Pluripolar multifunctions are analytic

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Abstract. Let Ω be a domain in $\mathbb{C}^n_z$ and let $\mathcal{E}$ be a subset of $\Omega \times \mathbb{C}_w \subset \mathbb{C}^{n+1}_{z,w}$ defined by a Weierstrass pseudopolynomial, i.e. $\mathcal{E} = \{(z, w) \in \Omega \times \mathbb{C}_w : w^m + a_1(z)w^{m-1} + ... + a_m(z) = 0\}$, where $a_1(z)$, $a_2(z)$,..., $a_m(z)$ are continuous complex-valued functions in Ω. We prove that the set $\mathcal{E}$ is a pluripolar subset of $\mathbb{C}^{n+1}_{z,w}$ if and only if the functions $a_1(z)$, $a_2(z)$,..., $a_m(z)$ are holomorphic in Ω.

1. Introduction

A function φ defined on a domain $U \subset \mathbb{C}^n$ with values in $[-\infty, +\infty)$ is called *plurisubharmonic* in U if φ is upper semicontinuous and its restriction to the components of the intersection of any complex line with U is subharmonic.

A set $\mathcal{E} \subset \mathbb{C}^n$ is called *pluripolar* if there is a neighbourhood U of $\mathcal{E}$ and a plurisubharmonic function φ on U such that $\mathcal{E} \subset \{\varphi = -\infty\}$. By a result of B. Josefson [J], the function φ in this definition can be chosen to be plurisubharmonic in the whole of $\mathbb{C}^n$ (i.e. $U = \mathbb{C}^n$).

In 1963 T. Nishino raised the following question in connection to his paper [N1]:

Let Δ be the unit disk in $\mathbb{C}_z$ and let $f : \Delta \to \mathbb{C}_w$ be a continuous function such that its graph $\Gamma(f)$ is a pluripolar subset of $\mathbb{C}^2_{z,w}$. Does it follow that f is holomorphic?

A positive answer to Nishino's question in the case of an arbitrary continuous function f (when the mentioned above methods cannot be applied) was given in [S1]. In this paper we prove a more general result for multi-valued functions defined by a Weierstrass pseudopolynomial.

Theorem. *Let Ω be a domain in $\mathbb{C}^n_z$ and let $\mathcal{E}$ be a subset of $\Omega \times \mathbb{C}_w \subset \mathbb{C}^{n+1}_{z,w}$ of the form*

$$\mathcal{E} = \{(z, w) \in \Omega \times \mathbb{C}_w : w^m + a_1(z)w^{m-1} + ... + a_m(z) = 0\}, \qquad (1)$$

where $a_1(z), a_2(z), ..., a_m(z)$ are continuous complex-valued functions in Ω. Then $\mathcal{E}$ is a pluripolar subset of $\mathbb{C}^{n+1}_{z,w}$ if and only if the functions $a_1(z), a_2(z), ..., a_m(z)$ are holomorphic in Ω.

Note, that our assumption that the set $\mathcal{E}$ has the form (1) is essential for topological arguments used in the proof of Lemma 3, namely, for the fact that the first homology group $H_1(\Omega\times\mathbb{C}_w\setminus\mathcal{E},\mathbb{Z})$ is nontrivial. If we instead assume that the fibers $\mathcal{E}(z)=\{w\in\mathbb{C}_w:(z,w)\in\mathcal{E}\}$ of $\mathcal{E}$ are finite and depend continuously on $z\in\Omega$ in the Hausdorff metric, then this homology group can be equal to zero. In the last section of the paper we construct an example of a compact subset $\mathcal{E}$ of $\bar{\Delta}\times\mathbb{C}_w\subset\mathbb{C}^2_{z,w}$ ($\Delta=\{z:|z|<1\}$) with finite fibers $\mathcal{E}(z)$ depending continuously on $z\in\bar{\Delta}$ in the Hausdorff metric such that $H_1(\Delta\times\mathbb{C}_w\setminus\mathcal{E},\mathbb{Z})=0$. In particular, there is a neighbourhood $U(\mathcal{E})$ of $\mathcal{E}$ which does not contain any subset of $\bar{\Delta}\times\mathbb{C}_w$ defined by a Weierstrass pseudopolynomial.

Acknowledgements. Part of this work was done while the author was a visitor at the Max Planck Institute of Mathematics (Bonn). It is my pleasure to thank this institution for its hospitality and excellent working conditions. I would like to thank T. Ohsawa for informing me that the problem stated above was first raised in 1963 by T.Nishino and E.L.Stout for pointing me out reference on the paper [A].

2. Preliminaries

For a compact set K in $\mathbb{C}^n$, the *polynomial hull* $\hat{K}$ of K is defined as

$$\hat{K}=\{z\in\mathbb{C}^n:|P(z)|\leq\sup_{w\in K}|P(w)| \text{ for all holomorphic polynomials } P \text{ in } \mathbb{C}^n\}.$$

The set K is called *polynomially convex* if $\hat{K}=K$.

The following statement proved in [S1] will be essential in the next section.

Lemma 1 *Let K be a polynomially convex compact set in $\mathbb{C}^n$ and let $\mathcal{E}$ be a pluripolar compact set in $\mathbb{C}^n$. Then the set $(\widehat{K\cup\mathcal{E}})\setminus K$ is pluripolar.*

For bounded nonempty sets $\mathcal{E}_1$ and $\mathcal{E}_2$ in $\mathbb{C}_w$, the Hausdorff distance is defined as

$$d(\mathcal{E}_1,\mathcal{E}_2)=\sup_{w_2\in\mathcal{E}_2}\inf_{w_1\in\mathcal{E}_1}|w_1-w_2|+\sup_{w_2\in\mathcal{E}_1}\inf_{w_1\in\mathcal{E}_2}|w_1-w_2|.$$

A family of compact sets $\mathcal{E}(z)$ in $\mathbb{C}_w$ parametrized by $z\in\Omega\subset\mathbb{C}_z$ is said to be continuously dependent on z in the Hausdorff metric if for each sequence $\{z_n\}$ of points in Ω converging to a point $z_0\in\Omega$ one has $d(\mathcal{E}(z_n),\mathcal{E}(z_0))\to 0$ as $n\to\infty$. In particular, if Ω is a domain in $\mathbb{C}_z$ and $\mathcal{E}$ is a nonempty closed subset of $\Omega\times\mathbb{C}_w$ with bounded fibers $\mathcal{E}(z)=\{w\in\mathbb{C}_w:(z,w)\in\mathcal{E}\}$ depending continuously on $z\in\Omega$ in the Hausdorff metric, then $\mathcal{E}(z)$ are nonempty for all $z\in\Omega$.

Lemma 2 *Let Ω be a domain in $\mathbb{C}_z$ and let $\mathcal{E}$ be a closed subset of $\Omega\times\mathbb{C}_w \subset \mathbb{C}^2_{z,w}$ such that the fibers $\mathcal{E}(z) = \{w \in \mathbb{C}_w : (z, w) \in \mathcal{E}\}$ of $\mathcal{E}$ are bounded and depend continuously on $z \in \Omega$ in the Hausdorff metric. Suppose that the set $\mathcal{E}$ is not pseudoconcave. Then there is a point $P = (z^*, w^*) \in \mathcal{E}$ and a biholomorphic map $\mathbb{F} : (z, w) \to (z', w')$ of a neighbourhood $\mathcal{V}_P$ of P to a neighbourhood $\mathcal{V}_O$ of the origin O which has the form*

$$z' = z - z^*, w' = F(z, w) \tag{2}$$

such that

$$\mathbb{F}(\mathcal{E} \cap \mathcal{V}_P) \subset \{\operatorname{Re} w' \geq 0\} \tag{3}$$

and

$$\mathbb{F}(\mathcal{E} \cap \mathcal{V}_P) \cap \{\operatorname{Re} w' = 0\} = \mathbb{F}(P) = O. \tag{4}$$

Proof. Consider the function $\phi(z, w) = -\log R(z, w)$ on the domain $\Omega\times\mathbb{C}_w\setminus\mathcal{E}$, where $R(\cdot, \cdot)$ is the Hartogs radius of the domain $\Omega \times \mathbb{C}_w \setminus \mathcal{E}$ with respect to the axis w, i.e.

$$R(z_0, w_0) = \inf_{(z_0,w)\in\mathcal{E}} |w - w_0|.$$

for each $(z_0, w_0) \in \Omega \times \mathbb{C}_w \setminus \mathcal{E}$. From the assumption that the fibers $\mathcal{E}(z)$ depend continuously on $z \in \Omega$ in the Hausdorff metric we conclude that the function ϕ is continuous in $\Omega \times \mathbb{C}_w \setminus \mathcal{E}$. The assumption that the domain $\Omega \times \mathbb{C}_w \setminus \mathcal{E}$ is not pseudoconvex implies that for some neighbourhood V of the set $\mathcal{E}$ the restriction of ϕ to $V \setminus \mathcal{E}$ is not plurisubharmonic. Since the restriction of ϕ to each complex line $\{z = z_0\}$

$$\phi(z_0, w) = \sup_{(z_0,\tilde{w})\in\mathcal{E}} \{-\log |w - \tilde{w}|\}$$

is subharmonic being the supremum of the subharmonic functions $-\log |w - \tilde{w}|$, it follows that for some point $(z_1, w_1) \in \Omega \times \mathbb{C}_w \setminus \mathcal{E}$ and some complex line $\{w = w_1 + a(z - z_1)\}$ the restriction $\psi(z)$ of the function $\phi(z, w)$ to this line is not subharmonic near the point $z = z_1$. This means that there exists a small enough disk $\Delta_r(z_1) = \{z \in \mathbb{C}_z : |z - z_1| < r\}$ (so small that the closed disk $\{(z, w) \in \bar{\Delta}_r(z_1) \times \mathbb{C}_w : w = w_1 + a(z - z_1)\}$ is contained in $\Omega \times \mathbb{C}_w \setminus \mathcal{E}$) and a function $\tilde{\psi}$ continuous on $\bar{\Delta}_r(z_1)$ and harmonic on $\Delta_r(z_1)$ such that

$$\psi < \tilde{\psi} \quad \text{on} \quad b\Delta_r(z_1) \tag{5}$$

and

$$\tilde{\psi}(z_2) < \psi(z_2) \quad \text{for some} \quad z_2 \in \Delta_r(z_1). \tag{6}$$

Consider the one-parameter family of functions

$$g_t(z) = t\left(1 - \frac{|z - z_1|^2}{r^2}\right)$$

strictly superharmonic for $t > 0$. It follows from (5) and continuity of ψ and $\tilde{\psi}$ that $\psi(z) - \tilde{\psi}(z) < g_t(z)$ for all $z \in \bar{\Delta}_r(z_1)$ if t is big enough. Since, by (6), for $t = 0$ one has $\psi(z_2) - \tilde{\psi}(z_2) > g_0(z_2)$, we conclude that there exists $t^* > 0$ such that

$$\psi(z) - \tilde{\psi}(z) \leq g_{t^*}(z) \quad \text{for all} \quad z \in \bar{\Delta}_r(z_1) \tag{7}$$

and

$$\psi(z^*) - \tilde{\psi}(z^*) = g_{t^*}(z^*) \quad \text{for some} \quad z^* \in \Delta_r(z_1). \tag{8}$$

Consider the Hartogs domain

$$H = \{(z, w) \in \Delta_r(z_1) \times \mathbb{C}_w : |w - w_1 - a(z - z_1)| < e^{-\tilde{\psi}(z) - g_{t^*}(z)}\}.$$

It follows from harmonicity of $\tilde{\psi}$ and strict superharmonicity of g_{t^*} that the domain H is strictly pseudoconcave on the part $bH \cap (\Delta_r(z_1) \times \mathbb{C}_w)$ of its boundary. Moreover, (7) and the definition of ψ imply that $H \cap \mathcal{E} = \emptyset$ and (8) implies that there exists a point $(z^*, w^*) \in bH \cap \mathcal{E} \cap (\Delta_r(z_1) \times \mathbb{C}_w)$. To exclude the other points of the set $\mathcal{E} \cap \{z = z^*\}$ from bH we deform H in the following way.

Denote by ω the complex number with the same argument as $w^* - w_1 - a(z^* - z_1)$ and such that $|\omega| = 1$. Consider the Hartogs domain

$$H' = \{(z, w) \in \Delta_{r_1}(z_1) \times \mathbb{C}_w : |w - w_1 - a(z - z_1) - \varepsilon\omega| < e^{-\tilde{\psi}(z) - g_{t^*}(z)} - \varepsilon\},$$

where $r_1 < r$ is fixed and such that $z^* \in \Delta_{r_1}(z_1)$. Since the property of a function to be strictly superharmonic is stable under a small perturbation in C^2-metric, the function $h(z) = -\log(e^{-\tilde{\psi}(z) - g_{t^*}(z)} - \varepsilon)$ will be strictly superharmonic on $\Delta_{r_1}(z_1) \subset\subset \Delta_r(z_1)$ for $\varepsilon > 0$ small enough. Therefore, for such ε the domain H' is strictly pseudoconcave on the part $bH' \cap (\Delta_{r_1}(z_1) \times \mathbb{C}_w)$ of its boundary. It follows from the definition of H' that $H' \subset H$ and, hence, $H' \cap \mathcal{E} = \emptyset$. Moreover, it also follows that

$$bH' \cap \mathcal{E} \cap \{z = z^*\} = (z^*, w^*). \tag{9}$$

Strict superharmonicity of the function h implies that for some $r^* > 0$ such that $\bar{\Delta}_{r^*}(z^*) \subset \Delta_{r_1}(z_1)$ there exists a harmonic function u defined in $\bar{\Delta}_{r^*}(z^*)$ such that

$$h(z) < u(z) \quad \text{for} \quad z \in \bar{\Delta}_{r^*}(z^*) \setminus \{z^*\} \tag{10}$$

and

$$h(z^*) = u(z^*) \tag{11}$$

(one can actually choose $u(z)$ to be a quadratic polynomial in $\operatorname{Re} z$ and $\operatorname{Im} z$). Consider the Hartogs domain

$$H'' = \{(z, w) \in \Delta_{r^*}(z^*) \times \mathbb{C}_w : |w - w_1 - a(z - z_1) - \varepsilon\omega| < e^{-u(z)}\}.$$

It follows from (10) that $H'' \subset H'$ and, therefore,

$$H'' \cap \mathcal{E} = \emptyset. \tag{12}$$

Moreover, from (9), (10) and (11) we conclude that

$$bH^{''} \cap \mathcal{E} = (z^*, w^*). \tag{13}$$

Denote by f the function holomorphic in $\Delta_{r^*}(z^*)$ such that $\operatorname{Re} f(z) = u(z)$ and $\operatorname{Im} f(z^*) = -\arg(w^* - w_1 - a(z^* - z_1) - \varepsilon\omega)$. Define the map $\mathbb{F}$ as $z^{'} = z - z^*$, $w^{'} = \operatorname{Log}(w - w_1 - a(z - z_1) - \varepsilon\omega) + f(z)$ choosing the branch of the function Log in such a way that $\mathbb{F}(z^*, w^*) = (0, 0) = O$. Then $\mathbb{F}$ is obviously biholomorphic in a small enough neighbourhood $\mathcal{V}_P$ of the point $P = (z^*, w^*)$ and, by (12) and our choice of the function f, it satisfies the condition $\mathbb{F}(\mathcal{E} \cap \mathcal{V}_P) \subset \{\operatorname{Re} w^{'} \geq 0\}$, while by (13) it satisfies the condition $\mathbb{F}(\mathcal{E} \cap \mathcal{V}_P) \cap \{\operatorname{Re} w^{'} = 0\} = \mathbb{F}(P) = O$. This completes the proof of Lemma 2. □

3. Proof of Theorem

If the set $\mathcal{E}$ is defined by a holomorphic Weierstrass pseudopolynomial, i.e.

$$\mathcal{E} = \{(z, w) \in \Omega \times \mathbb{C}_w : w^m + a_1(z)w^{m-1} + ... + a_m(z) = 0\},$$

where the functions $a_1(z)$, $a_2(z)$,..., $a_m(z)$ are holomorphic in Ω, then the function

$$\varphi(z, w) = \ln |w^m + a_1(z)w^{m-1} + ... + a_m(z)|$$

is plurisubharmonic in $\Omega \times \mathbb{C}_w$ and $\mathcal{E} = \{\varphi = -\infty\}$. Therefore, the set $\mathcal{E}$ is pluripolar in $\mathbb{C}^{n+1}_{z,w}$.

Suppose now that the set $\mathcal{E}$ is pluripolar. It follows from Oka's theorem (see [N2], Th. 4.9, p.136) that if the set $\mathcal{E}$ is pseudoconcave, then all the functions $a_1(z)$, $a_2(z)$,..., $a_m(z)$ are holomorphic in Ω. Hence, we only need to prove pseudoconcavity of $\mathcal{E}$. For proving it we consider two cases.

1. The special case $n = 1$. We argue by contradiction and assume that the set $\mathcal{E}$ is pluripolar, has the form (1) and is not pseudoconcave. Then, in view of Lemma 2, the set $\mathcal{E}^{'} = \mathbb{F}(\mathcal{E} \cap \mathcal{V}_P)$ satisfies properties (3) and (4). Using the special form (2) of $\mathbb{F}$ and shrinking, if necessary, the neighbourhood $\mathcal{V}_O = \mathbb{F}(\mathcal{V}_P)$ of the origin O, we can assume that the set $\mathcal{E}^{'}$ has the same form (1) with, maybe, smaller value of m and some other continuous functions $a_1(z), a_2(z), ..., a_m(z)$ and, moreover, that the domain $\Omega^{'}$ in the representation (1) of the set $\mathcal{E}^{'}$ is a disk with center at $z = 0$. Without loss of generality we can assume that $\Omega^{'} = \Delta = \{z \in \mathbb{C}_z : |z| < 1\}$ is the unit disk. Since, by our assumptions, the set $\mathcal{E}$ is pluripolar, and since $\mathcal{E}^{'}$ is obtained from $\mathcal{E} \cap \mathcal{V}_P$ by a biholomorphic map $\mathbb{F}$, it follows that the set $\mathcal{E}^{'}$ is also pluripolar. With a slight abuse of notations (like the use of $\mathcal{E}$ instead of $\mathcal{E}^{'}$) we can summarize our assumptions as follows:

(i) $\mathcal{E}$ *is a subset of* $\bar{\Delta} \times \mathbb{C}_w \subset \mathbb{C}^2_{z,w}$ *of the form*

$$\mathcal{E} = \{(z,w) \in \bar{\Delta} \times \mathbb{C}_w : w^m + a_1(z)w^{m-1} + ... + a_m(z) = 0\},$$

where $a_1(z), a_2(z), ..., a_m(z)$ *are continuous complex-valued functions in* $\bar{\Delta}$.

(ii) $\mathcal{E}$ *is pluripolar.*

(iii) $\mathcal{E}$ *satisfies the properties*

$$\mathcal{E} \subset \{\operatorname{Re} w \geq 0\} \tag{14}$$

and

$$\mathcal{E} \cap \{\operatorname{Re} w = 0\} = (0,0) = O. \tag{15}$$

Define the function $\mathcal{F}$ on $\bar{\Delta}$ by

$$\mathcal{F}(z) = \min\{u : u = \operatorname{Re} w, (z,w) \in \mathcal{E}(z)\}.$$

Then (i) implies that $\mathcal{F}$ is continuous on $\bar{\Delta}$, while (14) and (15) imply that $\mathcal{F} \geq 0$ on $\bar{\Delta}$ and $\mathcal{F}(z) = 0$ if and only if $z = 0$. Therefore, one has

$$A \stackrel{\text{def}}{=} \min_{z \in b\Delta} \mathcal{F}(z) > 0.$$

Let $\mathcal{U}$ be the connected component of the set

$$\{(z,u) \in \Delta \times \mathbb{R}_u : \mathcal{F}(z) < u < \frac{A}{2}\}$$

such that the origin O of $\mathbb{C}_z \times \mathbb{R}_u$ is contained in $b\mathcal{U}$. It follows from our choice of A that $\mathcal{U}$ is a nonempty domain in $\mathbb{C}_z \times \mathbb{R}_u$.

Denote $C = \max\{|w| : (z,w) \in \mathcal{E}\}$ and consider the following polynomially convex subset of $\mathbb{C}^2_{z,w}$:

$$K = \{(z,w) \in \bar{\Delta} \times \mathbb{C}_w : \frac{A}{2} \leq u \leq 3C, |v| \leq C\}.$$

Let $\pi : \mathbb{C}^2_{z,w} \to \mathbb{C}_z \times \mathbb{R}_u$ be the projection.

Lemma 3 $\quad \mathcal{U} \subset \pi(\widehat{K \cup \mathcal{E}} \setminus K)$.

Proof. Consider an approximation of the set $\mathcal{E}$ in the Hausdorff metric by a sequence $\{\mathcal{E}_n\}$ of smooth surfaces in $\bar{\Delta} \times \mathbb{C}_w$ having the form (1), transversal to $\{u = \frac{A}{2}\}$ and such that for the corresponding domains $\mathcal{U}_n \subset \mathbb{C}_z \times \mathbb{R}_u$ one has $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U}_3 \subset \cdots \subset \mathcal{U}$ and $\cup_{n=1}^{\infty} \mathcal{U}_n = \mathcal{U}$ (one can get such approximation using suitable approximation of the functions $a_1(z)$, $a_2(z)$,..., $a_m(z)$ defining the set $\mathcal{E}$ by smooth functions in general position). It follows then from the definition of polynomial hull that $\limsup_{n\to\infty} \widehat{K\cup\mathcal{E}_n} \subset \widehat{K\cup\mathcal{E}}$, where convergence is understood to be also in the Hausdorff metric. Hence, it is enough to prove the statement of Lemma 3 in the case when $\mathcal{E}$ is smooth and transversal to $\{u = \frac{A}{2}\}$.

Now we argue by contradiction and suppose that there is a point $(z_0, u_0) \in \mathcal{U} \setminus \pi(\widehat{K\cup\mathcal{E}} \setminus K)$. Let $\tilde{\mathcal{E}}$ be the connected component of the set $\mathcal{E} \cap \{u \leq \frac{A}{2}\}$ such that the origin O of $\mathbb{C}^2_{z,w}$ is contained in $\tilde{\mathcal{E}}$. Consider the image $\tilde{\tilde{\mathcal{E}}}$ of the set $\tilde{\mathcal{E}}$ under the reflection with respect to the plane $\{u = \frac{A}{2}\}$, i.e.

$$\tilde{\tilde{\mathcal{E}}} = \{(z, u+iv) \in \mathbb{C}^2_{z,w} : (z, (A-u)+iv) \in \tilde{\mathcal{E}}\}.$$

Then the set $\mathcal{M}_{\mathcal{E}} = \tilde{\mathcal{E}} \cup \tilde{\tilde{\mathcal{E}}}$ is an oriented compact closed piece-wise smooth manifold of dimension 2 in $\mathbb{C}^2_{z,w}$ (with the orientation of $\tilde{\mathcal{E}}$ being the pull-back of the standard orientation of $\mathbb{C}_z$ under the projection $\mathbb{C}^2_{z,w} \to \mathbb{C}_z$ and the orientation of $\tilde{\tilde{\mathcal{E}}}$ being the inverse of the pull-back of the orientation of $\mathbb{C}_z$ under the same projection). Consider the curve $\gamma_R \subset \mathbb{C}^2_{z,w}$ consisting of the segment $\{z = z_0, u = u_0, -R \leq v \leq R\}$ and the half-circle $\{z = z_0, w = (u_0 + R\cos\theta) + iR\sin\theta, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}$. It follows from the construction of $\mathcal{M}_{\mathcal{E}}$ and the choice of the point (z_0, u_0) that γ_R does not intersect $\mathcal{M}_{\mathcal{E}}$ for R big enough. The curve γ_R bounds the half-disk $D = \{z = z_0, w = (u_0 + r\cos\theta) + ir\sin\theta, 0 \leq r \leq R, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}$ which has a natural orientation induced from the standard orientation of $\mathbb{C}_w$. If we orient γ_R as the boundary of D, then the linking number $l(\gamma_R, \mathcal{M}_{\mathcal{E}})$ of γ_R and $\mathcal{M}_{\mathcal{E}}$ will be positive, since D and $\mathcal{M}_{\mathcal{E}}$ (more precisely, D and $\tilde{\mathcal{E}}$) have a nonempty intersection by the definition of D, $\tilde{\mathcal{E}}$ and the choice of (z_0, u_0), and since all the intersection indexes of D and $\tilde{\mathcal{E}}$ are positive. From positivity of the linking number $l(\gamma_R, \mathcal{M}_{\mathcal{E}})$ we conclude that the curve γ_R represent a nontrivial element of the first homology group $H_1(\mathbb{C}^2_{z,w} \setminus \mathcal{M}_{\mathcal{E}}, \mathbb{Z})$. Hence, by Alexander [A, Th. 1], $\widehat{\mathcal{M}_{\mathcal{E}}} \cap \gamma_R \neq \emptyset$. Since this argument can be applied to γ_R for all R big enough, it follows that $\widehat{\mathcal{M}_{\mathcal{E}}} \cap \{(z_0, u_0+iv) : v \in \mathbb{R}\} \neq \emptyset$, i.e. $(z_0, u_0) \in \pi(\widehat{\mathcal{M}_{\mathcal{E}}})$. By the construction of $\mathcal{M}_{\mathcal{E}}$ and the definition of K one has $\mathcal{M}_{\mathcal{E}} \subset K \cup \mathcal{E}$. Therefore, $\widehat{\mathcal{M}_{\mathcal{E}}} \subset \widehat{K\cup\mathcal{E}}$ and, hence, $(z_0, u_0) \in \pi(\widehat{\mathcal{M}_{\mathcal{E}}}) \subset \pi(\widehat{K\cup\mathcal{E}})$. Since $(z_0, u_0) \notin \pi(K)$, we conclude that $(z_0, u_0) \in \pi(\widehat{K\cup\mathcal{E}} \setminus K)$. This contradicts the choice of (z_0, u_0) and the lemma follows. $\square$

Since, by our assumption (ii), the set $\mathcal{E}$ is pluripolar, and since the compact set K is polynomially convex, we conclude from Lemma 1 that the set $\widehat{K\cup\mathcal{E}} \setminus K$ is pluripolar, i.e.

$$\widehat{K\cup\mathcal{E}} \setminus K \subset \{\varphi = -\infty\} \tag{16}$$

for some plurisubharmonic function φ.

Let a be a point in $\mathbb{C}_z$ such that $\mathcal{U} \cap \{z = a\} \neq \emptyset$. For each $a' \in \mathbb{C}_z$ consider the complex line $\ell_{a'} = \{(z,w) \in \mathbb{C}^2_{z,w} : z = a'\}$ and the set

$$E_{a'} = (\widehat{K \cup \mathcal{E}} \setminus K) \cap \ell_{a'}.$$

Since the set $\mathcal{U} \subset \mathbb{C}_z \times \mathbb{R}_u$ is open, it follows from Lemma 3 and the choice of the point a that for all $a' \in \mathbb{C}_z$ close enough to a the projection of $E_{a'}$ on the real line $\ell_{a'} \cap \{v = 0\}$ contains an open segment. A polar set in $\mathbb{C}$ has Hausdorff dimension zero (see, for example [T, Th.III.19, p.65]), hence it cannot be projected on an open segment in $\mathbb{R}$. Therefore, the set $E_{a'}$ is not polar. It follows then from (16) that $\varphi \equiv -\infty$ on $\ell_{a'}$. Since this argument holds true for all $a' \in \mathbb{C}_z$ close enough to a, we conclude that $\varphi \equiv -\infty$ on $\mathbb{C}^2_{z,w}$. This contradiction proves the Theorem in the case $n = 1$.

2. The general case. Consider, as in the proof of Lemma 2, the function $\phi(z,w) = -\log R(z,w)$ on the domain $\Omega \times \mathbb{C}_w \setminus \mathcal{E} \subset \mathbb{C}^{n+1}_{z,w}$, where $R(\cdot,\cdot)$ is the Hartogs radius of the domain $\Omega \times \mathbb{C}_w \setminus \mathcal{E}$ with respect to the axis w, i.e.

$$R(z_0, w_0) = \inf_{(z_0,w)\in\mathcal{E}} |w - w_0|.$$

for each $(z_0, w_0) \in \Omega \times \mathbb{C}_w \setminus \mathcal{E}$. Then $\phi(z,w) \to +\infty$ as $(z,w) \to \mathcal{E}$ and, in view of the form (1) of the set $\mathcal{E}$, $\phi(z,w)$ is continuous on $\Omega \times \mathbb{C}_w \setminus \mathcal{E}$.

Let k be one of the numbers $1, 2, \ldots, n$. For each $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \Omega$ consider the set

$$\mathcal{E}^{\mathbf{a}}_k = \mathcal{E} \cap \{z_1 = a_1, \ldots, z_{k-1} = a_{k-1}, z_{k+1} = a_{k+1}, \ldots, z_n = a_n\} \subset \Omega^{\mathbf{a}}_k \times \mathbb{C}_w \subset \mathbb{C}^2_{z_k,w},$$

where

$$\Omega^{\mathbf{a}}_k = \Omega \cap \{z_1 = a_1, \ldots, z_{k-1} = a_{k-1}, z_{k+1} = a_{k+1}, \ldots, z_n = a_n\} \subset \mathbb{C}_{z_k}.$$

Since, by our assumptions, the set $\mathcal{E}$ is pluripolar, there is a function φ, plurisubharmonic in $\mathbb{C}^{n+1}_{z,w}$, such that $\mathcal{E} \subset \{\varphi = -\infty\}$. For all points $\mathbf{a}$ except for a pluripolar set in $\mathbb{C}^n_z$ one obviously has that the function

$$\varphi^{\mathbf{a}}_k(z_k, w) = \varphi(a_1, \ldots, a_{k-1}, z_k, a_{k+1}, \ldots, a_n, w)$$

is not identically equal to $-\infty$ in $\mathbb{C}^2_{z_k,w}$. For all such points $\mathbf{a}$ we can use the argument from case 1 and conclude from the inclusion $\mathcal{E}^{\mathbf{a}}_k \subset \{\varphi^{\mathbf{a}}_k = -\infty\}$ that the set $\mathcal{E}^{\mathbf{a}}_k$ is pseudoconcave. Therefore, the function

$$\phi^{\mathbf{a}}_k(z_k, w) = \phi(a_1, \ldots, a_{k-1}, z_k, a_{k+1}, \ldots, a_n, w)$$

is plurisubharmonic in $\Omega^{\mathbf{a}}_k \times \mathbb{C}_w \setminus \mathcal{E}^{\mathbf{a}}_k$. Since the complement of a pluripolar set is everywhere dense, it follows from continuity of ϕ that the functions $\phi^{\mathbf{a}}_k$ are plurisubharmonic for all $\mathbf{a} \in \Omega$. This argument holds true for any $k = 1, 2, \ldots, n$, hence the whole function ϕ is plurisubharmonic and, therefore, the set $\mathcal{E}$ is pseudoconcave. The proof of the Theorem is now completed. $\square$

4. Example

We prove first the following simple lemma.

Lemma 4 *Let f and g be holomorphic functions, defined in a neighbourhood U of a point $a \in \mathbb{C}_z$, such that $f(a) = g(a)$ and $f^{'}(a) \neq g^{'}(a)$. Let r be a positive number such that $\bar{\Delta}_r(a) = \{z \in \mathbb{C}_z : |z-a| \leq r\} \subset U$ and $f(z) \neq g(z)$ for $z \in \bar{\Delta}_r(a) \setminus \{a\}$. Then for all sufficiently small $\varepsilon > 0$ the complex curve $\Sigma \subset \Delta_r(a) \times \mathbb{C}_w$ defined by the equation*

$$G(z,w) \stackrel{\text{def}}{=} (w - f(z))(w - g(z)) - \varepsilon = 0 \tag{17}$$

is a branched covering over the disk $\Delta_r(a)$ with 2 branches and 2 branching points

$$b^{\pm} = a \pm \frac{2i}{f'(a) - g'(a)}\sqrt{\varepsilon} + O(\varepsilon). \tag{18}$$

Proof. Equation (17) is quadratic with respect to w, hence Σ is a branched covering over $\Delta_r(a)$ with 2 branches. A point b is a branching point of Σ if for some w_b one has $0 = G_w^{'}(b, w_b) = 2w_b - f(b) - g(b)$. Therefore, $w_b = \frac{1}{2}(f(b) + g(b))$ and then (17) implies that $-\frac{1}{4}(f(b) - g(b))^2 - \varepsilon = 0$, i.e.

$$f(b) - g(b) = \pm 2i\sqrt{\varepsilon}. \tag{19}$$

Hence, in view of our choice of r, $b \to a$ as $\varepsilon \to 0$. Then, using Taylor expansions of f and g at the point a, we conclude from (19) and the assumption $f(a) = g(a)$ that $(f^{'}(a) - g^{'}(a))(b-a) + O(|b-a|^2) = \pm 2i\sqrt{\varepsilon}$. Finilly, the assumption $f^{'}(a) \neq g^{'}(a)$ implies that

$$b - a = \pm\frac{2i}{f'(a) - g'(a)}\sqrt{\varepsilon} + O(|b-a|^2) = \pm\frac{2i}{f'(a) - g'(a)}\sqrt{\varepsilon} + O(\varepsilon).$$

$\square$

Construction of the set $\mathcal{E}$. Let ρ be a smooth real-valued function defined on the segment $[0,1]$ such that

$$\rho(t) = \begin{cases} 1, & \text{for} \quad 0 \leq t \leq \frac{1}{3}, \\ \text{decreasing}, & \text{for} \quad \frac{1}{3} < t < \frac{2}{3}, \\ 0, & \text{for} \quad \frac{2}{3} \leq t \leq 1. \end{cases}$$

Consider the set

$$\mathcal{E}_1 = \{(z,w) \in \bar{\Delta} \times \mathbb{C}_w : w^2 = \rho(|z|)z\},$$

where, as above, $\Delta = \{z \in \mathbb{C}_z : |z| < 1\}$ is the unit disk. This set has 2 branches over the disk $\Delta_{\frac{2}{3}}(0)$ with one branching point at $z = 0$. The branches are glued to each other along the circle $\mathcal{A} = \{|z| = \frac{2}{3}, w = 0\}$ and become one branch $\{w = 0\}$ for $\frac{2}{3} \leq |z| \leq 1$. Consider some points $A_1 = (a_1, 0)$ and $A_3 = (a_3, \sqrt{a_3})$ of $\mathcal{E}_1$ and a point $A_2 = (a_2, C)$ with a_1, a_2, a_3 and C real and positive such that $\frac{2}{3} < a_1 < 1$, $0 < a_3 < \frac{1}{3}$ and $a_3 < a_2 < a_1$. Further, consider the complex line $\mathcal{L}'$ passing through the points A_2 and A_1 and the complex line $\mathcal{L}''$ passing through the points A_2 and A_3. Let a_1, a_2, a_3 be already chosen and consider C so big that the line $\mathcal{L}''$ intersects $\mathcal{E}_1$ in two points A_3 and $A_3' = (a_3', -\sqrt{a_3'})$ with a_3' real such that $0 < a_3' < a_3$, and the line $\mathcal{L}'$ intersects $\mathcal{E}_1$ only at the point A_1. The set $\mathcal{E}$ will be constructed as a small deformation of the set $\mathcal{E}_1 \cup ((\mathcal{L}' \cup \mathcal{L}'') \cap (\bar{\Delta} \times \mathbb{C}_w))$ near the points A_k, $k = 1, 2, 3$, that creates, as in Lemma 4, 2 branching points instead of each selfintersection point.

Let $r > 0$ be so small that the disks $\bar{\Delta}_1 = \bar{\Delta}_r(a_1)$, $\bar{\Delta}_2 = \bar{\Delta}_r(a_2)$ and $\bar{\Delta}_3 = \bar{\Delta}_r(a_3)$ do not intersect each other and the circle $\{|z| = \frac{2}{3}\}$ and, moreover, do not contain the point a_3'. Denote by E_1 the set $(\mathcal{E}_1 \cup \mathcal{L}') \cap (\Delta_1 \times \mathbb{C}_w)$, by E_2 the set $(\mathcal{L}' \cup \mathcal{L}'') \cap (\Delta_2 \times \mathbb{C}_w)$ and by E_3 the connected component of the set $(\mathcal{E}_1 \cup \mathcal{L}'') \cap (\Delta_3 \times \mathbb{C}_w)$ containing the point A_3. Then each of the sets E_k, $k = 1, 2, 3$, is the union of the graphs of two holomorphic functions f_k^j, $j = 1, 2$, having the same value and different derivatives, both of them real (which is easy to check by direct calculation), at the center of the respective disk Δ_k. Therefore, we can apply Lemma 4 to each of these sets and, if ε is small enough, we will get branched coverings Σ_1, Σ_2 and Σ_3 over the disks Δ_1, Δ_2 and Δ_3, respectively, with 2 branches and 2 branching points contained in the smaller disks $\Delta_1' = \Delta_{\frac{r}{3}}(a_1)$, $\Delta_2' = \Delta_{\frac{r}{3}}(a_2)$ and $\Delta_3' = \Delta_{\frac{r}{3}}(a_3)$. Moreover, since for each $k = 1, 2, 3$ the derivatives at the center of the disk Δ_k of the functions f_k^j, $j = 1, 2$, are real, we conclude from (18) that one point in the pair of the branching points contained in Δ_k' is contained in the half-disk $\{z \in \Delta_k' : \operatorname{Im} z > 0\}$, while the other point in this pair of branching points is contained in the half-disk $\{z \in \Delta_k' : \operatorname{Im} z < 0\}$. Since both branching points of each set Σ_k are contained in the respective disk Δ_k', the set $\Sigma_k \cap ((\Delta_k \setminus \Delta_k') \times \mathbb{C}_w)$ will be the union of the graphs of 2 holomorphic functions $\tilde{f}_k^j$, $j = 1, 2$, defined on $\Delta_k \setminus \Delta_k'$ and, moreover, if ε is small enough, then each function $\tilde{f}_k^j$ will be close enough to the corresponding function f_k^j. Define the functions

$$\tilde{\tilde{f}}_k^j(z) = \rho\left(\frac{|z - a_k|}{r}\right) \tilde{f}_k^j(z) + \left(1 - \rho\left(\frac{|z - a_k|}{r}\right)\right) f_k^j(z),$$

for $z \in \Delta_k \setminus \Delta_k'$, $k = 1, 2, 3$, $j = 1, 2$. Let $\tilde{\Sigma}_k$ be the union of the graphs of $\tilde{\tilde{f}}_k^1$ and $\tilde{\tilde{f}}_k^2$. Now we can define the set $\mathcal{E}$ as

$$\mathcal{E} = \left(\left(\mathcal{E}_1 \cup \left((\mathcal{L}' \cup \mathcal{L}'') \cap (\bar{\Delta} \times \mathbb{C}_w) \right) \right) \setminus \bigcup_{k=1}^{3} E_k \right) \cup \bigcup_{k=1}^{3} \left(\tilde{\Sigma}_k \cup \left(\Sigma_k \cap (\bar{\Delta}_k' \times \mathbb{C}_w) \right) \right).$$

Define also the set $\mathcal{E}^{\text{reg}}$ as $\mathcal{E}$ with the circle $\mathcal{A}$, the point A_3' of the transversal selfintersection of $\mathcal{E}$ and all the branching points of $\mathcal{E}$ being removed. Then, by our construction, $\mathcal{E}^{\text{reg}}$ is a smooth connected 2-dimensional surface transversal to the w-direction.

Note, that each fiber $\mathcal{E}(z)$ of the set $\mathcal{E}$ has at most 4 points and that the fibers $\mathcal{E}(z)$ depend continuously on $z \in \bar{\Delta}$ in the Hausdorff metric.

Claim 1. $H_1(\Delta \times \mathbb{C}_w \setminus \mathcal{E}, \mathbb{Z}) = 0$.

Proof. Let a be a real positive number such that $a_3 \le a < \frac{1}{3}$. Consider the point $A = (a, -\sqrt{a}) \in \mathcal{E}$ and a disk $\bar{\mathcal{D}}_s = \{(z,w) : z = a, |w + \sqrt{a}| \le s\}$ so small that it intersects the set $\mathcal{E}$ only at the point A. We prove first that the circle $\mathcal{C}_s = b\mathcal{D}_s$ is homological to zero in $\Delta \times \mathbb{C}_w \setminus \mathcal{E}$.

Consider the curve $z(t)$ in $\mathbb{C}_z$ defined as

$$z(t) = \begin{cases} a(1-t) + (a_1 + r)t, & \text{for} \quad 0 \le t \le 1, \\ a_1 + re^{\pi i(t-1)}, & \text{for} \quad 1 < t \le 2, \\ (a_1 - r)(3-t) + (a_3 + r)(t-2), & \text{for} \quad 2 < t \le 3, \\ a_3 + re^{\pi i(t-3)}, & \text{for} \quad 3 < t \le 4, \\ (a_3 - r)(5-t) + \frac{2}{3}(t-4), & \text{for} \quad 4 < t \le 5. \end{cases}$$

If $\pi_z : \mathbb{C}^2_{z,w} \to \mathbb{C}_z$ is the projection, then the curve $z(t)$ admits a uniquely defined lifting by π_z^{-1} to the piece-wise smooth curve γ in $\mathcal{E}$ with the initial point A.

The curve γ is transversal to the w-direction and has one point of selfintersection, namely, the endpoint $(\frac{2}{3}, 0)$, where two smooth curves on the side $\{|z| < \frac{2}{3}\}$ meet each other.

Geometric description of the curve γ looks as follows. We start from the point $A = (a, -\sqrt{a})$ and then, over the segment $\{a \le \operatorname{Re} z < \frac{2}{3}, \operatorname{Im} z = 0\}$ the curve γ is contained in the "lower" branch of the set $\mathcal{E}_1$, while over the segment $\{\frac{2}{3} \le \operatorname{Re} z \le a_1 - r, \operatorname{Im} z = 0\}$ γ is contained in the only branch $\{w = 0\}$ of $\mathcal{E}_1$ for $\{|z| > \frac{2}{3}\}$. Since both branching points of Σ_1 are contained in $\Delta_1 = \{|z - a_1| < r\}$, and since only one of them is contained in the half-disk $\{z \in \Delta_1 : \operatorname{Im} z > 0\}$, we conclude that over the segment $\{a_1 - r \le \operatorname{Re} z \le a_1 + r, \operatorname{Im} z = 0\}$ the curve γ will "change from the branch $\mathcal{E}_1$ to the branch $\mathcal{L}'$". Then, over the half-circle $\{|z - a_1| = r, \operatorname{Im} z > 0\}$ and the segment $\{a_2 + r \le \operatorname{Re} z \le a_1 - r, \operatorname{Im} z = 0\}$ γ is contained in $\mathcal{L}'$. After that, applying the same argument as we used for the segment $\{a_1 - r \le \operatorname{Re} z \le a_1 + r, \operatorname{Im} z = 0\}$, we conclude that over the segment $\{a_2 - r \le \operatorname{Re} z \le a_2 + r, \operatorname{Im} z = 0\}$ the curve γ will "change from the branch $\mathcal{L}'$ to the branch $\mathcal{L}''$". Then, over the segment $\{a_3 + r \le \operatorname{Re} z \le a_2 - r, \operatorname{Im} z = 0\}$ and the half-circle $\{|z - a_3| = r, \operatorname{Im} z > 0\}$ γ is contained in $\mathcal{L}''$. After that, the same argument as above shows that over the segment $\{a_3 - r \le \operatorname{Re} z \le a_3 + r, \operatorname{Im} z = 0\}$ the curve γ will "change from the branch $\mathcal{L}''$ to the branch $\mathcal{E}_1$". And finally, over the segment $\{a_3 + r \le \operatorname{Re} z \le \frac{2}{3}, \operatorname{Im} z = 0\}$ the curve γ is contained in the "upper" branch of $\mathcal{E}_1$ up to the endpoint $(\frac{2}{3}, 0)$, where we meet the first part of the curve γ which was (for $|z| < \frac{2}{3}$) contained in the "lower" branch of $\mathcal{E}_1$.

For each $z_0 \in \pi_z(\gamma)$ and each $s > 0$ consider the sets

$$\Gamma_s(z_0) = \{(z_0, w) : \min_{(z_0, w')\in\gamma} |w - w'| = s\}$$

and

$$\Omega_s(z_0) = \{(z_0, w) : \min_{(z_0, w')\in\gamma} |w - w'| < s\}.$$

Then, for s small enough, each set $\Omega_s(z_0)$ is the union of finitely many (at most 3) disks in $\{z_0\} \times \mathbb{C}_w$ which are disjoint, if z_0 is far enough from the circle $\{|z| = \frac{2}{3}\}$, and is the union of two connected components, one of which is a disk and the other one is the union of two disks having nonempty intersection, if $\{|z_0| < \frac{2}{3}\}$ and z_0 is close enough to the circle $\{|z| = \frac{2}{3}\}$. As $|z_0| \to \frac{2}{3}$ from the side $\{|z| < \frac{2}{3}\}$, the centers of two disks constituting the second connected component of $\Omega_s(z_0)$ becomes closer to each other and for $\{|z_0| \geq \frac{2}{3}\}$ this component becomes just one disk. Each set $\Omega_s(z_0)$ has a natural orientation induced from $\mathbb{C}_w$ and, hence, $\Gamma_s(z_0) = b\Omega_s(z_0)$ has also a natural orientation.

Consider the set

$$T_s = \bigcup_{z_0\in\pi_z(\gamma)} \Gamma_s(z_0).$$

Since the curve γ is piece-wise smooth, it follows from the definition of $\Gamma_s(z_0)$ that the set T_s is a piece-wise smooth surface of dimension 2 in $\Delta \times \mathbb{C}_w$ with the boundary on the chosen above circle $\mathcal{C}_s$. Moreover, since γ is oriented, and since each set $\Gamma_s(z_0)$ is oriented, we can also orient the surface T_s. Topologically, T_s is a torus with a disk removed, $\mathcal{C}_s$ being the boundary of this disk. Since the curve $\gamma \subset \mathcal{E}$ is transversal to the w-direction, we conclude that $T_s \subset \Delta \times \mathbb{C}_w \setminus \mathcal{E}$ for s sufficiently small. This implies that the homology class $[\mathcal{C}_s]$ of the circle $\mathcal{C}_s$ in $H_1(\Delta \times \mathbb{C}_w \setminus \mathcal{E}, \mathbb{Z})$ is trivial.

Now we observe that for each point $(z, w) \in \mathcal{E}^{\text{reg}}$ the circle $\mathcal{C}_s(z, w) = \{(z, w') : |w - w'| = s\}$ is homological to zero, if $s > 0$ is small enough. Indeed, since the set $\mathcal{E}^{\text{reg}}$ is connected, there is a smooth curve $\tilde{\gamma} \subset \mathcal{E}^{\text{reg}}$ connecting the points A and (z, w). Then, for $s > 0$ small enough, the set $\mathcal{M}_s = \{(z, w') : |w - w'| = s, (z, w) \in \tilde{\gamma}\}$ is a smooth "cylinder" of dimension 2 which is contained in $\Delta \times \mathbb{C}_w \setminus \mathcal{E}$ and has the boundary on $\mathcal{C}_s(z, w)$ and $\mathcal{C}_s$. Therefore, the circles $\mathcal{C}_s(z, w)$ and $\mathcal{C}_s$ represent the same homology class in $H_1(\Delta \times \mathbb{C}_w \setminus \mathcal{E}, \mathbb{Z})$. Since $\mathcal{C}_s$ is already proved to be homological to zero in $\Delta \times \mathbb{C}_w \setminus \mathcal{E}$, it follows that $\mathcal{C}_s(z, w)$ is also homological to zero in $\Delta \times \mathbb{C}_w \setminus \mathcal{E}$.

Finally, let $\mathcal{C}$ be any smooth closed curve in $\Delta \times \mathbb{C}_w \setminus \mathcal{E}$. Then, there is a 2-dimensional disk $\mathcal{D}$ smoothly embedded into $\Delta \times \mathbb{C}_w$ such that $\mathcal{C} = b\mathcal{D}$. We can assume that the disk $\mathcal{D}$ is in general position, in particular, that $\mathcal{D}$ intersects $\mathcal{E}$ in finitely many points $\{(z_p, w_p)\}_{p=1}^n$ which are contained in $\mathcal{E}^{\text{reg}}$. Without restriction of generality, we can also assume that $\mathcal{D}$ is parallell to the w-direction in a neighbourhood of each point (z_p, w_p). Then the disks $\mathcal{D}_s(z_p, w_p) = \{(z_p, w') : |w_p - w'| \leq s\}$ are contained in $\mathcal{D}$ for $s > 0$ small enough. Therefore, $\mathcal{C} = b\mathcal{D}$ is homological to $\bigcup_{p=1}^{n} b\mathcal{D}_s(z_p, w_p)$ in $\Delta \times \mathbb{C}_w \setminus \mathcal{E}$, $\mathcal{D} \setminus \bigcup_{p=1}^{n} \mathcal{D}_s(z_p, w_p)$ being the homology. Since each circle $\mathcal{C}_s(z, w) = b\mathcal{D}_s(z_p, w_p)$ is already proved to be homological to zero in $\Delta \times \mathbb{C}_w \setminus \mathcal{E}$, we conclude that $\mathcal{C}$ is also homological to zero. The proof of the claim is now completed. $\square$

As an application of Claim 1 we show the following property of the set $\mathcal{E}$.

Claim 2. *There exists a neighbourhood $U(\mathcal{E})$ of the set $\mathcal{E}$ which does not contain any subset of $\bar{\Delta} \times \mathbb{C}_w$ defined by a Weierstrass pseudopolynomial.*

Proof. Assume, to get a contradiction, that every neighbourhood $U(\mathcal{E})$ of $\mathcal{E}$ contains a subset defined by a Weierstrass pseudopolynomial. For R big enough consider the circle $\mathcal{C}_R = \{z = 0, |w| = R\} \subset \Delta \times \mathbb{C}_w \setminus \mathcal{E}$ oriented counterclockwise in w-variable. Then, in view of Claim 1, there is a 2-chain S such that $bS = \mathcal{C}_R$ and $\operatorname{supp} S \subset \Delta \times \mathbb{C}_w \setminus \mathcal{E}$. The last inclusion implies that there exists a neighbourhood $U(\mathcal{E})$ of $\mathcal{E}$ such that $\operatorname{supp} S \cap U(\mathcal{E}) = \emptyset$. By our assumption, there is a subset $\tilde{\mathcal{E}}$ of $U(\mathcal{E})$ which is defined by a Weierstrass pseudopolynomial, i.e. it has the form (1). Since $\operatorname{supp} S \cap \tilde{\mathcal{E}} = \emptyset$, the homology class $[\mathcal{C}_R]$ of the circle $\mathcal{C}_R$ in $H_1(\Delta \times \mathbb{C}_w \setminus \tilde{\mathcal{E}}, \mathbb{Z})$ is trivial. Consider the continuous map $\Phi : \Delta \times \mathbb{C}_w \setminus \tilde{\mathcal{E}} \to S^1$ defined by

$$\Phi(z, w) = \frac{w^m + a_1(z)w^{m-1} + ... + a_m(z)}{|w^m + a_1(z)w^{m-1} + ... + a_m(z)|}. \tag{20}$$

Then, on one hand, $[\mathcal{C}_R] = 0$ in $H_1(\Delta \times \mathbb{C}_w \setminus \tilde{\mathcal{E}}, \mathbb{Z})$ and, hence, $\Phi_* ([\mathcal{C}_R]) = 0$ in $H_1(S^1, \mathbb{Z})$. On the other hand, the term w^m in the nominator of formula (20) will be dominating for $(z, w) \in \mathcal{C}_R$, if R is big enough. Therefore, the degree of the restriction of Φ to $\mathcal{C}_R$ (it is a map from S^1 to S^1) is equal to m. Hence, $\Phi_* ([\mathcal{C}_R]) = m\,[S^1] \neq 0$ in $H_1(S^1, \mathbb{Z})$. This gives the desired contradiction and proves the claim. $\square$

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