Automorphisms and moduli spaces of varieties with ample canonical class via deformations of abelian covers

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Abstract

By a recent result of Viehweg, projective manifolds with ample canonical class have a coarse moduli space, which is a union of quasiprojective varieties. In this paper, we prove that there are manifolds with ample canonical class that lie on arbitrarily many irreducible components of the moduli; moreover, for any finite abelian group G there exist infinitely many components M of the moduli of varieties with ample canonical class such that the generic automorphism group G_M is equal to G.

In order to construct the examples, we use abelian covers. Let Y be a smooth complex projective variety of dimension ≥ 2. A Galois cover $f : X \to Y$ whose Galois group is finite and abelian is called an abelian cover of Y; by [Pa1], it is determined by its building data, i.e. by the branch divisors and by some line bundles on Y, satisfying appropriate compatibility conditions. Natural deformations of an abelian cover are also introduced in [Pa1].

In this paper we prove two results about abelian covers: first, that if the building data are sufficiently ample, then the natural deformations surject on the Kuranishi family of X; second, that if the building data are sufficiently ample and generic, then $Aut(X) = G$.

These results, although in some sense "expected", are in fact rather powerful and enable us to construct the required examples. Finally, note that it is essential for our applications to be able to deal with general abelian covers and not only with cyclic ones.

1 Introduction

Coverings of algebraic varieties are a classical theme in algebraic geometry, since Riemann's description of curves as branched covers of the projective line. Double covers were used by the Italian school to construct examples that shed

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light on the theory of surfaces and to describe special classes of surfaces, as in the case of Enriques surfaces.

More recently, cyclic coverings have been extensively applied by several authors to the study of surfaces of general type; it will be enough to recall the work of Horikawa, Persson and Xiao Gang. Abelian covers have been used by Hirzebruch to give examples of surfaces of general type on and near the line $c_1^2 = 3c_2$; Catanese and Manetti have used bidouble and iterated double covers, respectively, of $\mathbf{P}^1 \times \mathbf{P}^1$ to construct explicitly connected components of the moduli space of surfaces of general type.

In [Pa1], the second author has given a complete description of abelian covers of algebraic varieties in terms of the so-called building data, namely of certain line bundles and divisors on the base of the covering, satisfying suitable compatibility relations. Natural deformations of an abelian cover $f : X \to Y$ are also introduced there and it is shown that they are complete, if Y is rigid, regular and of dimension ≥ 2, and if the building data are sufficiently ample. (Natural deformations are obtained by modifying the equations defining X inside the total space of the bundle $f_*\mathcal{O}_X$).

In this paper we study natural deformations of an abelian cover $f : X \to Y$ and prove that they are complete for varieties of dimension at least two if the branch divisors are sufficiently ample. The result requires no assumption on Y, and in particular also holds when the cover has obstructed deformations; this is a key technical step towards the moduli space constructions described below.

We then turn to the study of the automorphism group of the cover. Since the automorphism group of a variety of general type is finite, one would expect that in the case of a Galois cover it coincides with the Galois group, at least if the cover is generic. Our main theorem 4.6 shows that this is indeed the case for an abelian cover, if the branch divisors are generic and sufficiently ample.

We construct explicitly coarse moduli spaces of abelian covers and complete families of natural deformations for a fixed base of the cover Y; this is useful if one wants to investigate the birational structure of the components of the moduli obtained by the methods of this paper.

The main application of the results described so far is the study of moduli of varieties with ample canonical class. Recently Viehweg proved the existence of a coarse moduli space for varieties with ample canonical class of arbitrary dimension, generalizing Gieseker's result for surfaces. Given an irreducible component M of the moduli space of varieties with ample canonical class, the automorphism group G_M of a generic variety in M is well-defined. In contrast with the case of curves (where this group is trivial for $g \geq 3$), it was already known in the case of surfaces that there exist infinitely many components M of the moduli with nontrivial automorphism group G_M; it is easy to construct examples such that G_M contains an involution, and Catanese gave examples where G_M contains a subgroup isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$. There are also, of course, easy examples of components M where G_M is trivial (for instance the hypersurfaces of degree $d \geq 5$ in $\mathbf{P}^3$).

As a first application of theorem 4.6 we prove that for any finite abelian group G there are infinitely many irreducible components M of the moduli of varieties with ample canonical class such that $G_M = G$; notice that we precisely determine G_M instead of just bounding it from below.

We also prove that there are varieties with ample canonical class lying on arbitrarily many irreducible components of the moduli. We distinguish these components by means of their generic automorphism group; there are examples both in the equidimensional and in the non-equidimensional case. In the surface case, this answers a question raised by Catanese in [Ca1].

Let S be a surface of general type; Xiao has given explicit upper bounds both for the cardinality of $Aut(S)$ and of an abelian subgroup of $Aut(S)$, in terms of the invariants of S ([Xi1], [Xi2]). Some upper bounds are also known for a higher-dimensional variety X with ample canonical class, although sharp bounds are still lacking. It seems interesting to ask whether these bounds can be improved by considering instead of $Aut(X)$ the group $Aut_{\text{gen}}(X)$, namely the intersection in $Aut(X)$ of the images of the generic automorphism groups G_M of all irreducible components M of the moduli space containg X (in particular, if X lies in a unique component M, then $Aut_{\text{gen}}(X) = G_M$).

As a first step towards the computation of a sharp bound for $\#Aut_{\text{gen}}$, we show that such a bound cannot be too small; for example, we construct a sequence of surfaces of general type X such that K_X^2 tends to infinity and $\#Aut_{\text{gen}}(X)$ grows asymptotically as $K_X^2(\log_2 K_X^2)^{-2}$.

The paper goes as follows: in section 2 we collect some results from the literature and set up the notation. In section 3 we prove that, if the branch divisors are sufficiently ample, then infinitesimal natural deformations are complete. In section 4 we prove (theorem 4.6) that the automorphism group of an abelian cover coincides with the Galois group if the building data are sufficiently ample and generic. To do this, we prove some results on extensions of automorphisms, which we believe should be of independent interest. The proof of 4.6 is based on a degeneration argument and requires an explicit partial desingularization, contained in section 7. Section 5 contains the construction of a coarse moduli space for abelian covers of a given variety Y and of a complete family of natural deformations. Finally, in section 6 we apply the results of sections 3 and 4 to the study of moduli spaces of varieties with ample canonical class, as stated above.

Acknowledgements. This work was supported by the italian MURST 60% funds. The first author would also like to thank the Max-Planck-Institut für Mathematik (Bonn) for hospitality and the italian CNR for support.

2 Notation and conventions

All varieties will be complex, and smooth and projective unless the contrary is explicitly stated.

For a projective morphism of schemes $Y \to S$, $Hilb_S(Y)$ will be the relative Hilbert scheme (see [FGA], exposé 221). When Y is smooth over S, $Hilb_S^{div}(Y)$ will be the (open and closed) subscheme of $Hilb_S(Y)$ parametrizing divisors (see [Fo] for a proof of this). When S is a point, it will be omitted from the notation.

For Y a smooth projective variety, let $NS(Y)$ be the image in $H^2(Y, \mathbf{Z})$ of $Pic(Y)$, and $Pic^{\xi}(Y)$ the inverse image of $\xi \in NS(Y)$; let $q(Y) = \dim H^1(Y, \mathcal{O}_Y)$ be the dimension of $Pic^0(Y)$.

Let $\mathcal{X} \to B$ be any flat family, with integral fibres. Then there are open subschemes $Aut_{\mathcal{X}/B}$ and $Bir_{\mathcal{X}/B}$ of the relative Hilbert scheme $Hilb_B(\mathcal{X} \times_B \mathcal{X})$ parametrizing fibrewise the (graphs of) automorphisms and birational automorphisms of the fibre ([FGA], [Ha]).

We denote the cardinality of a (finite) set S by $\#S$; for each integer $m \geq 2$, let $\zeta_m = e^{2\pi i/m}$.

Notation for abelian covers. The following notation will be used freely throughout the paper: we collect it here for the reader's convenience.

G will be a finite abelian group, G^* its dual; the order of an element g will be denoted by $o(g)$. Let I_G be the set of all pairs (H, ψ) where H is a cyclic subgroup of G with at least two elements and ψ is a generator of H^*. There is a bijection between I_G and $G \setminus 0$ given by $(H, \psi) \mapsto g$ where $g \in H$ is such that $\psi(g) = \zeta_{\#H}$. For $\chi \in G^*$, $i = (H_i, \psi_i) \in I_G$, let a^i_χ be the unique integer such that $0 \leq a^i_\chi < m_i$ (where $m_i = \#H_i$) and $\chi_{|H_i} = \psi_i^{a^i_\chi}$ (cfr. [Pa1], remark 1.1 on p. 195, where a^i_χ is denoted by $f_{H,\psi}(\chi)$). Let $\varepsilon^i_{\chi,\chi'} = [(a^i_\chi + a^i_{\chi'})/m_i]$, where $[r]$ is the integral part of a real number r; note that $\varepsilon^i_{\chi,\chi'}$ is either 0 or 1.

A basis of G will be a sequence of elements of G, $(e_1, \ldots, e_s)$, such that G is the direct sum of the (cyclic) subgroups generated by the e_j's, and such that $o(e_j)$ divides $o(e_{j+1})$ for each $j = 1, \ldots, s-1$. Given a basis $(e_1, \ldots, e_s)$ of G, we will call dual basis of G^* the s-tuple $(\chi_1, \ldots, \chi_s)$, where $\chi_j(e_i) = 1$ if $i \neq j$ and $\chi_i(e_i) = \zeta_{o(e_i)}$. We will write a^i_j instead of $a^i_{\chi_j}$, for all $j = 1, \ldots, s$; for $\chi = \chi_1^{\alpha_1} \cdots \chi_s^{\alpha_s}$, let

$$q^i_\chi = \left[\sum_{j=1}^{s} \frac{\alpha_j a^i_j}{m_i} \right].$$

Note that, unlike a^i_χ, q^i_χ depends on the choice of the basis and not only on χ and i.

Lemma 2.1 *Let G be as above, and let $I \subset I_G$ be a subset with k elements (which we denote by $1, \ldots, k$) such that the natural map $H_1 \oplus \ldots \oplus H_k \to G$ is surjective. Then the $k \times s$ matrix (a^i_j) has rank s over $\mathbf{Q}$.*

PROOF. Let g_i be the element corresponding to (H_i, ψ_i) via the bijection $I_G \leftrightarrow G \setminus 0$ described above. Then, for any $i = 1, \ldots, k$ and for any $j = 1, \ldots, s$, one has $a^i_j/m_i = \lambda_{ij}/n_j$, where $n_j = o(e_j)$ and $g_i = \sum \lambda_{ij} e_j$, with $0 \leq \lambda_{ij} < n_j$ and

$\lambda_{ij} \in \mathbf{Z}$. So the matrix (a^i_j) has the same rank over $\mathbf{Q}$ as the matrix λ_{ij}. On the other hand λ_{ij} is the matrix associated to the natural map $H_1 \oplus \ldots \oplus H_k \to G$, which is surjective. Let p be a prime factor of n_1, hence of all of the n_j's. Then the map $\mathbf{Z}_p^k \to \mathbf{Z}_p^s$ represented by the matrix (λ_{ij}) mod p is also surjective, hence the matrix (λ_{ij}) has an $s \times s$ minor whose determinant is nonzero modulo p. This implies that the determinant is nonzero, hence the result. □

Let X be any projective variety. A *deformation* of X over a pointed analytic space (T, o) will be a flat, proper map $\mathcal{X} \to T$, together with an isomorphism of the special fibre $\mathcal{X}_o$ with X.

Deformations modulo isomorphism are a contravariant functor Def_X from the category $Ansp_0$ of pointed analytic spaces to the category *Sets*, where the functoriality is given by pullback.

More generally, given a contravariant functor $F : Ansp_0 \to Sets$, we will use the same letter F to denote the induced functor on the categories *Germs* of germs of analytic spaces and Art^* of finite length spaces supported in a point (i.e. *Spec*'s of local Artinian C-algebras). For the properties of functors on Art^*, we refer the reader to [Schl].

Let M be an irreducible component of the moduli space of (projective) manifolds with ample canonical class. As the automorphism group is semicontinuous (see corollary 4.5), it makes sense to speak of the automorphism group of a generic manifold in M; we will denote it by G_M. Note that for any X such that $[X] \in M$, there is a natural identification of G_M with a subgroup of $Aut(X)$. If X is a minimal surface of general type, we denote the intersection in $Aut(X)$ of G_M for all components M containing $[X]$ by $Aut_{\text{gen}}(X)$; it is the largest subgroup H of $Aut(X)$ such that the action of H extends to any small deformation of X.

3 Deformations of abelian covers

In this section we introduce natural deformations of a smooth abelian cover and prove that infinitesimal natural deformations are complete, if the branch divisors are sufficiently ample and the dimension is at least two.

We start by recalling from [Pal] some fundamental results on abelian covers; the reader will find there a more detailed exposition and proofs of the following statements.

Let G be a finite abelian group and let I be a subset of I_G: we will use freely throughout the paper the notation introduced in section 2. Let Y be a smooth projective variety: a (G, I)-cover of Y is a normal variety X and a Galois cover $f : X \to Y$ with Galois group G and branch divisors D_i (for $i \in I$) having (H_i, ψ_i) as inertia group and induced character (see [Pal] for details). X is smooth if and only if the D_i's are smooth, their union is a normal crossing divisor, and, whenever $D_{i_1}, \ldots, D_{i_k}$ have a common point, the natural map

$H_{i_1} \oplus \ldots \oplus H_{i_k} \to G$ is injective. The cover is said to be *totally ramified* if the natural map $\bigoplus_{i \in I} H_i \to G$ is surjective. Note that each abelian cover can be factored as composition of a totally ramified with an unramified cover.

Let $M_i = \mathcal{O}_Y(D_i)$. The vector bundle $f_*\mathcal{O}_X$ on Y splits naturally as sum of eigensheaves L_χ^{-1} for $\chi \in G^*$, and multiplication in the $\mathcal{O}_Y$-algebra $f_*\mathcal{O}_X$ induces isomorphisms

$$L_\chi \otimes L_{\chi'} = L_{\chi\chi'} \otimes \bigotimes_{i \in I} M_i^{\otimes \epsilon^i_{\chi,\chi'}} \qquad \text{for all } \chi, \chi' \in G^* \setminus 1. \tag{3.0.1}$$

Denote L_{χ_j} by L_j, and let $n_j = o(\chi_j)$. The isomorphisms above induce isomorphisms

$$L_j^{\otimes n_j} = \bigotimes_{i \in I} M_i^{\otimes (n_j a^i_j)/m_i} \qquad \text{for all } j = 1, \ldots, s. \tag{3.0.2}$$

The (D_i, L_χ) are the *building data* of the cover; the (D_i, L_j) are the *reduced building data*. The sheaves L_χ can be recovered from the reduced building data by setting, for $\chi = \chi_1^{\alpha_1} \cdots \chi_s^{\alpha_s}$,

$$L_\chi = \bigotimes_{j=1}^{s} L_j^{\alpha_j} \otimes \bigotimes_{i \in I} M_i^{-q^i_\chi}. \tag{3.0.3}$$

Conversely, for each choice of (D_i, L_χ) (resp. (D_i, L_j)) satisfying equation (3.0.1) (resp. (3.0.2)), there exists a unique cover having these as (reduced) building data. Note that equations (3.0.2) have a solution in $Pic(Y)$ (viewing the line bundles M_i's as parameters and the L_j's as variables) if and only if their images via c_1 have a solution in $NS(Y)$.

Assumption 3.1 *In this paper all (G, I)-covers will be totally ramified. Unless otherwise stated, $f : X \to Y$ will be a (G, I)-cover, with reduced building data (D_i, L_j). We will also assume that X and Y are smooth, of dimension ≥ 2, and that X has ample canonical class.*

We say that a property holds whenever a line bundle L (or a divisor D) is sufficiently ample if it holds whenever $c_1(L)$ (or $c_1(D)$) belongs to a (given) suitable translate of the ample cone. It is easy to see that assumption 3.1 implies the following: if all of the D_i's are sufficiently ample then so is L_χ for any $\chi \neq 1$. Moreover, if V is a vector bundle, $V \otimes L$ is ample for any sufficiently ample L.

Let $S = \{(i, \chi) \in I \times G^* | \chi_{|H_i} \neq \psi_i^{-1}\}$. Given a (G, I)-cover $X \to Y$ as above, together with sections $s_{i,\chi}$ of $H^0(M_i \otimes L_\chi^{-1})$ for all $(i, \chi) \in S$, a natural deformation of X was defined in [Pal], §5. We now give a functorial (and more general) version of that definition in order to be able to apply standard techniques from deformation theory.

Definition 3.2 A *natural deformation of the reduced building data* of $f : X \to Y$ over $(T, o) \in Ansp_0$ is $(\mathcal{Y}, \mathcal{M}_i, \mathcal{L}_j, s_{i,\chi}, \varphi_j)$ where:

1. $i \in I$, $j = 1, \ldots, r$, and $(i, \chi) \in S$;
2. $\mathcal{Y} \to T$ is a deformation of Y over T;
3. $\mathcal{L}_j$ and $\mathcal{M}_i$ are line bundles on $\mathcal{Y}$ such that $\mathcal{L}_j$ restricts to L_j and $\mathcal{M}_i$ to M_i over o;
4. $\varphi_j : \mathcal{L}_j^{\otimes n_j} \to \bigotimes \mathcal{M}_i^{\otimes (n_j a_j^i)/m_i}$ is an isomorphism whose restriction to $\mathcal{Y}_o$ coincides with the isomorphism $L_j^{\otimes n_j} \to \bigotimes M_i^{\otimes (n_j a_j^i)/m_i}$ given by multiplication;
5. $s_{i,\chi}$ is a section of $\mathcal{L}_\chi^{-1} \otimes \mathcal{M}_i$, where $\mathcal{L}_\chi = \bigotimes_{j=1}^{s} \mathcal{L}_j^{\alpha_j} \otimes \bigotimes_{i \in I} \mathcal{M}_i^{-q_\chi^i}$;
6. $s_{i,\chi}$ restricts over $\mathcal{Y}_o$ to $s_{i,\chi}^0$, where $s_{i,\chi}^0 = 0$ if $\chi \neq 1$, and $s_{i,1}^0$ is a section of M_i defining D_i.

We will say that a deformation is *Galois* if $s_{i,\chi} = 0$ for $\chi \neq 1$.

Natural deformations modulo isomorphism define a contravariant functor $\mathrm{Dnat}_X : Ansp_0 \to Sets$, and Galois deformations are a subfunctor Dgal_X. Note that the inclusion $\mathrm{Dgal}_X \hookrightarrow \mathrm{Dnat}_X$ is naturally split. We now extend formulas in §5 of [Pa1] to define a natural transformation of functors $\mathrm{Dnat}_X \to \mathrm{Def}_X$.

Definition 3.3 Let T be a germ of an analytic space, and let

$$(\mathcal{Y}, \mathcal{L}_j, \mathcal{M}_i, \varphi_j, s_{i,\chi}) \in \mathrm{Dnat}_X(T).$$

Let V be the total space of the vector bundles $\bigoplus_{\chi \in G^*} \mathcal{L}_\chi$, and let $\pi : V \to \mathcal{Y}$ be the natural projection. For a line bundle $\mathcal{L}$ on $\mathcal{Y}$, denote its pullback to V by $\bar{\mathcal{L}}$, and analogously for sections and isomorphisms. Each of the line bundles $\bar{\mathcal{L}}_\chi$ has a tautological section σ_χ.

For each pair $(\chi, \chi') \in G^* \times G^*$, the isomorphisms φ_j induce isomorphisms

$$\varphi_{\chi,\chi'} : \mathcal{L}_\chi \otimes \mathcal{L}_{\chi'} \to \mathcal{L}_{\chi\chi'} \otimes \bigotimes \mathcal{M}_i^{\epsilon^i_{\chi,\chi'}}.$$

Let $\tau_i \in H^0(V, \bar{\mathcal{M}}_i)$ be defined by

$$\tau_i = \sum_{\{\chi | (i,\chi) \in S\}} \bar{s}_{i,\chi} \sigma_\chi.$$

Define a section $\rho_{\chi,\chi'}$ of $\bar{\mathcal{L}}_\chi \otimes \bar{\mathcal{L}}_{\chi'}$ by

$$\rho_{\chi,\chi'} = \sigma_\chi \sigma_{\chi'} - \bar{\varphi}^*_{\chi,\chi'}(\sigma_{\chi\chi'} \prod \tau_i^{\epsilon^i_{\chi,\chi'}}).$$

Then the zero locus of all the $\rho_{\chi,\chi'}$ is naturally a deformation $\mathcal{X} \to T$ of X over T (in particular X can be naturally identified with the fibre of $\mathcal{X} \to T$ over the closed point). This is proven in [Pa1] in the case where the deformation of Y, L_j and M_i is the trivial one, but it is easy to see that the same proof works in our generalized setting. The deformation $\mathcal{X} \to T$ so obtained is called the *natural deformation of* X associated to the given natural deformation of the reduced building data.

It is now clear why Galois deformations were called that way:

Remark 3.4 *Let $\mathcal{X} \to T$ be a deformation of X induced by a Galois deformation $(\mathcal{Y}, \ldots)$ of the reduced building data; $\mathcal{X}$ has a canonical structure of (G, I)-cover of $\mathcal{Y}$, induced by the action of G on the total space of the line bundle $\mathcal{L}_\chi$ given by the character χ.*

The restrictions to the category Art^* of the functors Dnat_X and Dgal_X satisfy Schlessinger's conditions for the existence of a projective hull (see [Schl]); in fact, they can be described (as usual in deformation theory) in terms of tangent and obstruction spaces. If $F : Art^* \to Sets$ is a contravariant functor, then we denote its tangent (resp. obstruction) space by $T^1(F)$ (resp. $T^2(F)$), when this makes sense.

Lemma 3.5 *There is a natural action of G on Dnat_X, whose invariant locus is Dgal_X; the decomposition of $T^l(\mathrm{Dnat}_X)$ according to characters, for $l = 1, 2$, is the following:*

$$T^l(\mathrm{Dgal}_X) = T^l(\mathrm{Dnat}_X)^{\mathrm{inv}} = H^l(Y, T_Y(-\log \textstyle\sum D_i)); \qquad (3.5.1)$$

$$T^l(\mathrm{Dnat}_X)^\chi = \bigoplus_{i \in S_\chi} H^{l-1}(Y, \mathcal{O}_Y(D_i) \otimes L_\chi^{-1}) \qquad \textit{for } \chi \neq 1; \quad (3.5.2)$$

where $S_\chi = \{i \in I | (i, \chi) \in S\}$.

Proof. An element $g \in G$ acts by

$$(\mathcal{Y}, \mathcal{M}_i, \mathcal{L}_j, s_{i,\chi}, \varphi_j) \mapsto (\mathcal{Y}, \mathcal{M}_i, \mathcal{L}_j, \chi(g)s_{i,\chi}, \varphi_j).$$

It is clear that Dgal_X is contained in the invariant locus. It is not difficult to show the other inclusion using the fact that the cover is totally ramified.

We now study separately tangent and obstructions spaces corresponding to the different characters. For the trivial character, i.e. Dgal_X, the functor is isomorphic to the deformation functor of the data (Y, M_i, s_i); (3.5.1) is then well known (see [We]).

Fix a nontrivial character χ. Then the problem reduces to studying the deformations of the zero section of a line bundle, given a deformation of the base and of the bundle. The statement can then be proven by applying the following lemma. □

Lemma 3.6 *Let $o \in B' \subset B \in Art^*$ be schemes of length $1, n, n+1$ respectively for some n; for schemes, etc. over B denote the restriction to B' by a prime and the restriction to o by $_o$. Let $\mathcal{Y} \to B$ be a smooth projective morphism, $\mathcal{L}$ a line bundle on $\mathcal{Y}$; let s' be a section of $\mathcal{L}'$, such that $s'_o = 0$. Then the obstruction to lifting s' to a section s of $\mathcal{L}$ lies in $H^1(\mathcal{Y}_o, \mathcal{L}_o)$, and two liftings differ by an element of $H^0(\mathcal{Y}_o, \mathcal{L}_o)$.*

PROOF. Let U_α be an affine open cover of $Y = \mathcal{Y}_o$ such that L is trivial on each U_α. Let $U_{\alpha\beta}$ be $U_\alpha \cap U_\beta \subset U_\alpha$. As Y is smooth, we have that $\mathcal{Y}$ is covered by open subsets V_α isomorphic to $U_\alpha \times B$, glued via B-isomorphisms $\varphi_{\alpha\beta} : U_{\alpha\beta} \times B \to U_{\beta\alpha} \times B$ satisfying the cocycle condition and restricting to the identity over o. Let $g_{\alpha\beta}$ be transition functions for $\mathcal{L}$ with respect to the open cover V_α.

The section s' can be described by functions s'_α on $U_\alpha \times B'$ such that, on $U_{\alpha\beta} \times B'$,

$$s'_\alpha = g'_{\alpha\beta}(s'_\beta \circ \varphi_{\alpha\beta}).$$

Extend s'_α arbitrarily to a function s_α on $U_\alpha \times B$; any other extension is of the form $s_\alpha + \varepsilon\sigma_\alpha$, where $\varepsilon = 0$ is an equation of B' in B and σ_α is a function on U_α (as $\varepsilon f = 0$ for any function f in the ideal of o in B). If an extension s of s' exists, there must be functions σ_α on U_α such that, on $U_{\alpha\beta} \times B$,

$$s_\alpha + \varepsilon\sigma_\alpha = g_{\alpha\beta}((s_\beta + \varepsilon\sigma_\beta) \circ \varphi_{\alpha\beta}).$$

Let $u_{\alpha\beta} = s_\alpha - g_{\alpha\beta}(s_\beta \circ \varphi_{\alpha\beta})$. The restriction of $u_{\alpha\beta}$ to $U_{\alpha\beta} \times B'$ is zero, hence $u_{\alpha\beta}$ is divisible by ε: let $u_{\alpha\beta} = \varepsilon v_{\alpha\beta}$. One can verify, using the fact that $s_o = 0$, that $v_{\alpha\beta}$ is a cocycle in $H^1(Y, \mathcal{L}_o)$: it is enough to check that

$$u_{\alpha\beta} + g_{\alpha\beta}(u_{\beta\gamma} \circ \varphi_{\alpha\beta}) = u_{\alpha\gamma}$$

on $U_{\alpha\beta\gamma}$; for all triples α, β, γ of indices of the cover. It is then immediate to verify that $v_{\alpha\beta}$ is the obstruction to lifting s' to $\mathcal{Y}$, and the statement about the difference of two liftings can be proven in a similar way. □

We now recall some properties of Def_X. Let $\mathrm{Def}_X^G : Ansp_0 \to Sets$ be the functor of deformations of X together with the G action.

Lemma 3.7 *There is a natural action of G on Def_X, whose invariant locus is Def_X^G.*

PROOF. Let $\mathcal{X} \to T$ be a deformation of X over (T, o); there is a given isomorphism $i : X \to \mathcal{X}_o$. The action of an element $g \in G$ is given by replacing i with $i \circ \varphi(g)$, where $\varphi : G \to Aut(X)$ is the natural action.

It is clear that if G acts on a deformation $\mathcal{X} \to T$, then this belongs to Def_X^G. The other implication follows from [Ca1], §7 or directly from the fact that the automorphisms of X and of its deformations are rigid. □

Note that, as X is of general type, the G-action on Def_X induces an action on the Kuranishi family $\mathcal{X} \to B$ of X; the restriction of the Kuranishi family to the fixed locus B^G is universal for the functor Def_X^G (compare ([Pi], (2.8) p. 19, [Cal], §7).

Recall the following result from [Pal].

Lemma 3.8 *Let X be a smooth (G, I)-cover of Y with building data (D_i, L_χ). Then the decomposition according to characters of $H^l(X, T_X)$ is as follows:*

$$H^l(X, T_X)^{\mathrm{inv}} = H^l(T_Y(-\log \sum_{i \in I} D_i)) \qquad (3.8.1)$$

$$H^l(X, T_X)^\chi = H^l(T_Y(-\log \sum_{i \in S_\chi} D_i) \otimes L_\chi^{-1}) \qquad \text{if } \chi \neq 1 \qquad (3.8.2)$$

where S_χ is the same as in lemma 3.5.

PROOF. This follows immediately from proposition 4.1. in [Pal]. □

Corollary 3.9 *Assume that, for all $\chi \in G^* \setminus 1$, the bundles L_χ and $\Omega^1_Y \otimes L_\chi$ are ample. Then there are natural exact sequences, for all $\chi \in G^* \setminus 1$:*

$$0 \to \bigoplus_{i \in S_\chi} H^0(Y, \mathcal{O}(D_i) \otimes L_\chi^{-1}) \to H^1(X, T_X)^\chi \to 0. \qquad (3.9.1)$$

$$0 \to \bigoplus_{i \in S_\chi} H^1(Y, \mathcal{O}(D_i) \otimes L_\chi^{-1}) \to H^2(X, T_X)^\chi. \qquad (3.9.2)$$

PROOF. Fix $\chi \neq 1$, let $D = \sum_{i \in S_\chi} D_i$, and consider the following diagram of sheaves with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \bigoplus_{i \in S_\chi} \mathcal{O}_Y & = & \bigoplus_{i \in S_\chi} \mathcal{O}_Y & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_Y(-\log D) & \longrightarrow & \mathcal{P}^* & \longrightarrow & \bigoplus_{i \in S_\chi} \mathcal{O}_Y(D_i) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_Y(-\log D) & \longrightarrow & T_Y & \longrightarrow & \bigoplus_{i \in S_\chi} \mathcal{O}_{D_i}(D_i) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $\mathcal{P}$ is the prolongation bundle associated to the normal crossing divisor D. By the previous lemma, it is enough to prove that the first two cohomology groups of $\mathcal{P}^* \otimes L_\chi^{-1}$ vanish; this follows from the corresponding vanishing for L_χ^{-1} and $T_Y \otimes L_\chi^{-1}$, and the latter is just Kodaira vanishing (it is here that one needs the assumption $\dim Y \geq 2$). □

The natural transformation of functors $\mathrm{Dnat}_X \to \mathrm{Def}_X$ defined in 3.3 is equivariant with respect to the natural actions of G on these functors. Therefore, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Dgal}_X & \longrightarrow & \mathrm{Def}_X^G \\ \downarrow & & \downarrow \\ \mathrm{Dnat}_X & \longrightarrow & \mathrm{Def}_X \end{array}$$

where the vertical arrows are injections. The following theorem shows that the horizontal arrows are smooth morphisms of functors when the branch divisors are sufficiently ample.

This was proven in [Pal] under the hypothesis that Y be rigid and regular; in this case natural deformations are unobstructed, and it is enough to check the surjectivity of the Kodaira-Spencer map. In the general case one has to take into account the obstructions as well.

Theorem 3.10 *Let $f : X \to Y$ be a totally ramified (G, I)-cover with building data D_i, L_χ, such that X and Y are smooth of dimension ≥ 2 and that X is of general type. Assume that for all $\chi \in G^* \setminus 1$ the bundles L_χ and $\Omega^1_Y \otimes L_\chi$ are ample. Then the natural map of functors (from Art* to Sets) $\mathrm{Dnat}_X \to \mathrm{Def}_X$ is smooth, and so is the induced map Dgal_X and Def_X^G.*

PROOF. By a well-known criterion, smoothness of a natural transformation of functors is implied by surjectivity of the induced map on tangent spaces, and injectivity on obstruction spaces.

This is immediate by lemma 3.5 and corollary 3.9, and by the fact that the map between tangent (obstruction) spaces induced by the map of functors is the natural one. □

4 Main theorem

In this section we will prove that the automorphism group of an abelian cover is precisely the Galois group, provided that the branch divisors are sufficiently ample and generic. The proof depends on the construction of an explicit partial resolution of some singular covers, which will be given in section 7.

Although the result is in some sense expected, the proof is rather involved and the techniques applied are, we believe, of independent interest.

The following lemma is inspired by a similar result of McKernan ([McK]).

Lemma 4.1 *Let Δ be the unit disc in $\mathbf{C}$, $\Delta^* = \Delta \setminus \{0\}$. Let $p : \mathcal{X} \to \Delta$ be a flat map, smooth over Δ^*, whose fibres are integral projective varieties of non* *negative Kodaira dimension. Assume we are given a section σ of $Aut_{\mathcal{X}/\Delta^*}$. If there exists a resolution of singularities $\varepsilon : \tilde{\mathcal{X}} \to \mathcal{X}$ such that each divisorial component of the exceptional locus has Kodaira dimension $-\infty$, then σ can be (uniquely) extended to a section of $Bir_{\mathcal{X}/\Delta}$.*

PROOF. The section σ induces a birational map $\varphi : \mathcal{X} \dashrightarrow \mathcal{X}$ over Δ; the uniqueness of the extension follows from this. Let $\tilde{\varphi} : \tilde{\mathcal{X}} \dashrightarrow \tilde{\mathcal{X}}$ be the induced birational map, and let Γ be a resolution of the closure of the graph of $\tilde{\varphi}$; let p_1, p_2 be the natural projections of Γ on $\tilde{\mathcal{X}}$ (such that $p_2 = \tilde{\varphi} \circ p_1$), and let $q_i = \varepsilon \circ p_i$.

The strict transform $\mathcal{X}_0'$ of $\mathcal{X}_0$ in Γ via q_1^{-1} has positive Kodaira dimension, hence it cannot be contracted by p_2, which is a birational morphism with smooth image. Therefore the restriction of p_2 to $\mathcal{X}_0'$ is birational (because $\mathcal{X}_0'$ is not contained in the exceptional locus of p_2) onto some irreducible divisor $\mathcal{X}_0''$ in $\tilde{\mathcal{X}}$.

As $\mathcal{X}_0''$ is birational to $\mathcal{X}_0$ it cannot be of Kodaira dimension $-\infty$; hence it is not contained in the exceptional locus of ε. Therefore $\varepsilon(\mathcal{X}_0'')$ is a divisor contained in $\mathcal{X}_0'$, hence it is $\mathcal{X}_0'$ by irreducibility, and the map $\varepsilon : \mathcal{X}_0'' \to \mathcal{X}_0$ is birational.

So the birational map φ can be extended to $\mathcal{X}_0$ by the birational map $q_2 \circ \left(q_{1|\mathcal{X}_0'}\right)^{-1}$. □

Lemma 4.2 *In the same hypotheses of lemma 4.1, assume moreover that there is a line bundle L on $\mathcal{X}$, flat over Δ, whose restriction to $\mathcal{X}_t$ is very ample for all t, and such that $h^0(\mathcal{X}_t, L_{|\mathcal{X}_t})$ is constant in t. If the action of σ can be lifted to an action on L, then σ can be uniquely extended to a section of $Aut_{\mathcal{X}/\Delta}$.*

PROOF. Let N be the rank of the vector bundle p_*L on Δ; choosing a trivializing basis yields an embedding $\mathcal{X} \hookrightarrow \mathbf{P}^{N-1} \times \Delta$. The automorphisms φ_t of $\mathcal{X}_t$ are restrictions to $\mathcal{X}_t$ of nondegenerate projectivities of $\mathbf{P}^{N-1}$; their limit, as $t \to 0$, is a well-defined, possibly degenerate projectivity φ_0. This gives an extension of φ to an open set of $\mathcal{X}_0$; this must now be birational by the previous lemma, which in turn implies that φ_0 is nondegenerate (as $\mathcal{X}_0$ is not contained in a hyperplane), and therefore that φ_0 is a morphism. Applying the same argument to φ^{-1} concludes the proof. □

Remark 4.3 *The hypothesis that σ acts on L is obviously verified if $L_{|\mathcal{X}_t}$ is a pluricanonical bundle for all $t \neq 0$.*

Proposition 4.4 *Let $p : \mathcal{X} \to \Delta$ be a flat family of integral projective varieties of general type, smooth over Δ^*. Assume that there is a line bundle L on $\mathcal{X}$, flat over Δ, with $L_t := L_{|\mathcal{X}_t}$ ample on $\mathcal{X}_t$, and $Aut(\mathcal{X}_t)$ acts on L_t for $t \neq 0$. Assume moreover that for any m-th root base change $\rho_m : \Delta \to \Delta$ the pullback $\rho_m^*\mathcal{X}$ admits a resolution having only divisors of negative Kodaira dimension in the exceptional locus. Then $Aut_{\mathcal{X}/\Delta}$ is proper over Δ, and the cardinality of the fibre is an upper semi-continuous function.*

PROOF. After replacing L with a suitable multiple and maybe shrinking Δ, we can assume that L_t is very ample on $\mathcal{X}_t$, and that $h^0(\mathcal{X}_t, L_t)$ is constant in t. The map $Aut_{\mathcal{X}/\Delta} \to \Delta$ is obviously quasi-finite (because the fibres are of general type) and the fibres are reduced (because automorphism groups are always reduced in char. 0). It is enough to prove that given a map of a pointed curve (C, P) to Δ and a lifting of the map to $Aut_{\mathcal{X}/B}$ out of P, the lifting can be extended to P.

Via restriction to an open set we can assume that C is the unit disc Δ, P is the origin and $\Delta \to \Delta$ is the map $z \to z^m$; we can then apply lemma 4.2 to conclude the proof. □

Corollary 4.5 *Let $\mathcal{X} \to B$ be a smooth family of varieties having ample canonical bundle. Then the scheme $Aut_{\mathcal{X}/B}$ is proper over B, and the cardinality of the fibre is an upper semi-continuous function.*

PROOF. We can apply the previous proposition with $L = K_{\mathcal{X}/\Delta}$. □

Theorem 4.6 *Let Y be a smooth projective variety, and X a smooth (G, I)-cover with ample canonical bundle, with covering data L_χ, D_i. Let $H = \mathcal{O}_Y(1)$ for some embedding of Y in $\mathbf{P}^{N-2}$; assume that the linear system*

$$|D_1 - m_1 NH|$$

is base-point-free (where H is the line bundle giving the embedding). Assume also that the $\mathbf{Q}$-divisor

$$M = K_Y - (m_1 - 1)NH + \sum_{i \in I} \frac{(m_i - 1)}{m_i} D_i$$

is ample on Y. Then, for a generic choice of D_1 in its linear system, X has automorphism group isomorphic to G.

PROOF. Let d be the number of automorphisms of a generic cover with the given covering data (cfr. corollary 4.5). It is enough to show that $d \leq \#G$, the other inequality being obvious.

Let H be as in the statement of the theorem, and let $\mathcal{H} \subset |H|$ be the (not necessarily complete) linear system giving the embedding; let $H_1, \ldots, H_N$ be N projectively independent divisors in $\mathcal{H}$. Assume that the H_i's are generic, in particular that they are smooth and that their union with all of the D_i's has normal crossings. Let $m = m_1$, $D = D_1$.

The strategy of the proof is the following: start from a generic cover X of Y, and construct a sequence of manifolds $X_1, \ldots, X_N$ and of subgroups G_k of $Aut(X_k)$ such that

$$\#Aut(X) \leq \#G_1 \leq \cdots \leq \#G_N \qquad \text{and} \qquad G_N = G.$$

In fact, X_k will be a (G, I)-cover of Y with covering data $D^{(k)}, D_2, \ldots, L_X^{(k)}$, where $L_X^{(k)} = L_X - ka_X^1 H$ and $D^{(k)}$ is a generic divisor in $|D - kmH|$ (recall that a_X^i was defined as the unique integer a satisfying $0 \leq a \leq m_i - 1$ and $\chi_{|H_i} = \psi_i^a$). We let G_k be the group of automorphisms of X_k preserving the inverse images of the curves $H_1, \ldots, H_k$ in Y.

We therefore want to prove the following:

1. $\#Aut(X) \leq \#G_1$;
2. $\#G_k \leq \#G_{k+1}$;
3. $G_N = G$.

FIRST STEP: $\#Aut(X) \leq \#G_1$. Let $D^{(1)}$ be a generic divisor in $|D - mH|$, and choose equations f_1, g and h_1 for H_1, D and $D^{(1)}$ respectively. Define divisors $\mathcal{D}_i$ on $Y \times \mathbf{C}$ by $\mathcal{D}_i = D_i \times \mathbf{C}$ for $i \neq 1$, $\mathcal{D}_1 = \{(1-t)f_1^m h_1 + tg = 0\}$; let $\mathcal{X}^1$ be the corresponding abelian cover. $\mathcal{X}_0^1$ is a singular variety (singular along the inverse image of the curve H_1 in Y), with smooth normalization X_1 (see [Pal], step 1 of normalization algorithm of p. 203). Note that X_1 is of general type by the ampleness assumption on M.

By proposition 7.3, the family $\mathcal{X}^1$ and each n-th root base change of $\mathcal{X}^1$ admit a resolution with only divisors of Kodaira dimension $-\infty$ in the exceptional locus. Moreover, the pull-back of M restricts to the g-canonical bundle on the smooth fibres of $\mathcal{X}^1$ (cfr the proof of prop. 4.2 in [Pal], p. 208). Applying proposition 4.4 gives that $Aut_{\mathcal{X}^1/\mathbf{C}}$ is proper over $\mathbf{C}$, and hence that $\#Aut(X) \leq Aut(\mathcal{X}_0^1)$ (as we assumed X to be generic). On the other hand it is clear that each automorphism of $\mathcal{X}_0^1$ lifts to the normalization X_1, yielding an automorphism which maps to itself the inverse image of the singular locus, i.e., the inverse image of the curve H_1.

SECOND STEP: $\#G_{k-1} \leq \#G_k$. We use a similar construction; let X_{k-1} be as above, let h_{k-1} be an equation of $D^{(k-1)}$, f_k an equation of H_k, and h_k an equation of $D^{(k)}$. Define a (G, I)-cover $\mathcal{X}^k$ of $Y \times \mathbf{C}$ branched over $D_i \times \mathbf{C}$ for $i \neq 1$, and over $\mathcal{D}_1^{(k)} = \{(1-t)f_k^m h_k + th_{k-1} = 0\}$; $\mathcal{X}_0^k$ is singular along the inverse image C_k of H_k, and its normalization is X_k; again X_k is of general type.

Again by proposition 7.3 the family $\mathcal{X}^{(k)}$ and all its n-th root base changes have a resolution with only uniruled components in the exceptional locus; the same argument as before proves the result.

FINAL STEP: $G_N = G$. Let $\pi : X_N \to Y$ be the covering map: G_N is the group of automorphisms of Y fixing the inverse images of the curves $H_1, \ldots, H_N$. Every element of G_N preserves $\pi^*(\mathcal{H})$, hence induces an automorphism of Y; this automorphism must be the identity as it induces the identity on $\mathcal{H}$. Therefore G_N must coincide with G. □

Remark 4.7 *In theorem 4.6 we can replace the assumption that the linear system $|D_1 - m_1 N H|$ be base point free by asking that for each $i \in I$*

$$|D_i - m_i N_i H|$$

be base point free, with N_i nonnegative integers with sum N; we then get that, for a generic choice of the D_i's such that $N_i \neq 0$, $Aut(X) = G$.

Example 4.8 One might wonder whether it is always true that a generic abelian cover of general type has no "extra automorphisms". Here is an easy example where this is not the case. Consider a $\mathbf{Z}_3$-cover of $\mathbf{P}^1$, branched over two pairs of distinct points, with opposite characters. A generic such cover is a smooth genus 2 curve, hence its automorphism group cannot be $\mathbf{Z}_3$.

Example 4.9 Here is a slightly more complicated example of extra automorphisms, which works in any dimension. Let Y be a principally polarized abelian variety, and let L be a principal polarization; assume that L is symmetric, i.e. invariant under the natural involution $\sigma(y) = -y$ on Y. The sections of $L^{\otimes 2}$ are all symmetric, and the associated linear system has no base points. Let $G = \mathbf{Z}_2^s$, with the canonical basis $e_1, \ldots, e_s$. Choose $I = \{1, \ldots, s\}$, and let H_i be the subgroup generated by e_i, for $i = 1, \ldots, s$.

The equations for the reduced building data become $L_j^{\otimes 2} = \mathcal{O}_Y(D_j)$; we choose the solution $L_j = L$, $M_i = L^{\otimes 2}$ for all i, j. We are in fact constructing a fibred product of double covers. Choose the D_i's to be generic divisors in the linear system $|L^{\otimes 2}|$. Each of them must be symmetric; this implies that the involution σ can be lifted to an involution of X, which is an automorphism not contained in the Galois group of the cover.

Note that in this case the total branch divisor can become arbitrarily large, still all (G, I)-covers have an automorphism group bigger than G.

5 Moduli spaces of abelian covers and global constructions

In this section we will explicitly construct a coarse moduli space for abelian covers of a smooth variety Y and a complete space of natural deformations. Although some of the material in this section is implicit in [Pal], we find it important to state it in a precise and explicit way. In particular we will apply theorem 3.10 to construct (under suitable ampleness assumptions) a family of natural deformations which maps dominantly to the moduli (theorem 5.12).

Let Y be a smooth, projective variety, G an abelian group, I a subset of I_G. A *family of smooth (G, I)-covers* of Y over a base scheme T is a smooth, proper map $\mathcal{X} \to T$ and an action of G on $\mathcal{X}$ compatible with the projection on T, together with a T-isomorphism of the quotient $\mathcal{X}/G$ with $Y \times T$, such that for each $t \in T$ the induced cover $\mathcal{X}_t \to Y$ is a (G, I)-cover. Two families over T are *(strictly) isomorphic* if there is a G-equivariant isomorphism inducing on the quotient $Y \times T$ the identity map.

A (coarse) moduli space $\mathcal{Z}$ for smooth (G, I)-covers of Y is a scheme structure on the set of smooth (G, I)-covers modulo isomorphisms, such that for any family of (G, I)-covers of Y with base T the induced map $T \to \mathcal{Z}$ is a morphism.

Theorem 5.1 *There is a coarse moduli space of (G, I)-covers of Y, which is a Zariski open set $\mathcal{Z} = \mathcal{Z}(Y, G, I)$ in the closed subvariety of*

$$\prod_{\chi \in G^* \setminus 1} Pic(Y) \times \prod_{i \in I} Hilb^{\mathrm{div}}(Y)$$

of all the (L_χ, D_i) satifying the relations (3.0.1). *The open set $\mathcal{Z}$ is the set of (L_χ, D_i)'s which satisfy the additional conditions:*

1. *each D_i is smooth and the union of the D_i's is a divisor with normal crossings;*
2. *whenever $D_{i_1}, \ldots, D_{i_k}$ meet, the natural map $H_{i_1} \oplus \cdots \oplus H_{i_k} \to G$ is injective.*

PROOF. The set $\mathcal{Z}$ parametrizes the smooth abelian covers of Y by [Pa1], theorem 2.1. The fact that the induced maps from a family of abelian covers to $\mathcal{Z}$ are morphisms follows from the corresponding property of the Hilbert schemes and Picard groups. □

Proposition 2.1 of [Pa1] implies:

Remark 5.2 *For any basis $\chi_1, \ldots, \chi_s$ of G^*, the natural map*

$$\mathcal{Z} \to \prod_{j=1}^{s} Pic(Y) \times \prod_{i \in I} Hilb^{\mathrm{div}}(Y)$$

induced by projection is an isomorphism with its image.

$\mathcal{Z}$ decomposes as the disjoint union of infinitely many quasiprojective varieties $Z(\xi_i, \eta_\chi) = Z(\xi_i, \eta_\chi)(Y, G, I)$, where η_χ, ξ_i are the Chern classes of L_χ and $\mathcal{O}(D_i)$, respectively. We now give an explicit description of $Z(\xi_i, \eta_\chi)$ under the assumption that the ξ_i's are sufficiently ample.

Proposition 5.3 *Let ξ_i, η_χ be cohomology classes satisfying the following relations (compare* (3.0.1)*):*

$$\eta_\chi + \eta_{\chi'} = \eta_{\chi\chi'} + \sum_{i \in I} \varepsilon^i_{\chi,\chi'} \xi_i \qquad \textit{for all } \chi, \chi' \in G^* \setminus 1. \tag{5.3.1}$$

Assume moreover that $\xi_i - c_1(K_Y)$ is the class of an ample line bundle for all $i \in I$. Then $Z(\xi_i, \eta_\chi)$ is an open set in a smooth fibration (with fibre a product of projective spaces) over an abelian variety $A(\xi_i, \eta_\chi)$ isogenous to $Pic^0(Y)^{\# I}$. $Z(\xi_i, \eta_\chi)$ is nonempty iff there are smooth effective divisors D_i, with $c_1(D_i) = \xi_i$, such that their union has normal crossings.

PROOF. Let $A = A(\xi_i, \eta_\chi) \subset \prod_{i \in I} Pic^{\xi_i}(Y) \times \prod_{\chi \in G^* \setminus 1} Pic^{\chi}(Y)$ be the image of $Z(\xi_i, \eta_\chi)$; by equations (3.0.2) the natural map $A \to \prod_{i \in I} Pic^{\xi_i}(Y)$ is a finite étale cover of degree $(2q)^{\# G}$, where q is the irregularity of Y. So each connected component of A is an abelian variety, isogenous to $Pic^0(Y)^{\# H}$. The fact that A is connected is a consequence of the covering being totally ramified. In fact, choose a basis $\chi_1, \ldots, \chi_s$ of G^*, and consider the diagram

$$\begin{array}{ccc} A & \longrightarrow & \prod_{i \in I} Pic^{\xi_i}(Y) \\ \downarrow & & \downarrow \\ \prod_{j=1}^{s} Pic^{\eta_j}(Y) & \longrightarrow & \prod_{j=1}^{s} Pic^{o(\chi_j)\eta_j} \end{array}$$

with maps given by

$$\begin{array}{ccc} (M_i, L_j) & \mapsto & (M_i) \\ \downarrow & & \downarrow \\ (L_j) & \mapsto & (L_j^{\otimes n_j} = \otimes M_i^{(n_j a_j^i)/m_i}). \end{array}$$

The diagram is a fibre product of (connected) abelian varieties; to prove that A is connected is equivalent to proving that $\pi_1(\prod_{i \in I} Pic^{\xi_i}(Y))$ surjects on

$$\pi_1(\prod_{j=1}^{s} Pic^{o(\chi_j)\eta_j}) / \pi_1(\prod_{j=1}^{s} Pic^{\eta_j}(Y));$$

this is in turn equivalent to proving that G^* injects in $\oplus_{i \in I} H_i^*$, which follows by dualizing from assumption 3.1.

Let $\mathcal{P}_i$ on $A \times Y$ be the pullback of the Poincaré line bundles from $Pic^{\xi_i}(Y) \times Y$; the pushforward of $\mathcal{P}_i$ to A is a vector bundle E_i because of the ampleness condition (the rank of E_i can be computed by Riemann-Roch). The moduli space $Z(\xi_i, \eta_\chi)$ is an open set of the fibred product of the $\mathbf{P}(E_i)$. □

Remark 5.4 If $q(Y)$ is not zero, then the components $Z(\xi_i, \eta_\chi)$ are uniruled, but not unirational.

Remark 5.5 In general $Z(\xi_i, \eta_\chi)$ is a coarse but not a fine moduli space, i.e., it does not carry a universal family. Keeping the notation of proposition 5.3, let $\mathcal{V}$ be the total space of the fibred product of the E_i's, and let $\mathcal{V}^o$ the inverse image of $Z(\xi_i, \eta_\chi)$; we have a natural abelian cover of $Y \times \mathcal{V}^o$, which is a complete family of smooth covers of Y with the given data.

There is a natural action of $Aut(Y)$ on the moduli space of (G, I)-covers $\mathcal{Z}$, given by

$$\varphi(D_i, L_\chi) = (\varphi(D_i), (\varphi^{-1})^* L_\chi) \qquad \text{for } \varphi \in Aut(Y).$$

The automorphism group of G acts naturally on G^* (by $\Phi(\chi) = \chi \circ \Phi^{-1}$) and on I_G (by $\Phi(H, \psi) = (\Phi(H), \psi \circ \Phi^{-1})$); given a subset I of I_G, let $Aut_I(G)$ be the set of automorphisms of G preserving I. There is a natural action of $Aut_I(G)$ on $\mathcal{Z}$, induced by the natural action of this group on the indexing sets $G^* \setminus 1$ and I.

Proposition 5.6 *If the classes ξ_i's are ample enough (so that theorem 4.6 applies to some cover in $Z(\xi_i, \eta_j)$), then the quotient of $Z(\xi_i, \eta_j)$ by the natural action of $Aut(Y) \times Aut_I(G)$ maps birationally to its image in the moduli of manifolds with ample canonical class.*

PROOF. That the natural map to the moduli factors via this action is clear. Viceversa, given a generic cover X in $Z(\xi_i, \eta_j)$, by theorem 4.6 its automorphism group is isomorphic to G; so it can be identified uniquely as a (G, I)-cover up to isomorphisms of G and of Y. □

Definition 5.7 Let $\mathcal{Y} \to T$ be a deformation of Y over a simply connected pointed analytic space (T, o). As T is simply connected, the cohomology of every fibre $\mathcal{Y}_t$ is canonically isomorphic with that of Y. Then the varieties $Z(\xi_i, \eta_\chi)(\mathcal{Y}_t, G, I)$ (resp. $A(\xi_i, \eta_\chi)(\mathcal{Y}_t, G, I)$) for $t \in T$ glue to a global variety $\mathcal{Z}_T(\xi_i, \eta_\chi) = \mathcal{Z}_T(\xi_i, \eta_\chi)(\mathcal{Y}, G, I)$ (resp. $\mathcal{A}_T(\xi_i, \eta_\chi)$), surjecting on the locus on T where the classes ξ_i (and hence also the η_χ) stay of type $(1,1)$. The global varieties are constructed by replacing the Hilbert and Picard schemes in the construction of $Z(\xi_i, \eta_\chi)$ and $A(\xi_i, \eta_\chi)$ with their relative versions. The previous results can all be extended to this relative setting.

For each smooth (G, I)-cover $f : X \to Y$, the natural deformations of the reduced building data such that the induced deformations of (Y, L_j, M_i) is trivial are parametrized naturally by $\prod_{(i,\chi) \in S} H^0(Y, M_i \otimes L_\chi^{-1})$, as in §5 of [Pal].

Theorem 5.8 *Let $\mathcal{Y} \to T$ be a deformation of Y over a germ (T, o), and assume that the ξ_i's stay of type $(1,1)$ on T. Then there is a quasiprojective morphism $\mathcal{W}_T(\xi_i, \eta_\chi) \to \mathcal{A}_T(\xi_i, \eta_\chi)$ whose fibre over a point parametrizing line bundles (L_j, M_i) on $\mathcal{Y}_t$ is canonically isomorphic to $\prod_{(i,\chi) \in S} H^0(Y, M_i \otimes L_\chi^{-1})$.*

PROOF. The theorem follows, by taking suitable fibre products, from the following two lemmas. □

Lemma 5.9 *Let Y be a smooth projective variety, and $\xi \in NS(Y)$. Then there exists a morphism of schemes $\pi : W^\xi(Y) \to Pic^\xi(Y)$ such that the fibre over a point $[L]$ is naturally isomorphic to the vector space $H^0(Y, L)$. For any choice of the Poincaré line bundle $\mathcal{P}$ on $Y \times Pic^\xi(Y)$, there exists such a $W^\xi(Y)$ with the property that the line bundle $\pi^*\mathcal{P}$ on $Y \times W^\xi(Y)$ has a tautological section.*

Let $\mathcal{P}$ be the Poincaré line bundle on $Y \times Pic^{\xi}(Y)$, and let $p : Y \times Pic^{\xi}(Y) \to Pic^{\xi}(Y)$ and $q : Y \times Pic^{\xi}(Y) \to Y$ be the projections; if $p_*(\mathcal{P})$ is a vector bundle, it is enough to take W to be the total space of this vector bundle.

It is also clear that if $\xi - c_1(K_Y)$ is an ample class, then $p_*(\mathcal{P})$ is indeed a vector bundle. For the general case, let A be a line bundle on Y such that $c_1(A) + \xi - c_1(K_Y)$ is ample, and such that there exists an $s \in H^0(Y, A)$ defining an effective, smooth divisor D. Let $\pi : V \to Pic^{\xi}(Y)$ be the total space of the vector bundle $p_*(\mathcal{P} \otimes q^*A)$, and let $\sigma : \mathcal{O}_{Y\times V} \to \pi^*(\mathcal{P} \otimes q^*A)$ be the tautological section. For every $y \in D$, let σ_y be the induced section of $\pi^*(\mathcal{P} \otimes q^*A)|_{\{y\}\times V}$; let $W_y \subset V$ be the divisor defined by σ_y. Let $W = W^{\xi}(Y)$ be the intersection of all W_y's for $y \in D$: then σ/s is regular on W, and defines the required tautological section. □

Lemma 5.10 *Let $\mathcal{Y} \to T$ be a deformation of Y over a germ of analytic space T, and assume that ξ stays of type $(1,1)$ over T. Then, after maybe replacing T with a Zariski-open subset, the spaces $W^{\xi}(\mathcal{Y}_t)$ glue together to a quasiprojective morphism $W^{\xi}_T(\mathcal{Y}) \to Pic^{\xi}_T(\mathcal{Y})$.*

PROOF. After eventually restricting T, we can extend A to a line bundle $\mathcal{A}$ over $\mathcal{Y}$, and s to a section of $\mathcal{A}$. The rest of the proof remains the same, using the fact that the relative Picard scheme exists and carries a Poincaré line bundle. □

We now want to describe explicitly $W^{\xi}(Y)$ in the case $\xi = 0$, which we will use repeatedly later.

Remark 5.11 For any deformation $\mathcal{Y} \to T$ over a germ of analytic space, $W^0_T(\mathcal{Y})$ is naturally isomorphic to the union in $Pic^0_T(\mathcal{Y}) \times \mathbf{C}$ of $j(T) \times \mathbf{C}$ and $Pic^0_T(\mathcal{Y}) \times \{0\}$, where $j : T \to Pic^0_T(\mathcal{Y})$ is the zero section.

In particular $W^0(Y)$ is reducible when $q(Y) \neq 0$; this reflects the fact that the deformations, as pair (line bundle, section), of $(\mathcal{O}_Y, 0)$ are obstructed; one can either deform the line bundle or the section, but not both at the same time. This remark will be used to construct examples of manifolds lying in several components of the moduli in section 6.

Theorem 5.12 (i) *Let Y be a smooth projective variety, and let $X \to Y$ be a smooth (G, I)-cover such that theorem* 3.10 *holds. Then there exists a pointed analytic space $(\mathcal{W}, w)$ and a natural deformation of the reduced building data of X over $\mathcal{W}$ such that the induced map of germs from $(\mathcal{W}, w)$ to the Kuranishi family of X (defined as in* 3.3*) is surjective.*
(ii) *One can choose $\mathcal{W}$ to be a quasi-projective scheme, and then the induced rational map from $\mathcal{W}$ to the moduli of manifolds with ample canonical class is dominant onto each component of the moduli containing $[X]$.*

PROOF. (i) Let $\mathcal{Y} \to T$ be the restriction of the Kuranishi family of Y to the locus where all the ξ_i's stay of type $(1,1)$. Let $\mathcal{W} = \mathcal{W}_T(\chi_i, \eta_\chi)$, $\mathcal{Y}_\mathcal{W} = \mathcal{Y} \times_T \mathcal{W}$. Over $\mathcal{Y}_\mathcal{W}$ there are tautological line bundles $\mathcal{L}_j$, $\mathcal{M}_i$ and tautological sections $s_{i,\chi}$ of $\mathcal{M}_i \otimes \mathcal{L}_\chi^{-1}$ (where $\mathcal{L}_\chi$ is defined as in 3.2); moreover $\mathcal{L}_j^{\otimes n_j}$ is isomorphic to $\bigotimes \mathcal{M}_i^{(n_j a_j^i)/m_i}$. $\mathcal{W}$ parametrizes data $(\mathcal{Y}_t, L_j, M_i, s_{i,\chi})$ such that $t \in T$, L_j and M_i are line bundles on $\mathcal{Y}_t$ satisfying (3.0.2) and having Chern classes η_j, ξ_i, and $s_{i,\chi}$ are sections of $L_\chi \otimes M_i^{-1}$. Let $w \in \mathcal{W}$ be a point corresponding to the reduced building data of X: that is, assume that w corresponds to the data $(\mathcal{Y}_o, L_j, M_i, s_{i,\chi})$, where $s_{i,\chi} = 0$ for all $\chi \neq 1$, o is the chosen point in T, and the sections $s_{i,0}$ define divisors D_i such that (L_j, D_i) are the reduced building data of X.

Choose arbitrarily isomorphisms $\Phi_j : \mathcal{L}_j^{\otimes n_j} \to \bigotimes \mathcal{M}_i^{\otimes(n_j a_j^i)/m_i}$, extending the isomorphism over w induced by multiplication in $\mathcal{O}_X$.

By theorem 3.10, together with Artin's results on approximation of analytic mappings (see [Ar]), it is enough to show that every natural deformation of the reduced building data of X over a germ of analytic space can be obtained as pullback from $(\mathcal{W}, w)$.

It is clear that all small deformations of the data $(Y, L_j, M_i, s_{i,\chi})$ can be obtained as pullback from W. So it is enough to prove that, up to isomorphism of natural deformations, we can choose the φ_j's arbitrarily. This is proven in lemma 5.13.

(ii) Start by noting that one can construct a deformation $\mathcal{Y} \to B$ of Y over a pointed quasi-projective variety (B, o), such that the germ of B at o maps surjectively to the locus in the Kuranishi family of Y where the classes ξ_i's stay of type $(1,1)$. In fact, choose any $\chi \in G^* \setminus 1$, and let L be a sufficiently big multiple of L_χ; assume in particular that L is very ample and that all its higher cohomology groups vanish. Let $N = \dim H^0(Y, L) - 1$; choosing a basis of $H^0(Y, L)$ gives an embedding of Y in $\mathbf{P}^N$. Take the union of the irreducible components of the Hilbert scheme of $\mathbf{P}^N$ containing $b = [Y]$, and consider inside it the open locus B' of points corresponding to smooth subvarieties. Then the natural map from the germ of B' at b to the Kuranishi family of X surjects on the locus where η_χ stays of type $(1,1)$. Let B be the closed subscheme of B' where also the classes ξ_i stay of type $(1,1)$.

Let $\mathcal{Y} \to B$ be the universal family; by replacing B with an étale open subset we can assume that $\mathcal{Y} \to B$ has a section. Then (compare for instance [M-F], p. 20) there exists a global projective morphism $\mathcal{A} \to B$ and line bundles $\mathcal{M}_i$, $\mathcal{L}_j$ on $\mathcal{Y} \times_B \mathcal{A}$, such that $\mathcal{A}_b$ parametrizes line bundles (M_i, L_j) on $\mathcal{Y}_b$ such that firstly, they satisfy the usual compatibility conditions, and secondly, the Chern classes of (M_i, L_j) lie in the orbit of (ξ_i, η_j) via the monodromy action of $\pi_1(B, b)$.

Mimicking the proof in the germ case, and replacing B by an étale open subset if necessary, one can find a quasi-projective morphism $\mathcal{W} \to \mathcal{A}$ whose fibre over a point corresponding to line bundles (M_i, L_j) on $\mathcal{Y}_b$ is isomorphic to

$\prod H^0(\mathcal{Y}_b, M_i \otimes L_\chi^{-1})$ for $(i,\chi) \in S$ via tautological sections $\sigma_{i,\chi}$ of the pullbacks to $\mathcal{Y} \times_B \mathcal{W}$ of $\mathcal{M}_i \otimes \mathcal{L}_\chi^{-1}$.

Let $w \in \mathcal{W}$ be a point corresponding to the building data of X as before. Again one can extend the multiplications isomorphisms φ_j to isomorphisms $\Phi_j : \mathcal{L}_j^{\otimes n_j} \to \bigotimes \mathcal{M}_i^{\otimes (n_j a_j^i)/m_i}$.

Putting everything together, we have a natural deformation of the building data of X over $(\mathcal{W}, w)$; this induces by (3.3) a rational map to the moduli of manifolds with ample canonical class, which is a morphism on the open subset of $\mathcal{W}$ where the natural deformation of X is smooth. Applying the same methods as in (i) implies that the map from $\mathcal{W}$ to the moduli is dominant on each irreducible component containing $[X]$. □

Lemma 5.13 *Let T be a germ of analytic space. For any $(\mathcal{Y}, \mathcal{M}_i, \mathcal{L}_j, s_{i,\chi}, \varphi_j) \in$ $\mathrm{Dnat}_X(T)$, and for any other admissible choice of isomorphisms $\varphi'_j : \mathcal{L}_j^{\otimes n_j} \to \bigotimes M_i^{\otimes a_j^i}$, there exist sections $s'_{i,\chi}$ such that $(\mathcal{Y}, \mathcal{M}_i, \mathcal{L}_j, s_{i,\chi}, \varphi_j)$ is isomorphic to $(\mathcal{Y}, \mathcal{M}_i, \mathcal{L}_j, s'_{i,\chi}, \varphi'_j)$.*

PROOF. It is enough to show that there are automorphisms ψ_i of $\mathcal{M}_i$ such that the composition $(\bigotimes \psi_i^{\otimes (n_j a_j^i)/m_i}) \circ \varphi_j$ equals φ'_j; in fact in this case one can choose $s'_{i,\chi} = \psi_i^*(s_{i,\chi})$, for all $(i,\chi) \in S$.

As both φ_j and φ'_j are isomorphisms, $\varphi_j = f_j \varphi'_j$, where f_j is an invertible function on $\mathcal{Y}$ restricting to 1 on the central fibre. Finding the ψ_i's is equivalent to finding functions g_i's on $\mathcal{Y}$ such that g_i restricts to 1 on the central fibre and $f_j = \prod g_i^{(n_j a_j^i)/m_i}$, for all $j = 1, \ldots, s$. The existence of such g_i's follows from the fact that the matrix a_j^i has rank equal to s, which in turn is implied by the cover being totally ramified (see lemma 2.1). □

Remark 5.14 There is a natural action of $(C^*)^{\#I}$ on the functor of natural deformations, which is the identity on $(\mathcal{Y}, \mathcal{L}_j, \mathcal{M}_i, \varphi_j)$ and acts on $\sigma_{i,\chi}$ by

$$(\lambda_i)_{i \in I}(\sigma_{j,\chi}) = \prod_{i \in I} \lambda_i^{\delta_{ij} m_i - a_\chi^i} \cdot \sigma_{j,\chi}; \tag{5.14.1}$$

This action has the property that the induced flat maps $\mathcal{X} \to T$ are invariant under it; in particular the natural map from $\mathcal{W}(\xi_i, \eta_j)$ to the moduli factors through the corresponding action.

6 Applications to moduli

In this chapter we want to apply the results on deformation theory together with theorem 4.6 to study the generic automorphism group of some components of the moduli spaces of manifolds with ample canonical class, components containing suitable abelian covers with sufficiently ample branch divisors.

To begin with, we study the case of simple cyclic covers (i.e., those for which the Galois group G is cyclic and there is only one irreducible branch divisor).

Proposition 6.1 *Let $f : X \to Y$ be a smooth simple cyclic cover, with Galois group $\mathbf{Z}_m$, and reduced building data D and L (where D is a smooth divisor and L is a line bundle satisfying $mL \equiv D$). Assume that D is sufficiently ample. Let M be an irreducible component of the moduli space of surfaces of general type containing X. Then G_M is trivial if $m \geq 3$, and $G_M = G$ if $m = 2$.*

PROOF. In case $m = 2$, it is easy to check that $H^i(X, T_X)$ is G-invariant for $i = 1, 2$; hence the natural map $\mathrm{Dgal}_X \to \mathrm{Dnat}_X$ is surjective, and all deformations are Galois. By theorem 4.6, $Aut(X) = G$ for a generic choice of D in its linear system.

If $m \geq 3$, assume without loss of generality that D is generic in its linear system. Let (G, χ) be the element of I_G corresponding to the only nonempty branch divisor. Then the natural deformations of X such that Y and $\mathcal{O}(D)$ are fixed are parametrized by

$$\bigoplus_{i=0}^{m-2} H^0(Y, L^{-i}(D)) = \bigoplus_{i=0}^{m-2} H^0(Y, L^{m-i});$$

in particular they are unobstructed. Moreover, given any nontrivial element g of the Galois group G, it acts on the (necessarily nonzero) summand $H^0(Y, L^{m-1})$ as multiplication by $\chi(g)$, hence nontrivially; therefore g does not extend to the generic deformation. By genericity however $Aut(X) = G$, hence by semicontinuity of the automorphism group the proof is complete. □

Hence, to get nontrivial examples, and to prove the results on the moduli claimed in the introduction, it is necessary to study more general abelian covers.

Construction 6.2 Let s be an integer ≥ 2. Let $d_1, \ldots, d_s$ be integers ≥ 2; denote their least common multiple by d_0, and define integers b_i by requiring that $b_i d_i = d_0$, for all $i = 1, \ldots, s$. Let $G = \mathbf{Z}_{d_1} \times \cdots \times \mathbf{Z}_{d_s}$, and let $e_1, \ldots, e_s$ be the canonical basis of G; let $\chi_1, \ldots, \chi_s$ be the dual basis of G^*.

Let $e_0 := -(e_1 + \ldots + e_s)$, and let H_i be the subgroup generated by e_i; for $i = 0, \ldots, s$, let $\psi_i \in H_i^*$ be the unique character such that $\psi_i(e_i) = \zeta_{d_i}$; note that, for each $j = 1, \ldots, s$ and $i \neq 0$ we have $a_j^i = \delta_{ij}$, while $a_j^0 = b_j(d_j - 1)$. Moreover $o(e_i) = d_i$, for $i = 0, \ldots, s$. Let $I = \{0, \ldots, s\}$; identify I with a subset of I_G via $i \mapsto (H_i, \psi_i)$.

Fix a smooth projective variety Y of dimension d, and assume that $s \geq d \geq 2$. Let $f : X \to Y$ be a (G, I)-cover of Y, with branch divisors D_i. Equations (3.0.2) become

$$L_j^{\otimes d_j} = M_j \otimes M_0^{\otimes(d_j - 1)}$$

for all $j = 1, \ldots, s$, hence they can be solved by letting $L_j = M_0 \otimes F_j$, $M_j = M_0 \otimes F_j^{\otimes d_j}$, for all $j = 1, \ldots, s$.

We compute explicitly L_χ for $\chi \in G^*$, using equation (3.0.3). Let $\chi \in G^*$, and write $\chi = \chi_1^{\alpha_1} \cdots \chi_s^{\alpha_s}$, with $0 \leq \alpha_j < d_j$. One gets

$$L_\chi = \bigotimes_{j=1}^{s} F_j^{\otimes \alpha_j} \otimes M_0^{\otimes N_\chi},$$

where $N_\chi = -[(-\alpha_1 b_1 - \ldots - \alpha_s b_s)/d_0]$. In particular N_χ is an integer ≥ 0; $N_\chi = 0$ if and only if $\chi = 0$, $N_\chi = 1$ if and only if $\sum(\alpha_i b_i) \leq d_0$.

In the following we will always assume that $c_1(F_j) = 0$, for $j = 1, \ldots, s$; let $\xi = c_1(M_0)$. Assume also that X is a smooth (G, I)-cover, that is that the divisors D_i are smooth and their union has normal crossings.

In the surface case, one can compute the Chern invariants of the cover X:

$$\begin{aligned} K_X^2/\#G &= \left(K_Y + (s - (d_0^{-1} + \ldots + d_s^{-1}))\xi\right)^2 \\ c_2(X)/\#G &= c_2(Y) - ((s+1) - (d_0^{-1} + \ldots + d_s^{-1}))\xi K_Y + \\ &\quad \left(\binom{s+2}{2} + \sum_{i=0}^{s} d_i^{-1} + \sum_{0 \leq i < j \leq s} d_i^{-1} d_j^{-1}\right)\xi^2. \end{aligned}$$

The first equality follows from [Pal], proposition 4.2; the second from the additivity of the Euler characteristic, by decomposing Y in locally closed subsets according to whether a point lies in 2, 1 or no branch divisor. Note that no other possibilities can occur, as we assume that the union of the branch divisors has normal crossings. The second equality could also be derived by Noether's formula and proposition 4.2 in [Pal].

Lemma 6.3 *Let $f : X \to Y$ be a (G, I)-cover as in construction 6.2. Assume that $q(Y)$ is nonzero, that $\xi \in NS(Y)$ is sufficiently ample, that $F_j = \mathcal{O}_Y$ (for $j = 1, \ldots, s$), and that $D_i \in |M_i| = |M_0|$ is generic (for $i = 0, \ldots, s$). Then, for each $k = 0, \ldots, s$, there exists a component M_k of the moduli of manifolds with ample canonical class, containing X, such that the generic automorphism group $G_{M_k} \subset G$ is equal to $G_k = \mathbf{Z}_{d_1} \times \ldots \times \mathbf{Z}_{d_k}$.*

PROOF. By assumption X has ample canonical class, $Aut(X) = G$ and the natural deformations of X are complete. Assume first that Y is rigid. Let $\chi \in G^*$ be such that $N_\chi = 1$, and let $(i, \chi) \in S$; these are the only values of i, χ (with χ nontrivial) for which $M_i \otimes L_\chi^{-1}$ can have sections, i.e. can contribute to non-Galois deformations. In fact $c_1(M_i \otimes L_\chi^{-1}) = 0$, hence it has sections if and only if it is trivial (compare remark 5.11). The condition that the line bundle $M_i \otimes L_\chi^{-1}$ be trivial can be expressed, in terms of the F_j's, as

$$\sum_j \alpha_j F_j = d_i F_i. \tag{6.3.1}$$

Let $T_k \subset Pic^0(Y)^s$ be the locus where $F_i = 0$ for all $i > k$. Note that $F_i = 0$ for all $i > k$ implies that $M_i = M_0$ for all $i > k$, and that $M_i \otimes L_\chi^{-1}$ is trivial for any (i, χ) such that $N_\chi = 1$, $i > k$ and χ restricted to G_k is trivial.

For a generic choice of $(F_j) \in T_k$, the line bundles $M_i \otimes L_\chi^{-1}$ are nontrivial for each χ such that $\chi_{|G_k} \neq 1$; in fact, for any such χ there exists $j_0 \leq k$ such that $\alpha_{j_0} > 0$, hence the coefficient of F_{j_0} in (6.3.1) is nonzero (being either $\alpha_{j_0} > 0$ or $\alpha_{j_0} - d_{j_0} < 0$).

On the other hand, for each $j > k$, one has $(0, \chi_j) \in S$ and $M_0 \otimes L_j^{-1}$ is trivial (in fact one has to exclude here the case where d_0 is equal to 2, and hence all d_i's are; this case needs a slightly different analysis, see below). Hence for every $g \in G \setminus G_k$, and for any (G, I)-cover with building data in T_k, there are natural deformations of the cover to which the action of g does not extend.

Therefore the (G, I)-covers whose building data are in T_k, together with their natural deformations such that $s_{i,\chi} = 0$ for all χ acting nontrivially on G_k, form an irreducible component of the Kuranishi family of X; in fact they are parametrized by an irreducible variety, and at some point they are complete (at least at all points corresponding to (G, I)-covers with a generic choice of the F_j's for $j \leq k$). The generic element of this component has therefore automorphism group G_k.

In the case where $d_0 = 2$, $(0, \chi_j) \notin S$; however, if $k \neq s - 1$, we can consider $M_{j'} \otimes L_j^{-1}$ instead of $M_0 \otimes L_j^{-1}$, where j' is any index $> k$ and different from j. If $k = s - 1$, let $\chi = \chi_1 + \chi_s$; then $N_\chi = 1$ (as $s \geq 2$), and $(0, \chi) \in S$. As $\chi(e_s) \neq 0$, there are natural deformations to which the action of G does not extend.

The same argument applies if Y is non-rigid, by replacing $Pic^0(Y)$ with $Pic^0_T(\mathcal{Y})^s$, where $\mathcal{Y} \to T$ is the restriction of the Kuranishi family of Y to the locus where ξ stays of type $(1, 1)$. □

Remark 6.4 We can find a Y of arbitrary dimension and an ample class ξ such that deformations of Y for which ξ stays of type $(1, 1)$ are unobstructed; for instance, by taking Y a product of curves of genus at least two and ξ the canonical class.

Theorem 6.5 *Let $d \geq 2$ be an integer. Given any integer N, there exists a point in the moduli space of manifolds of dimension d with ample canonical class which is contained in at least N distinct irreducible components.*

PROOF. Without loss of generality, assume that $N \geq d$. Choose arbitrarily integers $d_1, \ldots, d_N$, each of them ≥ 2. Let (Y, L) be as in lemma 6.3; then for each $k = 1, \ldots, N$ there exists a component of the moduli containing X and having generic automorphism group isomorphic to $\mathbf{Z}_{d_1} \times \ldots \times \mathbf{Z}_{d_k}$. Hence X lies in at least N different irreducible components of the moduli. □

In the case of surfaces, this result gives a strong negative answer to the open problem (ii) on page 485 of [Ca2].

Theorem 6.6 *Let G be a finite abelian group, and $d \geq 2$ an integer. Then there exist infinitely many components M of the moduli space of manifolds of dimension d with ample canonical class such that $G_M = G$.*

PROOF. Let $G = \mathbf{Z}_{d_1} \times \ldots \times \mathbf{Z}_{d_k}$. If $k \geq d$, let $s = k$; if $k < d$, let $s = d$ and choose arbitrarily $d_{k+1}, \ldots, d_s$. Choose (Y, ξ) as in lemma 6.3. Applying the lemma to $(Y, i\xi)$ for $i \geq 1$ gives the claimed result. □

In the case of surfaces, another natural question concerns the cardinality of the automorphism group. Xiao proved in [Xi1] that if X is a minimal surface of general type, $\#G \leq 52K_X^2 + 32$ for all abelian subgroups G of $Aut(X)$; it is not known whether this bound is sharp, but he gives examples to the effect that any better bound must still be linear in K_X^2. It seems natural to ask if there is a smaller bound if one replaces $Aut(X)$ by $Aut_{\text{gen}}(X)$, the intersection in $Aut(X)$ of G_M, for each irreducible component M containg X. Notice that in Xiao's examples the generic automorphism group is obviously smaller, so a better bound should in principle be possible. In particular, it is an open question as to whether such a bound could be less than linear in K_X^2. We prove here that it cannot be "much" less than linear.

Proposition 6.7 *There exists a sequence S_n of minimal surfaces of general type such that*

1. *$k_n = K_{S_n}^2$ tends to infinity with n;*
2. *S_n lies on a unique irreducible component, M_n;*
3. *$\#G_{M_n} \geq k_n(\log_2 k_n)^{-2}(1 + O(\log_2 \log_2 k_n / \log_2 k_n))$.*

PROOF. Let $n \geq 2$ be an integer. Apply contruction 6.2 with $s = n$, Y a principally polarized abelian surface with $NS(Y) = \mathbf{Z}$ and ξ equal to the double of the class of the principal polarization. Choose S_n to be a cover branched over divisors D_i whose linear equivalence classes are generic; then all infinitesimal deformations must be Galois, and the Kuranishi family of S_n is smooth. So G_{M_n} must contain $\mathbf{Z}_2^n$, hence $\#G_{M_n} \geq 2^n$. On the other hand, by [Pa1], proposition 4.2, we have

$$k_n = 2^n (n+1)^2 \xi^2 / 4.$$

A simple computation yields the result. Note that as we only want to bound G_M from below, we don't need to apply theorem 4.6, which would have forced us to choose as class ξ a higher multiple of the principal polarization. □

Remark 6.8 Using the computation of Chern numbers for construction 6.2, one can determine where the examples constructed so far lie in the geography of surfaces of general type. For instance by setting all d_i's equal to m and letting s and m go to infinity, one gets a sequence of examples where K^2/c_2 tends to 2 from below.

7 Resolution of singularities

Remark 7.1 *Let $\pi : X \to Y$ be a (G, I)-cover with Y smooth and branch locus with normal crossings. Let $Z \to X$ be a resolution of singularities; then the exceptional locus of Z has uniruled divisorial components.*

Proof. The question is local on Y, so we can assume that Y is affine and that the line bundles L_χ and $\mathcal{O}(D_i)$ are trivial. Let G' be the abelian group with $\#I$ generators $e_1, \ldots, e_s$, and relations $m_i e_i = 0$ (where $m_i = \#H_i$). There exists a smooth G'-cover X' of Y branched over the D_i such that the inertia subgroup of D_i is generated by e_i, and such that the map $V \to Y$ factors via X. Let Z' be a resolution of singularities of the fibre product $Z \times_X X'$; we have a commutative diagram

$$\begin{array}{ccc} Z' & \to & X' \\ \downarrow & & \downarrow \\ Z & \to & X \end{array}$$

Let E be an irreducible divisorial component of the exceptional locus of $Z \to X$; its strict transform E' in Z' must be contracted in X' as $X' \to X$ is finite. As X' is smooth and $Z' \to X'$ is birational, E' must be ruled by [Ab], therefore E must be uniruled. □

Lemma 7.2 *Let $\mathcal{Y} \to \Delta$ be a family of smooth manifolds, $\mathcal{X} \to \mathcal{Y}$ an abelian cover branched on divisors which are all smooth except D, of branching order n, which has local equation $f^n h + tg = 0$ with f, t, h, g local coordinates on Y (and t coordinate on Δ). Then there exists a morphism $\tilde{\mathcal{Y}} \to \mathcal{Y}$ such that:*

1. *$\tilde{\mathcal{Y}} \to \mathcal{Y}$ is a composition of blowups with smooth center;*
2. *the normalization $\tilde{\mathcal{X}}$ of the induced cover of $\tilde{\mathcal{Y}}$ is an abelian cover of $\tilde{\mathcal{Y}}$ branched over a normal crossing divisor;*
3. *the exceptional divisors of $\tilde{\mathcal{X}} \to \mathcal{X}$ have Kodaira dimension $-\infty$.*

Proof. We will construct $\tilde{\mathcal{Y}}$ by successive blowups; a local coordinate and its strict transform after the blowup will be denoted by the same letter. At each blowing-up step one checks that the normalization of the last introduced exceptional divisor has Kodaira dimension $-\infty$ (further blowups change the situation only up to birational maps).

The strategy of the proof is as follows; each blowup introduces a divisor which is a $\mathbf{P}^r$ bundle (for $r = 1, 2$), and we prove that the induced cover of the generic $\mathbf{P}^r$ has Kodaira dimension $-\infty$. We can assume that the Galois group coincides with the inertia subgroup H of D; if this is not the case, consider the factorization $\mathcal{X} \to \mathcal{X}/H \to Y$, and note that the map $\mathcal{X}/H \to Y$ is unramified near generic points of D, hence after blowing up the inverse image of the generic $\mathbf{P}^r$ is an unramified cover, which is therefore a disjoint union of copies of $\mathbf{P}^r$.

We first prove the result on the locus where $h \neq 0$ (this is all one needs if $\mathcal{Y}$ is a threefold). By changing local coordinates one can assume $h = 1$. Let n be the order of H. We distinguish two cases: n even and n odd. Let $E_1, E_2, \ldots$ be the subsequent exceptional divisors.

CASE OF n EVEN. Blow up at each step the singular locus $t = f = g = 0$ and look at the f chart. At the first step one obtains

$$z^n = f^2(f^{n-2} + tg)$$

and the total transform of the branch locus D is $D+2E_1$. The covering restricted to E_1 is the composition of a totally ramified cover of degree $n/2$ and of a double cover ramified over $D \cap E_1$ which is (on each $\mathbf{P}^2$ in E_1) a (possibly reducible) conic. Hence the cover of E_1 is fibered in two-dimensional quadrics (maybe singular).

At the k-th step $(1 < k \leq n/2)$ we have

$$z^n = f^{2k}(f^{n-2k} + tg)$$

and the total transform of D is

$$D + 2E_1 + \ldots + 2kE_k.$$

Again D cuts out a (possibly reducible) conic on the $\mathbf{P}^2$ fibration of E_k; moreover, $E_k \cap E_i = \emptyset$ if $i < k-1$, and $E_k \cap E_{k-1}$ is (fibrewise) a line which is not contained in D.

If ξ is a generator of the group H, the induced cover of E_k is the composite of a totally ramified cover and of a cyclic cover of degree r, where r is the cardinality of $H/\langle\xi^{2k}\rangle$; the cover is ramified on each $\mathbf{P}^2$ on a conic and on a line. The pairs (inertia group, character) for the branch divisors correspond, via the bijection defined in §2, to ξ for the conic and to ξ^{-2} for the line.

The canonical bundle of the cover is (fibrewise) the pullback of a multiple of a line in $\mathbf{P}^2$, the multiple being

$$-3 + 2\left(\frac{r-1}{r}\right) + \left(\frac{r/2-1}{r/2}\right) < 0$$

if r is even and

$$-3 + 2\left(\frac{r-1}{r}\right) + \left(\frac{r-1}{r}\right) < 0$$

if r is odd; in both cases the anticanonical bundle of the cover is ample and the surface must be of Kodaira dimension $-\infty$.

CASE OF n ODD. Start by blowing up the singular locus $t = f = g = 0$. At the first step the total transform of D is $D + 2E_1$ and the cover of E_1 is totally ramified (as 2 is prime with n), hence the cover is again E_1. If $2k < n$ the same formulas as before hold; we can repeat the previous argument where r is necessarily odd.

Look now at the $k = (n-1)/2$ case. The total transform of D is

$$D + 2E_1 + \ldots + (n-1)E_{(n-1)/2}.$$

The strict transform of D is now smooth; $E_{(n-1)/2} \cap D$ is fibered in singular conics, and we blow up the singular locus. The center of this blowup does not meet E_k for $k < (n-1)/2$, and D and $E_{(n-1)/2}$ have the same tangent space there. Therefore after blowing one gets an exceptional divisor $E_{(n+1)/2}$ intersecting both $E_{(n-1)/2}$ and D in the same line. The equation (in the g chart) becomes

$$z^n = f^{n-1}g^n(f + tg).$$

The cover of $E_{(n+1)/2}$ is a $\mathbf{P}^1$-bundle ramified on a generic $\mathbf{P}^1$ with opposite characters on the same divisor, hence when normalizing it splits completely. The components of the total transform of D are smooth, but they meet non-transversally along the $\mathbf{P}^1$-bundle $f = g = 0$.

We now blow up the locus $f = g = 0$ and call the exceptional divisor F; the total transform of D is

$$D + 2E_1 + \ldots + (n-1)E_{(n-1)/2} + nE_{(n+1)/2} + 2nF,$$

and F is a $\mathbf{P}^1$-bundle over a $\mathbf{P}^1$-bundle. The covering of the generic $\mathbf{P}^1$-fibre of F is ramified of degree n over two points (corresponding to $F \cap D$ and $F \cap E_{(n+1)/2}$) with opposite characters, hence is again isomorphic to $\mathbf{P}^1$.

In both cases the fact that the divisors are smooth and transversal can be checked at each step out of the center of the next blowup.

We now work in the neighborhood of a point where $h = 0$. If n is even, one can perform the same blowups as in the previous case and check that the same arguments work. If n is odd, one can perform the first $(n-1)/2$ blowups as before. After them, the total transform of D has equation $f^{n-1}(fh + tg)$. In particular (the strict transform of) D is not smooth any more; we blow up its singular locus, and get a smooth exceptional divisor $\bar{E}$. The total transform of D is

$$D + 2E_1 + \ldots + (n+1)\bar{E}$$

and is given (in local equations in the h chart) by

$$f^{n-1}h^{n+1}(f + tg).$$

Let ξ be a generator of H; the induced cover of $\bar{E}$ is cyclic with group $H/\langle \xi^{n+1} \rangle$, hence it is totally ramified and therefore of Kodaira dimension $-\infty$, being a $\mathbf{P}^2$-bundle. We are not done because the divisors D and $E_{(n-1)/2}$ are not transversal along $f = g = t = 0$; but now we can apply the previous blowup procedure again. □

Proposition 7.3 *Let $\mathcal{X} \to \mathcal{Y} \to \Delta$ be an abelian cover, branched over all smooth divisors except one, which has local equation $f^m h + tg$, where f, t, g are coordinates and m is the order of branching (where t is the coordinate on Δ). Then $\mathcal{X}$ and all its transforms via an n-th root base change admit a resolution of singularities such that the divisorial components of the exceptional divisor all have Kodaira dimension $-\infty$.*

PROOF. The statement without the base change has already been proved; let $\tilde{\mathcal{X}}$ be such a resolution. By Hironaka's resolution of singularities ([Hi], p. 113, lines 8–4 from the bottom) we can assume that $\tilde{\mathcal{X}}_0$ is a normal crossing divisor. Let now $\rho_n : \Delta \to \Delta$ be the map $t \mapsto t^n$. There is a natural birational mapping $\rho_n^* \tilde{\mathcal{X}} \to \rho_n^* \mathcal{X}$; moreover $\rho_n^* \tilde{\mathcal{X}}$ is a cyclic cover of the manifold $\tilde{\mathcal{X}}$ ramified over $\tilde{\mathcal{X}}_0$, which has normal crossings, hence by remark 7.1 $\rho_n^* \tilde{\mathcal{X}}$ has a resolution such that the divisorial components of the exceptional divisor all have Kodaira dimension $-\infty$. □

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