta,t}^N - u_{\alpha,t}^N}{u_{\beta,t} - u_{\alpha,t}}$$

whenever $u_{\alpha,t} \neq u_{\beta,t}$ and

$$v_{\alpha,\beta,t} = N u_{\alpha,t}^{N-1}$$

whenever $u_{\alpha,t} = u_{\beta,t}$. The map $v_{\alpha,\beta,t}$ is smooth on $M \times (0, \infty)$ and for all $(x,t) \in M \times \infty$, we have

$$N \leq v_{\alpha,\beta,t} \leq N \left| \frac{R_{min}}{R_{max}} \right|^{\frac{n+2}{4}-1}, \tag{6}$$

because we have seen in (4) that

$$1 \leq u_{\alpha,t} \leq u_{\beta,t} \leq \left| \frac{R_{min}}{R_{max}} \right|^{\frac{n-2}{4}}.$$

Since $u_\beta^N - u_\alpha^N = (u_{\beta,t} - u_{\alpha,t})v_{\alpha,\beta,t}$, equation (5) becomes

$$\begin{aligned} v_{\alpha,\beta,t}\frac{\partial}{\partial t}(u_{\beta,t} - u_{\alpha,t}) = & \tfrac{n+2}{4}\left(R_{max}(u_{\beta,t}^N - u_{\alpha,t}^N) - R_g(u_{\beta,t} - u_{\alpha,t})\right) \\ & - \tfrac{(n+2)(n-1)}{n-2}\Delta_g(u_{\beta,t} - u_{\alpha,t}) - (u_{\beta,t} - u_{\alpha,t})\,\tfrac{\partial v_{\alpha,\beta,t}}{\partial t}, \end{aligned} \tag{7}$$

since we have

$$(u_{\beta,t} - u_{\alpha,t}) \frac{\partial v_{\alpha,\beta,t}}{\partial t} = \frac{\partial}{\partial t}\left(u^N_{\beta,t} - u^N_{\alpha,t}\right) - \frac{\partial}{\partial t}\left(u_{\beta,t} - u_{\alpha,t}\right) v_{\alpha,\beta,t}.$$

Therefore, just as was done in [SST11], because of the bounds given by equation (6), Schauder estimates applied to equation (2) imply that for any compact set $K \subset M$, and any $r > 0$, there exists a constant $B_{K,r}$ such that for all $t \in M$ and $x \in K$,

$$\left|(u_{\beta,t} - u_{\alpha,t}) \frac{\partial v_{\alpha,\beta,t}}{\partial t}\right|(x) \leq B_{K,r} \sup_{t-r \leq s \leq t} \left(|u_{\beta,s}| + |u_{\alpha,s}|\right) \leq B'_K.$$

Note that this B'_K only depends on the initial metric g and the compact set K, and not on $t \in (0,\infty)$ nor on $\alpha, \beta \in \mathbb{N}$. This implies that equation (7) is a parabolic equation for $u_{\beta,t} - u_{\alpha,t}$, uniformly parabolic on any compact set. The constant of parabolicity of this equation is independent of the values of α and β. Therefore, Schauder estimates for equation (7) imply that for all $k \in \mathbb{N}, r > 0$, and all compact sets $K \subset M$, there exists $C_{K,k,r} > 0$ such that for all $\alpha, \beta \in \mathbb{N}$ and for all $(x,t) \in M \times [0,\infty)$,

$$\sup_{x \in K} \left\|\nabla^k u_{\alpha+\beta,t} - \nabla^k u_{\alpha,t}\right\|_K (x) \leq C_{K,k} \sup_{t-r \leq s \leq t} \sup_{x \in K} \left|u_{\alpha+\beta,s} - u_{\alpha,s}\right|(x).$$

Now, we have seen that $u_{\alpha,t}$ converges increasingly when $\alpha \to \infty$ to a positive limit u_t. This implies that for all $(x,t) \in M \times [0,\infty)$,

$$\left\|\nabla^k u_{\alpha+\beta,t} - \nabla^k u_{\alpha,t}\right\|_K (x) \leq C'_{K,k} \sup_{t-r \leq s \leq t} |u_s - u_{\alpha,s}|$$

for some $C'_{K,k} > 0$. Therefore, $(\nabla^k u_{\alpha,t})_{\alpha \in \mathbb{N}}$ is a Cauchy sequence and hence $(u_{\alpha,t})_{\alpha \in \mathbb{N}}$ converges in the $\mathcal{C}^k$ topology. Then the map $u_t : M \to \mathbb{R}$ is $\mathcal{C}^k$. Since this is valid for all $k \geq 0$, equation (2) implies that $\frac{\partial u_{\alpha,t}}{\partial t}$ also converges smoothly to $\frac{\partial u_t}{\partial t}$ and that u_t is a solution to the CYF$^+$ on M. ☐

This concludes the proof of Theorem 2.1. ☐

As $R_{g_t} \leq R_{max}$, and by definition of the CYF$^+$ we have $\frac{\partial g_t}{\partial t} = (R_{max} - R_{g_t}) g_t$, we obtain the following.

Corollary 2.4. *The Curvature-normalized increasing Yamabe flow is pointwise increasing:*

$$\forall\, x \in M, \quad \forall\, t \geq s \geq 0, \quad g_t \geq g_s.$$

Provided that the Riemannian curvature tensor of the initial metric is uniformly bounded, the CYF$^+$ converges to a smooth metric:

Theorem 2.5. *Let (M,g) be a complete Riemannian manifold, such that $R_{min} \leq R_g \leq R_{max}$. We assume moreover that the norm of the Riemannian curvature tensor is uniformly bounded. Let $(g_t)_{t \geq 0}$ be the solution of the CYF$^+$ constructed in Theorem 2.1. Then on any compact set $K \subset M$ and for all $k \in \mathbb{N}$, the metric g_t converges uniformly exponentially fast in the $\mathcal{C}^k$ topology to the unique metric g_{max} in the conformal class of g with constant scalar curvature $R_{g_{max}} = R_{max}$.*

Proof. Let (M,g) be a complete Riemannian manifold, such that $R_{min} \leq R_g \leq R_{max}$ with uniformly bounded Riemannian curvature tensor. We have seen that there exists a unique metric g_{max} conformally equivalent to g with $R_{g_{max}} = R_{max}$.

Lemma 2.6. *The scalar curvature R_{g_t} of g_t converges uniformly exponentially fast on M to $R_{g_{max}} \leq R_{max}$ when $t \to \infty$.*

Proof. From Theorem (2.1) we know that $R_{g_t} \leq R_{max}$ for all $t \geq 0$. Therefore, the evolution equation of the scalar curvature gives for all $(x,t) \in M \times [0,\infty)$,

$$\begin{aligned} \frac{\partial(R_{g_t} - R_{max}(g_t))}{\partial t} &= -(n-1)\Delta_{g_t}(R_{g_t} - R_{max}) + R_{g_t}(R_{g_t} - R_{max}) \\ &\geq -(n-1)\Delta_{g_t}(R_{g_t} - R_{max}) + R_{max}(R_{g_t} - R_{max}). \end{aligned}$$

Using the variable form of the Maximum Principle (Proposition A.3) given in the Appendix, we get for all $(x,t) \in M \times [0,\infty)$:

$$0 \geq R_{g_t} - R_{max} \geq (R_{min} - R_{max})e^{R_{max}t} \tag{8}$$

Since $R_{max} < 0$, the curvature converges exponentially fast to R_{max} at all points of M. □

Now, let $K \subset M$ be a fixed compact set. Repeating the proof of Theorem 10 of [SST11], Schauder estimates show that on the compact set K, the flow converges uniformly exponentially fast to a smooth metric g_{max} with constant scalar curvature $R_{g_{max}} = R_{max}$. More precisely, for all integers $k \geq 0$, there exists a positive $C_{K,k} > 0$ such that

$$\|g_t - g_{max}\|_{\mathcal{C}^k(K)} \leq C_{K,k}e^{R_{max}t}.$$

Since this is valid for any compact set K, it concludes the proof of Theorem 2.5. □

Remark 2. In Theorem 1.1, we assume that the Riemannian curvature tensor is uniformly bounded to get the convergence of the flow $(g_t)_{t\geq 0}$. This hypothesis is not optimal since we only need a global maximum principle as stated in Proposition A.3. Some exponential lower bounds on Ric_{g_0} may be sufficient to get it. We did not focus on this technical issue to limit the size of our exposition.

Let (M,g) be a complete Riemannian manifold satisfying $R_{min} \leq R_g \leq R_{max} < 0$. We will call ***Curvature-normalized decreasing Yamabe Flow*** (in short, CYF$^-$) a family of Riemannian metrics $(g_t^-)_{t\geq 0}$ on M satisfying the following equations:

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= (R_{min} - R_{g_t})g_t \\ g_0 &= g \end{aligned}$$

A proof similar to the one we have just presented above gives the following result.

Theorem 2.7. *Let (M,g) be a Riemannian manifold such that $R_{min} \leq R_g \leq R_{max} < 0$. Then there exists a solution $g_t^- = v_t g$ to the CYF$^-$, with $v_0 = 1$, defined for all time $t \geq 0$. Moreover, for all $t \geq 0$, it satisfies*

$$R_{min} \leq R_{g_t^-} \leq R_{max} \text{ and } v_t \leq \left|\frac{R_{min}}{R_{max}}\right|,$$

and $t \mapsto g_t^-(x)$ is decreasing in time for all $x \in M$.

When the Riemannian curvature tensor of g is uniformly bounded, then g_t^- converges uniformly exponentially fast on every compact set to the unique metric g_{min} in the conformal class of g which satisfies $R_{g_{min}} = R_{min}$.

Theorems 2.5 and 2.7 imply the following result, originally due to Yau.

Corollary 2.8 (Conformal Schwarz Lemma)**.** *Let (M,g) be a Riemannian manifold with $R_{min} \leq R_g \leq R_{max} < 0$ and with bounded Riemann curvature tensor. Then there exists a Yamabe metric g_Y conformally equivalent to g, with uniformly bounded conformal factor, such that*

$$g_{min} = \frac{g_Y}{|R_{min}|} \leq g \leq g_{max} = \frac{g_Y}{|R_{max}|}.$$

The proof presented above is independent of the arguments in [Yau73], from which we had only deduced uniqueness of the Yamabe metric.

We now turn to the proof of Theorem 1.3, showing that the CYF preserve pinched negative sectional curvatures for a small time, provided the gradient of the scalar curvature of the initial metric and its Laplacian are bounded. We are going to prove the more precise following result.

Theorem 2.9. *Let (M, g) be a complete Riemannian n-manifold, $n \geq 3$, with pinched negative sectional curvatures K_g:*

$$\frac{R_{min}}{n(n-1)} = -b^2 \leq K_g \leq -a^2 = \frac{R_{max}}{n(n-1)} < 0.$$

Assume moreover that $\|\nabla_g R_g\|$ and $|\Delta_g R_g|$ are uniformly bounded. Let $(g_t^+)_{t\geq 0}$ (resp. $(g_t^-)_{t\geq 0}$) be the solution of the CYF$^+$ (resp. the CYF$^-$) with initial metric g. Then there exist $C > 0$ and $\beta \in (0, \frac{1}{2})$ such that forall $t \in [0, \epsilon]$, we get

$$\left| K_{g_t^+} - K_g \right| \leq Ct^\beta \quad \text{and} \quad \left| K_{g_t^-} - K_g \right| \leq Ct^\beta.$$

Theorem 2.9 can be restated as follows: starting at a metric satisfying the hypotheses of Theorem 2.9, the sectional curvatures of g_t^+ and g_t^- are uniformly continuous in time around $t = 0$, and their modulus of uniform continuity is at most Ct^β for some $\beta \in (0, \frac{1}{2})$ and some constant $C > 0$.

Proof of Theorem 2.9. We will show this theorem for the CYF$^+$, the proof for the CYF$^-$ is similar. Let (M, g) be a smooth Riemannian n-manifold, $n \geq 3$, pinched negative sectional curvatures $-b^2 \leq K_g \leq -a^2 < 0$, and such that $\|\nabla_g R_g\|_g$ and $\Delta_g R_g$ are uniformly bounded. We set $R_{min} = -n(n-1)b^2$, $R_{max} = -n(n-1)a^2$ and denote by $(g_t)_{t\geq 0}$ the solution of the CYF$^+$ given by Theorem 2.1. Therefore, we have $g_t = u_t^{\frac{4}{n-2}} g$, here u_t is a uniformly bounded solution to equation (2) with initial condition $u_0 = 1$.

Consider the lifts of the metrics $(g_t)_{t\geq 0}$ to the universal cover $\tilde{M}$ of M. We will still denote them by $(g_t)_{t\geq 0}$. This family of metrics on $\tilde{M}$ is by construction equivariant with respect to the action of the fundamental group of M. We also lift the family $(u_t)_{t\geq 0}$ of conformal factors to an equivariant family of conformal factors on $\tilde{M}$, which we will still write $(u_t)_{t\geq 0}$. Since $(\tilde{M}, g_0) = (\tilde{M}, g)$ is a simply connected manifold with negative sectional curvatures, it has infinite injectivity radius. For the rest of the proof, our argument will implicitly take place in $\tilde{M}$.

The curvatures of g_t can be obtained from the curvature of g and the derivatives of u_t, see Theorem 1.159 p. 58 of [Bes08]. Therefore, it is enough to show that there exists $C > 0$ and $\beta \in (0, \frac{1}{2})$ such that for all $t \geq 0$,

$$\|u_t(.) - 1\|_{\mathcal{C}^2(M,g)} := \sup_M |u_t - 1| + \sup_M \|\nabla^g u_t\|_g + \sup_M \|\nabla^g \nabla^g u_t\|_g \leq Ct^\beta. \tag{9}$$

Since we only deal with maps on $\tilde{M}$ which are lifts from maps on M, there is no difference in taking these norms on M or on $\tilde{M}$.

From equation (1), we have for all $t > 0$,

$$\frac{4(n-1)}{n-2} \Delta_g u_t = R_{g_t} u_t^{\frac{n+2}{n-2}} - R_g u_t. \tag{10}$$

The sectional curvatures of g are pinched and its injectivity radius is positive (here infinite), so it follows from the work of Anderson and Cheeger [AC92] that in each point $x \in \tilde{M}$, there exists a chart of harmonic coordinates (for the *fixed* metric g) centered in x with radius $\rho > 0$ depending only on R_{min} and R_{max}. Therefore, equation (10) is uniformly elliptic on $\tilde{M}$, with ellipticity constants depending only on R_{min} and R_{max}.

We will need the following Hölder norms on functions on M (extended to equivariant maps on $\tilde{M}$). For any smooth map $f : M \to \mathbb{R}$ and any $\alpha \in (0,1)$, we write

$$\|f\|_{\mathcal{C}^\alpha} := \sup_{x\in M} |f(x)| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{d_g(x,y)^\alpha}$$

where $d_g(x,y)$ is the Riemannian distance induced by g between x and y, and

$$\|f\|_{\mathcal{C}^{2,\alpha}} := \sup_{x\in M} |f(x)| + \sup_{x\in M} \|df(x)\|_g + \left\| \|\nabla^g df\|_g \right\|_{\mathcal{C}^\alpha}.$$

By Schauder estimates applied to Equation (10), see e.g. Theorem 6.2 p. 90 of [GT01], for any $\alpha \in (0,1)$, there exists a constant $C > 0$ depending only on n, α and the curvature bounds such that

$$\|u_t - 1\|_{\mathcal{C}^{2,\alpha}} \leq C \left(\sup_{x\in M} |u_t - 1| + \left\| R_{g_t} u_t^{\frac{n+2}{n-2}} - R_g u_t \right\|_{\mathcal{C}^\alpha} \right). \tag{11}$$

To estimate the right-hand side of (11), we will use the following lemma which is an immediate consequence of the variable form of Maximum Principle given in Proposition A.3.

Lemma 2.10. *Let $v : M \times [0,T) \to \mathbb{R}$ be a smooth solution to the following PDE:*

$$\frac{\partial v}{\partial t}(x,t) = -h(x,t)\Delta_g v(x,t) + f(x,t),$$

where Δ_g is the Laplacian induced by the metric g, and the functions f and h satisfy the following: there exist $C_1, C_2, C_3 > 0$ such that $C_1 \leq h \leq C_2$ and $|f| \leq C_3$. Assume that at $t = 0$, $v(.,0) \equiv 0$ and that v is uniformly bounded on $M \times (0,T)$. Then for all $x, t \in M \times [0,T)$, we have

$$\sup_{x\in M} |v(x,t) - v(x,0)| \leq C_3 t.$$

Let us go back to the study of the conformal factor u_t. It follows from the bounds established in Theorem 2.1 that equation (2) which gives the evolution of u_t is precisely of the form

$$\frac{\partial u_t}{\partial t}(x) = -h(x,t)\Delta u_t + f(x,t), \tag{12}$$

where h and f satisfy the hypotheses of Lemma 2.10. Therefore, there exists $C > 0$ such that for all $t > 0$ small enough,

$$|u_t(x) - 1| \leq Ct. \tag{13}$$

Therefore, because of equation (11), to conclude the proof of Theorem 2.9 it is enough to control

$$\left\| R_{g_t} u_t^{\frac{n+2}{n-2}} - R_g u_t \right\|_{\mathcal{C}^\alpha}$$

for some $\alpha > 0$. We have

$$R_{g_t} u_t^{\frac{n+2}{n-2}} - R_g u_t = u_t^{\frac{4}{n-2}} (R_{g_t} u_t - R_g) + R_g (u_t^{\frac{4}{n-2}} - u_t),$$

which can be restated as

$$R_{g_t} u_t^{\frac{n+2}{n-2}} - R_g u_t = u_t^{\frac{4}{n-2}} (R_{g_t} u_t - R_g) + R_g u_t^{\frac{4}{n-2}} (1 - u_t) \frac{u_t^{1-\frac{4}{n-2}} - 1}{u_t - 1}.$$

The bounds established in Theorem 2.1 together with the previous equation imply that there exists a constant $C_R > 0$ such that for all $\alpha \in (0,1)$, we get

$$\left\| R_{g_t} u_t^{\frac{n+2}{n-2}} - R_g u_t \right\|_{\mathcal{C}^\alpha} \leq C_R \left(\|R_{g_t} u_t - R_g\|_{\mathcal{C}^\alpha} + \|u_t - 1\|_{\mathcal{C}^\alpha} \right). \tag{14}$$

Equation (13) controls the $\mathcal{C}^0$ norm of $u_t - 1$. To control its $\mathcal{C}^\alpha$ norm, we will use the following lemma.

Lemma 2.11. *Let $v : \tilde{M} \times [0,T) \to \mathbb{R}$ be a smooth solution to the following PDE:*

$$\frac{\partial v}{\partial t}(x,t) = -h(x,t)\Delta_g v(x,t) + f(x,t),$$

where Δ_g is the Laplacian induced by the metric g, and the functions f and h satisfy the following: there exist $C_1, C_2, C_3 > 0$ such that $C_1 \leq h \leq C_2$ and $|f| \leq C_3$. Assume moreover that at $t = 0$, $|v(.,0)|$ and $\|\nabla v(.,0)\|_g$ are bounded by C_4, and that for all $t > 0$, $|v(.,t)| \leq C_5$. Then there exists $\theta \in (\frac{1}{2}, 1)$ and $C_\theta > 0$ depending on the C_i such that for all $(x,t) \in \tilde{M} \times (0,T)$, we have

$$\|\nabla v(x,t)\|_g \leq \frac{C_\theta}{t^\theta}.$$

This is merely an adaptation to our context of Theorem 11.3 p. 269 of [Lie96].

We have already seen that $v(x,t) = u_t(x) - 1$ satisfies the hypothesis of Lemma 2.11. Therefore, we have for all $(x,t) \in \tilde{M} \times (0,T)$,

$$\|\nabla u_t(x)\|_g \leq \frac{C_\theta}{t^\theta}$$

for some fixed $\theta \in (\frac{1}{2}, 1)$ and $C_\theta > 0$. Moreover, for all $\alpha \in (0,1)$, all $t > 0$ and all $x, y \in \tilde{M}$ such that $x \neq y$, we have

$$\frac{|u_t(x) - u_t(y)|}{d_g(x,y)^\alpha} = \frac{|u_t(x) - u_t(y)|}{d_g(x,y)} d_g(x,y)^{1-\alpha} \leq \|\nabla u_t\|_g d_g(x,y)^{1-\alpha}.$$

Let us now fix $\alpha \in (0, \frac{1}{2})$ small enough so that $1 - \alpha > \theta$. Due to Lemma 2.10, we have for all $t > 0$,

$$\begin{aligned}
\|u_t - 1\|_{\mathcal{C}^\alpha} &= \sup_{x \in M} |u_t(x) - 1| + \sup_{x \neq y} \frac{|u_t(x) - u_t(y)|}{d_g(x,y)^\alpha} \\
&\leq Ct + \sup_{x \neq y, d_g(x,y) \leq t} \frac{|u_t(x) - u_t(y)|}{d_g(x,y)^\alpha} + \sup_{x \neq y, d_g(x,y) > t} \frac{|u_t(x) - u_t(y)|}{d_g(x,y)^\alpha} \\
&\leq Ct + \sup_{d_g(x,y) \leq t} \|\nabla u_t\|_g d_g(x,y)^{1-\alpha} + 2\frac{sup_{x \in M} |u_t(x) - 1|}{t^\alpha} \\
&\leq Ct + C_\theta t^{1-\alpha-\theta} + 2Ct^{1-\alpha}.
\end{aligned}$$

Eventually, for some $C' > 0$, we have for all $t > 0$,

$$\|u_t - 1\|_{\mathcal{C}^\alpha} \leq C' t^{\beta_1}, \tag{15}$$

where

$$\beta_1 = 1 - \alpha - \theta.$$

We will now show an analogous estimate for $\|R_{g_t} u_t - R_g\|_{\mathcal{C}^\alpha}$. From equation (3), we get that for all $t \geq 0$,

$$\frac{\partial R_{g_t}}{\partial t} = -(n-1)\Delta_{g_t} R_{g_t} + R_{g_t}(R_{g_t} - R_{max}). \tag{16}$$

Moreover, it follows from equation (1) that for every smooth map $f : M \to \mathbb{R}$ and all $t \geq 0$, we have

$$\frac{4(n-1)}{n-2}\Delta_{g_t} f + R_{g_t} f = u_t^{-\frac{n+2}{4}} \left(\frac{4(n-1)}{n-2}\Delta_g(u_t f) + R_g u_t f \right). \tag{17}$$

Due to this invariance, the operator $\Delta_g + \frac{n-2}{4(n-1)} R_g$ is known as the *conformal Laplacian.* Equations (16) and (17) imply that for all $t \geq 0$,

$$\begin{aligned} \frac{\partial R_{g_t}}{\partial t} &= -\frac{n-2}{4}\left(\frac{4(n-1)}{n-2}\Delta_{g_t}R_{g_t} + R_{g_t}^2\right) + \frac{n-2}{4}R_{g_t}^2 + R_{g_t}(R_{g_t} - R_{max}) \\ &= -\frac{n-2}{4}u_t^{-\frac{n+2}{4}}\left(\frac{4(n-1)}{n-2}\Delta_g(u_tR_{g_t}) + R_g u_t R_{g_t}\right) + \frac{n-2}{4}R_{g_t}^2 + R_{g_t}(R_{g_t} - R_{max}), \end{aligned}$$

which yields

$$\begin{aligned} \frac{\partial(u_tR_{g_t})}{\partial t} - R_{g_t}\frac{\partial u_t}{\partial t} &= -\frac{n-2}{4}u_t^{1-\frac{n+2}{4}}\left(\frac{4(n-1)}{n-2}\Delta_g(u_tR_{g_t}) + R_{g_0}u_tR_{g_t}\right) \\ &\quad + u_t\left(\frac{n-2}{4}R_{g_t}^2 + R_{g_t}(R_{g_t} - R_{max})\right). \end{aligned} \tag{18}$$

Moreover, by definition of CYF$^+$ and $g_t = u_t^{\frac{4}{n-2}} g_0$, we have

$$\frac{\partial u_t}{\partial t} = \frac{\partial\left(u_t^{\frac{4}{n-2}}\right)}{\partial t}u_t^{1-\frac{4}{n-2}} = (R_{max} - R_{g_t})u_t^{\frac{4}{n-2}}u_t^{1-\frac{4}{n-2}} = (R_{max} - R_{g_t})u_t.$$

Therefore, equation (18) becomes

$$\begin{aligned} \frac{\partial(u_tR_{g_t})}{\partial t} &= -\frac{n-2}{4}u_t^{1-\frac{n+2}{4}}\left(\frac{4(n-1)}{n-2}\Delta(u_tR_{g_t}) + R_g u_t R_{g_t}\right) \\ &\quad + u_t\left(\frac{n-2}{4}R_{g_t}^2 + R_{g_t}(R_{g_t} - R_{max})\right) + R_{g_t}(R_{max} - R_{g_t})u_t, \end{aligned}$$

where, as before, we have written $\Delta = \Delta_g$.

Since $|\Delta_g R_g|$ is uniformly bounded, this can be restated as the following :

$$\frac{\partial(u_tR_{g_t} - R_g)}{\partial t} = -h(x,t)\Delta(u_tR_{g_t} - R_g) + f(x,t), \tag{19}$$

where h and f satisfy the hypotheses of Lemmas 2.10 and 2.11. Therefore, repeating the proof of the estimate (15) for $u_t - 1$, there exists $C'' > 0$, and $\alpha, \beta_2 \in (0, \frac{1}{2})$ such that for all $t > 0$,

$$\|R_{g_t}u_t - R_g\|_{\mathcal{C}^\alpha} \leq C'' t^{\beta_2}. \tag{20}$$

The bounds (15) and (20) together with the estimates (14) and (11) conclude the proof of Theorem 2.9.

□

Remark 3. G. Carron has pointed out to us that, adapting the techniques of Grigor'yan in [Gri94] to our context, it might be possible to suppress the hypothesis that $|\Delta_g R_g|$ is uniformly bounded from Theorem 2.9 (and its application in Theorem 1.4).

3. Extremal entropy and Yamabe flows on convex-cocompact manifolds

Let us first recall some basic facts about the topological entropy and convex-cocompact manifolds. Let (M, g) be a complete Riemannian n-manifold, $n \geq 2$, let $T_g^1 M$ its unit tangent bundle, and $(\phi_t^g)_{t\in\mathbb{R}}$ its geodesic flow. We denote by $\pi : TM \to M$ the canonical projection. For any $T > 0$ and $v, w \in T_g^1 M$, we write

$$d_T^g(v,w) := \sup_{t\in[0,T]} d\left(\phi_t^g(v), \phi_t^g(w)\right).$$

If $\Omega \subset T^1M$ is a *compact* subset invariant by the flow (ϕ_t^g), let $\mathcal{N}(\Omega, \epsilon, T)$ be the cardinality of a maximal ϵ-separated set in Ω for the distance d_T^g.

Definition 3.1. *The topological entropy of the geodesic flow restricted to Ω is defined by*

$$h_{top}(\phi^g|_\Omega) := \limsup_{\epsilon \to 0} \lim_{T \to \infty} \frac{\log \mathcal{N}(\Omega, \epsilon, T)}{T}.$$

In this section, we will focus on the following case.

Definition 3.2. *We will say that a complete Riemannian manifold with strictly negative sectional curvatures is* convex-cocompact *if all its closed geodesics are contained in a compact set.*

This notion is also known in the case of pinched negative sectional curvature as *geometrically finite manifolds without cusps*, cf [Bow95] for more details on this terminology and other equivalent definitions.

Let (M, g) be a convex-cocompact manifold with negative sectional curvatures. We will identify each closed geodesic in M with the unique closed orbit for the geodesic flow in T_g^1M to which it corresponds. Let $\Omega_g \subset T_g^1M$ be the closure of the set of closed geodesics, which is invariant by the geodesic flow. Since (M, g) is convex-cocompact, Ω_g is compact. It is called the ***non-wandering set*** of the geodesic flow, see [Eb73] for a justification of this terminology. We will call the topological entropy of the geodesic flow restricted to Ω_g the ***topological entropy of*** (M, g).

Let (M, g) be a complete manifold with uniformly pinched negative sectional curvatures. It is well known that the geodesic flow on T_g^1M is uniformly hyperbolic, as shown for instance in the classical book of Klingenberg [Kl82], Chap. 9. It was shown by Eberlein in [Eb73] that there is an orbit for the geodesic flow which is dense in the non-wandering set Ω_g. Assume that (M, g) is convex-cocompact: together with the previous facts, it implies that Ω_g is a *basic hyperbolic set* in the sense of [Bow72]. Let us write for all $R > 0$,

$$\mathcal{P}_g(R) := \{\gamma \text{ closed geodesics for } g : \ell_g(\gamma) \leq R\},$$

where $\ell_g(\gamma)$ denotes the length of γ with respect to the metric g. It follows from Theorem 4.11 of [Bow72] that the topological entropy of the geodesic flow restricted to Ω_g is determined by the counting functions $\#\mathcal{P}_g(R)$ on the set of closed geodesics:

$$h_{top}(g) = \lim_{R \to \infty} \frac{\log(\#\mathcal{P}_g(R))}{R}. \tag{21}$$

Proposition 3.1. *Let M be a complete manifold of dimension $n \geq 2$, and g_1, g_2 be two Riemannian metrics on M with pinched negative sectional curvatures such that (M, g_1) and (M, g_2) are convex-cocompact.*

(1) *If the closures of the sets of all closed geodesics $\pi\Omega_{g_1}$ and $\pi\Omega_{g_2}$ coincide, and if*

$$g_1|_{\pi\Omega_{g_1}} = g_2|_{\pi\Omega_{g_2}} \text{ then } h_{top}(g_1) = h_{top}(g_2).$$

(2) *If $g_1 \leq g_2$ at all points, then*

$$h_{top}(g_1) \geq h_{top}(g_2).$$

Proof. Let M be a complete manifold, and g_1, g_2 be two Riemannian metrics on M with pinched negative sectional curvatures such that (M, g_1) and (M, g_2) are convex-cocompact. Let us prove item (1). Assume that $\pi\Omega_{g_1}$ and $\pi\Omega_{g_2}$ coincide and $g_1|_{\pi\Omega_{g_1}} = g_2|_{\pi\Omega_{g_1}}$. Let γ_1 be a closed geodesic for g_1, and γ_2 be the unique closed geodesic of g_2 in the same free homotopy class as γ_1. Assume that γ_1 and γ_2 are disjoint. By uniqueness of the geodesic, we have

$$\ell_{g_1}(\gamma_1) < \ell_{g_1}(\gamma_2) \quad \text{and} \quad \ell_{g_2}(\gamma_2) < \ell_{g_2}(\gamma_1).$$

This is a contradiction since g_1 and g_2 coincide on their non-wandering set. Therefore, the closed geodesics of g_1 and g_2 coincide and have same length, which implies (1) due to equation (21).

Let us now prove (2), assuming that $g_1 \leq g_2$. It follows from Theorem 3.9.5 of [Kl82] that, in each free homotopy class of M, there is exactly one closed geodesic. Let γ_1 be a closed geodesic for g_1. There exists a unique closed geodesic γ_2 for g_2 in the same free homotopy class. Moreover, since the geodesic minimizes the length in this homotopy class, and since $g_1 \leq g_2$, we have

$$\ell_{g_1}(\gamma_1) \leq \ell_{g_1}(\gamma_2) \leq \ell_{g_2}(\gamma_2).$$

Each closed geodesic γ_1 for g_1 will correspond to a different free homotopy class, therefore to a different geodesic γ_2 for g_2. Therefore, for all $R > 0$,

$$\#\mathcal{P}_{g_1}(R) \geq \#\mathcal{P}_{g_2}(R),$$

which implies item (2). □

Remark 4. Let us point out that, in general, when the sectional curvatures are not negative, the inequality $g_1 \leq g_2$ *does not imply* that $h_{top}(g_1) \geq h_{top}(g_2)$, even for compact manifolds. Indeed, it was shown by G. Contreras and G. Paternain in [CP02] that on $\mathbb{S}^2$, a smooth metric generically has positive topological entropy. Let g_h be such a metric with positive topological entropy. There exists a standard round metric g_R on the two sphere, with radius $R > 0$ big enough so that $g_R \leq g_h$ at every point. However,

$$h_{top}(g_h) > 0 = h_{top}(g_R).$$

Together with the study of the Curvature-Normalized Yamabe flows carried in Section 2, Proposition 3.1 implies the following result.

Corollary 3.2. *Let (M, g) be a complete manifold such that $R_{min} \leq R_g \leq R_{max} < 0$, and let g_Y be the unique Yamabe metric in the conformal class of g with scalar curvature $R_{g_Y} \equiv -1$. Assume that the sectional curvatures of g and g_Y are negative. Then (M, g) is convex-cocompact if and only if (M, g_Y) is. If this is the case,*

$$\sqrt{|R_{max}|}h_{top}(g_Y) \leq h_{top}(g) \leq \sqrt{|R_{min}|}h_{top}(g_Y).$$

Moreover, if $\pi\Omega_g = \pi\Omega_{g_Y}$ and $g = \frac{g_Y}{|R_{max}|}$ (resp. $g = \frac{g_Y}{|R_{min}|}$) on $\pi\Omega_{g_Y}$, then

$$h_{top}(g) = \sqrt{|R_{max}|}h_{top}(g_Y) \left(\text{resp. } h_{top}(g) = \sqrt{|R_{min}|}h_{top}(g_Y) \right).$$

Proof. Let (M, g) satisfy the hypotheses of the Theorem. It follows from Theorems 2.1 and 2.7 that we have

$$\frac{R_{max}}{R_{min}} g \leq \frac{g_Y}{|R_{min}|} \leq g \leq \frac{g_Y}{|R_{max}|} \leq \frac{R_{min}}{R_{max}} g. \tag{22}$$

Therefore, Theorem 1.7 p.401 of [Br-H99] implies that g is convex-cocompact if and only g_Y is convex-cocompact. The rest of our statement follows from Proposition 3.1 and the bounds (22). □

We now study the local behaviour of the topological entropy for convex-cocompact manifolds along CYF.

Proposition 3.3. *Let (M, g) be a convex-cocompact manifold of dimension $n \geq 3$, with $K_g \leq -a^2 < 0$, and $R_{min} \leq R_g \leq R_{max} < 0$, and such that $\|\nabla_g R_g\|$ and $|\Delta_g R_g|$ are uniformly bounded. Let $(g_t^+)_{t \geq 0}$ and $(g_t^-)_{t \geq 0}$ be the solutions of CYF$^+$ and CYF$^-$ respectively. Then at $t = 0$, the maps $t \mapsto h_{top}(g_t^+)$ and $t \mapsto h_{top}(g_t^-)$ are $\mathcal{C}^1$, and satisfy the following:*

(1) *At $t = 0$, we have*

$$\left.\frac{dh_{top}(g_t^+)}{dt}\right|_{t=0} \leq 0 \quad \text{and} \quad \left.\frac{dh_{top}(g_t^-)}{dt}\right|_{t=0} \geq 0;$$

(2) *unless* $R_g = R_{max}$ *on* $\pi\Omega_g$, *we have*

$$\left.\frac{dh_{top}(g_t^+)}{dt}\right|_{t=0} < 0;$$

(3) *unless* $R_g = R_{min}$ *on* $\pi\Omega_g$, *we have*

$$\left.\frac{dh_{top}(g_t^-)}{dt}\right|_{t=0} > 0.$$

Proof. Let (M, g) satisfy the hypotheses of the proposition, and let $(g_t^+)_{t\geq 0}$ and $(g_t^-)_{t\geq 0}$ be the solutions of CYF$^+$ and CYF$^-$ respectively. It follows from Theorem 2.9 that for some $\epsilon > 0$, the sectional curvatures g_t^+ and g_t^- are strictly negative for $t \in [0, \epsilon)$. Moreover, since g, g_t^+ and g_t^- are quasi-isometric, it follows again from Theorem 1.7 p.401 of [Br-H99] that the CYF solutions are convex-cocompact as long as their sectional curvature remain negative. Therefore, since these flows are smooth, Theorem 1 of [Tap10] implies that $t \mapsto h_{top}(g_t^+)$ is $\mathcal{C}^1$ on $t \in [0, \epsilon)$, and its derivative in $t = 0$ is given by

$$\left.\frac{\partial h_{top}(g_t^+)}{\partial t}\right|_{t=0} = -\frac{h_{top}(g)}{2}\int_{\Omega_g} \left.\frac{\partial g_t(v,v)}{\partial t}\right|_{t=0} d\mu_g(v),$$

where μ_{g_0} is the Bowen-Margulis probability measure for the geodesic flow of (M, g) restricted to Ω_g. This formula can also be derived from Proposition 2.5.2 of the paper by Flaminio [Fla95]. By definition of the CYF$^+$, for every g-unit vector, we have

$$\left.\frac{\partial g_t(v,v)}{\partial t}\right|_{t=0} = (R_{max} - R_g)g(v,v) = (R_{max} - R_g).$$

Therefore,

$$\left.\frac{\partial h_{top}(g_t^+)}{\partial t}\right|_{t=0} = -\frac{h_{top}(g)}{2}\int_{\Omega_g} (R_{max} - R_g)d\mu_g(v) \leq 0.$$

Similarly, $t \mapsto h_{top}(g_t^-)$ is $\mathcal{C}^1$ on $[0, \epsilon)$, and

$$\left.\frac{\partial h_{top}(g_t^-)}{\partial t}\right|_{t=0} = -\frac{h_{top}(g)}{2}\int_{\Omega_g} (R_{min} - R_g)d\mu_g(v) \geq 0.$$

Moreover, it follows from Corollary 4.6 of [BR75] that μ_g gives positive weight to any open set which intersect Ω_g. This implies item (2) and (3) of Proposition 3.3. □

Proposition 3.3 implies the monotonicity of the entropy along the CYF, as stated in Theorem 1.4. We now study the extrema of the topological entropy for manifolds assuming that the sectional curvatures remain negative along the flow. Due to Proposition 3.1, prescribing the entropy can at most impose the locus of closed geodesics (the non-wandering set) and determine the metric on this non-wandering set. For manifolds of dimension $n \geq 3$, we get the following result.

Theorem 3.4 (Minimal entropy and maximal curvature)**.** *Let (M, g) be a convex-cocompact manifold of dimension $n \geq 3$ such that $R_{min} \leq R_g \leq R_{max} < 0$, and $(g_t)_{t\geq 0}$ be the solution of CYF$^+$ with initial metric g. Assume that there exists $\epsilon > 0$ such that for all $t \geq 0$, the sectional curvatures of (M, g_t) satisfy $K_{g_t} \leq -\epsilon$ and let g_Y be the unique Yamabe metric in the conformal class of g with scalar curvature $R_{g_Y} \equiv -1$. Then we have the following.*

(1) *The entropy of g satisfies*

$$h_{top}(g) \geq \sqrt{|R_{max}|}h_{top}(g_Y) = h_{top}(g_{max}).$$

(2) *If* $h_{top}(g) = h_{top}(g_{max})$*, then*

$$g = \frac{g_Y}{|R_{max}|} \text{ and } R_g = R_{max} \text{ on } \pi\Omega_{g_Y} = \pi\Omega_{g_{max}}.$$

Proof. Let (M, g) be a complete convex-cocompact manifold such that $R_{min} \leq R_g \leq R_{max} < 0$, and $(g_t)_{t\geq 0}$ be the solution of CYF$^+$ with initial metric g. Assume that there exists $\epsilon > 0$ such that for all $t \geq 0$, the sectional curvatures of (M, g_t) satisfy $K_{g_t} \leq -\epsilon$. It follows from Corollary 2.4 that the CYF$^+$ is increasing. Therefore, Proposition 3.1 implies that the map $t \mapsto h_{top}(g_t)$ is decreasing. So we have established point (1).

We assume now that $h_{top}(g) = h_{top}(g_{max})$. Let us show that $g = g_{max}$ along the non-wandering set $\Omega_{g_{max}} \subset T^1_{g_{max}}M$ of (M, g_{max}). It follows from Proposition 3.2 of [Tap10] that

$$h_{top}(g_{max}) \leq h_{top}(g) \int_{\Omega_{g_{max}}} \|v\|_g \, d\mu_{g_{max}}(v),$$

where $\mu_{g_{max}}$ is the Bowen-Margulis probability measure for the geodesic flow of (M, g_{max}) restricted to $\Omega_{g_{max}}$, cf Section 5 of [Bow72] for a detailed construction of this measure. Since $h_{top}(g) = h_{top}(g_{max})$, this inequality becomes

$$h_{top}(g_{max}) \leq h_{top}(g) \int_{\Omega_{g_{max}}} \|v\|_g \, d\mu_{g_{max}}(v) \leq h_{top}(g) \int_{\Omega_{g_{max}}} \|v\|_{g_{max}} \, d\mu_{g_{max}}(v) = h_{top}(g),$$

which gives eventually

$$\int_{\Omega_{g_{max}}} \|v\|_g \, d\mu_{g_{max}}(v) = 1. \tag{23}$$

Since for every g_{max}-unit tangent vector $v \in T^1_{g_{max}}M$, $\|v\|_g \leq 1$, equation (23) implies that $\|v\|_g = 1$ $\mu_{g_{max}}$-almost everywhere. on $\Omega_{g_{max}} \subset T^1_{g_{max}}$. Moreover, it follows from Corollary 4.6 of [BR75] that $\mu_{g_{max}}$ gives positive weight to any open set which intersects $\Omega_{g_{max}}$. Therefore, $\|v\|_g = 1$ on a dense set in $\Omega_{g_{max}}$. Since g is smooth, this implies that $\|v\|_g = 1$ on $\Omega_{g_{max}} \subset T^1_{g_{max}}M$. □

Theorem 3.4 is not a complete reciprocal to Proposition 3.1. Indeed, we do not know whether, when the entropy of g is minimal, the sets of closed geodesics $\pi\Omega_g$ and $\pi\Omega_{g_{max}}$ coincide. We will see in Section 4 that such a complete ***entropy-rigidity theorem*** can be obtained from CYF on *convex-cocompact surfaces*, but the proof does not adapt to higher dimensions. It is in general very hard to check whether sectional curvatures remain negative along the Yamabe flow for manifolds of dimension $n \geq 3$.

We also get the analogous result for maximal entropy, whose proof from the CYF$^-$ is similar to the proof of Theorem 3.4.

Theorem 3.5 (Maximal entropy and minimal curvature)**.** *Let* (M, g) *be a complete convex-cocompact manifold of dimension* $n \geq 3$ *such that* $R_{min} \leq R_g \leq R_{max} < 0$*, and* $(g_t^-)_{t\geq 0}$ *be the solution of CYF*$^-$ *with initial metric* g*. Assume that there exists* $\epsilon > 0$ *such that for all* $t \geq 0$*, the sectional curvatures of* (M, g_t^-) *satisfy* $K_{g_t^-} \leq -\epsilon$ *and let* g_Y *be the unique Yamabe metric in the conformal class of* g *with scalar curvature* $R_{g_Y} \equiv -1$*.*

(1) *The entropy of* g *satisfies*

$$h_{top}(g) \leq \sqrt{|R_{min}|}\, h_{top}(g_Y) = h_{top}(g_{min}).$$

(2) *If* $h_{top}(g) = h_{top}(g_{min})$*, then*

$$g = \frac{g_Y}{|R_{min}|} \text{ and } R_g = R_{min} \text{ on } \pi\Omega_{g_Y}.$$

4. Entropy-rigidity through Yamabe flows on convex-cocompact surfaces

We now focus on the case of convex-cocompact surfaces, where we will establish a full entropy-rigidity statement, analogous to the result of Katok in [Kat82]. Let us first give the analogue of Theorem 2.1 and Theorem 2.5 for surfaces, since the proofs we developped in the Section 2 are no longer valid in dimension 2. Recall that for surfaces all notions of curvatures coincide, since $R_g g = 2Ric_g = 2K_g g$. Therefore, on surfaces, Yamabe metrics are multiples of hyperbolic metrics.

Theorem 4.1. *Let (S, g) be a complete convex-cocompact Riemannian surface whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$. Then there is a unique solution $(g_t)_{t\geq 0}$ to the equation*

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= (R_{max} - R_{g_t}) g_t \\ g_0 &= g \end{aligned}$$

We call this solution CYF$^+$. For all $t \in [0, T)$, the scalar curvature of g_t satisfies :

$$\frac{\partial R_{g_t}}{\partial t} = -\Delta_{g_t} R_{g_t} + R_{g_t}(R_{g_t} - R_{max}) \text{ and } R_{min} \leq R_{g_t} \leq R_{max},$$

and g_t^+ converges uniformly exponentially fast on every compact set to the unique metric g_{max} conformally equivalent to g with $R_{g_{max}} \equiv R_{max}$.

Sketch of proof. On surfaces, the Yamabe flow and the Ricci flow coincide. The short-time existence for the Ricci flow on complete manifolds has been established by Shi in [Shi89]. However, for surfaces with bounded curvature, the heavy machinery of Shi is not needed. This has been shown by Ji, Mazzeo and Sesum for finite volume surfaces in [JMS09] Let us sketch the argument for completeness.

Let (S, g) be a complete surface with $R_{min} \leq R_g \leq R_{max} < 0$, and g_{max} the Yamabe metric conformally equivalent to g with scalar curvature R_{max}. A smooth family $g_t = e^{v_t} g$ is solution to CYF$^+$ for $t \in (0, \epsilon)$ if and only if

$$\frac{\partial v_t}{\partial t} = (R_{max} - R_{g_t}). \tag{24}$$

By a well known computation (see [Bes08] p 59), the scalar curvature of $g_t = e^{v_t} g$ is given by

$$R_{g_t} = e^{-v_t}(R_g + \Delta_g v_t). \tag{25}$$

Therefore, the evolution equation (24) becomes

$$\frac{\partial v_t}{\partial t} = R_{max} - e^{-v_t}(R_g + \Delta_g v_t). \tag{26}$$

This is a uniformly parabolic equation as long as v_t is finite and uniformly bounded, which gives short time existence. The proof of the variational formula for the scalar curvature given in equation (3) is still valid for surfaces, which establishes our second claim. For the smooth convergence to the Yamabe metric, the reader can either repeat the Schauder Theory argument given in Section 1 to analyze equation (26), or refer to the manuscript of Albin, Aldana and Rochon [AAR] where this is shown in full detail. □

Of course, the analogous theorem holds for the decreasing Yamabe Flow CYF$^-$, as in Theorem 2.7.

For any Riemannian surface (S, g), we write

$$\pi\Omega_g := \overline{\{\gamma \text{ closed geodesic for } g\}}.$$

By definition, if the curvature of g is negative, then $\pi\Omega_g$ is compact if and only if (S, g) is convex-cocompact. Given a complete surface, searching for an absolute minimum of the topological entropy among all convex-cocompact metrics gives no information, as shown by the following classical result.

Theorem 4.2 (Patterson)**.** *Let (S,g) be a convex-cocompact surface with infinite volume, and $\mathcal{G}_H(S)$ be the set of hyperbolic metrics on S. Then*

$$\{h_{top}(g_h); g_h \in \mathcal{G}_H(S)\} = (0,1).$$

This follows from Theorem 5 p.312 of [Pat76] and the continuity of the entropy with respect to the Teichmueller metric topology, see [McM99] and references given there. The idea of the proof is the following. A convex-cocompact surface is made of a compact convex-core, bounded by closed geodesics, on which funnels are attached. Deform a given hyperbolic metric in the Teichmueller space by changing the length of these closed geodesics. When all the lengths go to infinity, entropy goes to 0. When these lengths go to 0, entropy goes to 1. The global infimum of the topological entropies is always 0 for such surfaces. However, in each conformal class, there is a unique hyperbolic metric. Theorem 1.6 will show that this hyperbolic metric minimizes the entropy in its conformal class as soon as an upper bound for the curvature is given. It also gives a complete entropy-rigidity result for convex-cocompact surfaces.

The rest of this section is devoted to the proof of Theorem 1.6. On surfaces, the curvatures remain automatically negative along the Curvature-normalized Yamabe flows. When the entropy is extremal, Theorem 1.6 gives two pieces of information: the set of closed geodesics of the metric g coincides with the set of closed geodesics of the hyperbolic metric, and on these, the metrics coincide.

Proof of Theorem 1.6. We will prove (1) using the increasing flow CYF$^+$; the proof of (2) is similar using the decreasing flow CYF$^-$.

Let (S,g) be a complete convex-cocompact Riemannian surface whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$ and g_Y be the unique Yamabe metric conformally equivalent to g with $R_{g_Y} \equiv -1$. The metric $g_Y = 2g_H$, where g_H is the hyperbolic metric conformally equivalent to g. However, considering g_Y instead of g_H will lighten the notation.

We have already shown in Proposition 3.1 that $h_{top}(g) \geq \sqrt{|R_{max}|}h_{top}(g_Y)$. Let us assume that $h_{top}(g) = \sqrt{|R_{max}|}h_{top}(g_Y)$. It follows from Theorem 3.4 that $g = \frac{g_Y}{|R_{max}|}$ and $R_g = R_{max}$ on Ω_{g_Y}. Therefore, we are only left with showing that $\pi\Omega_g = \pi\Omega_{g_Y}$, i.e. that the closed geodesics of (S,g) coincide with those of (S,g_Y).

Let $(g_t)_{t\geq 0}$ be the solution to the CYF$^+$ with initial metric g. It follows from Proposition 3.1 that $t \mapsto h_{top}(g_t)$ is decreasing. Since $h_{top}(g) = \sqrt{|R_{max}|}h_{top}(g_Y) = h_{top}(g_{max})$, we have for all $t \geq 0$,

$$h_{top}(g_t) = h_{top}(g).$$

A key step is given by the following lemma.

Lemma 4.3. *The scalar curvature of g satisfies $R_g = R_{max}$ on $\pi\Omega_{g_t}$ for all $t \geq 0$.*

Proof. By contradiction, assume that there exists $t \geq 0$ and $x \in \Omega_{g_t}$ such that $R_{g_t}(x) < R_{max}$. In some open neighbourhood $\mathcal{O}_x \subset M$, we have $R_{g_t} < R_{max}$. Moreover, since $(M, g_s)_{s\geq 0}$ is a smooth path of convex-cocompact metrics, it follows from Theorem 1 of [Tap10] (or from [Fla95]) that the topological entropy $h_{top}(g_t)$ is $\mathcal{C}^1$, and its derivative is given by

$$\frac{\partial h_{top}(g_s)}{\partial s}\bigg|_{s=t} = -\frac{h_{top}(g_t)}{2}\int_{\Omega_{g_t}} \frac{\partial g_s(v,v)}{\partial s}\bigg|_{s=t} d\mu_{g_t}(v),$$

where μ_{g_t} is the Bowen-Margulis probability measure for the geodesic flow of (M, g_t) restricted to Ω_{g_t}. By definition of the CYF$^+$, for every g_t-unit vector, we have

$$\frac{\partial g_s(v,v)}{\partial s}\bigg|_{s=t} = (R_{max} - R_{g_t})g_t(v,v) = (R_{max} - R_{g_t}).$$

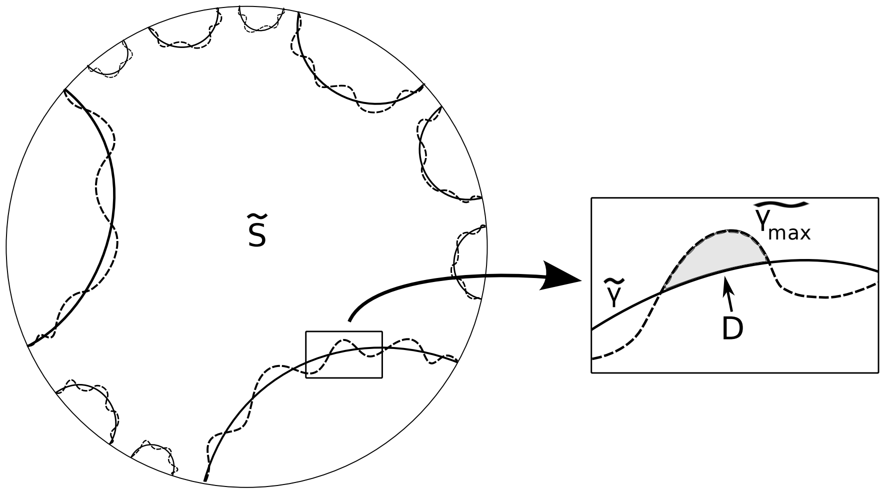

FIGURE 2. Lifts of γ and γ_{max} in the universal cover.

Moreover, it follows from Corollary 4.6 of [BR75] that μ_{g_t} gives positive weight to $\mathcal{O}_x$. Therefore, since $R_{g_t} \leq R_{max}$ on M and $R_{g_t} < R_{max}$ on $\mathcal{O}_x$, we have $\left.\frac{dh_{top}(g_s)}{ds}\right|_{s=t} < 0$. This contradicts the fact that $h_{top}(g_t) = h_{top}(g_{max})$ for all $t \geq 0$. □

Let γ be a closed geodesic of (S, g), and γ_t (resp. γ_{max}) be the unique closed geodesic of (S, g_t) (resp. (S, g_{max})) isotopic to γ. We will show that γ and γ_{max} coincide. Choose a lift $\tilde{\gamma} \subset \tilde{S}$ of γ in the universal cover $\tilde{S}$, and let $\tilde{\gamma_t}$ (resp. $\widetilde{\gamma_{max}}$) be the lift of γ_t (resp. γ_{max}) whose endpoints at infinity coincide with those of $\tilde{\gamma}$ (see Figure 2). If γ and γ_{max} do not intersect, then $\tilde{\gamma}$ and $\widetilde{\gamma_{max}}$ bound a contractible open set. Otherwise, they bound a countable number of open sets whose boundary are made of two smooth curves and have two singular points. The geodesics γ and γ_{max} coincide if and only if there is no such non-empty open region.

By contradiction, assume that there exists such an open region D, bounded by two segments of $\tilde{\gamma}$ and $\widetilde{\gamma_{max}}$. We call the common endpoints of these segments x_- and x_+: they are either on the geodesic or on the boundary at infinity. Let us write $g = e^f g_{max}$. Up to normalization, we can choose g_{max} to have curvature $R_{g_{max}} = -1$. We know by Corollary 2.4 that $f \leq 0$ on D, and it follows from Theorem 3.4 that $f = 0$ on $\widetilde{\gamma_{max}}$. The domain D is contained in $\cup\{\tilde{\gamma_t}, t \geq 0\}$: the curves *sweep* D as t runs from 0 to ∞. Therefore, it follows from Lemma 4.3 that $R_g = -1$ on D. By equation (25), this implies that f satisfies on D the following PDE :

$$e^{-f}\left(R_{max} + \Delta_{g_{max}} f\right) = -1 \Leftrightarrow \Delta_{g_{max}} f = 1 - e^f. \tag{27}$$

By Proposition 1.7 p. 95 of [Tay III], there is a unique solution f to (27) as soon as the value of f is fixed on $\tilde{\gamma}$ and $\widetilde{\gamma_{max}}$. Let us assume that $f < 0$ at some point of $\tilde{\gamma} \cup \partial D$. It follows from the Strong Maximum Principle (see Lemma 3.4 p. 34 of [GT01]) that at each point of $\widetilde{\gamma_{max}} \cup \partial D$ (except the singular points of ∂D), the normal derivative of f is non-zero. Since $f \leq 0$ on D and $f = 0$ on $\widetilde{\gamma_{max}} \cup \partial D$, this implies that in a half-neighbourhood of $\widetilde{\gamma_{max}} \cup \partial D$ outside D, we have $f > 0$. So there is a point $x \in S$ such that $g(x) > g_{max}(x)$, which is a contradiction. Therefore, $f = 0$ on the whole ∂D. As the solution of (27) is unique, we obtain that $f = 0$ on D, i.e. $g = g_{max}$ on D. Since $\tilde{\gamma}$ is a geodesic for g, and $\widetilde{\gamma_{max}}$ is a geodesic for g_{max}, this implies that there are two geodesic rays for g connecting x_- and x_+. This cannot happen because the curvature of g is strictly negative, therefore $\tilde{\gamma} = \widetilde{\gamma_{max}}$ and $\gamma = \gamma_{max}$.

The reciprocal implication in point (1) of Theorem 1.6 is given by Proposition 3.1. Point (2) of Theorem 1.6 can be shown by a proof similar to the one we have just presented, using CYF$^-$. □

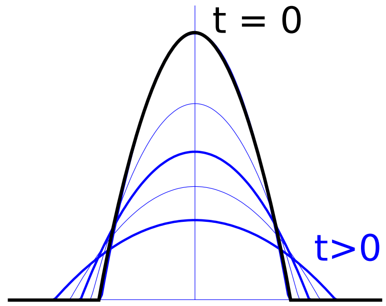

FIGURE 3. A solution to the porous-medium diffusion PDE concentrated in a growing compact set

5. CONCLUDING REMARKS

A complete entropy-rigidity theorem analogous to Theorem 1.6 for convex-cocompact manifolds of higher dimension would be implied by a positive answer to the two following questions.

Question 5.1. *Let (M,g) be a convex-cocompact manifold with pinched negative curvature, and (g_t) the solution of the increasing (resp. decreasing) Curvature-normalised Yamabe flow. Do the sectional curvatures remain negative along the flow ? Or, at least, does the entropy decrease (resp. increase) along the flow ?*

Preservation of negative sectional curvature by Yamabe flows is an open question even for compact manifolds (see, for example [SST11]).

Question 5.2. *Let (M,g) be a convex-cocompact manifold with $R_g \leq -1$, and g_Y be the unique metric in the conformal class of g with $R_{g_Y} = -1$. We assume that g_Y has negative sectional curvatures, and that $h_{top}(g) = h_{top}(g_Y)$. Do the sets of closed geodesics of (M,g) and the set of closed geodesics of (M,g_Y) coincide ?*

In the case of surfaces, the answer to the first question follows automatically from Theorem 1.1. The PDE argument which we have used in Theorem 1.6 to answer the second question is specific to surfaces. In higher dimensions, the difficulty of this second question comes from the fact that the equation (2), which gives the evolution of the conformal factor along the CYF, is a porous-medium diffusion type PDE. Such kind of equations admit solutions which are concentrated inside a compact domain which grows as time flows, see Figure 3. This rules out most Strong Maximum Principle arguments.

One can also wonder whether our results for convex-cocompact manifolds extend to more general complete manifolds with negative sectional curvatures, such as geometrically finite manifolds. Since the Curvature-normalized Yamabe flow construction is valid on any complete manifold, it only remains to study the variations of the entropy. When the non-wandering set is non-compact, many difficulties arise. However, one might reasonably hope to extend our results to all manifolds with negative sectional curvatures whose geodesic flow has a *finite Bowen-Margulis measure*, which include geometrically finite hyperbolic manifolds.

The case of manifolds with some non-negative sectional curvatures seems much more mysterious. It is not clear whether the topological entropy is monotone along the Curvature-normalized Yamabe flow. For example, topological entropy is not monotone under the Volume-normalized Ricci-Yamabe flow for certain compact surfaces found by Jane in [Jan07]. In [SST11], we had avoided this difficulty by considering the volume entropy, which always decreases when the volume increases. Nevertheless, when the volume of the manifold is infinite, volume entropy loses its interest since it is no longer a dynamical invariant, it only depends on the universal cover.

Appendix A. Maximum principles for complete manifolds

We have used in Section 2 the two following forms of the Maximum Principle.

Proposition A.1 (Weak Maximum Principle on a compact set with boundary)**.** *Let M be a smooth manifold, $(g_t)_{t\in I}$ a smooth time-dependent family of metrics on M defined on $I=[0,T)$, and $O\subset M$ be an open set with compact closure. Let $b,\eta>0$ and $C\in\mathbb{R}$ be three constants. Let $f:M\times I\to\mathbb{R}$ be a smooth map.*

(1) *Assume that for all (x,t) with $C<f(x,t)\leq C+\eta$, we have*
$$\frac{\partial f}{\partial t}(x,t)\leq -b\Delta_{g_t}f(x,t),$$
that for all $(x,t)\in\partial O\times(0,T)$ we have $f(x,t)\leq C$ and that $f(.,0)\leq C$. Then for all $(x,t)\in O\times I$ we have $f(x,t)\leq C$.

(2) *Assume that for all (x,t) with $C-\eta\leq f(x,t)<C$, we have*
$$\frac{\partial f}{\partial t}(x,t)\geq -b\Delta_{g_t}f(x,t)$$
that for all $(x,t)\in\partial O\times(0,T)$ we have $f(x,t)\geq C$ and that $f(.,0)\geq C$. Then for all $(x,t)\in O\times I$ we have $f(x,t)\geq C$.

Proof. We shall prove the first item, the case of an upper bound (the other proof is analogous). Under the assumption of item (1), let $0<\epsilon<\frac{\eta}{1+T}$ be fixed, we set for all $(x,t)\in O\times I$:
$$v_\epsilon(x,t)=f(x,t)-C-\epsilon(1+t).$$
We want to prove that v_ϵ is always non-positive on $O\times I$. We have $v_\epsilon(.,0)=f(.,0)-C-\epsilon<0$ and for all $(x,t)\in\partial O\times(0,T)$ we have
$$v(x,t)\leq-\epsilon(1+t)<0. \tag{28}$$
Assume there exists $t\in I$ and $x\in O$ such that $v_\epsilon(x,t)=0$, which implies that $C<f(x,t)=C+\epsilon(1+t)\leq C+\eta$ by construction of ϵ. Since the closure $\bar{O}$ is compact, there exists $t_0\in I$ and $x_0\in\bar{O}$ such that $v_\epsilon(x_0,t_0)=0$ and t_0 is minimal for this property. It follows from (28) that $x_0\notin\partial O$. The function v_ϵ must be increasing in time at (x_0,t_0) to reach 0, so we have
$$0\leq\frac{\partial v_\epsilon}{\partial t}(x_0,t_0)=\frac{\partial f}{\partial t}(x_0,t_0)-\epsilon\leq-b\Delta_{g_{t_0}}f(x_0,t_0)-\epsilon.$$
Moreover, by construction $v_\epsilon(.,t_0)$ is maximal in x_0. Hence $f(.,t_0)$ is also maximal in x_0 and $\Delta_{g_{t_0}}f(x_0,t_0)\geq 0$ (because the Hessian of f is non-positive at an interior point where f is maximum). The previous equality becomes $0\leq-\epsilon$, a contradiction. Hence, for all $(x,t)\in O\times I$ we obtain
$$v_\epsilon(x,t)=f(x,t)-C-\epsilon(1+t)\leq 0.$$
As this is valid for all $\epsilon\in(0,\eta/(1+T))$, it implies that $f\leq C$ on $O\times I$. □

A slight variation of the previous proof provides the following non-linear version of the Weak Maximum Principle.

Proposition A.2 (Non linear Weak Maximum Principle on a compact set with boundary)**.** *Let (M,g) be a smooth manifold with $R_g\leq R_{max}$, and $O\subset M$ be an open set with compact closure. Let $u,v:O\times[0,T)$ be two positive maps satisfying*
$$\frac{\partial u^N}{\partial t}=-C_1\Delta_{g_t}u+C_2(R_{max}u^N-R_gu)\ \textit{and}\ \frac{\partial v^N}{\partial t}=-C_1\Delta_{g_t}v+C_2(R_{max}v^N-R_gv)$$
for some fixed $C_1,C_2,N>0$. Assume that $u(.,0)\leq v(.,0)$ and for all $(x,t)\in\partial O\times(0,T)$ we have $u(x,t)\leq v(x,t)$. Then for all $(x,t)\in O\times(0,T)$, we have $u(x,t)\leq v(x,t)$.

Proof. Let u, v satisfy the hypotheses of the proposition, and $\epsilon > 0$ be fixed. We consider the map $f(x,t) = u(x,t) - v(x,t) - \epsilon(1+t)$ defined on $\bar{O} \times [0,T)$. The same proof as for Proposition A.1 shows that for all $(x,t) \in \bar{O} \times [0,T)$, we have $f(x,t) < 0$. This concludes the proof of our proposition. □

The following variable form of the Maximum Principle on complete manifolds is also a key tool in the study of the Curvature-normalized Yamabe flow.

Proposition A.3 (Variable Maximum Principle on complete manifolds)**.** *Let g_0 be a complete metric on M whose Riemannian curvature tensor satisfies $\|\mathrm{Rm}_{g_0}\| \leq k_0$ for some $k_0 > 0$, and let $g_t, t \in [0,T]$ be a family of complete metrics on M such that $g_t \geq \delta g$ for some fixed $\delta > 0$. Let $F : \mathbb{R} \times [0,T] \to \mathbb{R}$ be a locally Lipschitz map in the $\mathbb{R}$ factor and continuous in the $[0,T]$ factor. Let $h : M \times [0,T] \to \mathbb{R}$ be a smooth bounded map : $0 < C_1 \leq h \leq C_2$ on $M \times [0,T]$.*

Let $c \in \mathbb{R}$ be fixed, and $U : [0,T] \to \mathbb{R}$ be the solution of

$$\frac{dU}{dt} = F(U,t) \text{ with } U(0) = c.$$

Let $u : M \times [0,T]$ be a $\mathcal{C}^2$ bounded.

(1) *If $\forall x \in M, u(x,0) \leq c$ and satisfies on $M \times [0,T]$*

$$\frac{\partial u}{\partial t}(x,t) \leq -h(x,t)\Delta_{g(t)} u + F(u,t),$$

then $\forall (x,t) \in M \times [0,T]$, $u(x,t) \leq U(t)$ as long as the solution U to the ODE exists.

(2) *If $\forall x \in M, u(x,0) \geq c$ and satisfies on $M \times [0,T]$*

$$\frac{\partial u}{\partial t}(x,t) \geq -h(x,t)\Delta_{g(t)} u + F(u,t),$$

then $\forall (x,t) \in M \times [0,T]$, $u(x,t) \geq U(t)$ as long as the solution U to the ODE exists.

Proof. Let (M, g_0) and $(g_t)_{t\in[0,T]}$ satisfy the hypotheses of the proposition. Let $h : M \times [0,T] \to \mathbb{R}$ be a smooth map such that $0 < C_1 \leq h \leq C_2$. We first need to show the existence of a so-called *barrier function* adapted to our problem. This is given by the following lemma which follows from an adaptation of the proof of Lemma 12.9 p. 146 of [CLN06].

Lemma A.4. *Let $O \in M$ be a fixed point. For all $a, A > 0$, there exists a positive map ϕ : $M \times [0,T] \to \mathbb{R}$ and $b < 0$ such that $\forall (x,t) \in M \times [0,T]$,*

$$\left(\frac{\partial \phi}{\partial t} - h(x,t)\Delta_{g_t}\right)\phi(x,t) \geq A\phi(x,t),$$

and

$$exp(ad_{g_t}(O,x)) \leq \phi(x,t) \leq exp(bd_{g_t}(O,x) + 1).$$

Using this barrier function ϕ, the end of the proof of Proposition A.3 is now a repetition of the proof of Theorem 12.14 p. 148 of [CLN06]. □

Note that in Proposition A.3 we have asked u to be a priori bounded, since we do not need a more general result in this paper. However, as follows from the proofs of [CLN06], it would be enough that u grows at most exponentially.

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