On Convex Hamiltonians

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Abstract

Let H be a convex Hamiltonian on T^*M. We show that if the Hamiltonian flow of H admits on the compact regular energy surface $H^{-1}(\sigma)$ a continuous invariant Lagrangian subbundle, then there are no conjugate points provided the energy σ is sufficiently high. If in addition $H^{-1}(\sigma)$ is symmetric we show that for any values of the energy σ, $H^{-1}(\sigma)$ projects over the whole configuration space M. These results generalize a well know theorem of Klingenberg [19] for riemannian metrics.

In a different direction we show that the topological entropy of the Hamiltonian flow of a convex Hamiltonian on a compact regular energy

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surface (with sufficiently high energy) is positive as long as $\pi_1(M)$ has exponential growth or if $\pi_1(M)$ is finite and for some coefficient field the loop space homology of M grows exponentially.

1 *Introduction*

The purpose of the present paper is to extend various geometric and topological results that hold for riemaniann metrics to the wider open class of convex Hamiltonians. Let M be a manifold without boundary and let $H : T^*M \to \mathbf{R}$ be a smooth Hamiltonian. We will say that H is *convex* if for each $q \in M$ the function $H(q, .)$ regarded as a function on the linear space T_q^*M has positive definite Hessian; equivalently the second fibre derivative $D_F^2 H$ is positive definite.

As Anosov points out in [1] it is important to distinguish among the problems of riemannian geometry those which are related to their variational character alone and do not depend on the remaining specific peculiarities of the riemannian metric. For several of the results we will describe the common denominator is precisely the variational calculus in the large i.e. the Morse theory of the loop space.

In [19] Klingenberg proved that if the geodesic flow of a compact riemannian manifold M is Anosov, then M has no conjugate points. Mañé [21] generalized Klingenberg's result to the non-compact case and gave at the same time a new proof of it using a variant of the Maslov-Arnold index adapted to the use of the Riccati equation. Moreover, Mañé showed that if M has finite volume and the geodesic flow admits a continuous invariant Lagrangian subbundle, then M has no conjugate points. When the geodesic flow is Anosov the stable and unstable subbundles are Lagrangian.

In the present paper we will extend Klingenberg and Mañé's results to the case of a convex Hamiltonian under adecuate hypotheses on the energy surface. Our extensions will include in particular, non-symmetric Finsler metrics. Anosov in [1] suggested a possible proof to handle the Finsler case but no details were given. Here we approach the problem using Mañé's ideas in [21].

Before stating our theorems precisely let us recall some well known facts and give a few definitions. Let H be Hamiltonian on T^*M and let ϕ_t denote its associated Hamiltonian flow. It is well known that ϕ_t preserves the canonical symplectic form of T^*M and leaves all the energy surfaces $H^{-1}(\sigma)$ invariant. From now on we shall assume that σ is a regular value of H and that the flow ϕ_t on $H^{-1}(\sigma)$ is complete.

We will say that ϕ_t has a *continuous invariant Lagrangian subbundle* if there exists a continuous subbundle E of $TH^{-1}(\sigma)$ such that for all $\theta \in H^{-1}(\sigma)$ the fibre $E(\theta)$ is a Lagrangian subspace of $T_\theta T^*M$ and $E(\phi_t(\theta)) = d\phi_t(E(\theta))$ for all $t \in \mathbf{R}$.

Let $\pi : T^*M \to M$ be the canonical projection. For $s \in (-\epsilon, \epsilon)$ consider a curve $s \to p_s \in T_q^*M$ with $p_0 \in H^{-1}(\sigma)$. Let $Y(t)$ be the variational field

$$Y(t) = \frac{d}{ds} \,|_{s=0}\, \pi \circ \phi_t(p_s).$$

We will say that $H^{-1}(\sigma)$ has *no conjugate points* if for all fields Y as above we have that $Y(t) \neq 0$ for all $t \neq 0$.

Theorem 1.1 *Let M be a compact manifold and let H be a convex Hamiltonian on T^*M. Suppose the Hamiltonian flow of H on the compact regular energy surface $H^{-1}(\sigma)$, possesses a continuous invariant Lagrangian subbundle. Then $H^{-1}(\sigma)$ has no conjugate points provided the energy σ is stricly bigger than the maximum of H over the zero section of T^*M.*

The energy surface $H^{-1}(\sigma)$ is said to be *symmetric* if it is invariant under the involution $(q,p) \to (q,-p)$. The next theorem complements Theorem 1.1 under the additional assumption of symmetry.

Theorem 1.2 *Let H be a convex Hamiltonian on T^*M. Suppose the Hamiltonian flow of H on the regular energy surface $H^{-1}(\sigma)$ possesses a continuous invariant Lagrangian subbundle. Then if $H^{-1}(\sigma)$ is symmetric and every point in $H^{-1}(\sigma)$ is non-wandering we have that $\pi H^{-1}(\sigma) = M$ and $H^{-1}(\sigma)$ has no conjugate points.*

Theorem 1.2 is false -even in the geodesic flow case- if we drop the hypothesis that every point in $H^{-1}(\sigma)$ is non-wandering. As an example take the paraboloid of revolution $z = x^2 + y^2$. The obvious circle action together with the field of the geodesic flow span a continuous invariant Lagrangian subbundle but there are conjugate points.

As in the case of the geodesic flow, if the Hamiltonian flow of H on $H^{-1}(\sigma)$ is Anosov, the stable and unstable bundles give rise to continuous invariant Lagrangian subbundles (cf. Lemma 2.3), so all the previous results hold if we assume the Hamiltonian flow to be Anosov. Consider for example $H = T + U$ where T is the kinetic energy associated with some riemannian metric and U is a smooth function on the compact manifold M. It follows from Theorem 1.2 that if the Hamiltonian flow of H on $H^{-1}(\sigma)$ is Anosov then σ is strictly bigger than the maximum of U on M.

We briefly sketch now the ideas involved in the proofs of the previous theorems. By the help of a riemannian metric we obtain in Section 3 a global Riccati equation associated to any Hamiltonian on T^*M. The coefficients of this equation are operators that naturally extend the curvature operator in the riemannian case. Next we define a Maslov-Arnold index following very closely Arnold [3] and Mañé [21]. This index is associated to continuous closed curves on an energy surface of a Hamiltonian provided a continuous Lagrangian subbundle E is given. The way the index is defined implies automatically its invariance under homotopies.

In Section 5 we show that if H is convex and E is invariant under the flow of H, then the Maslov-Arnold index of a closed pseudo-orbit of the Hamiltonian flow is positive as long as E touches the vertical non-trivially. This result is crucial for proving Theorems 1.1 and 1.2 and is acomplished by making strong use of the convexity and the Riccati equation mentioned above. The "spirit" of this result is already contained in Klingenberg's paper [19]. Various authors have proved it in slightly different settings, cf. [4, 5, 6, 7, 10, 12] (we thank M.L. Bialy for calling our attention to these references).

In the case of Theorem 1.1 a reduction to the Finsler case allows us to use Morse theory and show that in fact the Maslov-Arnold index of any closed pseudo-orbit is zero. In this way we deduce that the subbundle E cannot touch the vertical non-trivially. The convexity implies now via a Sturm-Liouville type of result that there are no conjugate points. For Theorem 1.2 the symmetry is enough to show that the Maslov-Arnold index of any closed pseudo-orbit is zero. Theorem 1.2 is completed by showing that points that project onto the boundary of $\pi H^{-1}(\sigma)$ give rise to closed pseudo-orbits with positive index. All these is done in Sections 5 and 6.

We observe that the previous theorems allows us to extend many other properties that hold for Anosov geodesic flows to Anosov convex Hamiltonians. Once the non-existence of conjugate points has been established one can show easily, for example, that $\pi_1(M)$ has exponential growth and that M is cover by euclidean space as well as other properties like topological equivalence as in [13, Theorem B].

In [25, 26] the second author showed that if the geodesic flow on a compact surface M is expansive, then there are no conjugate points. Recall that a flow $\varphi_t : X \to X$ on a compact metric space (X, d) is said to be *expansive* if given $\epsilon > 0$ there exists $\delta > 0$ such that if there is an homeomorphism $\tau : \mathbf{R} \to \mathbf{R}$, $\tau(0) = 0$, such that

$$d(\phi_{\tau(t)}(y), \phi_t(x)) < \delta$$

then $y = \phi_{\bar{t}}$ where $|\bar{t}| < \epsilon$. Anosov flows are expansive flows. Results analogous to Theorems 1.1 and 1.2 also hold for convex expansive Hamiltonians, however the details are more involved and will be presented in a forthcoming paper.

In Section 7 we explore a different realm. We are concern here with the relations between topological entropy and topology . For a flow $\varphi_t : X \to X$ on a compact metric space X, let $h_{top}(\varphi_t, A)$ denote the topological entropy of φ_t with respect to the set $A \subset X$.

We will say that a compact manifold M^n with finite fundamental group is *rationally elliptic* if the total rational homotopy of M, $\pi_*(M) \otimes \mathbf{Q}$, is finite dimensional [16]. We will show:

Theorem 1.3 *Let M be a compact manifold with finite fundamental group. Let H be a convex Hamiltonian with Hamiltonian flow ϕ_t on the compact regular energy surface $H^{-1}(\sigma)$ with σ strictly bigger than the maximum of H over the zero section. If for some $p \in M$ we have that*

$$h_{top}(\phi_t, H^{-1}(\sigma) \cap T_p^*M) = 0,$$

then the loop space homology of M with coefficients on any field grows sub-exponentially. In particular M is rationally elliptic.

Theorem 1.4 *Let M be a compact manifold. Let H be a convex Hamiltonian with Hamiltonian flow ϕ_t on the compact regular energy surface $H^{-1}(\sigma)$ with σ strictly bigger than the maximum of H over the zero section. If for some $p \in M$ we have that*

$$h_{top}(\phi_t, H^{-1}(\sigma) \cap T_p^*M) = 0,$$

then $\pi_1(M)$ grows sub-exponentially.

Theorem 1.4 extends a classical result of Dinaburg [11] and Theorem 1.3 was proved in [23] for the riemannian case although Gromov implicitely uses it in [15]. Let us note that Theorems 1.3 and 1.4 show that the results of the first author in [23, 24] about topological obstructions for complete integrability of geodesic flows extend to convex Hamiltonians.

Finally in Section 8 we consider the more general situation of twisted cotangent bundles and we show that the key lemmas hold in this setting. We also apply Theorems 1.3 and 1.4 to Hamiltonians that contain terms that depend linearly in the velocities like the corresponding to the the motion of charged particle under the effect of an exact electromagnetic potential.

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2 *Preliminaries*

Let (N,ω) be a symplectic manifold. Given a smooth function $H : N \to \mathbf{R}$ (i.e. a Hamiltonian) its symplectic gradient is defined as the vector $J\nabla H(x)$ for which

$$dH_x(v) = \omega_x(J\nabla H(x), v),$$

for all v. As it is well known cotangent bundles are symplectic manifolds. Let M be a manifold and let T^*M denote the cotangent bundle of M with canonical projection π. The canonical 1-form ξ is defined by

$$\xi_{(q,p)}(v) = p(d\pi_{(q,p)}(v)),$$

where $p \in T_q^*M$ and $v \in T_{(q,p)}(T^*M)$. The canonical symplectic structure ω on T^*M is defined as

$$\omega = d\xi.$$

Let U be an open set of $\mathbf{R}^n$. We identify T^*U with the set of $2n$-uples $(q,p) = (q_1, ..., q_n, p_1, ..., p_n)$ where $q \in U$ and $p \in \mathbf{R}^n$. It is easily seen that

$$\xi = \sum_i p_i \wedge dq_i,$$

and

$$\omega = \sum_i dp_i \wedge dq_i.$$

Also if $H : T^*U \to \mathbf{R}$ is a smooth function we have that

$$J\nabla H = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}). \tag{1}$$

Let H_{pp}, H_{pq}, H_{qq} stand for the matrices of partial derivatives $(\frac{\partial^2 H}{\partial p_i \partial p_j})$, $(\frac{\partial^2 H}{\partial p_i \partial q_j})$ and $(\frac{\partial^2 H}{\partial q_i \partial q_j})$ respectively.

Next observe that if M_1 and M_2 are manifolds and $f : M_1 \to M_2$ is a diffeomorphism, then f induces a diffeomorphism $\tilde{f} : T^*M_1 \to T^*M_2$ such that

$$\tilde{f}^*\xi_2 = \xi_1. \tag{2}$$

Suppose now that N_1 and N_2 are symplectic manifolds and f is a symplectomorphism. Then if $H : N_2 \to \mathbf{R}$ is a smooth function,

$$J\nabla(H \circ f) = f^*(J\nabla H).$$

Hence if U is a open set of $\mathbf{R}^n$, M a manifold and $\varphi : U \to \varphi(U) \subset M$ a diffeomorphism, then for any smooth function $H : T^*M \to \mathbf{R}$ we obtain

$$\tilde{\varphi}^*(J\nabla H) = (\frac{\partial(H \circ \tilde{\varphi})}{\partial p}, -\frac{\partial(H \circ \tilde{\varphi})}{\partial q}).$$

Suppose now we fix on M a smooth riemannian metric $< , >$ and let $\flat : T^*M \to TM$ denote the obvious identification map given by the metric. This induces in a canonical way a riemannian metric g on T^*M. If $\theta \in T^*M$ then the riemannian connection on M gives rise to a direct sum decomposition $T_\theta(T^*M) = H(\theta) \oplus V(\theta)$. The subspace $H(\theta)$ is called the horizontal subspace and is the kernel of the connection map $K : T(T^*M) \to TM$ defined as follows: let $x \in T_\theta(T^*M)$ and let $Z : (-\epsilon, \epsilon) \to T^*M$ be a curve such that $Z(0) = \theta$, $Z'(0) = x$, then

$$Kx = \frac{D\flat Z}{dt} |_{t=0},$$

where $\frac{D}{dt}$ denotes covariant derivative along $\pi \circ Z(t)$.

The subspace $V(\theta)$ is called the vertical subspace and is nothing but the kernel of $d\pi_\theta$.

Each subspace $H(\theta)$ and $V(\theta)$ can be identified with $T_{\pi(\theta)}M$ using the maps $d\pi_\theta$ and K respectively. We provide T^*M with the unique riemannian metric g for which the summands are orthogonal and the restriction of the metric to each summand is the inner product given by the riemannian metric on M.

A subspace $E \subset T_\theta T^*M$ is said to be *Lagrangian* if the symplectic form ω_θ vanishes on it and $dimE = n$ ($n = dimM$). It is obvious that $V(\theta)$ is a Lagrangian subspace (this fact is independent of the metric). The horizontal subspace is also Lagrangian since one can show that ω_θ can be written as [20]:

$$\omega_\theta(x,y) = < d\pi_\theta x, Ky > - < Kx, d\pi_\theta y > .$$

Equivalently if we write $x = (h_1, v_1)$ and $y = (h_2, v_2)$ under the identifications described above then

$$\omega_\theta((h_1, v_1), (h_2, v_2)) = < h_1, v_2 > - < h_2, v_1 > .$$

Let $J_\theta : T_\theta T^*M \to T_\theta T^*M$ be the complex structure

$$J_\theta(h, v) = (-v, h).$$

Clearly

$$\omega_\theta(x,y) = g(J_\theta x, y).$$

Let $E \subset T_\theta(T^*M)$ be a subspace of dimension n and with the property $E \cap V(\theta) = \{0\}$. Then E is the graph of some linear map $S(\theta, E) : H(\theta) \to V(\theta)$. It can be easily checked that E is Lagrangian if and only if $S(\theta)$ is symmetric with respect to the product $< \, , \, >$.

Consider now a parametrization $\varphi : U \to M$ and the maps $\tilde{\varphi} : T^*U \to T^*M$ and $\hat{\varphi} \stackrel{\text{def}}{=} d\tilde{\varphi} : TT^*U \to TT^*M$. Identify T^*U with $U \times \mathbf{R}^n$ and TT^*U with $U \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$. For $\theta \in T^*M$ let $\hat{\varphi}_\theta$ be the map $d\tilde{\varphi}_{\tilde{\varphi}^{-1}(\theta)} : \mathbf{R}^n \times \mathbf{R}^n \to T_\theta T^*M$.

Lemma 2.1 *If* $\tilde{\varphi}(q,p) = \theta$ *then*

$$d\pi_\theta \circ \hat{\varphi}_\theta(q', 0) = d\varphi_q(q').$$

$$K \circ \hat{\varphi}_\theta(0, p') = \flat(d\varphi_q^*)^{-1}(p').$$

Proof:

$$d\pi_\theta \circ d\tilde{\varphi}_{(q,p)}(q',0) = d(\pi \circ \tilde{\varphi})_{(q,p)}(q',0) = d\varphi_q(q'),$$

where the last equallity follows from the fact that $\tilde{\varphi}(q,p) = (\varphi(q), (d\varphi_q^*)^{-1}(p))$.

Now observe that $\hat{\varphi}_\theta(0,p') \in V(\theta)$ and since the covariant derivative of vertical vectors is just ordinary derivative we have

$$K \circ \hat{\varphi}_\theta(0,p') = \frac{d}{dt}\,|_{t=0}\, \flat\tilde{\varphi}(q,p(t)),$$

where $p'(0) = p'$. Thus observing again that $\tilde{\varphi}(q,p) = (\varphi(q), (d\varphi_q^*)^{-1}(p))$ we have

$$K \circ \hat{\varphi}_\theta(0,p') = \flat(d\varphi_q^*)^{-1}(p').$$

◇

Now let H be a function on T^*M and let ϕ_t denote the flow of the vector field $J\nabla H$. It is well known that ϕ_t preserves ω and leaves all the level surfaces $H^{-1}(\sigma)$ invariant. Suppose now that σ is a regular value for H. Set

$$S^*M = H^{-1}(\sigma).$$

Let Q denote the symplectic vector bundle over S^*M whose fiber $Q(\theta)$ ($\theta \in S^*M$) is $T_\theta T^*M$. It is obvious that $d\phi_t$ acts on Q and preserves the symplectic form on the fibers. We recall the definition of a continuous invariant Lagrangian subbundle.

Definition 2.2 *A continuous invariant Lagrangian subbundle E is a continuous subbundle of TS^*M such that each fiber is a Lagrangian subspace of $Q(\theta)$ and*

$$E(\phi_t(\theta)) = d\phi_t E(\theta)$$

for all $t \in \mathbf{R}$ and $\theta \in S^*M$.

We now recall the definition of an Anosov flow. Let N be a compact riemannian manifold and let $\phi_t : N \to N$ be a flow with associated vector field X. The flow ϕ_t is said to be *Anosov* if there are numbers $c > 0$, $0 < \lambda < 1$ and continuous $d\phi_t$-invariant subbundles E^s and E^u of TN such that

$$TN = E^s \oplus E^u \oplus \mathbf{R}X,$$

$$\| d\phi_{-t} |_{E^u} \| \leq c\lambda^t \quad t \geq 0,$$
$$\| d\phi_t |_{E^s} \| \leq c\lambda^t \quad t \geq 0.$$

Next we will show:

Lemma 2.3 *Suppose the Hamiltonian flow $\phi_t : S^*M \to S^*M$ of the Hamiltonian H is an Anosov flow. Then the bundles $E^s \oplus \mathbf{R}X$ and $E^u \oplus \mathbf{R}X$ are continuous invariant Lagrangian subbundles.*

Proof: Take x and y in $E^s(\theta)$. Hence

$$\omega_\theta(x, y) = \omega_{\phi(\theta)}(d\phi_t x, d\phi_t y).$$

But $\omega(d\phi_t x, d\phi_t y)$ goes to zero as $t \to +\infty$ and consequently $\omega_\theta(x, y) = 0$. Therefore the symplectic form vanishes on $E^s(\theta) \oplus \mathbf{R}X(\theta)$ (obviously by the same argument $\omega_\theta(x, X(\theta)) = 0$ for $x \in E^s(\theta)$). Similarly the symplectic form vanishes on $E^u(\theta) \oplus \mathbf{R}X(\theta)$ and hence both have to be n-dimensional and therefore Lagrangian.

$\diamond$

Finally we give a few generalities about Finsler metrics. Let M be a differentiable manifold. A *Finsler metric* on M is a norm on each tangent space (possibly not symmetric) such that the unit sphere in each tangent space T_qM is a strictly convex submanifold which depends differentiably on q. Alternatively, a Finsler metric is a function $N : TM \to \mathbf{R}$, differentiable off the zero-section, such that $D_F^2(N^2)$ is positive definite and such that

$$N(\lambda v) = \lambda N(v),$$

for all $\lambda > 0$ and $v \in T_pM$. Here D_F^2 denotes the second derivative in the fibre direction. We call N symmetric if $N(-v) = N(v)$.

Finsler metrics are sometimes conviniently described on the cotangent bundle. We consider T^*M with its standard symplectic structure. Let H be a Hamiltonian with Hamiltonian vector field $J\nabla H$. If $D_F^2 H$ is positive definite, the Legendre transform

$$L_H = D_F H : T^*M \to TM,$$

is a local diffeomorphism. If H is positively homogeneous of degree two,

$$H(\lambda p) = \lambda^2 H(p),$$

for $\lambda > 0$, then L_H is a global diffeomorphism and

$$N^2 = H \circ L_H^{-1}$$

is a Finsler metric on M. The field $J\nabla H$ decribes the geodesics of the Finsler metric since the projection of the integral curves of $J\nabla H$ under $T^*M \to M$ are the geodesics of N. If $2N^2$ is a riemannian metric, $2H$ is the dual metric and $L_H = \flat$ is the canonical identification between TM and T^*M.

3 *The linear equation and the Riccati equation*

Let $\varphi : U \to \varphi(U) \subset M$ be a diffeomorphism where U is an open set in $\mathbf{R}^n$.

Take $\theta \in S^*M$ so that $\pi(\theta) \in \varphi(U)$ and take $x \in Q(\theta)$. Now consider a variation

$$\alpha_s(t) = (q_s(t), p_s(t))$$

such that for each $s \in (-\epsilon, \epsilon)$, α_s is a solution of the Hamiltonian $H \circ \tilde{\varphi}$ such that α_0 has energy σ and $\frac{d}{ds}\mid_{s=0} \alpha_s(0) = \hat{\varphi}^{-1}(x)$. Then if we write $(h(t), v(t)) = \hat{\varphi}^{-1}(d\phi_t(v))$ we have that h and v verify the following linear equation

$$\begin{aligned} h' &= H_{qp} h + H_{pp} v, \\ v' &= -H_{qq} h - H_{pq} v, \end{aligned} \tag{3}$$

where derivatives are evaluated along $\alpha_0(t)$. This follows directly from equation (1).

Fix a riemannain metric on M and using the splitting described in the previous section write $(d\phi_t)_\theta(x) = (\bar{h}(t), \bar{v}(t))$. Let $G(t) \stackrel{\text{def}}{=} \hat{\varphi}_{\phi_t(\theta)} : \mathbf{R}^{2n} \to T_{\pi(\phi_t(\theta))}M \oplus T_{\pi(\phi_t(\theta))}M$. Then clearly

$$G(t)(h(t), v(t)) = (\bar{h}(t), \bar{v}(t)).$$

Observation 3.1 Note that S^*M has no conjugate points if and only if there is no non-trivial $\bar{h}$ as above such that $\bar{h}$ vanishes for two different points. This is equivalent to saying that for all $\theta \in S^*M$ and non-zero $x \in Q(\theta)$ the set $\{t : \, d\phi_t(x) \in V(\phi_t(\theta))\}$ consists at most of one point.

Let $'$ denote usual derivatives and let a dot $\dot{}$ denote covariant derivatives. ¿From the last equation we get

$$(\dot{\bar{h}}, \dot{\bar{v}}) = \dot{G}(h,v) + G(h',v').$$

Using the linear equation (3) and evaluating everything in $t=0$ we obtain

$$(\dot{\bar{h}}, \dot{\bar{v}}) = \dot{G}G^{-1}(\bar{h},\bar{v}) + G\begin{pmatrix} H_{qp} & H_{pp} \\ -H_{qq} & -H_{pq} \end{pmatrix}(h,v) =$$

$$= [\dot{G}G^{-1} + G\begin{pmatrix} H_{qp} & H_{pp} \\ -H_{qq} & -H_{pq} \end{pmatrix} G^{-1}](\bar{h},\bar{v}) \stackrel{\text{def}}{=} \begin{pmatrix} \bar{H}_{qp} & \bar{H}_{pp} \\ -\bar{H}_{qq} & -\bar{H}_{pq} \end{pmatrix}(\bar{h},\bar{v}) = \tag{4}$$

$$\stackrel{\text{def}}{=} \mathcal{H}(\theta)(\bar{h},\bar{v}).$$

The operator $\mathcal{H}(\theta) : T_{\pi(\theta)}M \oplus T_{\pi(\theta)}M \to T_{\pi(\theta)}M \oplus T_{\pi(\theta)}M$ is exactly defined by the covariant derivatives in $t=0$ of the components of $d\phi_t(x)$ in the splitting, along the orbit defined by θ. The component $\bar{H}_{pp}$ of $\mathcal{H}$ is what we are interested in. We will now show:

Lemma 3.2 *Suppose H is convex. Then the operator $\bar{H}_{pp}(\theta) : T_{\pi(\theta)}M \to T_{\pi(\theta)}M$ is positive definite.*

Proof: Observe that $G(0,v)$ is always vertical. Hence $\dot{G}G^{-1}(0,\bar{v})$ has zero horizontal part. Therefore from equation (4) we have:

$$\bar{H}_{pp}(\theta)(\bar{v}) = d\pi_\theta \circ G\begin{pmatrix} H_{pq} & H_{pp} \\ -H_{qq} & H_{pq} \end{pmatrix} G^{-1}(0,\bar{v}).$$

Using Lemma 2.1 we get

$$\bar{H}_{pp}(\bar{v}) = d\varphi_q \circ H_{pp} \circ d\varphi_q^* \circ \flat^{-1}(\bar{v}).$$

Therefore

$$< \bar{H}_{pp}(\bar{v}), \bar{v} > = < d\varphi_q \circ H_{pp} \circ d\varphi_q^* \circ \flat^{-1}(\bar{v}), \bar{v} > =$$

$$= < H_{pp} \circ d\varphi_q^* \circ \flat^{-1}(\bar{v}), d\varphi_q^* \circ \flat^{-1}(\bar{v}) >_{flat},$$

where $< , >_{flat}$ denotes the flat metric of $\mathbf{R}^n$. Hence if H_{pp} is positive definite (H is convex) then $\bar{H}_{pp}$ is also positive definite and we deduce the lemma. The last equality follows from the following (elementary) linear algebra fact:

Let $(V_1, < , >_1)$ and $(V_2, < , >_2)$ be two vector spaces with inner products. Let $\flat_i : V_i^* \to V_i$ be the canonical identification by means of the inner product $< , >_i$ $(i = 1, 2)$. Let $A : V_1 \to V_2$ be a linear map. Then

$$< Av, \omega >_2 = < v, \flat_1 A^* \flat_2^{-1} \omega >_1 .$$

$\diamond$

Next we will obtain a Riccati equation. Let E be a Lagrangian subspace of $Q(\theta)$. Suppose for t in some interval $(-\epsilon, \epsilon)$, $d\phi_t(E) \cap V(\phi_t(\theta)) = \{0\}$. As we explained in Section 2 we can write $d\phi_t(E) = graph\ S(t)$ where $S(t) : T_{\pi(\phi_t(\theta))}M \to T_{\pi(\phi_t(\theta))}M$ is a symmetric map. That is, if $x \in E$ then

$$d\phi_t(x) = (\bar{h}(t), S(t)\bar{h}(t)).$$

By means of the equation (4) we obtain:

$$\dot{S}\bar{h} + S(\bar{H}_{qp}\bar{h} + \bar{H}_{pp}S\bar{h}) = -\bar{H}_{qq}\bar{h} - \bar{H}_{pq}S\bar{h}.$$

Since this works for every $x \in E$ we obtain the Riccati equation:

$$\dot{S} + S\bar{H}_{pp}S + S\bar{H}_{qp} + \bar{H}_{pq}S + \bar{H}_{qq} = 0. \qquad (5)$$

4 *The Maslov-Arnold index*

In this section we will define a Maslov-Arnold index associated to an energy surface of a Hamiltonian H and a continuous Lagrangian subbundle. We will follow the presentation in [21, Part III] quite closely.

For $\theta \in S^*M$ let $\Lambda(\theta)$ denote the set of Lagrangian subspaces of $Q(\theta)$. Denote by $\Lambda(S^*M)$ the bundle over S^*M whose fibre on θ is $\Lambda(\theta)$. Let $\Lambda_k(\theta)$ be the set of Lagrangian subspaces $E \in \Lambda(\theta)$ such that $dim(E \cap V(\theta)) = k$. Then $\Lambda_k(\theta)$ is a Lagrangian submanifold of $\Lambda(\theta)$ with codimension $\frac{k(k+1)}{2}$. Then if $\Lambda_k(S^*M)$ denotes the set of elements $(\theta, E) \in \Lambda(S^*M)$ such that $E \in$

$\Lambda_k(\theta)$, it follows that $\Lambda_k(S^*M)$ is a submanifold of $\Lambda(S^*M)$ with codimension $\frac{k(k+1)}{2}$. Define

$$\Gamma = \cup_{k\geq 2}\Lambda_k(S^*M)$$

and let Σ be the complement of Γ. Fix a riemannian metric on M. Define a function $m : \Sigma \to S^1$ as follows: if $(\theta, E) \in \Lambda_0(S^*M)$ take a symmetric map (as in Section 2) $S(\theta, E) : T_{\pi(\theta)}M \to T_{\pi(\theta)}M$ such that *graph* $S(\theta, E) = E$ and define

$$m(\theta, E) = \frac{1 - i\, trace(S(\theta, E))}{1 + i\, trace(S(\theta, E))}.$$

When $(\theta, E) \in \Lambda_1(S^*M)$ set

$$m(\theta, E) = -1.$$

The arguments in [21] show that m is continuous.

Next we define an index that to every continuous closed curve $\gamma : [0,1] \to \Lambda(S^*M)$ associates an integer $Ind\ \gamma$. If $\gamma : [0,1] \to \Sigma$ is a continuous closed curve, define $Ind\gamma$ as the degree of the map $m \circ \gamma$. Observe that if $\gamma_1 : [0,1] \to \Sigma$ and $\gamma_2 : [0,1] \to \Sigma$ are homotopic then $Ind\ \gamma_1 = Ind\ \gamma_2$. Given a continuous closed curve $\gamma : [0,1] \to \Lambda(S^*M)$ define $Ind\ \gamma$ as $Ind\ \tilde{\gamma}$ where $\tilde{\gamma} : [0,1] \to \Sigma$ is homotopic to γ. The arguments in [3, 21] show that this definition is consistent (the key fact here is that $cod(\Gamma) \geq 3$).

Now suppose that there is a continuous Lagrangian subbundle E in Q. Given a continuous closed curve $\alpha : [0,1] \to S^*M$ we define $\hat{\alpha} : [0,1] \to \Lambda(S^*M)$ by

$$\hat{\alpha}(t) = (\alpha(t), E(\alpha(t))),$$

and define the index of α by

$$ind\ \alpha = Ind\ \hat{\alpha}.$$

Remark 4.1 Let us show that in fact the index we defined is independent of the riemannian metric we fixed. Let g_0 and g_1 be two riemannian metrics on M and let let ind_0 and ind_1 be the two Maslov-Arnold indices corresponding to g_0 and g_1. Let $g_s = sg_1 + (1-s)g_0$ and let $m_s : \Sigma \to S^1$ be the associated functions. If $\gamma : [0,1] \to \Sigma$ is a continuous closed curve, then clearly $m_s \circ \gamma$ is a homotopy between $m_0 \circ \gamma$ and $m_1 \circ \gamma$. Hence $Ind_0\gamma = Ind_1\gamma$ and thus $ind_0 = ind_1$.

5 *Convex Hamiltonians and the Maslov-Arnold index*

In this section we will study various properties in which the convexity enters in a crucial manner. We will show that the Maslov-Arnold index we defined in the last section is positive as long as the Lagrangian subbundle is invariant and touches the vertical non-trivially. This is the key for Theorems 1.1 and 1.2.

Lemma 5.1 *Assume H is convex. If there is a Lagrangian subspace $E \subset Q(\theta)$ such that $V(\phi_t(\theta)) \cap d\phi_t(E) = \{0\}$ for $t \in [0,a]$ then the segment $\{\phi_t(\theta),\ t \in [0,a]\}$ does not have conjugate points i.e. the set of $t \in [0,a]$ such that $d\phi_t(x) \in V(\phi_t(\theta))$ consists at most of one point for all non-zero $x \in Q(\theta)$.*

Proof: Take the symmetric map $S(t)$ that gives $d\phi_t(E)$ as a graph and let $(\bar{h}(t), \bar{v}(t))$ represent $d\phi_t(x)$ ($x \in Q(\theta)$). Suppose $\bar{h}(c) = 0$ for some $c \in [0,a]$. Consider $Y(t) : T_{\pi(\theta)}M \to T_{\pi(\phi_t(\theta))}M$ a family of linear isomorphisms satisfying

$$\dot{Y} = (\bar{H}_{qp} + \bar{H}_{pp}S)Y,$$

$$Y(0) = id.$$

If we take ω such that $(\omega, S(0)\omega) \in E$ and define

$$h_1(t) = Y(t)\omega,$$

$$v_1(t) = S(t)Y(t)\omega,$$

we get that (h_1, v_1) is a solution of the equation (4). Since

$$\omega(x,y)_\theta = g(J_\theta x, y)$$

and $d\phi_t$ preserves the symplectic form on the fibres we get

$$< \bar{h}(t), v_1(t) > - < \bar{v}(t), h_1(t) > = - < \bar{v}(c), \omega >$$

and hence

$$< Y^*(t)S(t)\bar{h}(t), \omega > - < Y^*(t)\bar{v}(t), \omega > = - < \bar{v}(c), \omega > .$$

Therefore

$$\bar{v}(t) = S(t)\bar{h}(t) + (Y^*)^{-1}(t)\bar{v}(c).$$

Since

$$\dot{\bar{h}} = \bar{H}_{qp}\bar{h} + \bar{H}_{pp}v,$$

we get

$$\dot{\bar{h}} = (\bar{H}_{qp} + \bar{H}_{pp}S)\bar{h} + \bar{H}_{pp}(Y^*)^{-1}\bar{v}(c)$$

and hence

$$\bar{h}(t) = Y(t)\bar{h}(c) + Y(t)\int_c^t Y^{-1}(u)\bar{H}_{pp}(Y^*)^{-1}\bar{v}(c)du.$$

Since $\bar{h}(c) = 0$ we obtain

$$< Y^{-1}(t)\bar{h}(t), \bar{v}(c) >= \int_c^t < \bar{H}_{pp}(Y^*)^{-1}(u)\bar{v}(c), (Y^*)^{-1}(u)\bar{v}(c) > du.$$

Then the convexity of H implies via Lemma 3.2 that $\bar{h}(t) \neq 0$ for all $t \in [0, a]$ different from c and hence there are no conjugate points along the segment (cf. Observation 3.1).

◇

Next we will show:

Lemma 5.2 *Suppose H is convex. If $E \subset Q(\theta)$ is a Lagrangian subspace, the set of $t \in \mathbf{R}$ such that $d\phi_t(E) \cap V(\phi_t(\theta)) \neq \{0\}$ is discrete.*

Proof: Set $E_t = d\phi_t(E)$. We need to show that if $E \cap V(\theta) \neq \{0\}$ there is a neighborhood I of $t = 0$ such that $E_t \cap V(\phi_t(\theta)) = \{0\}$ for all $0 \neq t \in I$.

Let

$$P_t : Q(\phi_t(\theta)) \to H(\phi_t(\theta)),$$

be the orthogonal projection. As in Lemma III.2 in [21] we have that $P(E)$ is the orthogonal complement of $J_\theta(E \cap V(\theta))$ in $H(\theta)$.

For t close to 0 take a set of linearly independent vectors

$$\{\hat{z_1}(t), ..., \hat{z_m}(t)\},$$

such that in $t=0$ they span $\bar{H}_{pp}^{-1}P(E)$. Note that since H is convex, Lemma 3.2 implies that $\bar{H}_{pp}$ is positive definite and hence invertible. Define

$$z_i = \sqrt{\bar{H}_{pp}}\hat{z}_i \quad for\ 1 \leq i \leq m.$$

Let $\{\hat{\omega}_1, ..., \hat{\omega}_k\}$ be a basis of unitary vectors of $E \cap V(\theta)$, then $k+m = dim H(\theta)$. Define

$$\omega_i = \frac{\sqrt{\bar{H}_{pp}}\hat{\omega}_i}{\mid \sqrt{\bar{H}_{pp}}\hat{\omega}_i \mid}.$$

Observe that $< \bar{H}_{pp}\hat{z}_i(0), \hat{\omega}_j >= 0$ and hence $< \sqrt{\bar{H}_{pp}}\hat{z}_i(0), \sqrt{\bar{H}_{pp}}\hat{\omega}_j >= 0$. Thus

$$\mathcal{F} \stackrel{\text{def}}{=} \{z_1(0), ..., z_m(0), \omega_1, ..., \omega_k\},$$

is a basis of $H(\theta)$.

Now take $(\bar{h}_i, \bar{v}_i)$ solutions of the linear equation (4) so that

$$\bar{h}_i(0) = 0,$$

$$\bar{v}_i(0) = \hat{\omega}_i \quad for\ 1 \leq i \leq k.$$

Define

$$Y_i(t) = \frac{(\sqrt{\bar{H}_{pp}})^{-1}\bar{h}_i(t)}{\mid (\sqrt{\bar{H}_{pp}})^{-1}\bar{h}_i(t) \mid}.$$

Since $\dot{\bar{h}}_i(0) = \bar{H}_{pp}\bar{v}_i(0)$ we get

$$lim_{t\to 0}Y_i(t) = lim_{t\to 0}\frac{(\sqrt{\bar{H}_{pp}})^{-1}\frac{\bar{h}_i(t)}{t}}{\mid (\sqrt{\bar{H}_{pp}})^{-1}\frac{\bar{h}_i(t)}{t} \mid} = \omega_i.$$

Notice that

$$\{Y_1(t), ..., Y_k(t)\} \subset (\sqrt{\bar{H}_{pp}})^{-1}P_t(E_t).$$

Define

$$\mathcal{F}_t = \{z_1(t), ..., z_m(t), Y_1(t), ..., Y_k(t)\}.$$

Observe that

$$\mathcal{F}_t \subset (\sqrt{\bar{H}_{pp}})^{-1}P_t(E_t),$$

and notice that $\mathcal{F}_t$ converges to $\mathcal{F}$. Therefore $P(E_t) = H(\phi_t(\theta))$ which yields the desired lemma.

$\diamond$

Suppose now that E is a continuous invariant Lagrangian subbundle.

Definition 5.3 *A continuous closed curve $\alpha : [0,T] \to S^*M$ is called a pseudo-orbit of the Hamiltonian flow if for all $t_0 \in [0,T]$ where*

$$E(\alpha(t_0)) \cap V(\alpha(t_0)) \neq \{0\},$$

there exists $\epsilon > 0$ such that

$$\alpha(t + t_0) = \phi_t(\alpha(t_0)),$$

for $t \in (-\epsilon, \epsilon)$ (for $t_0 = 0$ or $t_0 = T$ we take the continuous periodic extension of α to the real line).

If S^*M is symmetric we require (in addition) the last equation to hold also for $t_0 \in [0,T]$ such that

$$E(\alpha^*(t_0)) \cap V(\alpha^*(t_0)) \neq \{0\},$$

where $\alpha^*(t) = (q(t), -p(t))$ if $\alpha(t) = (q(t), p(t))$.

Lemma 5.4 *Suppose H is convex. If $\alpha : [0,T] \to S^*M$ is a closed pseudo-orbit of the Hamiltonian flow such that the set*

$$K = \{s \in [0,T] : \ E(\alpha(s)) \cap V(\alpha(s)) \neq \{0\}\}$$

is not empty, then $\operatorname{ind} \alpha > 0$.

Proof: As in [21, Lemma III.1] consider the set:

$$P(\alpha, c) = \{t \in [0,T] : \ m \circ \hat{\alpha}(t) = c\},$$

Exactly the same arguments as in [21, Lemma III.1] imply that to prove the lemma it is enough to show that there is a c such that

$$\Sigma_{t \in P(\alpha,c)} sign(\frac{(m \circ \hat{\alpha})'(t)}{i(m \circ \hat{\alpha})(t)}) > 0. \tag{6}$$

On account of the previous lemma the set of points where $m \circ \hat{\alpha} = -1$ is finite. Let $t_1, ..., t_r$ be this set and let V_i be a neighborhood of t_i so that on each V_i, α is an orbit. Now take $-1 < c < 0$ so near -1 so that

$$P(\alpha, c) \subset \cup_i V_i.$$

Let t be a point in $P(\alpha, c)$ and suppose it belongs to V_i. Let $S(t)$ be the symmetric map that gives $E(\alpha(t))$ as a graph. Hence

$$\frac{(m \circ \hat{\alpha})'(t)}{i(m \circ \hat{\alpha})(t)} = -2\frac{tr\dot{S}}{1 + tr^2 S},$$

and then using the Riccati equation (5)

$$\frac{(m \circ \hat{\alpha})'(t)}{i(m \circ \hat{\alpha})(t)} = 2\frac{tr(\bar{H}_{pp}S^2) + tr(\bar{H}_{pq}S) + tr(\bar{H}_{qp}S) + tr(\bar{H}_{qq})}{1 + tr^2 S},$$

where tr denotes trace. Since $\bar{H}_{pp}$ is positive definite (Lemma 3.2), the matrix $\sqrt{\bar{H}_{pp}}$ is well defined and since S is symmetric we get

$$tr(\bar{H}_{pp}S^2) \geq \frac{tr^2(\sqrt{\bar{H}_{pp}}S)}{n-1}.$$

For a fixed matrix B define

$$\sigma_B = inf_{||A||=1} \mid tr(BA) \mid.$$

Then

$$\frac{(m \circ \hat{\alpha})'(t)}{i(m \circ \hat{\alpha})(t)} \geq \frac{\frac{\sigma^2_{\bar{H}_{pp}}}{n-1} tr^2 S - (\sigma_{\bar{H}_{pq}} + \sigma_{\bar{H}_{qp}}) trS + tr\bar{H}_{qq}}{1 + tr^2 S}.$$

Notice that $t \in P(\alpha, c)$ implies $trS = -i\frac{1-c}{1+c}$. So if we take c very near -1, then $\mid trS(t) \mid$ becomes very large and thus

$$\frac{(m \circ \hat{\alpha})'(t)}{i(m \circ \hat{\alpha})(t)} \geq \frac{1}{n-1}\sigma^2_{\bar{H}_{pp}}.$$

This clearly implies equation (6).

◇

Remark 5.5 We observe that if the set K in the last lemma is empty then we have that $ind\ \alpha = 0$ since in this case the map $m \circ \hat{\alpha}$ does not take the value -1 and hence it is no surjective.

Lemma 5.6 *If $E(\theta) \cap V(\theta) \neq \{0\}$ and every point in S^*M is non-wandering, then there exists a closed pseudo-orbit $\alpha : [0,T] \to S^*M$ such that $\alpha(0) = \theta$ and hence with positive index by the previous lemma.*

Proof: It follows from Lemma 5.2 exactly as in [21]. ◇

6 *Proofs of Theorems 1.1 and 1.2.*

Proof of Theorem 1.1:

Since H is convex and σ is strictly bigger than the maximum of H over the zero section, we deduce that for each $q \in M$, $H^{-1}(\sigma) \cap T_q^*M$ is a strictly convex hypersurface containing the origin in its interior. Thus we can define a Finsler metric F so that $F^{-1}(1) = S^*M$. Then on the level surface S^*M the orbits of the Hamiltonian flow of F are just reparametrizations of the orbits of the Hamiltonian flow of H [27]. Then one easily checks that H possesses a continuous invariant Lagrangian subbundle if and only if F does, and that S^*M has conjugate points if and only if the Finsler metric has conjugate points. So we only need to prove the theorem in the case of a Finsler metric.

By Lemma 5.1 it is enough to show that $E(\theta) \cap V(\theta) = \{0\}$ for all $\theta \in S^*M$. Suppose there exists $\theta \in S^*M$ such that $E(\theta) \cap V(\theta) \neq \{0\}$. Lemmas 5.4 and 5.6 imply that there exists a pseudo-orbit α with $ind\ \alpha > 0$. Note that since S^*M is a sphere bundle over M, homotopies in M can be lifted to homotopies in S^*M. Consider now the free homotopy class of the curve $\pi \circ \alpha$. If this class is trivial, clearly $ind\ \alpha = 0$, since the index is a homotopy invariant. Suppose the class is not trivial. Then on account of Morse theory, which also works for Finsler metrics (see for example [29]), we can find a geodesic $\pi \circ \beta$ freely homotopic to $\pi \circ \alpha$ and such that it minimizes the length in its homotopy class. Hence $\pi \circ \beta$ has zero Morse index and thus, it has no conjugate points within one period (also $\pi \circ \beta$ has no self-intersections). By [12, Equation (4.10)] we have that $E(\beta(t))$ never touches the vertical non-trivially and thus β must have zero Maslov-Arnold index (cf.

Remark 5.5). But this is a contradiction because $ind\ \beta = ind\ \alpha > 0$ since α and β are homotopic.

◇

Proof of Theorem 1.2:

The symmetry condition replaces the use of Morse theory in the proof of Theorem 1.1. Since the energy surface S^*M is symmetric, if $\alpha(t) = (q(t), p(t))$ is a closed pseudo-orbit of H , then the curve $\bar{\alpha}(t) = (q(-t), -p(-t))$ is also a closed pseudo-orbit (recall the definition of pseudo-orbit in the symmetric case, cf. Def. 5.3). Now observe that the curve α and the curve $\alpha^*(t) = (q(t), -p(t))$ are always homotopic and hence $ind\ \alpha = ind\ \alpha^*$. But clearly $ind\ \alpha^* = -ind\ \bar{\alpha}$. On account of Lemma 5.4 and Remark 5.5 the index of a closed pseudo-orbit is always non-negative. Since $\bar{\alpha}$ is also a closed pseudo-orbit we deduce that $ind\ \alpha = 0$. Lemmas 5.6 and 5.1 imply that there are no conjugate points.

To complete the proof of Theorem 1.2 we need to show that $\pi S^*M = M$. Suppose by absurd that πS^*M is strictly contained in M. Then πS^*M is a smooth n-dimensional manifold with boundary given by $U^{-1}(\sigma)$, where $U(q) = min_{p \in T_q^*M} H(q,p)$. But for $\theta \in S^*M$ such that $\pi\theta \in U^{-1}(\sigma)$ we have that $d\pi_\theta(T_\theta S^*M) = T_{\pi\theta}U^{-1}(\sigma)$. Since $E(\theta) \subset T_\theta S^*M$ and $dim\ E(\theta) = n$ we clearly have that $E(\theta) \cap ker\ d\pi_\theta \neq \{0\}$. This a contradiction since we showed before that E cannot touch the vertical non-trivially.

◇

Remark 6.1 We do not know how to drop the symmetry condition in Theorem 1.2.

7 *Topological entropy, loop space homology and the fundamental group*

Let M be a manifold endowed with a Finsler metric. Let ϕ_t denote its geodesic flow. Assume it is complete (this would be the case if M is compact). The flow ϕ_t acts on TM minus the zero section. Since N^2 is positively homogeneous of degree two we have the relation:

$$\phi_s(tv) = t\phi_{st}(v) \tag{7}$$

for all real s and $t > 0$.

As in riemannian geometry we set $exp = \pi \circ \phi_1$ where $\pi : TM \to M$ is the canonical projection and let exp_p be the restriction of exp to $T_pM - \{0\}$. Equation (7) implies that $t \to exp_p(tv)$ (for $t > 0$) is the geodesic through p with velocity v.

The proofs of the results that follow are similar to ones given in [23] for the riemannian case. One of the difficulties now is that the exponential map is defined on the punctured tangent space $T_pM - \{0\}$.

Assume from now on that M is compact. Fix a point $p \in M$ and a number $\lambda > 0$. Denote by $n_p(q, \lambda)$ the number of geodesics connecting p and q with length $< \lambda$. Let $B(\lambda)$ be the set of vectors $v \in T_pM - \{0\}$ with $N(v) < \lambda$. If q is a regular value of exp_p then the set $exp_p^{-1}(q) \cap B(\lambda)$ being discrete and closed is finite, hence since $n_p(q, \lambda)$ is the cardinality of this set, we obtain that $n_p(q, \lambda)$ is finite. Moreover, it is locally constant as a function on q. Endow M with a riemannian metric, call ω the associated volume form and μ the induced measure (if M is not orientable we work with its double covering). Set

$$I_p(\lambda) = \int_M n_p(q, \lambda) d\mu(q), \tag{8}$$

(Berger and Bott already considered this integral for the riemannian case in [8]).

Let X denote the set of regular values of exp_p. By Sard's theorem X has full measure and we saw that $n_p(q, \lambda)$ is finite and locally constant on X, thus $I_p(\lambda)$ is well defined.

Take now $\delta > 0$ and small and let $n_p(q, \delta, \lambda)$ denote the number of geodesics connecting p and q with length in $[\delta, \lambda)$. Then clearly

$$n_p(q, \lambda) = n_p(q, \delta) + n_p(q, \delta, \lambda),$$

and hence

$$I_p(\lambda) = \int_M n_p(q, \delta) d\mu(q) + \int_M n_p(q, \delta, \lambda) d\mu(q) \stackrel{\text{def}}{=} P_p(\delta) + Q_p(\delta, \lambda). \tag{9}$$

Now we state an important property of Finsler metrics: if δ is small enough, there exists a unique geodesic from p to q with length $< \delta$. (cf [9]). Thus $P_p(\delta) = \mu(B(p, \delta))$.

Next let $A(\delta,\lambda)$ denote the annulus in $T_pM-\{0\}$ given by those vectors v that verify $\delta \le N(v) < \lambda$. In view of the previous considerations the exponential map is a covering on $A(\delta,\lambda)\cap exp_p^{-1}(U_\alpha)$ where U_α is a connected component of X. Hence we get

$$Q_p(\delta,\lambda)=\sum_\alpha \int_{A(\delta,\lambda)\cap exp_p^{-1}(U_\alpha)} \mid exp_p^*\omega \mid,$$

where $exp_p^*\omega$ is the pull-back of the volume form ω under the exponential map. Therefore from equation (9) we obtain

$$I_p(\lambda) \le \mu(B(p,\delta)) + \int_{A(\delta,\lambda)} \mid exp_p^*\omega \mid . \tag{10}$$

As in [23] we define σ_p, the geodesic entropy at p by

$$\sigma_p = limsup_{\lambda\to\infty} \frac{1}{\lambda} log\ I_p(\lambda).$$

Denote by S_pM the set of vectors $v \in T_pM$ with $N(v)=1$. and let $h_{top}(\phi_t, S_pM)$ denote the topological entropy of the flow ϕ_t respect to the set S_pM .

Theorem 7.1 *For all $p\in M$ we have that $\sigma_p \le h_{top}(\phi_t, S_pM)$.*

Proof: Set $\Delta = limsup_{t\to\infty} \frac{1}{t}\ log\ Vol(\phi_t(S_pM))$ where Vol stands for the $(n-1)$-riemannian volume on TM induced by the metric we fixed on M. Yomdin's theorem implies [28] (see also [15]) that $h_{top}(\phi_t, S_pM) \ge \Delta$. Hence we only need to show that $\sigma_p \le \Delta$. Let $V(t) = Vol(\phi_t(S_pM))$.

Choose $K>0$ such that $\mid \frac{\partial \phi_t(v)}{\partial t} \mid_{t=0} \mid \le K$ for $v\in SM$.

Define $\psi : \mathbf{R}\times S_pM \to TM$ by $\psi(t,v)=\phi_t(v)$ and $g:\mathbf{R}\times S_pM \to TM$ by $g(t,v)=tv$. Observe that equation (7) implies

$$\pi\circ\phi_1\circ g = \pi\circ\psi.$$

A direct calculation shows that

$$\mid det(d\psi_{(t,v)}) \mid \le K \mid det(d(\phi_t)_v) \mid, \tag{11}$$

therefore

$$\int_{A(\delta,\lambda)} \mid exp_p^*\omega \mid = \int_{A(\delta,\lambda)} \mid det(d(\pi\circ\phi_1)_v) \mid = \int_{[\delta,\lambda]\times S_pM} \mid det(d(\pi\circ\phi_1\circ g)_{(t,v)}) \mid =$$

$$= \int_{[\delta,\lambda]\times S_pM} | det(d(\pi \circ \psi)_{(t,v)}) | \leq \int_{[\delta,\lambda]\times S_pM} | det(d\psi_{(t,v)}) | \leq$$

$$\leq K \int_{[\delta,\lambda]\times S_pM} | det(d(\phi_t)_v) | =$$

$$= K \int_\delta^\lambda V(t)dt,$$

where the last inequality follows from equation (11).

Therefore from equation (10)

$$I_p(\lambda) \leq \mu(B(p,\delta)) + K \int_\delta^\lambda V(r)dr.$$

But given $\epsilon > 0$ there exists $T(\epsilon)$ such that if $t \geq T(\epsilon)$ then $V(t) \leq e^{(\Delta+\epsilon)t}$. Thus

$$I_p(\lambda) \leq \mu(B(p,\delta)) + K \int_\delta^{T(\epsilon)} V(r)dr + K \int_{T(\epsilon)}^\lambda e^{(\Delta+\epsilon)r}dr.$$

This clearly implies that $\sigma_p \leq \Delta + \epsilon$.

$\diamond$

Proof of Theorem 1.3: Since for all $q \in M$ we have that $H^{-1}(\sigma) \cap T_q^*M$ is a compact strictly convex hypersurface containing the origin in its interior, we can define a Finsler metric F on T^*M -as we did in the proof of Theorem 1.1- so that $F^{-1}(1) = H^{-1}(\sigma)$. But then the curves of H on $H^{-1}(\sigma)$ are just reparametrizations of the curves of F (see [27]). Hence, since the topological entropy is a dynamical invariant we have that $h_{top}(\phi_t, S_pM) = 0$ where ϕ_t denotes the geodesic flow of the Finsler metric F. But now Theorem 7.1 implies that $\sigma_p = 0$.

In [14] Gromov proved in the riemannian case that there exists a constant c depending only on the geometry of M such that whenever p and q are not conjugate (i.e. if q is a regular value of exp_p) then

$$n_p(q,\lambda) \geq \sum_{i=1}^{c(\lambda-1)} b_i(\Omega M, K),$$

where $b_i(\Omega M, K)$ are the Betti numbers of the loop space ΩM respect to the field K. His proof extends directly to Finsler metrics, since Morse theory also works for Finsler metrics (see for example [29]). Thus if we integrate the above inequality respect to q we obtain

$$I_p(\lambda) \geq Vol(M) \sum_{i=1}^{c(\lambda-1)} b_i(\Omega M, K).$$

But if $\sigma_p = 0$, then $\sum_{i=1}^{m} b_i(\Omega M, K)$ grows sub-exponentially as we wanted to show. If $K = \mathbf{Q}$ this implies that M is rationally elliptic [16].

◇

Observation 7.2 In general it is not known whether sub-exponential growth implies polynomial growth for the loop space homology unless the field K has carachteristic zero or the carachteristic $p \neq 0$ verifies the condition $p > \frac{dim\, M}{r}$ where r is the least positive integer such that M has homology in degree r. The latter result is quite new and still unpublished and was obtained by Anich, Felix, Halperin and Thomas [18].

Let M be a manifold endowed with a Finsler metric. Recall that M naturally carries a distance function d (possibly not symmetric) that arises as the infimum of length of curves. Suppose now M is compact and let $\tilde{M}$ be the universal covering equipped with the induced Finsler metric. Pick a volume form on $\tilde{M}$ and set $V(p,r) = Vol(B(p,r))$ where

$$B(p,r) = \{y \in \tilde{M} : \tilde{d}(p,y) < r\}.$$

The same arguments as in [22] show now for the Finsler case that $r^{-1} log\ V(p,r)$ converges to a limit $\lambda \geq 0$ as $r \to +\infty$ and that λ is independent of p. Moreover since for a manifold with a complete Finsler metric is also true that two points can be joined by a minimal geodesic -this property can be easily deduced from Morse theory- the proof of Theorem 1 in [22] carries to the Finsler case to show

$$\lambda \leq h_{top}(\phi_t, S_pM).$$

Proof of Theorem 1.4: Arguing as in the proofs of Theorems 1.1 and 1.3 we obtain a Finsler metric with $h_{top}(\phi_t, S_pM) = 0$. Thus $\lambda = 0$ which in turn implies that $\pi_1(M)$ grows sub-exponentially.

◇

8 *Twisted cotangent bundles*

Part of the material in this section is taken from [17, Chapter 1]. Let M be a manifold so that T^*M denotes its cotangent bundle and ω the canonical 2-form on T^*M. Suppose that we are given an electromagnetic field F which is a closed 2-form. Via the projection $\pi : T^*M \to M$ we can pull back F to T^*M to obtain a 2-form on T^*M which we shall continue to denote F. We define

$$\omega_{e,F} = \omega + eF,$$

where e is an electric charge. Then $\omega_{e,F}$ is a symplectic form on T^*M and T^*M equipped with this form is called a *twisted cotangent bundle*. Any complete Lagrangian fibration with simply connected fibres can be viewed as a twisted cotangent bundle [2, p. 38]. Let H be a Hamiltonian on T^*M. If g is a riemannian metric on M, the Hamiltonian equations corresponding to the Hamiltonian $H(q,p) = \frac{1}{2} \mid p \mid_g^2$ relative to the symplectic form $\omega_{e,F}$ describe the motion of a charged particle of charge e in the presence of the electromagnetic potential field F.

Since F is closed, locally we can write $F = dA$, for some 1-form A. We can think of A as a section of T^*M and introduce a modified Hamiltonian $H_{e,A}$, where

$$H_{e,A}(q,p) = H(q, p - eA(q)).$$

If one defines $\varphi_{e,A} : T^*M \to T^*M$

$$\varphi_{e,A}(q,p) = (q, p + eA(q)),$$

then one checks that

$$\varphi_{e,A}^*\omega = \omega_{e,F}, \tag{12}$$

$$\varphi_{e,A}^* H_{e,A} = H. \tag{13}$$

Hence the solution curves of H relative to $\omega_{e,F}$ are images under $\varphi_{e,A}$ of the solution curves of $H_{e,A}$ relative to ω. Since $\pi \circ \varphi_{e,A} = \pi$ we obtain the same trajectories on M from one system as from another.

Now observe that if H is convex, then $H_{e,A}$ is also convex. Thus if a continuous invariant Lagrangian subbundle for the flow of H relative to $\omega_{e,F}$ is given, then equations (12) and (13) imply that the basic lemmas 5.3, 5.4 and 5.6 -which have a local nature- hold in the more general setting of twisted cotangent bundles. However we do not know how to extend Theorem 1.1 or Theorem 1.2 to this situation.

Finally suppose that F is exact and that M is compact. Let $H(q,p) = \frac{1}{2} \mid p \mid_g^2$ and set $K = max_{q \in M} \mid A(q) \mid_g$. Then $H_{e,A}$ verifies the hypotheses of Theorems 1.3 and 1.4 provided $\sigma > \frac{e^2 K^2}{2}$ and we have:

Theorem 8.1 *Let M be a compact manifold. If $\pi_1(M)$ is finite and for some coefficient field the loop space homology of M grows exponentially, or if $\pi_1(M)$ has exponential growth, then the motion of a charged particle under the effect of an exact electromagnetic potential with energy $\sigma > \frac{e^2 K^2}{2}$ has positive topological entropy.*

In general if we consider a Hamiltonian

$$H = \frac{1}{2} \mid p - A(q) \mid_g^2 + U(q),$$

then H will verify the hypotheses of Theorems 1.3 and 1.4 for an appropiate σ and thus all the previous results apply if we include conservative forces represented by the potential U.

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