Group actions on manifolds of positive sectional curvature

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There are only very few examples of compact Riemannian manifolds of positive sectional curvature known. In dimensions above 24 all known simply connected examples are diffeomorphic to rank 1 symmetric spaces, $\mathbb{S}^n$, $\mathbb{CP}^n$ or $\mathbb{HP}^n$. On the other hand there are only very few constrains known. For example there is not a single obstruction known that distinguishes the class of simply connected compact manifolds that admit positively curved metrics from the class of simply connected compact manifolds that admit non-negatively curved metrics, although the class of known examples is much bigger in the latter case.

Karsten Grove [1991] proposed to study isometric group actions on manifolds of positive sectional curvature and to find obstructions for such actions. One of the hopes is that a systematic study of the problem might give ideas how one can construct new examples.

The main aim of the talk is to present a few new results for isometric actions on manifolds of positive sectional curvature. One of the main new tools is

Theorem 1 *Let (M^n, g) be a closed oriented Riemannian manifold with positive sectional curvature, and let N^{n-k} be a totally geodesic compact oriented submanifold without boundary. Suppose that the codimension k satisfies $k \leq \frac{n}{3}$. Then there is a cohomology class $e \in H^k(M, \mathbb{Z})$ such that*

$$\cup e \colon H^i(M, \mathbb{Z}) \to H^{i+k}(M, \mathbb{Z})$$

is an isomorphism for $k \leq i \leq n - 2k$.

Given a Riemannian manifold with positive sectional curvature and with a 'large' amount of symmetry we can find totally geodesic submanifolds of low codimension by considering fixed point sets of isometries. By Theorem 1 this gives a period in the cohomology ring. This is a vital part in the proofs of the following results.

In order to measure the amount of symmetry of a Riemannian manifold (M^n, g) we consider its isometry group $\mathrm{Iso}(M, g)$ and assign to the pair the following three constants

$$\begin{aligned} \mathrm{symrank}(M,g) &:= \mathrm{rank}(\mathrm{Iso}(M,g)), \\ \mathrm{symdeg}(M,g) &:= \dim(\mathrm{Iso}(M,g)) \text{ and} \\ \mathrm{cohom}(M,g) &:= \dim((M,g)/\mathrm{Iso}(M,g)), \end{aligned}$$

where the last constant denotes the dimension of the orbit space or equivalently the codimension of a principal orbit. Our main results are

Theorem 2 *Let (M^n, g) be a simply connected Riemannian manifold of positive sectional curvature. Suppose that $n \geq 10$ and*

$$\mathrm{symrank}(M,g) \geq \begin{cases} \frac{n}{4}+2 & \text{if } n \equiv 3 \text{ mod } 4 \text{ or } n = 24, 25, 26. \\ \frac{n}{4}+1 & \text{otherwise} \end{cases}$$

Then M is homotopically equivalent to $\mathbb{S}^n$, $\mathbb{CP}^n$ or $\mathbb{HP}^{n/4}$.

In dimensions 12, 13 and 16 Theorem 2 turns out to be optimal.

Theorem 3 *Let (M^n, g) be a simply connected Riemannian manifold of positive sectional curvature. Suppose that*

$$\mathrm{symdeg}(M,g) \geq 2n - 5$$

Then M is homotopically equivalent to a rank 1 symmetric space or isometric to a homogeneous space of positive sectional curvature.

The simply connected homogeneous spaces of positive sectional curvature have been classified by Berger [1961], Wallach [1972] and Berard Bergery [1976]. Apart from the rank 1 symmetric spaces there are exceptional spaces in the dimensions 6, 7, 12, 13 and 24 and all of these satisfy the hypothesis of Theorem 3.

Theorem 4 *Let (M^n, g) be a simply connected Riemannian manifold of positive sectional curvature. Put $k := \mathrm{cohom}(M,g)$ and assume for the dimension n that $n \geq (30(k+1))^2$. Then M is homotopically equivalent to a rank 1 symmetric space.*