Automorphic forms: a physicist's survey

Boris Pioline[1] and Andrew Waldron[2]

[1] LPTHE, Universités Paris VI et VII, 4 pl Jussieu,
75252 Paris cedex 05, France, `pioline@lpthe.jussieu.fr`

[2] Department of Mathematics, One Shields Avenue,
University of California, Davis, CA 95616, USA, `wally@math.ucdavis.edu`

Automorphic forms play an important rôle in physics, especially in the realm of string and M-theory dualities. Notably, the hidden symmetries of 11-dimensional supergravity compactifications, discovered by Cremmer and Julia, motivate the study of automorphic forms for exceptional arithmetic groups $E_n(\mathbb{Z})$ (including their $n \leq 5$ classical A and D instances). These Notes are a pedestrian introduction to these (seemingly abstract) mathematical objects, designed to offer a concrete footing for physicists. The basic concepts are introduced via the simple $Sl(2)$ Eisenstein and theta series. The general construction of continuous representations and of their accompanying Eisenstein series is detailed for $Sl(3)$. Thereafter we present nilpotent representations and their theta series for arbitrary simply-laced groups based on our recent work with D. Kazhdan [1]. We include new results on the geometrical interpretation of minimal representations, implying e.g. that E_6 acts on $SO(5,5)$ pure spinors. We close with some comments about the physical applications of automorphic forms which motivated our research[3].

1 $Sl(2)$ Eisenstein and Theta series disembodied

The general mechanism underlying automorphic forms can be understood by taking a representation-theoretic tour of two familiar $Sl(2,\mathbb{Z})$ examples:

1.1 $Sl(2,\mathbb{Z})$ Eisenstein series

Our first example is the non-holomorphic Eisenstein series

$$\mathcal{E}_s^{Sl(2)}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\backslash(0,0)} \left(\frac{\tau_2}{|m+n\tau|^2}\right)^s , \tag{1}$$

[3] preprint `LPTHE-P03-22, MPI 03-76, hep-th/yymmnnn`

which appears in string theory as the description of the complete, non-perturbative, four-graviton scattering amplitude at low energies [2]. It is a function of the complex modulus τ, taking values on the Poincaré upper half plane, or equivalently points in the symmetric space $\mathcal{M} = K\backslash G = SO(2)\backslash Sl(2,\mathbb{R})$ with coset representative

$$e = \frac{1}{\sqrt{\tau_2}} \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} . \tag{2}$$

The Eisenstein series (1) is invariant under the modular transformation

$$\tau \to (a\tau + b)/(c\tau + d) , \tag{3}$$

which is the right action of $g \in Sl(2,\mathbb{Z})$ on $\mathcal{M}$. Invariance follows simply from that of the lattice $\mathbb{Z}^2$. This set-up may be formalized by introducing:

(i) The linear representation ρ of $Sl(2,\mathbb{R})$ in the space $\mathcal{H}$ of functions of two variables $f(x,y)$,

$$[\rho(g) \cdot f](x,y) = f(ax+by, cx+dy) \ , \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \quad ad - bc = 1 \ . \tag{4}$$

(ii) A distribution $\delta(x,y) = \sum_{(m,n)\in\mathbb{Z}^2\backslash(0,0)} \delta(x-m)\delta(y-n)$ in the dual $\mathcal{H}^*$.

(iii) A vector $f_\infty(x,y) = (x^2+y^2)^{-s}$ invariant under the maximal compact subgroup $K = SO(2) \subset G = Sl(2,\mathbb{R})$.

The Eisenstein series (1) may now be rewritten in a general notation for automorphic forms

$$\mathcal{E}_s^{Sl(2)}(e) = \langle \delta, \rho(e) \cdot f_\infty \rangle \ , \qquad e \in G \, . \tag{5}$$

The modular invariance of $\mathcal{E}_s^{Sl(2)}$ is now manifest: under the right action $e \to eg$ of $g \in Sl(2,\mathbb{R})$, the vector $\rho(e)\cdot f$ transforms by $\rho(g)$, which in turn hits the $Sl(2,\mathbb{Z})$ invariant distribution δ. Furthermore (5) is ensured to be a function of the *coset* $K\backslash G$ by invariance of the vector f_∞ under the maximal compact K. Such a distinguished vector is known as *spherical*. All the automorphic forms we shall encounter can be written in terms of a triplet (ρ, δ, f_∞).

Clearly any function of the $SO(2)$ invariant norm $|x,y|_\infty \equiv \sqrt{x^2+y^2}$ would be as good a candidate for f_∞. This reflects the reducibility of the representation ρ in (4). However, its restriction to homogeneous, even functions of degree $2s$,

$$f(x,y) = \lambda^{2s} f(\lambda x, \lambda y) = y^{-2s} f\left(\frac{x}{y}, 1\right) , \tag{6}$$

is irreducible. The restriction of the representation ρ acts on the space of functions of a single variable $\frac{x}{y}$ by weight $2s$ conformal transformations $z \to (az+b)/(cz+d)$ and admits $f_\infty(z) = (1+z^2)^{-s}$ as its unique spherical vector. In these variables, the distribution δ is rather singular as its support is on all rational values $z \in \mathbb{Q}$. A related problem is that the behavior of $\mathcal{E}_s^{Sl(2)}(\tau)$ at the cusp $\tau \to i\infty$ is difficult to assess – yet of considerable interest to physicists being the limit relevant to non-perturbative instantons [2].

These two problems may be evaded by performing a Poisson resummation on the integer $m \to \widetilde{m}$, after first separating out terms with $n = 0$. The result may be rewritten as a sum over the single variable $N = \widetilde{m}n$, except for two degenerate – or *"perturbative"* – contributions:

$$\begin{aligned}\mathcal{E}_s^{Sl(2)} &= 2\,\zeta(2s)\,\tau_2^s + \frac{2\sqrt{\pi}\,\tau_2^{1-s}\,\Gamma(s-1/2)\,\zeta(2s-1)}{\Gamma(s)} \\ &+ \frac{2\pi^s\sqrt{\tau_2}}{\Gamma(s)} \sum_{N\in\mathbb{Z}\backslash\{0\}} \mu_s(N)\, N^{s-1/2} K_{s-1/2}\left(2\pi\tau_2 N\right) e^{2\pi i\tau_1 N}\,. \end{aligned} \tag{7}$$

In this expression, the summation measure

$$\mu_s(N) = \sum_{n|N} n^{-2s+1}\,, \tag{8}$$

is of prime physical interest, as it is connected to quantum fluctuations in an instanton background [3, 4].

First focus on the non-degenerate terms in the second line. Analyzing the transformation properties under the Borel and Cartan $Sl(2)$ generators $\rho\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} : \tau_1 \to \tau_1 + t$ and $\rho\begin{pmatrix}t^{-1} & 0\\ 0 & t\end{pmatrix} : \tau_2 \to t^2\tau_2$, we readily see that they fit into the framework (5), upon identifying

$$f_\infty(y) = y^{s-1/2} K_{s-1/2}(y)\,, \quad \delta(y) = \sum_{N\in\mathbb{Z}\backslash\{0\}} \mu_s(N)\,\delta(y-N)\,, \tag{9}$$

and the representation ρ as

$$E_+ = iy\,, \quad E_- = i(y\partial_y + 2 - 2s)\partial_y, \quad H = 2y\partial_y + 2 - 2s\,. \tag{10}$$

This is of course equivalent to the representation on homogeneous functions (6), upon Fourier transform in the variable y. The power-like degenerate terms in (7) may be viewed as regulating the singular value of the distribution δ at $y = 0$. They may, in principle, be recovered by performing a Weyl reflection on the regular part. It is also easy to check that the spherical vector condition, $K \cdot f_\infty(y) \equiv (E_+ - E_-) \cdot f_\infty(y) = 0$, is the modified Bessel equation whose unique decaying solution at $y \to \infty$ is the spherical vector in (9).

While the representation ρ and its spherical vector f_∞ are easily understood, the distribution δ requires additional technology. Remarkably, the summation measure (8) can be written as an infinite product

$$\mu_s(y) = \prod_{p\ \mathrm{prime}} f_p(y)\,, \qquad f_p(y) = \frac{1 - p^{-2s+1}|y|_p^{2s-1}}{1 - p^{-2s+1}}\gamma_p(y)\,. \tag{11}$$

(A simple trial computation of $\mu_s(2\cdot 3^2)$ will easily convince the reader of this equality.) Here $|y|_p$ is the p-adic[4] norm of y, *i.e.* $|y|_p = p^{-k}$ with k the largest integer such that p^k divides y. The function $\gamma_p(y)$ is unity if y is a p-adic integer ($|y|_p \leq 1$) and vanishes otherwise. Therefore $\mu(y)$ vanishes unless y is an integer N. Equation (5) can therefore be expressed *adelically* as

$$\mathcal{E}_s^{Sl(2)}(e) = \sum_{y\in\mathbb{Q}} \rho(e)\cdot f_{\mathbb{A}}(y)\ , \quad f_{\mathbb{A}}(y) = \prod_{p\ =\ \text{prime},\infty} f_p(y)\,, \tag{12}$$

where the sum $y \in \mathbb{Q}$ is really over principle adeles[5]. The key observation now is that f_p (respectively $f_{\mathbb{A}}$) is in fact the spherical vector for the representation of $Sl(2,\mathbb{Q}_p)$ (resp. $Sl(2,\mathbb{A})$) naturally associated to that of $Sl(2,\mathbb{R})$ in (10). A convincing Exercise is to evaluate the p-adic Fourier transform of $f_p(y)$ on y, thereby returning to the $Sl(2)$ representation on homogeneous functions (6): the result being

$$\tilde{f}_p(z) = \int_{\mathbb{Q}_p} dy f_p(y) e^{iyz} = |1,z|_p^{-2s} \equiv \max(1,|z|_p)^{-2s}. \tag{13}$$

This is precisely the p-adic counterpart of the real spherical vector $f_\infty(z) = (1+z^2)^{-s} \equiv |1,z|_\infty^{-2s}$. It is easy to check that the formula (12) in this representation reproduces the Eisenstein series (1).

1.2 Symplectic theta series

Our next example, the Jacobi theta series, demonstrates the key rôle played by Fourier invariant Gaussian characters – *"the Fourier of the Gaussian is the Gaussian"*. Our later generalizations will involve cubic type characters invariant under Fourier transform.

In contrast to the Eisenstein series, the Jacobi theta series

$$\theta(\tau) = \sum_{m\in\mathbb{Z}} e^{i\pi\tau m^2}\,, \tag{14}$$

is a modular form for a congruence subgroup $\Gamma_0(2)$ of $Sl(2,\mathbb{Z})$ with modular weight $1/2$ and a non-trivial system of phases. It may, nevertheless, be cast in the framework (5), with a minor *caveat*. The representation ρ now acts on functions of a single variable x as

$$E_+ = i\pi\, x^2\,, \quad H = \frac{1}{2}\,(x\partial_x + \partial_x x)\,, \quad E_- = \frac{i}{4\pi}\partial_x^2\,, \tag{15}$$

[4] A useful physics introduction to p-adic and adelic fields is [5]. It is worth noting that a special function theory analogous to that over the complex numbers exists for the p-adics.

[5] Adeles are infinite sequences $(y_p)_{p=\text{prime},\infty}$ where all but a finite set of y_p are p-adic integers. Principle adeles are constant sequences $y_p = y \in \mathbb{Q}$, isomorphic to $\mathbb{Q}$ itself.

Here, the action of E_+ and H may be read off from the usual Borel and Cartan actions of $Sl(2)$ on τ while the generator E_- follows by noting that the Weyl reflection $S : \tau \to -1/\tau$ can be compensated by Fourier transform on the integer m. The invariance of the comb distribution $\delta(x) = \sum_{m\in\mathbb{Z}} \delta(x-m)$ under Fourier transform is just the Poisson resummation formula.

Finally (the *caveat*),the compact generator $K = E_+ - E_-$ is but the Hamiltonian of the harmonic oscillator, which notoriously does *not* admit a normalizable zero energy eigenstate, but rather the Fourier-invariant ground state $f_\infty(x) = e^{-\pi x^2}$ of eigenvalue $i/2$. This relaxation of the spherical vector condition is responsible for the non-trivial modular weight and phase system. Correspondingly, ρ does not represent the group $Sl(2,\mathbb{R})$, but rather its double cover, the metaplectic group.

Just as for the Eisenstein series, an adelic formula for the summation measure exists: note that the p-adic spherical vector must be invariant under the compact generator S which acts by Fourier transform. Remarkably, the function $f_p(x) = \gamma_p(x)$, imposing support on the integers only is Fourier invariant – it is the p-adic Gaussian ! One therefore recovers the comb distribution with uniform measure. Note that the $Sl(2) = Sp(1)$ theta series generalizes to higher symplectic groups under the title of Siegel theta series, relying also on Gaussian Poisson resummation.

2 Continuous representations and Eisenstein series

The two $Sl(2)$ examples demonstrate that the essential ingredients for automorphic forms with respect to an arithmetic group $G(\mathbb{Z})$ are (i) an irreducible representation ρ of G and (ii) corresponding spherical vectors over $\mathbb{R}$ and $\mathbb{Q}_p$. We now explain how to construct these representations by quantizing coadjoint orbits.

2.1 Coadjoint orbits, classical and quantum: $Sl(2)$

As emphasized by Kirillov, unitary representations are quite generally in correspondence with coadjoint orbits [6]. For simplicity, we restrict ourselves to finite, simple, Lie algebras $\mathfrak{g}$, where the Killing form $(\cdot,\cdot)$ identifies $\mathfrak{g}$ with its dual. Let $\mathcal{O}_j$ be the orbit of an element $j \in \mathfrak{g}$ under the action of G by the adjoint representation $j \to gjg^{-1} \equiv \tilde{j}$. Equivalently, $\mathcal{O}_j$ may be viewed as an homogeneous space $S\backslash G$, where S is the stabilizer (or commutant) of j.

The (co)adjoint orbit $\mathcal{O}_j$ admits a (canonical, up to a multiplicative constant) G-invariant Kirillov–Kostant symplectic form, defined on the tangent space at a point $\tilde{j}$ on the orbit by $\omega(x,y) = (\tilde{j},[x,y])$. Non-degeneracy of ω is manifest, since its kernel, the commutant $\tilde{S}$ of $\tilde{j}$, is gauged away in the quotient $S\backslash G$. Parameterizing $\mathcal{O}_j$ by an element e of $S\backslash G$, one may rewrite $\omega = d\theta$ where the contact one-form $\theta = (j, de\, e^{-1})$, making the closedness and G-invariance of ω manifest. The coadjoint orbit $\mathcal{O}_j = S\backslash G$ therefore yields a classical phase space with a G-invariant Poisson bracket and hence a set of canonical generators representing the action of G on functions of $\mathcal{O}_j$. The representation ρ associated to j follows by quantizing this classical action, *i.e.* by choosing a Lagrangian subspace $\mathcal{L}$ (a maximal commuting set of observables) and representing the generators of G as differential operators on functions of $\mathcal{L}$.

This apparently abstract construction is simply illustrated for $Sl(2)$: let us consider the coadjoint orbit of the element

$$j = \begin{pmatrix} \frac{l}{2} & \\ & -\frac{l}{2} \end{pmatrix} , \tag{16}$$

with stabilizer $S = \mathbb{R}j$. The quotient $S\backslash G$ may be parameterized as

$$e = \begin{pmatrix} 1 & \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix} . \tag{17}$$

The contact one-form is

$$\theta = \mathrm{tr}\; j\, de\, e^{-1} = -l\gamma d\beta \,. \tag{18}$$

The group G acts by right multiplication on e, followed by a compensating left multiplication by S maintaining the choice of gauge slice (17). The resulting infinitesimal group action is expressed in terms of Hamiltonian vector fields

$$E_+ = i\partial_\beta, \quad H = 2i\beta\partial_\beta - 2\gamma\partial_\gamma, \quad E_- = -i\beta^2\partial_\beta + i(1+2\beta\gamma)\partial_\gamma \,. \tag{19}$$

We wish to express these transformations in terms of the Poisson bracket determined by the Kirillov–Kostant symplectic form

$$\omega = d\theta = l\; d\gamma \wedge d\beta \,, \tag{20}$$

namely

$$\{\gamma, \beta\}_{PB} = \frac{1}{l} \,. \tag{21}$$

Indeed, it is easily verified that the generators (19) can be represented canonically

$$E_+ = il\gamma \;, \quad H = 2il\beta\gamma \;, \quad E_- = -il\beta(1+\beta\gamma) \,, \tag{22}$$

with respect to the Poisson bracket (21). The next step is to quantize this classical mechanical system

$$\gamma = \frac{1}{l}\, y \,, \qquad \beta = \frac{1}{i} \frac{\partial}{\partial y} \,. \tag{23}$$

The quantized coadjoint orbit representation follows directly by substituting (23) in (22) and the result is precisely the Eisenstein series representation (10). The physicist reader will observe that the parameter s appearing there arises from quantum orderings of the operators β and γ.

The construction just outlined, based on the quantization of an element j in the *hyperbolic* conjugacy class of $Sl(2,\mathbb{R})$, leads to the continuous series representation of $Sl(2,\mathbb{R})$. Recall that conjugacy classes of $Sl(2)$ are classified by the value [6] of $C \equiv 2\,\mathrm{tr} j^2 = l^2 > 0$. The elliptic case $C < 0$ with j conjugate to an antisymmetric matrix leads to discrete series representations and will not interest us in these Notes. However, the non-generic parabolic conjugacy class $C = 0$ is of considerable interest, being key to theta series for higher groups. There is only a single nilpotent conjugacy class with representative

$$j = \begin{pmatrix} & \\ 1 & \end{pmatrix}, \qquad j^2 = 0\,. \tag{24}$$

The stabilizer $S \subset Sl(2,\mathbb{R})$ is the parabolic group of lower triangular matrices so the nilpotent orbit $S\backslash G$ may be parameterized as

$$e = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}. \tag{25}$$

The contact and symplectic forms are now

$$\theta = \gamma d\beta\,, \qquad \omega = d\gamma \wedge d\beta\,, \tag{26}$$

and the action of $Sl(2)$ may be represented by the canonical generators

$$E_+ = i\gamma\ , \quad H = 2i\beta\gamma\ , \quad E_- = -i\beta^2\gamma \tag{27}$$

accompanied by Poisson bracket $\{\gamma,\beta\}_{PB} = 1$. This representation also follows by the contraction $l \to 0$ holding $l\gamma$ fixed in (22). The relation to theta series is exhibited by performing a canonical transformation $\gamma = y^2$ and $\beta = \frac{1}{2}p/y$ which yields

$$E_+ = iy^2\ , \quad H = ipy\ , \quad E_- = -\frac{i}{4}p^2\,. \tag{28}$$

Upon quantization, this is precisely the metaplectic representation in (15). In contrast to the continuous series, there is no quantum ordering parameter (although a peculiarity of $Sl(2)$ is that it appears as the $s = 1$ instance of the continuous series representation (10)).

[6] The geometry of the three coadjoint orbits is exhibited by parameterizing the $sl(2)$ Lie algebra as

$$\mathfrak{g} = \begin{pmatrix} k_1 & k_2 + k_0 \\ k_2 - k_0 & -k_1 \end{pmatrix}.$$

The orbits are then seen to correspond to massive, lightlike and tachyonic $2+1$ dimensional mass-shells $k_\mu k^\mu = -k_0^2 + k_1^2 + k_2^2 = -\frac{C}{4}$.

2.2 Coadjoint orbits: general case

For general groups G, the orbit method predicts the functional dimension[7] of the generic continuous irreducible representation to be $(\dim G - \text{rank } G)/2$: a generic non-compact element may be conjugated into the Cartan algebra, whose stabilizer is the Cartan (split) torus. There are, therefore, rank G parameters corresponding to the eigenvalues in the Cartan subalgebra. Non-generic elements arise when eigenvalues collide, and lead to representations of smaller functional dimension. When all eigenvalues degenerate to zero, there are a finite set of conjugacy class of nilpotent elements with non-trivial Jordan patterns, hence a finite set of parameter-less, nilpotent, representations. The *minimal* representation with smallest functional dimension plays a distinguished rôle as the analog of the $Sl(2)$ (Jacobi theta series) metaplectic representation [7].

2.3 Quantization by induction: $Sl(3)$

Given a symplectic manifold with G-action, there is no *general* method to resolve the quantum ordering ambiguities while maintaining the $\mathfrak{g}$-algebra. However, (unitary) induction provides a standard procedure to extend a representation ρ_H of a subgroup $H \subset G$ to the whole of G. Let us illustrate the first non-trivial case: the generic orbit of $Sl(3)$.

Just as for $Sl(2)$ in (17), the coadjoint orbit of a generic $sl(3)$ Lie algebra element

$$j = \begin{pmatrix} l_1 & & \\ & l_2 & \\ & & l_3 \end{pmatrix} , \tag{29}$$

can be parameterized by the gauge-fixed $Sl(3)$ group element

$$e = \begin{pmatrix} 1 & & \\ y & 1 & \\ w+yu & u & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x & v+xz \\ & 1 & z \\ & & 1 \end{pmatrix} , \tag{30}$$

whose six-dimensional phase space is equipped with the contact one-form

$$\theta = (l_2 - l_1) y dx + (l_3 - l_2) u dz + [(l_3 - l_1) w + (l_3 - l_2) yu](dv + x dz) . \tag{31}$$

(The canonical generators are easily calculated.) To quantize this orbit, a natural choice of Lagrangian submanifold is $w = y = u = 0$ so that $Sl(3)$ is realized on functions of three variables (x, z, v). These variables parameterize the coset $P\backslash G$, where $P = P_{1,1,1}$ is the (minimal) parabolic subgroup of lower triangular matrices (look at equation (30)). A set of one-dimensional representations on P are realized by the character

[7] The functional dimension counts the number of variables – being unitary, all these representations are infinite dimensional in the usual sense.

$$\chi(p) = \prod_{i=1}^{3} |a_{ii}|^{\rho_i} \mathrm{sgn}^{\epsilon_i}(a_{ii}) \ , \quad p = \begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in P \, , \tag{32}$$

where ρ_i are three constants (defined up to a common shift $\rho_i \to \rho_i + \sigma$) and $\epsilon_i \in \{0, 1\}$ are three discrete parameters. The representation of G on functions of $P\backslash G$ induced from P and its character representation (32) acts by

$$g : f(e) \mapsto \chi(p) f(eg^{-1}) \, , \tag{33}$$

where $eg^{-1} = pe'$ and $e' \in P\backslash G$ (coordinatized by $\{x, z, v\}$). It is straightforward to obtain the corresponding generators explicitly,

$$\begin{aligned}
&E_\beta = \partial_x - z\partial_v && E_{-\beta} = x^2\partial_x + v\partial_z + (\rho_2 - \rho_1)x \\
&E_\gamma = \partial_z && E_{-\gamma} = z^2\partial_z + vz\partial_v - (v + xz)\partial_x + (\rho_3 - \rho_2)z \\
&E_\omega = \partial_v && E_{-\omega} = v^2\partial_v + vz\partial_z + x(v + xz)\partial_x + (\rho_3 - \rho_1)v + (\rho_2 - \rho_1)xz
\end{aligned}$$

$$H_\beta = 2x\partial_x + v\partial_v - z\partial_z + (\rho_2 - \rho_1) \qquad H_\gamma = -x\partial_x + v\partial_v + 2z\partial_z + (\rho_3 - \rho_2) \, , \tag{34}$$

where $Sl(3)$ generators are defined by,

$$sl(3) \ni X = \begin{pmatrix} -\frac{2}{3}H_\beta - \frac{1}{3}H_\gamma & E_\beta & E_\omega \\ -E_{-\beta} & -\frac{1}{3}H_\gamma + \frac{1}{3}H_\beta & E_\gamma \\ -E_{-\omega} & -E_{-\gamma} & \frac{2}{3}H_\gamma + \frac{1}{3}H_\beta \end{pmatrix} . \tag{35}$$

For later use, we evaluate the action of the Weyl reflection A with respect to the root β which exchanges the first and second rows of e up to a compensating P transformation,

$$[A \cdot f](x, v, z) = x^{\rho_2 - \rho_1} f(-z, v, -1/x) \, . \tag{36}$$

The quadratic and cubic Casimir invariants $C_2 = \frac{1}{2}\mathrm{Tr}\, X^2$ and $C_3 = \frac{27}{2}\det X$,

$$C_2 = \frac{1}{6}\left[(\rho_1 - \rho_2)^2 + (\rho_2 - \rho_3)^2 + (\rho_3 - \rho_1)^2\right] + (\rho_1 - \rho_3) \, , \tag{37}$$

$$C_3 = -\frac{1}{2}\left[\rho_1 + \rho_2 - 2\rho_3 + 3\right]\left[\rho_2 + \rho_3 - 2\rho_1 - 3\right]\left[\rho_3 + \rho_1 - 2\rho_2\right] \, , \tag{38}$$

agree with those of the classical representation on the 6-dimensional phase space $\{x, y, z, u, v, w\}$, upon identifying $l_i = \rho_i$ and removing the subleading "quantum ordering terms".

The same procedure works in the case of a nilpotent coadjoint orbit. As an Exercise, the reader may show that the maximal nilpotent orbit of a single 3×3 Jordan block has dimension 6 and can be quantized by induction from the same minimal parabolic $P_{1,1,1}$. The nilpotent orbit corresponding to an $2+1$ block decomposition on the other hand has dimension 4, leading to a unitary, functional dimension 2, representation of $Sl(3)$ induced from the (maximal) parabolic $P_{2,1}$. This is the minimal representation of $Sl(3)$, or simpler, the $Sl(3)$ action on functions of projective $\mathbb{R}P^3$.

In fact, all irreducible unitary representations of $Sl(3,\mathbb{R})$ are classified as representations induced from (i) the maximal parabolic subgroup $P_{1,1,1}$ by the character $\chi(p)$ (with $\rho_i \in i\mathbb{C}$), or (ii) the parabolic subgroup $P_{1,2}$ by an irreducible unitary representation of $Sl(2)$ of the discrete, supplementary or degenerate series [8].

2.4 Spherical vector and Eisenstein series

The other main automorphic form ingredient, the spherical vector, turns out to be straightforwardly computable in the $Sl(n)$ representation unitarily induced from the parabolic subgroup P. We simply need a P-covariant, K-invariant function on G. For simplicity, consider again $Sl(3)$ and denote the three rows of the second matrix in (30) as e_1, e_2, e_3. Under left multiplication by a lower triangular matrix $p = (a_{i\leq j}) \in P$, $e_1 \mapsto a_{11}e_1$ and $e_2 \mapsto a_{21}e_1 + a_{22}e_2$. Therefore the norms of $|e_1|_\infty$ and $|e_1 \wedge e_2|_\infty$ are P-covariant and maximal compact $K = SO(3)$-invariant. The spherical vector over $\mathbb{R}$ is the product of these two norms raised to powers corresponding to the character χ in (32),

$$f_\infty = |1, x, v + xz|_\infty^{\rho_1-\rho_2} \quad |1, v, z|_\infty^{\rho_2-\rho_3} \; . \tag{39}$$

(Recall that $|\cdot\cdot|_\infty$ is just the usual orthogonal Euclidean norm.) Similarly, the spherical vector over $\mathbb{Q}_p$ is the product of the p-adic norms,

$$f_p = |1, x, v + xz|_p^{\rho_1-\rho_2} |1, v, z|_p^{\rho_2-\rho_3} \; . \tag{40}$$

The $Sl(3,\mathbb{Z})$, continuous series representation, Eisenstein series follows by summing over principle adeles,

$$\mathcal{E}^{Sl(3)}_{\rho_i}(e) = \sum_{(x,z,v)\in\mathbb{Q}^3} \left[\prod_{p \text{ prime}} f_p\right] \rho(e)\cdot f_\infty \; . \tag{41}$$

Writing out the adelic product in more mundane terms,

$$\mathcal{E}^{Sl(3)}_{\rho_i}(e) = \sum_{\substack{(m^i,n^i)\in\mathbb{Z}^6,\\ m^{ij}\neq 0}} \left[(m^{ij})^2\right]^{\frac{\rho_1-\rho_2}{2}} \left[(m^i)^2\right]^{\frac{\rho_2-\rho_3}{2}} \; , \tag{42}$$

where $m^{ij} = m^i n^j - m^j n^i$. As usual, the sum is convergent for $\mathrm{Re}(\rho_i - \rho_j)$ sufficiently large and can be analytically continued to complex ρ_i using functional relations representing the Weyl reflections on the weights (ρ_i). The above procedure suffices to describe Eisenstein series for all finite Lie groups.

2.5 Close encounters of the cube kind

Cubic characters are central to the construction of minimal representations and their theta functions for higher simply laced groups D_n and $E_{6,7,8}$. They can also be found in a particular realization of the $Sl(3)$ continuous series representation at $\rho_i = 0$: let us perform the following (mysterious) sequence of transformations: (i) Fourier transform over v, z, and call the conjugate variables $\partial_z = ix_0, \partial_v = iy$. (ii) Redefine $x = 1/(py^2) + x_0/y$. (iii) Fourier transform over p and redefine the conjugate variable[8] $p_p = x_1^3$. These operations yield generators,

$$\begin{aligned}
E_\beta &= y\partial_0 & E_{-\beta} &= -x_0\partial + i\tfrac{x_1^3}{y^2} \\
E_\gamma &= ix_0 & E_{-\gamma} &= -i(y\partial + x_0\partial_0 + x_1\partial_1)\partial_0 \\
&& &\quad + \tfrac{1}{27}y\partial_1^3 + \tfrac{4y\partial_1^2}{9x_1} + \tfrac{28y\partial_1}{27x_1^2} - 6i\partial_0 \\
E_\omega &= iy & E_{-\omega} &= -i(y\partial + x_0\partial_0 + x_1\partial_1)\partial \\
&& &\quad - \tfrac{1}{27}x_0\partial_1^3 - \tfrac{4x_0}{x_1}\partial_1^2 - \tfrac{28x_0}{27x_1^2}\partial_1 - 6i\partial \\
&& &\quad - \tfrac{x_1^3\partial_0}{y^2} - i\tfrac{10x_1}{3y}\partial_1 - i\tfrac{x_1^2}{3y}\partial_1^2 - 6\tfrac{i}{y} \\
H_\beta &= -y\partial + x_0\partial_0 & H_\gamma &= -y\partial - 2x_0\partial_0 - x_1\partial_1 - 2 - 4s\,.
\end{aligned} \tag{43}$$

where $\partial \equiv \partial_y$ and $\partial_0 \equiv \partial_{x_0}$. The virtue of this presentation is that the positive root Heisenberg algebra $[E_\beta, E_\gamma] = E_\omega$ is canonically represented. In addition, the Weyl reflection with respect to the root β is now very simple,

$$[A \cdot f](y, x_0, x_1) = e^{i\frac{x_1^3}{x_0 y}} f(-x_0, y, x_1) \tag{44}$$

and the phase is cubic ! Notice that the same cubic term appears in the expression for $E_{-\beta}$. Indeed, the spherical vector condition for the compact generator $K_\beta = E_\beta + E_{-\beta}$ has solution

$$f_\infty(y, x_0, x_1) = \exp\Big[- \frac{ix_0 x_1^3}{y(y^2 + x_0^2)} \Big]\, g(y^2 + x_0^2)\,, \tag{45}$$

which implies an automorphic theta series formula summing over cubic rather than Gaussian characters [9].

3 Nilpotent representations and theta series

The above construction of $Sl(3)$ Eisenstein series based on continuous series representations extends easily to $Sl(n)$ and (modulo some extra work) any simple Lie group: they generalize the non-holomorphic $Sl(2)$ Eisenstein series (1). However, the Jacobi theta series (14) and its generalizations, without any dependence on free parameters, is often more suited to physical applications. Theta series can be obtained as residues of Eisenstein series at special points in their parameter space. Instead, here we wish to take a representation theoretic approach to theta series, based on automorphic forms coming from nilpotent orbits.

[8] This sequence of transformations also makes sense at $\rho_2 \neq \rho_3$ as long as $\rho_1 = \rho_2$.

3.1 The minimal representation of (A)DE groups

The first step in gathering the various components of formula (5) is to construct the *minimal* representation ρ associated to a nilpotent orbit of simple Lie groups G other than A_n [10] (see also [11]). We will always consider the maximally split real form of G. Minimality is ensured by selecting the nilpotent orbit of smallest dimension: the orbit of any root will suffice but the longest root $E_{-\omega} = j$ is a canonical choice. This orbit can be described by grading the Lie algebra $\mathfrak{g}$ with the Cartan generator $H_\omega = [E_\omega, E_{-\omega}]$ (or equivalently studying the branching rule for the adjoint representation under the $Sl(2)$ subgroup generated by $\{E_\omega, H_\omega, E_{-\omega}\}$). The resulting 5-grading of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \tag{46}$$

where the one-dimensional spaces $\mathfrak{g}_{\pm 2}$ are spanned by the highest and lowest roots $E_{\pm\omega}$. Therefore the space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Heisenberg algebra of dimension $\dim \mathfrak{g}_1 + 1$ with central element E_ω. Furthermore, since $[\mathfrak{g}_0, \mathfrak{g}_{\pm 2}] = \mathfrak{g}_{\pm 2}$, we have $\mathfrak{g}_0 = \mathfrak{m} \oplus H_\omega$ where $[\mathfrak{m}, E_{\pm\omega}] = 0$. The Lie algebra $\mathfrak{m}$ generates the Levi subgroup M of a parabolic group $P = MU$ with unipotent radical[9] $U = \exp \mathfrak{g}_1$. Hence the coadjoint orbit of $E_{-\omega}$ is parameterized by $H_\omega \oplus \mathfrak{g}_1 \oplus E_\omega$, the orthogonal complement of its stabilizer. Its dimension is twice the dimension d of the minimal representation obtained through its quantization and is listed in Table 1.

G	d	M	L	$\mathfrak{g}_1$	I_3
$Sl(n)$	$n-1$	$Sl(n-2)$	$Sl(n-3)$	$[n-3]$	0
D_n	$2n-3$	$Sl(2) \times D_{n-2}$	D_{n-3}	$1 \oplus [2n-6]$	$x_1(\sum x_{2i} x_{2i+1})$
E_6	11	$Sl(6)$	$Sl(3) \times Sl(3)$	$(3,3)$	det
E_7	17	E_6	$Sl(6)$	15	Pf
E_8	29	E_7	E_6	27	$27^{\otimes_s 3}\vert_1$

Table 1. Dimension of minimal representations, canonically realized Levi subgroup M, linearly realized subgroup L, representation of $\mathfrak{g}_1$ under L and cubic L-invariant I_3.

To quantize the minimal nilpotent orbit, note that the symplectic vector space $\mathfrak{g}_1$ admits a canonical polarization chosen by taking as momentum variables the positive root β_0 attached to the highest root ω on the extended Dynkin diagram, along with those positive roots $\beta_{i=1,\dots,d-2}$ with

[9] Recall that a parabolic group P of upper block-triangular matrices (with a fixed given shape) decomposes as $P = MU$ where the unipotent radical U is the subgroup with unit matrices along the diagonal blocks while the Levi M is the block diagonal subgroup.

Killing inner products $(\beta_0,\beta_i)=1$. The conjugate position variables are then $\gamma_{i=0,\dots,d-2}=\omega-\beta_i$. These generators are given by the Heisenberg representation ρ_H acting on functions of d variables,

$$E_\omega = iy\ , \qquad E_{\beta_i} = y\ \partial_{x_0}\ , \qquad E_{\gamma_i} = ix_0\ , \quad i=0,\dots,d-2\,. \tag{47}$$

So far the generator y is central. By the Shale–Weil theorem [12], ρ_H extends to a representation of the double cover of the symplectic group $Sp(d-1)$. The latter contains the Levi M with trivial central extension of $Sp(2d)$ over M. In physics terms, the Levi M acts linearly on the positions and momenta by canonical transformations. In particular, the longest element S in the Weyl group of G is represented by Fourier transform,

$$[S\cdot f](y,x_0,\dots,x_{d-2}) = \int \left[\prod_{i=0}^{d-2}\frac{dp_i}{\sqrt{2\pi y}}\right] f(y,p_0,\dots,p_{d-2})\ e^{\frac{i}{y}\sum_{i=0}^{d-2}p_ix_i}\,. \tag{48}$$

The subgroup $L\subset M$ commuting with E_{β_0}, does not mix positions and momenta and therefore acts linearly on the variables $x_{i=1\dots d-2}$ while leaving (y,x_0) invariant. The representation of the parabolic subgroup P can be extended to $P_0=P\times\exp tH_{\beta_0}$ (where $\exp tH_{\beta_0}$ is the one-parameter subgroup generated by $H_{\beta_0}=[E_{\beta_0},E_{-\beta_0}]$) by defining

$$H_{\beta_0} = -y\partial + x_0\partial_0\,, \tag{49}$$

(here $\partial\equiv\partial_y$ and $\partial_i\equiv\partial_{x_i}$). Notice that the element y, which played the rôle of $\hbar$ before, is no longer central. To extend this representation to G, note that Weyl reflection with respect to the root β_0 acts just as in the $Sl(3)$ case (44),

$$[A\cdot f](y,x_0,x_1,\dots,x_{d-2}) = e^{-\frac{iI_3}{x_0y}} f(-x_0,y,x_1,\dots,x_{d-2})\,. \tag{50}$$

In this formula, $I_3(x_i)$ is the unique L-invariant (normalized) homogeneous, cubic, polynomial in the $x_{i=1,\dots,d-2}$ (see Table 1). Remarkably, the Weyl group relation

$$(AS)^3 = (SA)^3 \tag{51}$$

holds, thanks to the invariance of the cubic cocycle e^{-iI_3/x_0} under Fourier transform over $x_{i=0\dots d-2}$ [13] (see also [14]). This is the analog of the Fourier invariance of the Gaussian cocycle for the symplectic theta series. It underlies the minimal nilpotent representation and its theta series.

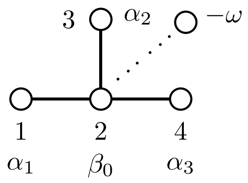
The remaining generators are obtained by applying the Weyl reflections A and S to the Heisenberg subalgebra (47). In particular, the negative root $E_{-\beta_0}$ takes the universal form,

$$E_{-\beta_0} = -x_0\partial + \frac{iI_3}{y^2} \tag{52}$$

which we first encountered in the $Sl(3)$ example (43).

3.2 D_4 minimal representation and strings on T^4

As illustration, we display the minimal representation of $SO(4,4)$ [15]. The extended Dynkin diagram is



and the affine root $-\omega$ attaches to the root β_0. The grade-1 symplectic vector space $\mathfrak{g}_1$ is spanned by $4+4$ generators

$$\left.\begin{array}{cc} \beta_0 & \gamma_0 = \beta_0 + \alpha_1 + \alpha_2 + \alpha_3 \\ \beta_i = \beta_0 + \alpha_i & \gamma_i = \beta_0 + \alpha_j + \alpha_k \end{array}\right\} \quad \{i,j,k\} = \{1,2,3\}\,. \tag{53}$$

The positive roots are represented as in (47), while the negative roots read

$$\begin{aligned} E_{-\beta_0} &= -x_0 \partial + \frac{i x_1 x_2 x_3}{y^2} \\ E_{-\beta_1} &= x_1 \partial + \frac{x_1}{y}\,(1 + x_2\partial_2 + x_3\partial_3) - i x_0 \partial_2 \partial_3 \\ E_{-\gamma_0} &= 3i\partial_0 + iy\partial\partial_0 - y\partial_1\partial_2\partial_3 + i(x_0\partial_0 + x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\,\partial_0 \\ E_{-\gamma_1} &= iy\partial_1\partial + i(2 + x_0\partial_0 + x_1\partial_1)\,\partial_1 - \frac{x_2 x_3}{y}\,\partial_0 \\ E_{-\omega} &= 3i\partial + iy\partial^2 + \frac{i}{y} + i x_0 \partial_0 \partial + \frac{x_1 x_2 x_3}{y^2}\,\partial_0 + x_0 \partial_1\partial_2\partial_3 \\ &\quad + \frac{i}{y}\,(x_1 x_2 \partial_1 \partial_2 + \text{cyclic}) + \ i(x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\,(\partial + \frac{1}{y})\,, \end{aligned} \tag{54}$$

as well as cyclic permutations of $\{1,2,3\}$. The Levi $M = [Sl(2)]^3$, obtained by removing β_0 from the extended Dynkin diagram, acts linearly on positions and momenta and has generators

$$E_{\alpha_i} = -x_0 \partial_i - \frac{i x_j x_k}{y}\ , \quad E_{-\alpha_i} = x_i \partial_0 + iy\partial_j\partial_k\,. \tag{55}$$

Finally, the Cartan generators are obtained from commutators $[E_\alpha, E_{-\alpha}]$,

$$H_{\beta_0} = -y\partial + x_0\partial_0 \tag{56}$$

$$H_{\alpha_i} = \qquad - x_0\partial_0 + x_i\partial_i - x_j\partial_j - x_k\partial_k - 1\,. \tag{57}$$

This representation also arises in a totally different context: the one-loop amplitude for closed strings compactified on a 4-torus! T-duality requires this amplitude to be an automorphic form of $SO(4,4,\mathbb{Z}) = D_4(\mathbb{Z})$. In fact, it may be written as an integral of a symplectic theta series over the fundamental domain of the genus-1 world-sheet moduli space,

$$\mathcal{A}(g_{ij}, B_{ij}) = \int_{SO(2)\backslash Sl(2,\mathbb{R})/Sl(2,\mathbb{Z})} \frac{d^2\tau}{\tau_2^2}\, \theta_{Sp(8)}(\tau, \bar{\tau}; g_{ij}, B_{ij})\,. \tag{58}$$

Here (g_{ij}, B_{ij}) are the metric and Neveu-Schwarz two-form on T^4 parameterizing the moduli space $[SO(4) \times SO(4)]\backslash SO(4,4,\mathbb{R})$. The symplectic theta series $\theta_{Sp(8,\mathbb{Z})}$ is exactly the partition function of the $4+4$ string world-sheet winding modes m^i_a, $i = 1, \ldots, 4, a = 1, 2$ around T^4. Like any Gaussian theta series, it is invariant under the (double cover of the) symplectic group over integers. Importantly, it contains $Sl(2,\mathbb{Z}) \times SO(4,4,\mathbb{Z}) \subset Sp(8,\mathbb{Z}) = C_8(\mathbb{Z})$ as maximal commuting subgroups. Therefore, after integrating over the $Sl(2)$ moduli space, an $SO(4,4,\mathbb{Z})$ automorphic form, based on the minimal representation remains.

To see the minimal representation emerge explicitly, note that $Sl(2)$-invariant functions of m^i_a must depend on the cross products,

$$m^{ij} = \epsilon^{ab} m^i_a m^j_b\ , \tag{59}$$

which obey the quadratic constraint

$$m^{[ij} m^{kl]} = 0\,, \tag{60}$$

and therefore span a cone in $\mathbb{R}^6$. The 5 variables (y, x_0, x_i) are mapped to this 5-dimensional cone by diagonalizing the action of the maximal commuting set of six observables $\mathcal{C} = (E_{\alpha_3}, E_{\beta_3}, E_{\gamma_1}, E_{\gamma_2}, E_{\gamma_0}, E_\omega)$ whose eigenvalues may be identified with the constrained set of six coordinates on the cone $i(m^{43}, m^{24}, m^{14},\ m^{23}, m^{13}, m^{12})$. The intertwiner between the two representations is a convolution with the common eigenvector of the generators $\mathcal{C}$ amounting to a Fourier transform over x_3. This intertwiner makes the hidden triality symmetry, which is crucial for heterotic/type II duality [16], of the $Sl(4)$-covariant string representation manifest.

An advantage of the covariant realization is that the spherical vector follows by directly computing the integral (58). The real spherical vector is read-off from worldsheet instanton contributions

$$f_\infty(m^{ij}) = \frac{e^{-2\pi\sqrt{(m^{ij})^2}}}{\sqrt{(m^{ij})^2}}\ , \tag{61}$$

while its p-adic counterpart follows from the instanton summation measure

$$f_p(m^{ij}) = \gamma_p(m_{ij}) \frac{1 - p\,|m_{ij}|_p}{1-p}\ . \tag{62}$$

Intertwining back to the triality invariant realization gives

$$f_\infty(y,x_0,x_i) = \frac{e^{-i\frac{x_0x_1x_2x_3}{y(y^2+x_0^2)}}}{\sqrt{y^2+x_0^2}}\, K_0\left(\frac{\sqrt{\prod_{i=1}^3(y^2+x_0^2+x_i^2)}}{y^2+x_0^2}\right) . \tag{63}$$

This is the prototype for spherical vectors of all higher simple Lie groups.

3.3 Spherical vector, real and p-adic

To find the spherical vector for higher groups, one may either search for generalizations of the covariant string representation in which the result is a simple extension of the world-sheet instanton formula (61) – see Section 3.5, or try and solve by brute force the complicated set of partial differential equations $(E_\alpha + E_{-\alpha})f = 0$ demanded by K-invariance. Fortunately, knowing the exact solution (63) for D_4 gives enough inspiration to solve the general case [1].

To see this, note that the phase in (63) has precisely the right anomaly under $(y,x_0) \to (-x_0,y)$ to cancel the cubic character of the Weyl generator (50), or equivalently the cubic term appearing in $E_{\beta_0}+E_{-\beta_0}$. The real part of the spherical vector therefore depends on (y,x_0) through their norm $R = \sqrt{y^2+x_0^2} = |y,x_0|_\infty$. Moreover, invariance under the linearly acting maximal compact subgroup of L restricts the dependence on x_i to its quadratic I_2, cubic I_3 and quartic I_4 invariants. Choosing a frame where all but three of the x_i vanish, the remaining equations are then essentially the same as for the known D_4 case. The universal result is

$$f_\infty(X) = \frac{1}{R^{s+1}}\, \mathcal{K}_{s/2}\left(|X, \nabla_X[I_3/R]|_\infty\right) \exp\left(-i\,\frac{x_0 I_3}{yR^2}\right) , \tag{64}$$

where $X \equiv (y, x_0, \dots x_{d-2})$ and I_3 is given in Table 1. Notice that the result depends on the pullback of the Euclidean norm to the Lagrangian subspace $(X, \nabla_X[I_3/R])$ of the coadjoint orbit. The function $\mathcal{K}_t(x)$ is related to the usual modified Bessel function by $\mathcal{K}_t(x) \equiv x^{-t}K_t(x)$, and the parameter $s = 0,1,2,4$ for $G = D_4, E_6, E_7, E_8$, respectively[10].

The p-adic spherical vector computation is much harder since the generators cannot be expressed as differential operators. It was nevertheless completed in [17] by very different techniques, again inspired by the D_4 result (62), intertwined to the triality invariant representation. The result mirrors the real case, namely for $|y|_p < |x_0|_p$,

$$f_p(X) = \frac{1}{R^{s+1}}\, \mathcal{K}_{p,s/2}\left(X, \nabla_X[I_3/R]\right) \exp\left(-i\,\frac{I_3}{x_0 y}\right) , \tag{65}$$

where $R = |y,x_0|_p = |x_0|_p$ is now the p-adic norm, and $\mathcal{K}_{p,t}$ is a p-adic analogue of the modified Bessel function,

[10] For $D_{n>4}$ the result is slightly more complicated, see [1]. It is noteworthy that the ratio I_3/R is invariant under Legendre transform with respect to all entries in X, although the precise meaning of this observation is unclear.

$$\mathcal{K}_{p,t}(x) = \frac{1-p^s|x|_p^{-s}}{1-p^s}\ \gamma_p(x)\,, \tag{66}$$

($\gamma_p(x)$ generalizes to a function of several arguments by $\gamma_p(X)=0$ unless $|X|_p \leq 1$). The result for $|y|_p > |x_0|_p$ follows by the Weyl reflection A.

3.4 Global theta series

Having obtained the real and p-adic spherical vectors for any p, one may now insert them in the adelic formula (12) to construct exceptional theta series. Equivalently, we may use the representation (5),

$$\theta_G(e) = \langle \delta, \rho_G(e) f_\infty \rangle\ , \qquad \delta(X) = \prod_{p\ \text{prime}} f_p(X)\,. \tag{67}$$

Thanks to the factor $\gamma_p(X, \nabla_X[I_3/R])$, the summation measure $\delta(X)$ will have support on integers X such that $\nabla_X[I_3/R]$ is also an integer.

While this expression is fine for generic X, it ceases to make sense when $y=0$, as the phase of the spherical vector (64) becomes singular. As shown in [17], the correct prescription for $y=0$ is to remove the phase and set $y=0$ in the rest of the spherical vector, thereby obtaining a new smooth vector

$$\overline{f}(\overline{X}) = \lim_{y\to 0}\left[\exp\left(i\,\frac{x_0 I_3}{yR^2}\right) f_\infty(y,x_0,x_i)\right]\,, \tag{68}$$

where $\overline{X}=(x_0,x_1,\dots x_{d-2})$, with a similar expression in the p-adic case.

However, there still remains a further divergence when $y=x_0=0$. It can be shown these terms may be regularized to give a sum of two terms, namely a constant plus a theta series based on the minimal orbit of the Levi subgroup M. Altogether, the global formula for the theta series in the minimal representation of G reads [17]

$$\begin{aligned}\theta_G(e) = &\sum_{X\in[\mathbb{Z}\backslash\{0\}]\times\mathbb{Z}^{d-1}} \mu(X)\ \rho(e)\cdot f_\infty(X)\\ &+ \sum_{\overline{X}\in[\mathbb{Z}\backslash\{0\}]\times\mathbb{Z}^{d-2}} \overline{\mu}(\overline{X})\ \rho(e)\cdot \overline{f}_\infty(\overline{X}) + \alpha_1 + \alpha_2\theta_M(e)\,.\end{aligned} \tag{69}$$

Notice that the degenerate contributions in the second line mix with the non-degenerate ones under a general right action of $G(\mathbb{Z})$.

3.5 Pure spinors, tensors, 27-sors, ...

We end the mathematical discussion by returning to the $Sl(4)$-invariant string representation of D_4. As we saw, a larger $Sl(4)$ symmetry was exhibited by a Fourier transform over x_3 and introduction of a redundant variable, related to the 5 others by the quadratic constraint (59).

This phenomenon also occurs for higher groups: for D_n, the D_4 construction yields a representation of D_n on antisymmetric $n \times n$ rank 2 matrices m^{ij}, subject to the same constraints (59). This leaves $2n-3$ independent variables, just as in the canonical realization of the minimal representation. The spherical vector is now a Bessel function of the norm $\sqrt{(m^{ij})^2}$, just as in (61).

For E_6, an $SO(5,5)$ symmetry appears upon Fourier transforming over one column of the 3×3 matrix of x_i's. As shown in [1], this yields a linear action of $Sl(5)$ on a 5×5 antisymmetric matrix X_{ij} and a singlet y; these $1 + 10$ coordinates in turn transform as a 16 dimensional Majorana–Weyl spinor Y of $SO(5,5)$, subject to the *pure spinor* condition

$$\overline{Y} \Gamma_\mu Y = 0 \,. \tag{70}$$

The minimal representation of E_6 can thus be viewed as the action of E_6 on $SO(5,5)$ pure spinors[11]. The spherical vector takes the remarkably simple form [1]

$$f_\infty(Y) = \mathcal{K}_1 \left(\sqrt{\overline{Y} Y} \right) \,. \tag{71}$$

Its p-adic counterpart, obtained by replacing orthogonal with p-adic norms, also simplifies accordingly. It would be interesting to see if the hidden E_6 symmetry of $SO(10)$ null spinors has implications for string field theory [19].

Similarly, for E_7 an E_6 symmetry appears upon Fourier transforming over one column of the antisymmetric 5×5 matrix of x_i's. As shown in [1], this yields a linear action of $SO(5,5)$ on a Majorana–Weyl spinor X and a singlet y; these $1+16$ coordinates are the 27 of E_6, denoted Y, subject to the quadratic condition that the $27 \otimes 27|_{\overline{27}}$ vanishes – in other words a *pure 27-sor*. Again the spherical vector [1] takes the simple form

$$f_\infty(Y) = \mathcal{K}_{3/2} \left(\sqrt{\overline{Y} Y} \right) \,. \tag{72}$$

While the canonical E_8 minimal representation can be realized with $Sl(8)$ acting linearly [20], it does not seem possible to maintain a larger symmetry acting on variables arranged as a single irreducible representation with quadratic constraint.

4 Physical applications

From a clearing in the dense forest of nilpotent representations and theta series, we raise our eyes skywards searching for the peaks we aimed to climb. Two mountains trace out the horizon: Mount Supermembrane and the Cosmological Singularity Aiguille.

[11] In contrast to the conformal realization of E_6 on 21 variables discussed in [18], this representation is irreducible.

4.1 The automorphic membrane

The primary motivation behind our study of exceptional theta series was the conjecture of [23]: the exact four-graviton R^4 scattering amplitude, predicted by U-duality and supersymmetry, ought be derivable from the eleven-dimensional quantum supermembrane – an obvious candidate to describe fundamental M-theory excitations (see [21] for a review of U-duality). For example, in eight dimensions, in analogy with the one-loop string amplitude, the partition function of supermembrane zero-modes should be a theta series of $E_6(\mathbb{Z})$, which subsumes both the U-duality group $Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$, and the toroidal membrane modular group $Sl(3,\mathbb{Z})$. Integrating the partition function over world-volume moduli $\mathbb{R}^+ \times Sl(3)$, yields by construction a U-duality invariant result which should reproduce the exact four-graviton R^4 scattering amplitude for M-theory on a T^3, including membrane instantons, namely a sum of $Sl(3)$ and $Sl(2)$ Eisenstein series [22, 24].

Having constructed explicitly the E_6 theta series, we may now test this conjecture [25]. Recall that in the canonical realization, the E_6 minimal representation contains an $Sl(3)\times Sl(3)$ group acting linearly from the left and right on a 3×3 matrix of integers m_M^A, together with two singlets y, x_0. In addition, there is an extra $Sl(3)$ built from the non-linearly acting generators $E_{\beta_0,\gamma_0,\omega}$, which further decomposes into the $\mathbb{R}^+ \times Sl(2)$ factors mentioned above. The integers m_M^A are interpreted as winding numbers of a toroidal membrane wrapping the target-space T^3, $X^M = m_A^M \sigma^A$. The two extra integers y, x_0 do not appear in the standard membrane action but may be interpreted as a pair of world-volume 3-form fluxes – an interesting prediction of the hidden E_6 symmetry.

The integration over the membrane world-volume $Sl(3)$ moduli amounts to decomposing the minimal representation with respect to the left acting $Sl(3)$ and keeping only invariant singlets. For a generic matrix m_M^A, the unique such invariant is its determinant, which we preemptively denote $x_1^3 = \det(M)$. This leaves a representation of the non-linear $Sl(3)$ acting on functions of three variables (y, x_0, x_1) (the right $Sl(3)$ acts trivially): this is precisely the representation studied in Section 2.5. In addition, non-generic matrices contribute further representations charged under both left and right $Sl(3)$s.

It remains to carry out the integration over the membrane world-volume factor $\mathbb{R}^+$ inside the non-linear $Sl(3)$. This integral is potentially divergent. Instead, a correct mathematical prescription is to look at the constant term with respect to a parabolic $P_{1,2} \subset Sl(3)_{NL}$: indeed we find that this produces the result predicted by the conjecture [25].

This is strong evidence that membranes are indeed the correct degrees of freedom of M-theory, although the construction only treats membrane zero-modes. It would be very interesting to see if the E_6 symmetry can be extended to fluctuations and in turn to quantization of the complete toroidal supermembrane.

4.2 Conformal Quantum Cosmology

The dynamics of spatially separated points decouple as a space-like singularity is approached. Only effective 0+1-dimensional quantum mechanical degrees of freedom remain at each point. Classically, these correspond to a particle on a hyperbolic billiard, whose chaotic motion translates into a sequence of Kasner flights and bounces of the spatial geometry [26]. Originally observed for 3+1-dimensional Einstein gravity, this chaotic behavior persists for 11-dimensional supergravity, whose the billiard is the Weyl chamber of a generalized E_{10} Kac-Moody group [27]. Upon accounting for off-diagonal metric and gauge degrees of freedom, the hyperbolic billiard can be unfolded onto the fundamental domain of the arithmetic group $E_{10}(\mathbb{Z})$. Automorphic forms for $E_{10}(\mathbb{Z})$ should therefore be relevant in to the wave function of the universe !

Automorphic forms for generalized Kac-Moody groups are out of our present reach. However, automorphic forms for finite Lie groups may still be useful in a cosmological context because their corresponding minimal representations can be viewed as conformal quantum mechanical systems of the type that arising near cosmological singularities [29]. Indeed, changing variables $y = \rho^2/2$, $x_i = \rho q_i/2$ in the canonical minimal representation, the generators of the grading $Sl(2)$ subalgebra become

$$E_\omega = \frac{1}{2}\rho^2 \ , \quad H_\omega = \rho p_\rho \ , \quad E_{-\omega} = \frac{1}{2}\left(p_\rho^2 + \frac{4\varDelta}{\rho^2}\right) . \tag{73}$$

Here $\varDelta$ is a quartic invariant of the coordinates and momenta $\{q_i, \pi_i\}$ corresponding (up to an additive constant) to the quadratic Casimir of the Levi M. Choosing $E_{-\omega}$ as the Hamiltonian, the resulting mechanical system has a dynamical, $d = 0 + 1$ conformal, $Sl(2) = SO(2, 1)$ symmetry. In contrast to the one-dimensional conformal quantum mechanics of [28], the conformal symmetry $Sl(2)$ is enlarged to a much larger group G mixing the radial coordinate ρ with internal ones x_i. It can be shown that these conformal systems appear upon dimensional reduction of Einstein's equations near a space-like singularity [29].

5 Conclusion

In this Lecture, we hope to have given a self-contained introduction to automorphic forms, based on string theory experience – rigor was jettisoned in favor of simplicity and utility. Our attempt will be rewarded if the reader is preempted to study other aspects of this rich field: higher nilpotent representations and integrable models, non-simply laced groups, non-maximally split real forms, cubic free field theories, not to mention the ever-elusive quantum membrane.

Acknowledgments: We would like to thank our mathematical muse D. Kazhdan and his colleagues S. Miller, C. Moeglin, S. Polischchuk for educating us and the Max Planck Institute für Mathematik Bonn and Gravitationsphysik – Albert Einstein Institut – in Golm for hospitality during part of this work. B.P. is also grateful for the organizers of Les Houches Winter School on "Frontiers in Number Theory, Physics and Geometry" for a wonderful session, and the kind invitation to present this work. Research supported in part by NSF grant PHY01-40365.

References

1. D. Kazhdan, B. Pioline and A. Waldron, "Minimal representations, spherical vectors, and exceptional theta series. I," Commun. Math. Phys. **226**, 1 (2002) [arXiv:hep-th/0107222].
2. M. B. Green and M. Gutperle, "Effects of D instantons," *Nucl. Phys.* **B498** (1997) 195–227, `hep-th/9701093`.
3. M. B. Green and M. Gutperle, "D-particle bound states and the D-instanton measure," *JHEP* **9801** (1998) 005 `hep-th/9711107`.
4. G. Moore, N. Nekrasov, and S. Shatashvili, "D-particle bound states and generalized instantons", *Commun. Math. Phys.* **209** (2000) 77 `hep-th/9803265`.
5. L. Brekke and P.G.O. Freund, "p-adic numbers in physics", Phys. Rep. **233** (1993) 1.
6. A. A. Kirillov, "Merits and demerits of the orbit method", *Bull. Am. Math. Soc.* **36** (1999) 433.
7. A. Joseph, "Minimal realizations and spectrum generating algebras," *Comm. Math. Phys.* **36** (1974) 325; "The minimal orbit in a simple Lie algebra and its associated maximal ideal," *Ann. Scient. Ecole Normale Sup.* 4ème série **9** (1976) 1-30.
8. I. Vahutinskii, "Unitary representations of $Gl(3,\mathbb{R})$", *Soviet Math. Dokl.* **7** (1966) 1123.
9. D. Bump and J. Hoffstein, "Cubic metaplectic forms on $Gl(3)$", *Invent. Math.* **84** (1986) 481.
10. D. Kazhdan and G. Savin, "The smallest representation of simply laced groups," Israel Math. Conf. Proceedings, Piatetski-Shapiro Festschrift **2** (1990) 209-223.
11. R. Brylinski and B. Kostant, "Minimal representations of E_6, E_7, and E_8 and the generalized Capelli identity", *Proc. Nat. Acad. Sci. U.S.A.* **91** (1994) 2469; "Minimal representations, geometric quantization, and unitarity", *Proc. Nat. Acad. Sci. U.S.A.* **91** (1994) 6026; "Lagrangian models of minimal representations of E_6, E_7 and E_8", in *Functional Analysis on the Eve of the 21st century*, Progress in Math., Birkhäuser (1995).
12. A. Weil, "Sur certains groupes d'opérateurs unitaires", Acta Math. **111** (1964) 143.
13. P. Etingof, D. Kazhdan, and A. Polishchuk, "When is the Fourier transform of an elementary function elementary?", `math.AG/0003009`.
14. B. Pioline, "Cubic free field theory", Proceedings of Cargese 2002 Summer School, arXiv:hep-th/0302043.
15. D. Kazhdan, "The minimal representation of D_4", in Operator algebras, Unitary representations, enveloping algebras and invariant theories, A. Connes, M. Duflo, A. Joseph, R. Rentschler eds., Progress in Mathematics **92**, Birkhäuser Boston (1990) 125.

16. E. Kiritsis, N. A. Obers and B. Pioline, "Heterotic/type II triality and instantons on K3," JHEP **0001**, 029 (2000), hep-th/0001083.
17. D. Kazhdan and A. Polishchuk, "Minimal representations: spherical vectors and automorphic functionals" arXiv:math.RT/0209315.
18. M. Gunaydin, K. Koepsell and H. Nicolai, "Conformal and quasiconformal realizations of exceptional Lie groups," Commun. Math. Phys. **221** (2001) 57 hep-th/0008063.
19. N. Berkovits, "Super-Poincare covariant quantization of the superstring," JHEP **0004** (2000) 018 [arXiv:hep-th/0001035].
20. M. Gunaydin, K. Koepsell and H. Nicolai, "The Minimal Unitary Representation of $E_{8(8)}$", hep-th/0109005.
21. N. A. Obers and B. Pioline, "U-duality and M-theory," *Phys. Rept.* **318** (1999) 113, hep-th/9809039; N. A. Obers and B. Pioline, "U-duality and M-theory, an algebraic approach," hep-th/9812139.
22. E. Kiritsis and B. Pioline, "On R^4 threshold corrections in IIB string theory and (p,q) string instantons," *Nucl. Phys.* **B508** (1997) 509–534, hep-th/9707018; B. Pioline and E. Kiritsis, "U-duality and D-brane combinatorics," *Phys. Lett.* **B418** (1998) 61, hep-th/9710078.
23. B. Pioline, H. Nicolai, J. Plefka and A. Waldron, "R^4 couplings, the fundamental membrane and exceptional theta correspondences," JHEP **0103**, 036 (2001), hep-th/0102123.
24. N. A. Obers and B. Pioline, "Eisenstein series and string thresholds," *Commun. Math. Phys.* **209** (2000) 275, hep-th/9903113; N. A. Obers and B. Pioline, "Eisenstein series in string theory," *Class. Quant. Grav.* **17** (2000) 1215, hep-th/9910115.
25. B. Pioline and A. Waldron, "The automorphic membrane", to appear.
26. V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, "A General Solution Of The Einstein Equations With A Time Singularity," Adv. Phys. **31**, 639 (1982); "Oscillatory Approach To A Singular Point In The Relativistic Cosmology," id, Adv. Phys. **19**, 525 (1970).
27. T. Damour and M. Henneaux, "Chaos in superstring cosmology," Phys. Rev. Lett. **85**, 920 (2000) [arXiv:hep-th/0003139]. T. Damour and M. Henneaux, "E(10), BE(10) and arithmetical chaos in superstring cosmology," Phys. Rev. Lett. **86**, 4749 (2001) [arXiv:hep-th/0012172].
28. V. de Alfaro, S. Fubini and G. Furlan, "Conformal Invariance In Quantum Mechanics," Nuovo Cim. A **34**, 569 (1976).
29. B. Pioline and A. Waldron, "Quantum cosmology and conformal invariance," Phys. Rev. Lett. **90** (2003) 031302 [arXiv:hep-th/0209044].