Scalar curvature of spheres

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It is known that, if a compact n-manifold M, $n \geq 3$, admits a metric of positive scalar curvature, then any smooth function of M is realized as the scalar curvature function of some metric of M (cf. [1]). This paper is an attempt to show this statement will be true even if we assume the metric has unit total volume. In the previous paper [2], this problem was solved except for positive constant functions. Therefore we have only to find metrics with unit volume and with scalar curvature equal to arbitrarily given positive constant. One difficulty is that we cannot apply the Yamabe problem because it provides only constant scalar curvature less than or equal to that of the standard sphere when the volume is normalized. On the other hand there are obvious cases in which we can easily get any positive constant scalar curvature under the volume constraint. That is, when M is a product manifold $M_1 \times M_2$ either of whose component admits a metric of positive scalar curvature, or when M is the total space of certain fiber bundle such that both fiber and the base admit positive scalar curvature. As for spheres S^n, it is the case when $n \equiv 3 \pmod 4$ and $n \geq 7$ by means of the Hopf fibering $S^{4k+3}/S^3 \cong \mathbb{H}P^k$. In this paper we shall construct metrics of large constant scalar curvature for even dimensional spheres with dimension at least 4. As a result, we get

Theorem. If $n \not\equiv 1 \pmod 4$ and $n \geq 4$, every smooth function of S^n is the scalar curvature of some metric of unit total volume.

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§1 Preliminaries.

We begin with a formula for the scalar curvature of a metric expressed in some special coordinates.

Lemma 1.1. Let $\{h_s\}_{s \in I}$ *be a 1-parameter family of metrics of an* $(n-1)$*-manifold* N*. Then the scalar curvature* R *of the metric* $ds^2 + h_s$ *on* $I \times N$ *is given as*

$$R = -(\mathrm{tr}_{h_s} \dot{h}_s)^{\cdot} - \frac{1}{4}(|\dot{h}_s|^2_{h_s} + (\mathrm{tr}_{h_s} \dot{h}_s)^2) + R_s ,$$

where R_s *is the scalar curvature of* h_s *and* $\cdot$ *stands for* $\partial/\partial s$.

The proof is a straightforward calculation and is omitted. We shall use this lemma in the following form.

Corollary 1.2. Let h_s *be as above and* R *be the scalar curvature of the metric* $ds^2 + u^{4/n} h_s$ *where* u *is a positive function of* $I \times N$*. Assume* $\mathrm{tr}_{h_s} \dot{h}_s = 0$*, namely the volume elements of* h_s *are the same. Then we have*

$$\ddot{u} + \frac{n}{4(n-1)}(R + \frac{1}{4}|\dot{h}_s|^2)u = \frac{n}{4(n-1)} u\, R(u^{4/n} h_s),$$

where $R(u^{4/n} h_s)$ *is the scalar curvature of* $u^{4/n} h_s$.

The following will be the starting point of the proof of our theorem which will be given in the next section.

Lemma 1.3. Suppose $n \geqq 4$ is an even integer. Then there is a smooth 1-parameter family of metrics h_s of S^{n-1} with the following properties:

(i) h_0 is the standard metric with the scalar curvature $(n-1)(n-2)$;

(ii) $tr_{h_s} \frac{\partial}{\partial s} h_s = 0$;

(iii) $\left| \frac{\partial}{\partial s} h_s \right| = 1$;

(iv) the scalar curvature $R(h_s)$ is constant for every s and $R(h_s) \leqq R(h_0)$.

Proof. Note that the Hopf fibering $S^{n-1}/S^1 \cong \mathbb{C}P^{n/2-1}$ induces a Killing vector field of unit length on (S^{n-1}, h_0). Let w be the 1-form associated to the Killing vector field. Then the family of metrics

$$h_s = \exp(-t/\sqrt{(n-1)(n-2)})\,(h_0 + ((\exp t\sqrt{(n-1)(n-2)}) - 1)\, w \otimes w)$$

satisfies the required conditions.

§2 Proof of Theorem.

Let ϕ be a smooth nonnegative function of $\mathbb{R}$ such that

$$\begin{cases} \phi(t) = 0 & \text{for } t \notin (0,\varepsilon), \\ |\dot{\phi}| < 1 & \text{and} \\ \dot{\phi} > 0 & \text{on } (0,\varepsilon/2], \end{cases} \tag{1}$$

where ε is a sufficiently small positive number. We then put

$$\phi_r(t) = \begin{cases} r\,\phi(t) & \text{for } 0 \leqq r \leqq 1 \\ \phi(rt) & \text{for } 1 \leqq r. \end{cases} \tag{2}$$

So the support of ϕ_r is contained in $(0,t_r)$, where

$$t_r = \begin{cases} \varepsilon & \text{for } 0 \leqq r \leqq 1 \\ \varepsilon/r & \text{for } 1 \leqq r. \end{cases} \tag{3}$$

Let h_s be the metrics of S^{n-1} as in Lemma 1.3 and define functions A_r and B_r as

$$\begin{cases} A_r(t) = \dfrac{n}{4(n-1)}\left(n(n-1) + \dfrac{1}{4}\left|\dot{h}_{\phi_r}(t)\right|^2\right), \\ B_r(t) = \dfrac{n}{4(n-1)} R(h_{\phi_r}(t)), \end{cases} \tag{4}$$

where $\dot{}$ is $\partial/\partial t$. Here we remark that A_r and B_r are functions in t because of (iii) and (iv) of Lemma 1.3, and that

$$\begin{cases} A_r(t) \geqq A_0 := n^2/4 \\ B_r(t) \leqq B_0 := n(n-2)/4. \end{cases} \tag{5}$$

Moreover the strict inequalities hold for t $(0,t_r/2)$.

Let $u_r(t)$ be the solution of the following equations

$$\begin{cases} \ddot{u}_r(t) + A_r(t)u_r(t) = B_r(t)u(t)^{1-4/n} \\ u_r(t) = (B_0/A_0)^{n/4} = ((n-2)/n)^{n/4} \quad \text{for } t \leqq 0 \\ u_r(t) > 0, \end{cases} \tag{6}$$

and $T_r > 0$ be the maximal time for which (6) is solvable for $t < T_r$.

Then from (5) we have

$$\ddot{u}_r(t) + A_0 u_r(t) \leqq B_0 u(t)^{1-4/n}. \tag{7}$$

Here the strict inequality holds for $t \in (0,t_r/2)$.

Lemma 2.1. *If* $A_0 t_r^{\,2} \leqq \pi^2/4$, *then* $\dot{u}_r < 0$ *on* $(0, \min\{t_r, T_r\})$.

Proof. Suppose $\dot{u}_r(a) = 0$ for some $a \in (0, \min\{t_r, T_r\})$. Then from (7) we can find $a_1 \in (0,a]$ such that $u_r(a_1) < (B_0/A_0)^{n/4}$ and $\dot{u}_r(a_1) = 0$. Since $0 < a_1 < \pi/2\sqrt{A_0}$, the graph of u_r is tangent from above to the graph of

$$v(t) = -k \sin\sqrt{A_0}\, t + (B_0/A_0)^{n/4}$$

at $t = a_2 \in (0, a_1)$ for some positive k. Therefore

$$\ddot{u}_r(a_2) \geqq \ddot{v}(a_2) = A_0((B_0/A_0)^{n/4} - u_r(a_2))$$

$$> B_0 u_r(a_2)^{1-4/n} - A_0 u_r(a_2),$$

which is contrary to (7). Hence u_r does not change its sign on $(0, \min\{t_r, T_r\})$, which implies, again by (7), that $\dot{u}_r < 0$ in this interval.

Lemma 2.2. *If* $u_r \leqq (B_0/A_0)^{n/4}$ *on* $(0,t)$, *then*

$$u_r(t) \geqq \left(1 - \frac{t^2}{2}\left(\max_{[0,t]} A_r\right)\right)(B_0/A_0)^{n/4} + \frac{t^2}{2}\left(\max_{[0,t]} A_r\right) \min_{[0,t]} (B_r/A_r)^{n/4}.$$

Proof. It follows immediately from (6) that

$$\ddot{u}_r(t) + \max\{A_r(t)(u_r(t) - (B_r(t)/A_r(t))^{n/4}), 0\} \geqq 0.$$

Since $u_r \leqq (B_0/A_0)^{n/4}$ and $B_0/A_0 \geqq B_r/A_r$, we have

$$\ddot{u}_r(t) + (\max A_r)((B_0/A_0)^{n/4} - (B_r(t)/A_r(t))^{n/4}) \geqq 0,$$

which yields the desired inequality simply by integration.

From the above two lemmas we get

Corollary 2.3. If $t_r^{\,2} \max A_r < 1$, then $t_r < T_r$. Moreover for $t \in (0,t_r)$, we have

$$\dot{u}(t) < 0 \tag{8}$$

and

$$\frac{1}{2}(B_0/A_0)^{n/4} < u_r(t). \tag{9}$$

It is easy to see that there is an $\varepsilon_0 > 0$ such that, if $\varepsilon < \varepsilon_0$, then $t_r^{\,2} \max A_r < 1$ for any r. So from now on, we assume $\varepsilon < \varepsilon_0$ and therefore the hypothesis of the above corollary is automatically satisfied.

Lemma 2.4. $\dot{u}_r(t_r/2) \to -\infty$ as $r \to \infty$.

Proof. First we observe that

$$\lim_{r\to\infty} A_r(kt_r) = \infty \quad \text{for } 0 < k \leqq 1/2. \tag{10}$$

Hence for any sufficiently large r we have

$$B_r(t)/A_r(t) < B_0/3A_0 \quad \text{for} \quad t \in [t_r/3, t_r/2].$$

Then we get from (9)

$$\ddot{u}_r \leqq -\frac{1}{6}(B_0/A_0)^{n/4} A_r .$$

Hence

$$\dot{u}_r(t_r/2) \leqq \dot{u}_r(t_r/3) - \frac{1}{6}(B_0/A_0)^{n/4} \int_{t_r/3}^{t_r/2} A_r(t)\,dt,$$

and the right hand side goes to $-\infty$ on account of (10).

Lemma 2.5. There exists an $R > 0$ such that $u_R(t) = \sin^{n/2}(T_R - t)$ for $t_r \leqq t < T_R$.

Proof. We put

$$E_r(t) = \frac{1}{2}(\dot{u}_r(t)^2 + A_0 u_r(t)^2 - \frac{n}{n-2} B_0 u_r(t)^{2-4/n}).$$

Then

$$\dot{E}_r(t) = \dot{u}_r(t)(\ddot{u}_r(t) + A_0 u_r(t) - B_0 u_r(t)^{1-4/n}).$$

Hence from (7) and (8) we have

$$\dot{E}_r(t) \geqq 0 \quad \text{for} \quad 0 \leqq t \leqq t_r . \tag{11}$$

From (8), $u_r(t) \leqq (B_0/A_0)^{n/4}$ for $0 \leqq t \leqq t_r$. This together with Lemma 2.4 yields

$$\lim_{r\to\infty} E_r(t_r/2) = \infty.$$

Therefore by (11) we get

$$\lim_{r\to\infty} E_r(t_r) = \infty.$$

Since $E_0(0) < 0$, we then get an $R > 0$ such that $E_R(t_R) = 0$, which implies

$$E_R(t) = 0 \quad \text{for } t_r \leqq t < T_r .$$

Then, the conclusion follows immediately.

Now consider the metric defined as

$$G = dt^2 + u_R(t)^{4/n} h_{\phi_R}(t)$$

on $[-L, T_R) \times S^{n-1}$ with $L \geqq 0$. By Corollary 1.2, this metric has constant scalar curvature $n(n-1)$. By Lemma 1.3 (i) and Lemma 2.5, this space can be smoothly closed up at $t = T_R$ by adding one point.

In this way we get a smooth Riemannian manifold M_L with boundary. Since u_R and ϕ_R are constant for $t \leqq 0$, we can take the double of M_L to get a family of Riemannian metrics G_L of S^n. Recall our construction depends on the choice of ε, and the argument here is valid for any sufficiently small $\varepsilon > 0$ in (1). Then choosing ε small, we easily see

$$\mathrm{Vol}(S^n, G_0) < \mathrm{Vol}(S^n(1)).$$

On the other hand

$$\lim_{L \to \infty} \mathrm{Vol}(S^n, G_L) = \infty.$$

Consequently, for even $n \geqq 4$, we obtain, by scaling, a metric of S^n with unit total volume whose scalar curvature is constant equal to any given positive number greater than $n(n-1)\mathrm{Vol}(S^n(1))^{2/n}$, which together with Theorem 3 of [2] completes the proof of Theorem.

References

[1] J.L. Kazdan and F.W. Warner, A direct approach to the determination of Gaussian and scalar curvature, Inv. Math. 28 (1975), 227-230.

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