On Kontsevich's integral for the Homfly polynomial and relations of mixed Euler numbers

Le Tu Quoc Thang
Jun Murakami

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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Abstract. Kontsevich's integral for the Homfly polynomial is studied by using representations of the cord diagram algebras via classical r-matrices for sl_m and via a Kauffman type state model. We compute the actual value of the image of $Z(\infty)$ by these representations, where $Z(\infty)$ is the normalization factor to construct invariant from the integral. This formula implies relations among mixed Euler numbers, which are values of generalized zeta functions at 1.

Introduction

Kontsevich defines a knot invariant by using iterated integral to get monodromy of the Knizhnik-Zamolodchikov(KZ)-equation [**2**], [**7**]. He considers KZ-equation with values in an algebra $\mathcal{A}$, where $\mathcal{A}$ is a linear span of cord diagrams with relations corresponding to the flatness of the KZ-equation. These relations are similar to the classical Yang-Baxter equation (CYBE) (without spectral parameters), and so we can construct a state model or a 'representation' of the algebra $\mathcal{A}$ by using a classical r-matrix, a solution of the CYBE [**2**]. This state model defines a mapping from $\mathcal{A}$ to $\mathbf{C}$ called a weight, and applying it to the integral invariant, we get a $\mathbf{C}$-valued invariant.

In this paper, we give an actual correspondence of the Homfly polynomial and invariants coming from the integral related to the classical r-matrix associated with the vector representation of sl_m. The Homfly polynomial is defined by the skein relation and coming from quantum R-matrix, associated with the vector representation of sl_m as in [**10**]. If r is the classical limit of a quantum R–matrix, then it is clear more or less from Drinfeld's work [**4**] that the Kontsevich integral, via weight of r, should be the same as the invariant coming from the R–matrix as in Reshetikhin-Turaev approach [**9**]. The reason is the corresponding quasi-Hopf algebras are gauge equivalent. But since Drinfeld's work [**4**] does not treat knot invariant thoroughly, and since literature on Kontsevich integral does not mention even the quasi-Hopf origin of the invariant, here we present a direct proof that these invariants are the same. For braids, Kohno [**6**] investigate such iterated integral and find skein relation in it and we generalize it for links. This is a partial answer of problem 4.9 posed by Bar-Natan in [**2**].

As an application of the correspondence, we get relations among mixed Euler numbers $\zeta(s_1, s_2, \cdots, s_k) = \displaystyle\sum_{m_1<m_2<\cdots<m_k} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_k^{s_k}}$. (Arnold calls $\displaystyle\sum_{m_1<m_2<\cdots<m_k}$

$\dfrac{z^{m_k}}{m_1^{s_1} m_2^{s_2} \cdots m_k^{s_k}}$ Zagier's zeta function, but here we use the terminology suggested by Zagier.) We compute the normalization factor $Z(\infty)$ of the integral invariant for the state model by two ways. One uses the actual correspondence of the integral and the Homfly polynomial. Another uses the expression of $Z(\infty)$ involving mixed Euler numbers. These two formulas give us relations among mixed Euler numbers. For example, we can compute the values of $\zeta(2, \cdots, 2)$ and can reproduce the famous theorem of Euler which explains $\zeta(2n)$ in terms of a Bernoulli number. Using this method to other invariants (e.g. the Kauffman polynomial), we may get more relations.

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1. Kontsevich's integral

In this section, we review some results we need later in [**7**] and [**2**].

1.1. Algebra of cord diagrams.

Definition 1.1.1. (generalization of Definition 1.5 in [**2**]) Let k be a positive integer. A *cord diagram* on k strings is k oriented numbered circles called Wilson loops with finitely many dashed cords marked on it, regarded up to orientation and component preserving diffeomorphisms of the circles. Here dashed cords means that two different cords does not meet and they just give pairings of points on Wilson loops. Denote the collection of all cord diagrams on k circles by $\mathcal{D}^{(k)}$. This collection is naturally graded by the number of cords in such a diagram. Denote the piece of degree d of $\mathcal{D}$ by $\mathcal{G}_d\mathcal{D}^{(k)}$. $\mathcal{G}_d\mathcal{D}^{(k)}$ is simply the collection of all cord diagram having precisely d cords.

To extend Kontsevich's integral to links, we generalize the notion of the algebra of cord diagrams in the above definition.

Definition 1.1.2. (generalization of Definition 1.7 in **[2]**) Let the vector space $\mathcal{A}'^{(k)}$ be the quotient

$$\mathcal{A}'^{(k)} = \mathrm{span}(\mathcal{D}^{(k)}) / \mathrm{span}(4 - \text{term relations}),$$

where 4-term relation is given in Figure 1 (a). $\mathcal{A}^{(1)}$ is also denoted by $\mathcal{A}$.

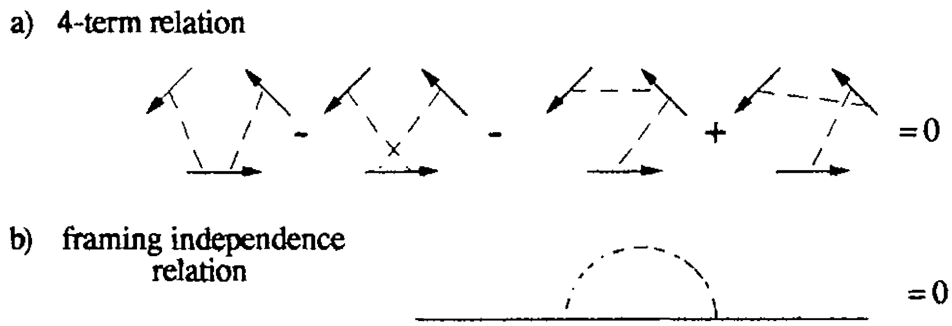


Figure 1. Relations for cord diagrams.

Definition 1.1.3 Let $\mathcal{A}_0'^{(k)}$ be the quotient of $\mathcal{A}'^{(k)}$ by the framing independence relation Figure 1 (b).

$$\mathcal{A}_0'^{(k)} = \mathcal{A}'^{(k)} / \text{framing independence relation}.$$

This means that a cord diagram with a part as in Figure 1(b) is considered to be 0. $\mathcal{A}_0^{(1)}$ is also denoted by $\mathcal{A}_0$.

Definition 1.1.4 (Completion) The module $\mathcal{A}'^{(k)}$ has a grading by the number of cords. Let $\mathcal{A}^{(k)}$ be the completion of $\mathcal{A}'^{(k,k)}$ by this grading. The module $\mathcal{A}_0'^{(k)}$ is also has similar grading and let $\mathcal{A}_0^{(k)}$ be the completion by this grading.

The 4-term relation implies the following. Let D_1 and D_2 be two cord diagrams in $\mathcal{A}^{(k_1)}$ and $A^{(k_2)}$, each with a noted string. Remove an arc on each noted string which does not contains any vertex and then using two lines to combine the two strings into one single string. We get a cord diagram in $\mathcal{A}^{(k_1+k_2-1)}$ called the product (or connected sum) of D_1 and D_2 along the noted strings. As in [**BarNatan**], this operation does not depend on the location of the arcs removed and $\mathcal{A}$ has an commutative algebra structure with this product.

1.2. Iterated integral for knots and links. Let L be a k-component link embedded in $\mathbf{R} \times \mathbf{C}$. We assume that L is in a general position. Then we can define $\mathcal{A}_0^{(k)}$-valued integral for L as in [**2**].

(1.2.1) $Z(L) =$
$$\sum_{n=0}^{\infty} \frac{1}{(2\pi i)^n} \int_{t_{\min}<t_1<\cdots<t_{\max}} \sum_{\substack{\text{applicable pairings} \\ P=\{(z_i, z_i')\}}} (-1)^{\#P_\downarrow} L_P \bigwedge_{i=1}^{n} \frac{dz_i - dz_i'}{z_i - z_i'} \in A_0^{(k,k')}.$$

In the above equation, $t_{\min}$ ($t_{\max}$) is the minimal (maximal) value of t on D, an 'applicable pairing' is a choice of an unordered pair (z_i, z_i') for every $1 \le i \le n$, for which (z_i, t_i) and (z_i, t_i) are *distinct* points on D, $\#P_\downarrow$ is the number of points of the form (z_i, t_i) or (z_i, t_i) at which D is decreasing, D_P is the image of the cord diagram in $\mathcal{A}_0^{(k)}$ naturally associated with D and P, and every pairing defines a locally homeomorphic map $\{t_i\} \mapsto \{(z_i, z_i')\}$. If L' is obtained from D by horizontal deformation, then $Z(L) = Z(L')$.

1.3. Invariant. Let L be a link with k numbered components. For $i = 1, \cdots, k$ let s_i be the number of maximal points of the i-th component. Let

$$\hat{Z}(L) = \gamma^{-s_1} \otimes \cdots \otimes \gamma^{-s_k} \cdot Z(T), \tag{1.3.1}$$

here in the right hand side, γ^{-s_i} acts on the i-th string. Then as in [**2**], we get

Theorem 1.3.2. *$\hat{Z}(L)$ is an isotopy invariant of oriented links.*

2. Weights for cord diagrams

2.1. Weights from the classical r-matrices. Let U be a finite dimensional vector space, $m = \dim U$, and $\{e_1, e_2, \cdots, e_m\}$ be a basis of U. Let $(R = R(q), \mu(q), \alpha(q), \beta(q))$ be an enhanced Yang-Baxter (EYB) operator of Turaev's sense ([**10**], 2.3) for U. We assume that $\mu(1) = \mathrm{id}$. Let R_{ij}^{st} be the matrix element of R with $R(e_i \otimes e_j) = \sum_{pq} R_{ij}^{st} e_s \otimes e_t$. Put $q = \exp(h)$, $R' = \alpha^{-1} R$ and $r = P \left. \dfrac{d(R' - R'^{-1})}{dh} \right|_{h=0}$, where $P(u_1 \otimes u_2) = u_2 \otimes u_1$. Comparing the degree two terms with respect to h of the braid relation for R, we get the following.

Lemma 2.1.1. *The matrix r satisfies the 4-term relation*

$$[r_{ij}, r_{ik} + r_{jk}] = 0 \ (\{i, j, k\} = \{1, 2, 3\}), \tag{2.1.2}$$

where $r_{ij} \in \mathrm{End}(U^{\otimes 3})$ acts on the i-th and j-th component of $U^{\otimes 3}$ by r.

Comparing the degree one terms with respect to h of the conditions of 2.3.1 in [**10**] for μ , we get the following.

Lemma 2.1.3. *For any i, $k \in \{1, 2, \cdots, m\}$,*

$$\sum_k r_{ik}^{kj} = 0. \tag{2.1.4}$$

Suppose r is an arbitrary matrix in $\mathrm{End}(U \otimes U)$ satisfying (2.1.2) and (2.1.4). As in [**2**], we construct a state model for cord diagrams as follows. This model is called the weight of cord diagrams associated with r. Let D be a cord diagram on k strings. A mapping $f : \{\text{arc of } D\} \to \{1, 2, \cdots, d\}$ is called a *state* of D. For every state of

D, we assign $r^{f(a_3)f(a_4)}_{f(a_1)f(a_2)}\,h$ for each cord c as in Figure 2. let $W_r(D)_{b_1,b_2,\cdots}$ be a state sum on D defined by

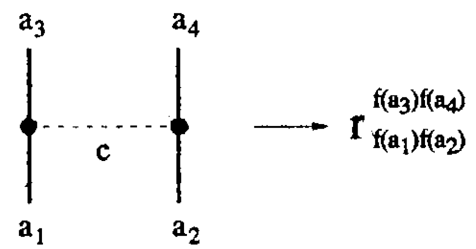


Figure 2

$$W_r(D) = \sum_{\substack{f: \\ \{\mathrm{arc}\}\to\{1,2,\cdots,m\}}} \prod_{\substack{\text{cord } c \\ \text{of } D}} h\, r^{f(a_3)f(a_4)}_{f(a_1)f(a_2)}. \tag{2.1.5}$$

Due to (2.1.2) and (2.1.4), W_r can be thought as a mapping from $\mathcal{A}_0^{(k)}$ to $\mathbf{C}[[h]]$. Especially, W_r gives a mapping from $\mathcal{A}_0$ to $\mathbf{C}[[h]]$.

Proposition 2.1.6. *Let r be a classical r-matrix associated with an irreducible representation of a Lie algebra on U. Then, for any cord diagrams D_1 and D_2, we have*

$$W_r(D_1 \# D_2) = W_r(D_1)\, W_r(D_2)/m, \tag{2.1.7}$$

where $D_1 \# D_2$ is a connected sum of D_1 and D_2 along arbitrary component.

Proof. After removing a small arc from D_1 and D_2, we get cord diagrams D_1' and D_2', each has two end points. We extend the definition of W so that $W_r(D_1')$ and $W_r(D_2')$ are matrices in $\mathrm{End}(U)$, where rows and columns are corresponding to the state of two arcs containing the end points. We have $W_r(D_i) = \mathrm{Tr} W_r(D_i')$ for

$i = 1, 2$. While $W_r(D_1 \# D_2) = \mathrm{Tr}(W_r(D_1')\,W_r(D_2'))$. Since the representation of the Lie algebra on U is irreducible and both $W_r(D_1')$ and $W_r(D_2')$ commute with this representation, we have $W_r(D_1') = \mathrm{const}_1\,\mathrm{id}$ and $W_r(D_2') = \mathrm{const}_2\,\mathrm{id}$. It follows that

$$\mathrm{Tr}(W_r(D_1')\,W_r(D_2')) = (\mathrm{Tr}W_r(D_1'))\,(\mathrm{Tr}W_r(D_2'))/m.$$

∎

For a link L, let

$$\kappa_r(L) = m^{s(L)-2}\,W_r(Z(L))\,W_r(Z(\infty))^{1-s(L)}, \tag{2.1.8}$$

where ∞ denote the knot diagram given by Figure 3. Since $Z(L)/Z(\infty)^{s(L)-1}$ is an invariant of L, (2.1.7) implies the following.

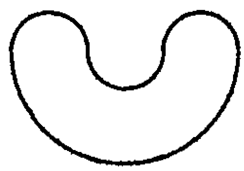


Figure 3

Theorem 2.1.9. *For a link L, $W_r(L)$ is an ambient isotopy invariant of L. It satisfies*

$$\kappa_r(\bigcirc) = 1,$$

for the trivial knot $\bigcirc$,

$$\kappa_r(L_1 \# L_2) = \kappa_r(L_1)\,\kappa_r(L_2),$$

for a connected sum of two links L_1, L_2 along arbitrary components, and

$$\kappa_r(L_1 \cup L_2) = \kappa_r(L_1)\,\kappa_r(L_2)\,W_r(Z(\infty))/m,$$

for the disjoint union of L_1, L_2.

2.2. sl_m **case.** Let U be the fundamental representation of sl_m. Let (R, μ, α, β) be the EYB-operator for U in **[10]**, §4.2 and let $q = \exp(h)$. Then $R = -q \sum_i E_{i,i} \otimes E_{i,i} - \sum_{i \neq j} E_{i,j} \otimes E_{j,i} + (q^{-1} - q) \sum_{i<j} E_{i,i} \otimes E_{j,j}$, $\mu = \mathrm{diag}(\mu_1, \cdots, \mu_m)$ where $\mu_i = q^{2i-m-1}$, $\alpha = -q^m$ and $\beta = 1$. Let

$$r = P \left.\frac{d(R' - R'^{-1})}{dh}\right|_{h=0}, \tag{2.2.1}$$

where $R' = \alpha^{-1} R$. Then r is given by $r = 2(P - m\ \mathrm{id})$. Another construction for r is the following. Let $\mathrm{Tr}(AB)$ be the usual scalar on sl_m, and I_ν is an orthonormal basis of sl_m. Then $r = 2(\sum_\nu I_\nu \otimes I_\nu)$. Sending Ω to hr, we get a representation of $\mathcal{A}^{(k,k')}$. By theorem 2.1.8, we get an isotopy invariant $\kappa_r(L) \in \mathbf{C}[[h]]$.

Next we will present a graphical algorithm to compute W_R for a cord diagram. Let r the matrix given by (2.2.1). Then $r = 2(P - m\ \mathrm{id})$ and is graphically presented by

$$W_r(\downarrow \cdots \downarrow) = 2\,h(W_r(m \downarrow \ \downarrow) - W_r(\swarrow\!\!\!\!\searrow)). \tag{2.2.2}$$

W_r also satisfies

$$\begin{aligned} W_r(D \cup \bigcirc) &= m\, W_r(D), \\ W_r(\bigcirc) &= m. \end{aligned} \tag{2.2.3}$$

This interpretation resembles Kauffman's state model for the Jones polynomial. For a cord diagram D, we can compute $W_r(D)$ from (2.2.2) and (2.2.3).

2.3. Equivalence of invariants from the integral and the Homfly polynomial. For sl_m case, Kohno already shows in Theorem 4.1 of **[6]** that the representation of the braid groups coming from the iterated integral satisfies the skein relation. We show that the invariant κ_r from the permutation model also satisfies the skein relation.

Theorem 2.3.2. *κ_r is equal to the Homfly polynomial satisfying the skein relation*

$$\exp(-mh)\,\kappa_r(L_+) - \exp(mh)\,\kappa_r(L_-) = (\exp(-h) - \exp(h))\,\kappa_r(L_0), \tag{2.3.3}$$

where L_+, L_-, L_0 are identical except with in a ball as in Figure 4.

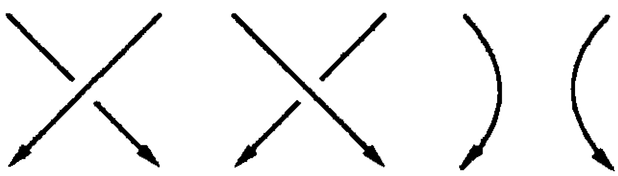


Figure 4

Proof. By isotopy, we can push the local part containing the difference of the three links far away as in Figure 5. In this figure the different part of the three links are in the box denoted T. The complement parts are the same and is denoted by X. We suppose that the end points of X are (0, 0), (0, 1), (1, 0), (1, 1). In Figure 5, L is decomposed into three tangles, the top is denoted by T_1, the middle by T_2, the bottom by T_3. The middle contains T and two extra lines parallel to the straight line $\mathbf{R}$. We suppose the upper end points of these two lines are $(\ell, 1)$, $(\ell+1, 1)$. We will consider the limit when $\ell \to \infty$, and write $T_1(\ell)$, $T_2(\ell)$, and $T_3(\ell)$. Let $Z(T_2(\ell)) = A + B(\ell)$ where $B(\ell)$ is the part containing all the cord diagrams with at least one "long" cord connecting a string of the left part of T_2 and a string of the right part of T_2, $A = Z(T)$ is the remaining. Of course A does not depend on ℓ. The coefficient of a diagram of $B(\ell)$ tends to zero when ℓ tends to infinity at least as fast as $\log(1 + 1/\ell)$. This also follows easily from the integral formula.

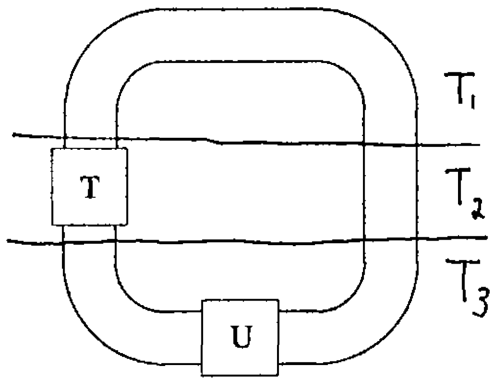


Figure 5

For all cord diagrams with less than k cords of $Z(T_1(\ell))$ or $Z(T_3(\ell))$, the coefficients tends to infinity when ℓ tends to infinity, but at most as fast as $(\log \ell)^k$. This also follows easily from the integral formula. Using $\lim_{\ell\to\infty} \log(1+1/\ell)(\log \ell)^k = 0$, we see that

$$Z(L) = \lim_{\ell\to 0} Z(T_1(\ell)) \times Z(T) \times Z(T_3(\ell)).$$

Now let T respectively the diagram of Figure 4. Then, by applying Kohno's result in [**6**] for the braid group B_2 on two strings, we get the skein relation. We can also get this relation by direct computation of W_r for two braids in Figure 4. ∎

3. Computation of $Z(\infty)$ and $W_r(Z(\infty))$

3.1. Computation of $Z(\infty)$. Our method to compute $Z(\infty)$ is suggested by [**1**]. First present the diagram ∞ as in Figure 6. For $I = (p_1, q_1, p_2, q_2, \cdots, p_g, q_g)$, let $\Omega(I)$ be the configuration as in Figure 6. Let $p(I) = \sum_{i=1}^{g} p_i$, $q(I) = \sum_{i=1}^{g} q_i$, $|I| = p(I) + q(I)$ and $g(I) = g$. Then the coefficient of the cord diagram $\Omega(I)$ in

$Z(\infty)$ is given by

$$\int_0^1 \int_0^{t_{|I|}} \cdots \int_0^{t_3} \int_0^{t_2} \underbrace{\frac{dt_{|I|}}{t_{|I|}} \cdots \frac{dt_{|I|-q_g+1}}{t_{|I|-q_g}+1}}_{q_g} \underbrace{\frac{dt_{|I|-q_g}}{1-t_{|I|-q_g}} \cdots \frac{dt_{|I|-q_g-p_g+1}}{1-t_{|I|-q_g-p_g+1}}}_{p_g} \cdots$$

$$\cdots \underbrace{\frac{dt_{p_1+q_1}}{t_{p_1+q_1}} \cdots \frac{dt_{|I|-p_1+1}}{t_{|I|-p_1+1}}}_{q_1} \underbrace{\frac{dt_{p_1}}{1-t_{p_1}} \cdots \frac{dt_1}{1-t_1}}_{p_1}.$$

To compute the above iterated integral, we introduce a function $F(s_1, s_2, \cdots, s_k; x)$ for positive integers $s_1, s_2, \cdots, s_k$.

(3.1.1)

$$F(1;x) = \int_0^x \frac{1}{1-y}\, dy = -\ln(1-x) = -\sum_{m_1 \in \mathrm{N}} \frac{x^{m_1}}{m_1},$$

$$\begin{aligned} F(s_1, \cdots, s_{k-1}, s_k, 1; x) &= \int_0^x \frac{F(s_1, \cdots, s_{k-1}, s_k; y)}{1-y}\, dy \\ &= (-1)^{k+1} \mathop{}_{0<m_1<m_2<\cdots<m_k<m_{k+1}} \frac{x^{m_{k+1}}}{{m_1}^{s_1} \cdots {m_k}^{s_k} m_{k+1}}, \\ F(s_1, \cdots, s_{k-1}, s_k; x) &= \int_0^x \frac{F(s_1, \cdots, s_{k-1}, s_k - 1; y)}{y}\, dy \\ &= (-1)^k \sum_{0<m_1<m_2<\cdots<m_k} \frac{x^{m_k}}{{m_1}^{s_1} \cdots {m_k}^{s_k}} \quad \text{for } s_k > 2. \end{aligned}$$

Remark 3.1.2 The function F is an extension of dilogarithmic functions.

Especially, $F(s_1, s_2, \cdots, s_k; 1) = \zeta(s_1, s_2, \cdots, s_k)$ for $s_k \geq 2$, where ζ is Zagier's mixed Euler numbers defined by

(3.1.3)
$$\zeta(s_1, s_2, \cdots, s_k) = \sum_{0<m_1<m_2<\cdots<m_k} \frac{1}{{m_1}^{s_1} \cdots {m_k}^{s_k}}.$$

The value of $Z(\infty)$ in $\mathcal{A}^{(1)}$ is a sum of iterated integrals for all the configurations of cord diagrams in Figure 6. Note that the integral is zero if $p_1 = 0$ or $q_g = 0$ and so we omit these cases. Let

$$\zeta(I) = \zeta(\underbrace{1, \cdots, 1}_{p_1-1}, q_1+1, \underbrace{1, \cdots, 1}_{p_2-1}, q_2+1, \cdots, \underbrace{1, \cdots, 1}_{p_g-1}, q_g+1).$$

By using F, the iterated integral for this configuration is equal to $\frac{(-1)^{p(I)}}{(2\pi i)^{|I|}}\zeta(I)$. Hence we get

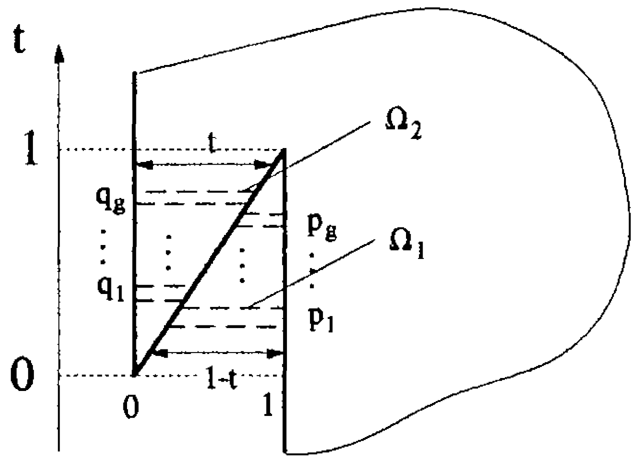


Figure 6

Theorem 3.1.4. $Z(\infty) = 1 + \sum_{I, g(I)\geq 1} \frac{(-1)^{p(I)}}{(2\pi i)^{|I|}}\zeta(I)\,\Omega(I).$

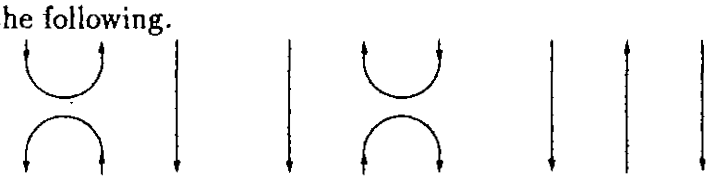
By computing the integral for this configuration from the top to the bottom, we get $(-1)^{|I|}(-1)^{q(I^*)}\zeta(I)/(2\pi i)^{|I|}$, where $I^* = (q_g, p_g, \cdots, q_2 p_2, q_1, p_1)$. Hence we get

Lemma 3.1.5. (inversion formula for ζ) $\zeta(I^*) = \zeta(I)$.

3.2. Computation of $W_r(Z(\infty))$. To apply the weight to the configuration $\Omega(I)$, we denote $\Omega(I)$ by ${\Omega_1}^{q_g}{\Omega_2}^{p_g}\cdots{\Omega_1}^{q_1}{\Omega_2}^{g_1}$.

Lemma 3.2.1. $W_r(\Omega(I)) = \frac{(2\,h\,m)^{|I|}}{m^{2\,g(I)-1}}(1-m^2).$

Proof. (Induction on $|I|$) Let P_1, P_2 and id be the configurations in Figure 7. Then the state model replaces Ω_i to $2\,h\,(P_i - m\ \mathrm{id})$ for $i = 1, 2$. The definition of the state model implies the following.



P_1 P_2 id

Figure 7

$$W_r(\Omega_2\cdots)=W_r(\cdots\Omega_1)=0,\quad W_r(\cdots P_2\,\Omega_2\cdots)=W_r(\cdots P_1\,\Omega_1\cdots)=0,$$
$$W_r(P_1\,P_2)=1,\quad W_r(P_1)=W_r(P_2)=m.$$

With these relations, we get

$$\begin{aligned}
W_r({\Omega_1}^{q_g}\,{\Omega_2}^{p_g}\cdots{\Omega_1}^{q_1}\,{\Omega_2}^{p_1}) &= -2\,h\,W_r({\Omega_1}^{q_g}\,{\Omega_2}^{p_g}\cdots{\Omega_1}^{q_1}\,{\Omega_2}^{p_1-1}\,(m-P_2))\\
&= -2\,h\,W_r(m{\Omega_1}^{q_g}{\Omega_2}^{p_g}\cdots{\Omega_1}^{q_1}\,{\Omega_2}^{p_1-1})\\
&= \cdots = (-2\,h)^{p_1-1}\,m^{p_1-1}\,W_r({\Omega_1}^{q_g}\,{\Omega_2}^{p_g}\cdots{\Omega_1}^{q_1}\,\Omega_2)\\
&= (-2\,h)^{p_1}\,m^{p_1-1}\,W_r({\Omega_1}^{q_g}\,{\Omega_2}^{q_g}\cdots{\Omega_1}^{q_1}\,(-P_2))\\
&= -(-2\,h)^{q_1+p_1-1}\,m^{q_1-1+p_1-1}\,W_r({\Omega_1}^{q_g}\,{\Omega_2}^{p_g}\cdots\Omega_1\,P_2)\\
&= (-1)^2\,(-2\,h)^{q_1+p_1}m^{q_1-1+p_1-1}\,W_r({\Omega_1}^{q_g}\,{\Omega_2}^{p_g}\cdots P_1\,P_2)\\
&= \cdots = (-1)^{2g-1}(-2\,h)^{p_g+\cdots+q_1+p_1}m^{p_g-1+\cdots+q_1-1+p_1-1}W_r({\Omega_1}^{p_1}\,P_2)\\
&= -(-2\,h)^{q_g+p_g+\cdots+q_1+p_1-1}m^{q_g-1+p_g-1+\cdots+q_1-1+p_1-1}W_r(\Omega_1\,P_2)\\
&= -(-2\,h)^{q_g+p_g+\cdots+q_1+p_1}m^{q_g-1+p_g-1+\cdots+q_1-1+p_1-1}W_r(-(m-P_1)\,P_2)\\
&= (-2\,h)^{q_g+p_g+\cdots+q_1+p_1}\,m^{q_g-1+p_g-1+\cdots+q_1-1+p_1-1}\,(1-m^2)\\
&= (-2\,h)^{q_g+p_g+\cdots+q_1+p_1}\,m^{q_g+p_g+\cdots+q_1+p_1-g}\,(1-m^2).
\end{aligned}$$

∎

From the above lemma, we have

$$\begin{aligned}
&W_r(Z(\infty))\\
=&1+\sum_{I,g(I)\geq 1}\ \sum_{p_1,q_1,\cdots,p_r,q_g\leq 1}\frac{(-1)^{p(I)}}{(2\,\pi\,i)^{|I|}}\,W_r(\Omega(i))\zeta(I)\\
=&1+\sum_{I,g(I)\geq 1}\ \sum_{p_1,q_1,\cdots,p_r,q_g\leq 1}\frac{(-1)^{p(I)}}{(2\,\pi\,i)^{|I|}}\,\frac{(2\,h\,m)^{|I|}}{m^{2\,g(I)}}\,(1-m^2)\zeta(I)\\
=&1+\sum_{n=1}^{\infty}\ \sum_{g(I)=1}^{[n/2]}\ \sum_{\substack{p_1,q_1,\cdots,p_g,q_g\geq 1\\ |I|=n}}(-h)^n\,(1-m^2)\,\frac{(-1)^{q(I)}}{(\pi\,i)^n}\,m^{n-2r}\zeta(I).
\end{aligned}$$

The inversion formula for ζ implies that the coefficient of h^n in the above formula is equal to 0 if n is odd. Hence we have

$$(3.2.2)\quad W_r(Z(\infty)) = 1 + \sum_{n=1}^{\infty}\sum_{g=1}^{n} \sum_{\substack{p_1,q_1,\cdots,p_g,q_g\geq 1 \\ |I|=2n}} (-1)^{g(I)-n}\, h^{2n}\, m^{2n-2g}\,(1-m^2)\frac{\zeta(I)}{\pi^{2n}}.$$

3.3. Another formula for $W_r(Z(\infty))$. Since the invariant κ_r satisfies the skein relation (2.3.3), we have

$$\kappa_r(\bigcirc \cup \bigcirc) = \frac{\exp(m\,h) - \exp(-m\,h)}{\exp(h) - \exp(-h)} = \frac{\sinh mh}{\sinh h}.$$

On the other hand, $\kappa_r\kappa_r(L) = m^{s(L)-2}\, W_r(Z(L)/Z(\infty)^{s(L)-1})$, $W_r(Z(\bigcirc \cup \bigcirc)) = m^2$ and $W_r(Z(\bigcirc)) = m$. Combining these relations, we get the following.

Theorem 3.3.1. $W_r(Z(\infty)) = m\,\dfrac{\sinh h}{\sinh m\,h}$.

3.4. Relations for mixed Euler numbers. Comparing (3.2.2) and Theorem 3.3.1 for $W_r(Z(\infty))$, we get relations for generalized zeta functions at one. To get these relation, we expand $m \sinh h/\sinh mh$. Since $t\ \exp(xt)/(\exp(t)-1) = \sum_{n=0}^{\infty} B_n(x)\, t^n/n!$ where $B_n(x)$ is the Bernoulli polynomial, we have

$$m\,\frac{\sinh h}{\sinh mh} = 1 + \sum_{n=1}^{\infty} B_{2n+1}\left(\frac{m+1}{2\,m}\right)\frac{(2m)^{2n+1}}{(2n+1)!}\, h^{2n}.$$

We also know that

$$B_{2n+1}\left(\tfrac{m+1}{2\,m}\right) = -\sum_{p=0}^{n} \begin{pmatrix} 2n+1 \\ 2p \end{pmatrix} (1-2^{1-2p})\,(2m)^{-2n-1+2p}\, B_{2p},$$

because $B_n(x+h) = \sum_{p=0}^{n} B_p(x)\, h^{n-p}$, $B_n(1/2) = -(1-2^{1-n})\, B_n$ and $B_{2\ell+1} = 0$ for any positive integer ℓ. Here B_n are the Bernoulli numbers. Hence comparing the coefficients of h^{2n} of (3.2.2) and Theorem 3.3.1, we have

$$1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \sum_{p=0}^{n} \binom{2n+1}{2p} \; (2-2^{2p})\,(m)^{2p}\, B_{2p}\, h^{2n} =$$

$$1 + \sum_{n=1}^{\infty} \sum_{g=1}^{n} \sum_{\substack{p_1,q_1,\cdots,p_g,q_g \geq 1 \\ |I|=2n}} (-1)^{q(I)-n}\, h^{2n}\, m^{2n-2g}\,(1-m^2) \frac{\zeta(I)}{\pi^{2n}}.$$

Comparing the coefficient of $h^{2n}\, m^{2p}$, we get

$$\frac{1}{(2n+1)!} \binom{2n+1}{2p} \; (2-2^{2p})\, B_{2p} =$$

$$\sum_{\substack{p_1,q_1,\cdots,p_{n-p},q_{n-p} \geq 1 \\ |I|=2n}} (-1)^{q(I)-n} \frac{\zeta(I)}{\pi^{2n}} - \sum_{\substack{p_1,q_1,\cdots,p_{n-p+1},q_{n-p+1} \geq 1 \\ |I|=2n}} (-1)^{p(I)-n} \frac{\zeta(I)}{\pi^{2n}}.$$

This relation implies the following.

$$\text{(3.4.2)} \qquad \sum_{\substack{p_1,q_1,\cdots,p_{n-p},q_g \geq 1 \\ |I|=2n}} (-1)^{p(I)-n} \frac{\zeta(I)}{\pi^{2n}} = \frac{1}{(2n+1)!} \sum_{r=0}^{n-g} \binom{2n+1}{2r} \; (2-2^{2r})\, B_{2r}$$

Examples. If $p = 0$ then

$$\text{(3.4.3)} \qquad \zeta(\underbrace{2,2,\cdots,2}_{k})/\pi^{2k} = 1/(2k+1)!.$$

If $p = n$ then

$$\text{(3.4.4)} \quad \frac{2-2^{2n}}{(2n)!} B_{2n} = -\sum_{\substack{p_1,q_1 \geq 1 \\ p_1+q_1=2n}} (-1)^{p_1-n} \frac{\zeta(\overbrace{1,\cdots,1}^{p_1-1},q_1+1)}{\pi^{2n}}$$

$$= (-1)^{p_1-n} \frac{1}{\pi^{2n}} [\zeta(2n) - \zeta(1,2n-1) + \cdots + \zeta(1,\cdots,1,2)].$$

By using $B_{2n} = 2\,(2n)!\,(-1)^{n-1}\,(2\,\pi)^{-2n}\,\zeta(2n)$, we get

$$\zeta(2n) - \zeta(1, 2n-1) + \cdots + \zeta(1, \cdots, 1, 2) = 2\,(1 - \frac{1}{2^{2n-1}})\,\zeta(2n).$$

Hence

$$(3.4.5) \qquad (\frac{1}{2^{2n-2}} - 1)\,\zeta(2n) - \zeta(1, 2n-1) + \cdots + \zeta(1, \cdots, 1, 2) = 0.$$

For example, if $n = 2$, $-\frac{3}{4}\,\zeta(4) - \zeta(1.3) + \zeta(1,1,2) = \frac{1}{4}\,\zeta(4) - \zeta(1.3) = 0$ since $\zeta(1,3) = \zeta(4)$, and so

$$(3.4.6) \qquad \zeta(1,3) = \frac{1}{4}\,\zeta(4) = \frac{1}{360}\,\pi^4.$$

Remark 3.4.7. The Euler's relation $B_{2n} = 2\,(2n)!\,(-1)^{n-1}\,(2\,\pi)^{-2n}\,\zeta(2n)$ can be obtained from (3.4.3) and relations like $\zeta(4) = \zeta(2)^2 - 2\,\zeta(2,2)$.

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