DERIVED CATEGORIES OF COHERENT SHEAVES AND MOTIVES.

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The bounded derived category of coherent sheaves $\mathbf{D}^b(X)$ is a natural triangulated category which can be associated with an algebraic variety X. It happens sometimes that two different varieties have equivalent derived categories of coherent sheaves $\mathbf{D}^b(X) \simeq \mathbf{D}^b(Y)$. There arises a natural question: can one say anything about motives of X and Y in that case? The first such example (see [4]) – abelian variety A and its dual $\widehat{A}$ – shows us that the motives of such varieties are not necessary isomorphic. However, it seems that the motives with rational coefficients are isomorphic in all known cases.

Recall a definition of the category of effective Chow motives $\mathrm{CH}^{\mathrm{eff}}(\mathsf{k})$ over a field k. The category $\mathrm{CH}^{\mathrm{eff}}(\mathsf{k})$ can be obtained as the pseudo-abelian envelope (i.e. as formal adding of cokernels of all projectors) of a category, whose objects are smooth projective schemes over k, and the group of morphisms from X to Y is the sum $\oplus_{X_i} A^m(X_i \times Y)$ (over all connected components X_i) of the groups of cycles of codimension $m = \dim Y$ on $X_i \times Y$ modulo rational equivalence (see [3, 1]). In [7] Voevodsky introduced a triangulated category of geometric motives $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathsf{k})$. He started with an additive category $\mathrm{SmCor}(\mathsf{k})$, objects of which are smooth schemes of finite type over k, and the group of morphisms from X to Y is the free abelian group generated by integral closed subschemes $Z \subset X \times Y$ such that the projection on X is finite and surjective onto a connected component of X. There is a natural embedding $[-] : \mathrm{Sm}(\mathsf{k}) \to \mathrm{SmCor}(\mathsf{k})$ of the category $\mathrm{Sm}(\mathsf{k})$ of smooth schemes of finete type over k. The category $\mathrm{SmCor}(\mathsf{k})$ is additive and one has $[X \coprod Y] = [X] \oplus [Y]$. Further, he considered the quotient of the homotopy category $\mathcal{H}^b(\mathrm{SmCor}(\mathsf{k}))$ of bounded complexes by minimal thick triangulated subcategory T, which contains all objects of the form $[X \times \mathbb{A}^1] \to [X]$ and $[U \cap V] \to [U] \oplus [V] \to [X]$ for any open covering $U \cup V = X$. Triangulated category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathsf{k})$ is defined as the pseudo-abelian envelope of the quotient category $\mathcal{H}^b(\mathrm{SmCor}(\mathsf{k}))/T$ (see [7, 1]).

There exists a canonical functor $\mathrm{CH}^{\mathrm{eff}}(\mathsf{k}) \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathsf{k})$, which is a full embedding if k admits resolution of singularities ([7, 4.2.6]). Thus, it doesn't matter in which category (in $\mathrm{CH}^{\mathrm{eff}}(\mathsf{k})$ or in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathsf{k})$) motives of smooth projective varieties are considered. Denote the

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motive of a variety X by $\mathrm{M}(X)$, and its motive in the category of motives with rational coefficients $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathsf{k})\otimes\mathbb{Q}$ (and in $\mathrm{CH}^{\mathrm{eff}}(\mathsf{k})\otimes\mathbb{Q}$) by $\mathrm{M}(X)_{\mathbb{Q}}$.

Conjecture 1. *Let X and Y be smooth projective varieties, and let $\mathbf{D}^b(X)\simeq\mathbf{D}^b(Y)$. Then the motives $\mathrm{M}(X)_{\mathbb{Q}}$ and $\mathrm{M}(Y)_{\mathbb{Q}}$ are isomorphic in $\mathrm{CH}^{\mathrm{eff}}(\mathsf{k})\otimes\mathbb{Q}$ (and in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathsf{k})\otimes\mathbb{Q}$)*

Conjecture 2. *Let X and Y be smooth projective varieties and let $F:\mathbf{D}^b(X)\to\mathbf{D}^b(Y)$ be a fully faithful functor. Then the motive $\mathrm{M}(X)_{\mathbb{Q}}$ is a direct summand of the motive $\mathrm{M}(Y)_{\mathbb{Q}}$.*

The category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathsf{k})$ has a tensor structure, and $\mathrm{M}(X)\otimes\mathrm{M}(Y)=\mathrm{M}(X\times Y)$. One defines the Tate object $\mathbb{Z}(1)$ to be the image of the complex $[\mathbb{P}^1]\to[\mathrm{Spec}(\mathsf{k})]$ placed in degree 2 and 3 and put $M(p)=M\otimes\mathbb{Z}(1)^{\otimes p}$ for any motive $M\in\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathsf{k})$ and $p\in\mathbb{N}$. The triangulated category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(\mathsf{k})$ is defined by formally inverting the functor $-\otimes\mathbb{Z}(1)$ on $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathsf{k})$. The important and nontrivial fact here is the statement that the canonical functor $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathsf{k})\to\mathrm{DM}_{\mathrm{gm}}(\mathsf{k})$ is a full embedding [7, 4.3.1]. Therefore, we can work in the category $\mathrm{DM}_{\mathrm{gm}}(\mathsf{k})$. Moreover (see [7]), for any smooth projective varieties X,Y and for any integer i there is an isomorphism

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(\mathsf{k})}(\mathrm{M}(X),\mathrm{M}(Y)(i)[2i])\cong A^{m+i}(X\times Y),\quad\text{where}\quad m=\dim Y.$$

Suppose, one has a fully faithful functor $F:\mathbf{D}^b(X)\to\mathbf{D}^b(Y)$ between derived categories of coherent sheaves of two smooth projective varieties X and Y of dimension n and m respectively. Any such functor has a right adjoint F^* by [2], and by Theorem 2.2 from [5] (see also [6, 3.2.1]) the functor F can be represented by an object on the product $X\times Y$, i.e. $F\cong\Phi_{\mathcal{A}}$, where $\Phi_{\mathcal{A}}=\mathbf{R}p_{2*}(p_1^*(-)\overset{\mathbf{L}}{\otimes}\mathcal{A})$ for some $\mathcal{A}\in\mathbf{D}^b(X\times Y)$. With any functor of the form $\Phi_{\mathcal{A}}:\mathbf{D}^b(X)\to\mathbf{D}^b(Y)$ one can associate an element $a\in A^*(X\times Y,\mathbb{Q})$ by the following rule

$$a=p_1^*\sqrt{\mathrm{td}_X}\cdot\mathrm{ch}(\mathcal{A})\cdot p_2^*\sqrt{\mathrm{td}_Y},\tag{1}$$

where td_X and td_Y are Todd classes of the varieties X and Y. The cycle a has a mixed type. Let us consider its decomposition on the components $a=a_0+\cdots+a_{n+m}$, where index is the codimension of a cycle on $X\times Y$. Each component a_q induces a map of motives

$$\alpha_q:\mathrm{M}(X)_{\mathbb{Q}}\to\mathrm{M}(Y)_{\mathbb{Q}}(q-m)[2(q-m)].$$

Thus the total cycle a gives a map $\alpha:\mathrm{M}(X)_{\mathbb{Q}}\to\bigoplus_{i=-m}^{n}\mathrm{M}(Y)_{\mathbb{Q}}(i)[2i]$. Now consider the object $\mathcal{B}\in\mathbf{D}^b(X\times Y)$, which represents the adjoint functor F^*, i.e. $F^*\cong\Psi_{\mathcal{B}}$, where $\Psi_{\mathcal{B}}=\mathbf{R}p_{1*}(p_2^*(-)\overset{\mathbf{L}}{\otimes}\mathcal{B})$. One attaches to the object $\mathcal{B}$ a cycle $b=b_0+\cdots+b_{n+m}$ defined by the same formula (1). The cycle b induces a map $\beta:\bigoplus_{i=-m}^{n}\mathrm{M}(Y)_{\mathbb{Q}}(i)[2i]\to\mathrm{M}(X)_{\mathbb{Q}}$.

Since the functor $\Phi_{\mathcal{A}}$ is fully faithful, the composition $\Psi_{\mathcal{B}} \circ \Phi_{\mathcal{A}}$ is isomorphic to the identity functor. Applying the Riemann-Roch-Grothendieck theorem, we obtain that the composition

$$\mathrm{M}(X)_{\mathbb{Q}} \xrightarrow{\alpha} \bigoplus_{i=-m}^{n} \mathrm{M}(Y)_{\mathbb{Q}}(i)[2i] \xrightarrow{\beta} \mathrm{M}(X)_{\mathbb{Q}}$$

is the identity map, i.e. $\mathrm{M}(X)_{\mathbb{Q}}$ is a direct summand of $\bigoplus_{i=-m}^{n} \mathrm{M}(Y)_{\mathbb{Q}}(i)[2i]$.

Assume now that $\dim X = \dim Y = n$ and, moreover, suppose that the support of the object A also has the dimension n. Therefore, $a_q = 0$ when $q = 0, \dots, n-1$, i.e. $a = a_n + \dots + a_{2n}$. It is easily to see that in this case $b = b_n + \dots + b_{2n}$ as well. This implies that the composition $\beta \cdot \alpha : \mathrm{M}(X)_{\mathbb{Q}} \to \mathrm{M}(X)_{\mathbb{Q}}$, which is the identity, coincides with $\beta_n \cdot \alpha_n$. Hence, $\mathrm{M}(X)_{\mathbb{Q}}$ is a direct summand of $\mathrm{M}(Y)_{\mathbb{Q}}$. Furthermore, since the cycles a_n and b_n are integral in this case we get the same result for integral motives, i.e. the integral motive $\mathrm{M}(X)$ is a direct summand of the motive $\mathrm{M}(Y)$ as well. Thus, we obtain

Theorem 1. *Let X and Y be smooth projective varieties of dimension n, and let $F : \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ be a fully faithful functor such that the dimension of the support of an object $\mathcal{A}$ on $X \times Y$, which represents F, is equal to n. Then the motive $\mathrm{M}(X)$ is a direct summand of the motive $\mathrm{M}(Y)$. If, in addition, the functor F is an equivalence, then the motives $\mathrm{M}(X)$ and $\mathrm{M}(Y)$ are isomorphic.*

Examples of such functors are known, they come from birational geometry (see e.g. [6]). In these examples one of the connected components of $\mathrm{supp}(\mathcal{A})$ gives a birational map $X \dashrightarrow Y$. Blow ups and antiflips induce fully faithful functors, and flops induce equivalences. Note that an isomorphism of motives implies an isomomorphism of any realization (singular cohomologies, l-adic cohomologies, Hodge structures and so on).

For arbitrary equivalence $\Phi_{\mathcal{A}} : \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ the map of motives $\alpha_n : \mathrm{M}(X)_{\mathbb{Q}} \to \mathrm{M}(Y)_{\mathbb{Q}}$, induced by the cycle $a_n \in A^n(X \times Y, \mathbb{Q})$, is not necessary an isomorphism (e.g. Poincare line bundle $\mathcal{P}$ on the product of abelian variety A and its dual $\widehat{A}$). However, the following conjecture, which specifies Conjecture 1, may be true.

Conjecture 3. *Let $\mathcal{A}$ be an object of $\mathbf{D}^b(X \times Y)$, for which $\Phi_{\mathcal{A}} : \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ is an equivalence. Then there exist line bundles L and M on X and on Y respectively such that the component a'_n of the object $\mathcal{A}' := p_1^* L \otimes \mathcal{A} \otimes p_2^* M$ gives an isomorphism between motives $\mathrm{M}(X)_{\mathbb{Q}}$ and $\mathrm{M}(Y)_{\mathbb{Q}}$.*

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