HECKE DUALITY OF IKEDA LIFTS

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Abstract. Ikeda lifts form a distinguished subspace of Siegel modular forms. In this paper we prove several global and local results concerning this space. We find that degenerate principal series representations (for the Siegel parabolic) of the symplectic group Sp_{2n} of even degree satisfy a Hecke duality relation which has applications to Ikeda lifts and leads to converse theorems. Moreover we apply certain differential operators to study pullbacks of Ikeda lifts.

1. Introduction and Statement of Results

In the two fundamental papers [14], [15], *On the lifting of elliptic cuspforms to Siegel cuspforms of degree* $2n$ and *Pullbacks of the lifting of elliptic cuspforms and Miyawaki's conjecture*, Tamotsu Ikeda proved the Duke-Imamoglu conjecture and main parts of the Miyawaki conjecture. For Siegel modular forms of degree 2 the Duke-Imamoglu conjecture coincides with the Saito-Kurokawa conjecture. These works raise many natural questions, about converse theorems, local-global principles, and interaction with differential operators. Let F be an Ikeda lift to the Siegel upper half-space $\mathfrak{H}_{2n}$, $F \circ j$ the pullback of F via an imbedding $j : \mathfrak{H}_{2m+r} \times \mathfrak{H}_r \hookrightarrow \mathfrak{H}_{2n}$, and f a Hecke eigenform on $\mathfrak{H}_r$. Ikeda [15] notes the interesting problem of determining when the restriction $F \circ j$ has non-vanishing projection to f, that is, determining when $\langle F \circ j, f \rangle \neq 0$.

This paper is aimed at these questions. One ingredient is a *Hecke duality relation* for degenerate principal series. This is a local result is applicable to $Sp_{2n}(k_v)$ for any ultrametric local field k_v, so also applies to Siegel-Hilbert modular forms, to not-necessarily holomorphic automorphic forms on symplectic groups, and a nearly identical argument applies to automorphic forms on other classical groups. For Siegel modular forms of degee 2 we obtain sharper results. Moreover, a combination of these results with [13] leads to converse theorems. Finally, we use differential operators to apply pullbacks of Ikeda lifts.

For a primitive Hecke newform $f \in S_{2\kappa}$, for every positive integer $n \equiv \kappa \pmod 2$, Ikeda explicitly constructed a Hecke eigenform $I^{2n}(f) \in S^{2n}_{\kappa+n}$, where S^n_κ is the space of Siegel cuspforms of weight κ on $\mathfrak{H}_n$. The standard L-function $L(s, I^{2n}(f), \mathrm{st})$ of the

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lift is

$$\zeta(s)\prod_{j=1}^{2n} L(s+\kappa+n-j,f) \tag{1.1}$$

where $L(s,f)$ is the standard GL_2 L-function of f.

Conversely, given a Hecke eigenform $F \in S^{2n}_{\kappa+n}$ whose standard L-function is the standard L-function of a lift, one can ask whether F is itself an Ikeda lift. In general there seem to be no systematic answer to this question. Nevertheless, we have

Theorem 1.1. *For $0 < \kappa \in 2\mathbb{Z}$, for f a primitive newform in $S_{2\kappa-2}$, and F a Hecke eigenform in S^2_κ with standard L-function*

$$L(s,F,\mathrm{st}) \;=\; \zeta(s)\, L(s+\kappa-1,f)\, L(s+\kappa-2,f). \tag{1.2}$$

Then F is a Saito-Kurokawa lift.

Remark. The question of whether a Siegel modular form on $\mathfrak{H}_2$ is a Saito-Kurokawa lift (here Ikeda lifts are Saito-Kurokawa lifts) was intensely studied around 1980 by Andrianov, Eichler, Zagier, Maass, Saito, Kurokawa and others (see [30]). For F on $\mathfrak{H}_2$, being a Saito-Kurokawa lift is reflected in satisfaction of the so-called *Maass relations* by the Fourier coefficients, and in properties of the Spinor L-function: F is a Saito-Kurokawa lift if and only if F satisfies the Maass Relations, if and only if the spinor L-function of F is

$$\zeta(s-\kappa+1)\,\zeta(s-\kappa+2)\,L(s,f), \tag{1.3}$$

for a primitive newform $f \in S_{2\kappa-2}$.

Let $\mathbb{I}^{2n}_\kappa \subseteq S^{2n}_{\kappa+n}$ with $\kappa, n \in \mathbb{N}$ with the same parity denote the space generated by Ikeda lifts. It is obvious that this space is Hecke invariant.

Let $F \in S^{2n}_{\kappa+n}$ be a Hecke eigenform and π the associated automorphic representation of $Sp_{2n}(\mathbb{A}_\mathbb{Q})$ over $\mathbb{Q}$, where $\mathbb{A}_k$ is the adele ring of a global field k. Then π factors over primes as $\pi = \bigotimes_v' \pi_v$, where π_∞ is a holomorphic discrete series representation (or *limit* of discrete series) and, for finite v, π_v is a spherical representation of $Sp_{2n}(\mathbb{Q}_v)$. Ikeda has proven that for lifts F all π_v for finite v are *degenerate principal series* representations. This is very special and suggests that Ikeda lifts can be better understood by studying degenerate principal series representations in a wider context. Indeed, it turns out that these representations satisfy a *Hecke duality relation* with several implications.

There is a local version of Hecke duality in the context of Siegel modular forms on $\mathfrak{H}_{2n}$. Fix the direct sum imbedding $j = j_1 \times j_2 : Sp_n \times Sp_n \longrightarrow Sp_{2n}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}, \tag{1.4}$$

and identify j_1, j_2 with the separate imbeddings. For an ultrametric local field k_v with ring of integers $\mathfrak{o}_v$, the standard (special) maximal compact subgroup of $Sp_n(k_v)$ is $Sp_n(\mathfrak{o}_v)$. The spherical Hecke algebra $\mathcal{H}_v^{sph}$ on $Sp_n(k_v)$ is the convolution algebra of left and right $Sp_n(\mathfrak{o}_v)$-invariant compactly supported $\mathbb{C}$-valued functions on $G = Sp_n(k_v)$. For $\eta \in \mathcal{H}_v^{sph}$ we have operators $j_1(\eta)$ and $j_2(\eta)$ on functions f on $Sp_{2n}(k_v)$ or on $Sp_{2n}(\mathbb{A}_k)$ defined by

$$(j_r(\eta) \cdot f)(g) \;=\; \int_{Sp_n(k_v)} \eta(h) \cdot f(g \cdot j_r(h))\, dh \quad (r = 1, 2). \tag{1.5}$$

Theorem 1.2. *Let π_v be a (spherical) degenerate principal series representation of $Sp_{2n}(k_v)$ with k_v an ultrametric local field. Then every spherical vector f in π_v satisfies the local Hecke duality relation:*

$$(j_1(\eta) \cdot f) = (j_2(\eta) \cdot f) \qquad \text{for all } \eta \in \mathcal{H}_v^{sph}. \tag{1.6}$$

The application of this Hecke duality identity to Ikeda lifts may be clarified by recollection of the comparison between classical global Hecke operators on automorphic forms on half-spaces, and convolution-algebra Hecke operators on automorphic forms on adele groups. (See [9], for example.) First, the global Hecke algebra $\mathcal{H}_n$ of the Hecke pair $(Sp_n(\mathbb{Z}), Sp_n(\mathbb{Q}))$ factors over primes, as $\mathcal{H}_n = \bigotimes_{v<\infty} \mathcal{H}_v^{sph}$, where $\mathcal{H}_v^{sph}$ can be identified with the local spherical Hecke algebra on $Sp_n(\mathbb{Q}_v)$. Second, for SL_n or Sp_n, strong approximation assures that the association $f \to \tilde{f}$ of classical automorphic forms to automorphic forms on adele groups (with suitable constraints) is a bijection, and, in particular, this bijection respects the correspondence of classical and adele-group Hecke operators. Let $|_\kappa$ be the Petersson slash operator of weight κ. For a classical Hecke operator $T = \sum_i a_i Sp_n(\mathbb{Z})\gamma_i \in \mathcal{H}_n$ with coefficients $a_i \in \mathbb{C}$ and $\gamma_i \in Sp_n(\mathbb{Q})$, and for $F \in S_k^{2n}$, define

$$T^{\uparrow}(F) = \sum_i a_i\, F|_\kappa\, j_1(\gamma_i) \text{ and } T^{\downarrow}(F) = \sum_i a_i\, F|_\kappa\, j_2(\gamma_i). \tag{1.7}$$

Then the previous comparison discussion yields:

Proposition 1.3. *By abuse of notation, let $\tilde{F}$ be the automorphic form on $Sp_{2n}(\mathbb{A}_\mathbb{Q})$ corresponding to a cuspform $F \in S_k^{2n}$. Then*

(1.8)
$$T^\uparrow(F) = T^\downarrow(F) \text{ for all } T \in \mathcal{H}_n \iff \eta^\uparrow(\tilde{F}) = \eta^\downarrow(\tilde{F}) \text{ for all } \eta \in \mathcal{H}_v^{sph} \text{ and finite } v.$$

Corollary 1.4. *Let $\mathbb{I}_\kappa^{2n}$ be the subspace of $S_{\kappa+n}^{2n}$ of Ikeda lifts from elliptic cuspforms of weight κ to Siegel modular forms of weight $\kappa+n$ on Sp_{2n} over $\mathbb{Q}$, with $\kappa \equiv n \mod 2$. For all $F \in \mathbb{I}_\kappa^{2n}$ there is the Hecke duality property*

$$T^\uparrow(F) = T^\downarrow(F) \tag{1.9}$$

for all $T \in \mathcal{H}_n$.

Remark. Note that *all* elements of $\mathbb{I}_\kappa^{2n}$, not only Hecke eigenforms, possess the Hecke duality property.

Remark. Recall that for $n = 1$ a converse theorem is also true: the second author has proven in [13] that a modular form F in S_{2k}^2 is an Ikeda lift if and only if F satisfies the Hecke duality relation (1.9). It would be interesting to know whether such a converse theorem is true for all Ikeda lifts. In the case of Siegel modular forms of degree 2 we also have

Theorem 1.5. *Let $\pi = \bigotimes_v \pi_v$ be an automorphic representation attached to a Hecke eigenform $F \in S_k^2$. Let π_v be a degenerate principal series for almost all finite v. Then F is a Saito-Kurokawa lift.*

In [15] Ikeda studied pullback integrals via imbeddings $\mathfrak{H}_{2m+r} \times \mathfrak{H}_r \hookrightarrow \mathfrak{H}_{2(m+r)}$ of the space $\mathbb{I}_\kappa^{2(m+r)}$. In honour of Miyawaki they are now called *Miyawaki lifts*. As an application he proved main parts of the Miyawaki conjecture. Our results on the Hecke duality relation of degenerate principal series leads to a generalization of parts of his results and finally to non-vanishing results, essential in studying Miyawaki lifts. For $F \in S_{\kappa+n}^{2n}$ with $n = m + r$ and $\Phi \in S_{\kappa+n}^r$ define

$$\mathcal{F}_{F,\Phi}(z) = \int_{Sp_r(\mathbb{Z})\backslash\mathfrak{H}_r} F\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \overline{\Phi(w)} \det(\operatorname{Im} w)^{\kappa+n} \frac{dw}{\det(\operatorname{Im} w)^{r+1}}. \tag{1.10}$$

Let $F \in \mathbb{I}_\kappa^{2n}$ and h be the associated modular form of half-integral weight involved in the Saito-Kurokawa and Shimura correspondance (see introduction of [15] for more details). Then the definition (1.10) coincides with Ikeda's notation $\mathcal{F}_{h,\Phi}$, since Φ can always be normalized to have totally real Fourier coefficients. Let $F = I^{2n}(f), \Phi \in S_{\kappa+n}^r$ Hecke eigenforms. Let $\mathcal{F}_{F,\Phi}(\tau) \neq 0$ then Ikeda proved that the Miyawaki lift $\mathcal{F}_{F,\Phi}$ is a Hecke eigenform whose standard L-function is equal to

$$L(s, \Phi, \mathrm{st}) \prod_{i=1}^{2n} L(s + \kappa + n - i, f). \tag{1.11}$$

For a $\mathbb{C}$-algebra homomorphism $\lambda : \mathcal{H}_n \to \mathbb{C}$, let $S^n_\kappa(\lambda)$ be the λ-th Hecke eigenspace in S^n_k. While for $GL(2)$ the multiplicities are at most 1, for $n > 1$ the multiplicities are not known. Let $\kappa, n \in \mathbb{N}$ and κ even. For each $0 \le \nu \in \mathbb{Z}$ there exist linear maps

$$\mathcal{D}_{2\nu} \ : \ C^\infty(\mathfrak{H}_{2n}) \longrightarrow C^\infty\left(\mathfrak{H}_n \times \mathfrak{H}_n\right)$$

which commute with operators $j_i(\eta)$, in the following sense: for $i = 1, 2$, in terms of the classical slash operator form of Hecke operators,

$$(1.12) \qquad \left(\mathcal{D}_{2\nu}\left(F\right)\right)|_{\kappa+2\nu}\, j_i(\gamma) \ = \ \mathcal{D}_{2\nu}\left(F|_\kappa\, j_i(\gamma)\right) \quad \text{and}$$

$$(1.13) \qquad \left(\mathcal{D}_{2\nu}(F)\right)|_{\kappa+2\nu} w \ = \ \mathcal{D}_{2\nu}\left((F)|_\kappa w\right),$$

where w is the Weyl element

$$(1.14) \qquad w = \left(\begin{smallmatrix} W & 0 \\ 0 & W \end{smallmatrix}\right) \quad \text{with} \quad W = \left(\begin{smallmatrix} 0 & 1_n \\ 1_n & 0 \end{smallmatrix}\right).$$

These maps will be explicitly exhibited as natural differential operators coming from the right action of the Lie algebra of the archimedan points of the group, which, therefore, have several further properties, such as mapping *holomorphic* automorphic forms to *holomorphic* automorphic forms. In particular,

$$(1.15) \qquad \mathcal{D}_{2\nu} : S^{2n}_\kappa \longrightarrow \mathrm{Sym}^2 S^n_{\kappa+2\nu}.$$

For $\nu = 0$ the definition will be $\left(\mathcal{D}_0 F\right)(\tau_1, \tau_2)) = F\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)$. Armed with these operators and their properties, we have

Theorem 1.6. *Let $\mathbb{I}^{2n}_\kappa$ be the space of Ikeda lifts of weight $\kappa + n$ and degree $2n$. Then, for all $0 \le \nu \in \mathbb{Z}$,*

$$(1.16) \qquad \mathcal{D}_{2\nu}\left(\mathbb{I}^{2n}_\kappa\right) \ \subseteq \ \bigoplus_\lambda \mathrm{Sym}^2\Big(S^n_{\kappa+n+2\nu}(\lambda)\Big).$$

From which we obtain

Corollary 1.7. *Let $F \in \mathbb{I}^{2r}_\kappa$ and $\Phi \in S^r_{\kappa+r+2\nu}$ be Hecke eigenforms. For $z \in \mathfrak{H}_r$, define $\widehat{\Phi}$ by*

$$(1.17) \qquad \widehat{\Phi}_{\mathcal{D}_{2\nu}F}(z) = \Big\langle \left(\mathcal{D}_{2\nu}F\right)(\, z\, , *),\ \Phi \Big\rangle.$$

Then $\widehat{\Phi} \in S^n_{\kappa+r+2\nu}$. If $\widehat{\Phi} \neq 0$ then $\widehat{\Phi}$ is a Hecke eigenform and the standard L-functions of $\widehat{\Phi}$ and Φ are equal. Moreover for $r = 1$ we have $\widehat{\Phi}_{\mathcal{D}_{2\nu}\Phi} = \beta(\nu)\Phi$, where $\beta(\nu) \in \mathbb{C}$. The eigenform Φ can be choosen such that either $\beta(0)$ or $\beta(1)$ is non-zero. Moreover let $r = 2$ and Φ a Saito-Kurokawa lift. Then again $\widehat{\Phi}_{\mathcal{D}_{2\nu}F}$ is proportional to Φ.

2. Degenerate principal series and the Hecke duality relation

In this section we recall basic facts on degenerate principal series representations of the symplectic group and prove Theorem 1.2.

Proof. [*Theorem 1.2*] Let F be an ultrametric local field, with ring of integers $\mathfrak{o}$. Let $Sp_n(F)$ be the usual symplectic group of the standard alternating form with matrix

$$J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}.$$

Let $G = Sp_{2n}(F)$ and let P^{Sieg} be the Siegel parabolic

$$P^{\mathrm{Sieg}} \;=\; \Big\{ \begin{pmatrix} A & * \\ 0 & (A^\top)^{-1} \end{pmatrix} \in G \Big\}.$$

Let χ be a character of P^{Sieg} of the form

$$\chi \begin{pmatrix} A & * \\ 0 & (A^\top)^{-1} \end{pmatrix} = |\det A|^s, \qquad (\text{with } A \in GL_{2n}(F),\ s \in \mathbb{C}) \tag{2.1}$$

where $|\ |$ is the norm on F. As usual, a function on the totally disconnected group G is *smooth* if is locally constant. A $\mathbb{C}$-valued function f on G is left χ*-equivariant* by P^{Sieg} if

$$f(p \cdot g) = \chi(p) \cdot f(g) \qquad (\text{for } p \in P^{\mathrm{Sieg}} \text{ and } g \in G).$$

The χ-th (unramified) degenerate principal series representation of G is

$$I_\chi^{\deg} = \big\{ \text{smooth } \mathbb{C}\text{-valued } \chi\text{-equivariant functions } f \text{ on } G \big\}. \tag{2.2}$$

Let

$$j = j_1 \times j_2 \ :\ Sp_n(F) \times Sp_n(F) \to Sp_{2n}(F)$$

as usual by

$$j = j_1 \times j_2 \ :\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \to \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.$$

We use the standard maximal compact subgroup $Sp_n(\mathfrak{o})$ of $Sp_n(F)$. The *spherical Hecke algebra* on $Sp_n(F)$ is the algebra $\mathcal{H}_v^{sph}$ of compactly supported $\mathbb{C}$-valued left and right $Sp_n(\mathfrak{o})$-invariant functions on $Sp_n(F)$. Functions $\eta \in \mathcal{H}_v^{sph}$ act on any smooth representation space (π, V) for $Sp_n(F)$ by

$$\pi(\eta)\, x = \eta \cdot x = \int_{Sp_n(F)} \eta(g)\, \pi(g)\, x\, dg = \int_{Sp_n(F)} \eta(g)\, g \cdot x\, dg \qquad (\text{for } x \in V).$$

This extension of the group action is compatible with convolution, namely

$$\eta_1 \cdot (\eta_2 \cdot v) = (\eta_1 * \eta_2) \cdot v \qquad \text{where} \qquad (\eta_1 * \eta_2)(g) = \int_{Sp_n(F)} \eta_1(gh^{-1})\, \eta_2(h)\, dh.$$

That is, $\pi(\eta_1) \circ \pi(\eta_2) = \pi(\eta_1 * \eta_2)$.

Let $H = Sp_n(F)$ and $G = Sp_{2n}(F)$. The two group homomorphisms $j_1, j_2 : H \to G$ make any smooth representation (π, V) of G into a smooth representation of H, in two obvious ways, by

$$(\pi \circ j_i)(h)\, v = \pi(j_i(h))\, v \qquad (\text{with } i = 1, 2).$$

Each of these two representations of H has associated actions of η in the spherical Hecke algebra $\mathcal{H}_v^{sph}$ of H, by the integrals above.

A local version of a space related to a *Spezialschar* $(\pi, V)^{spz}$ in a smooth representation (π, V) of G is

$$(\pi, V)^{spz} = \Big\{ v \in V \ :\ (\pi \circ j_1)(\eta)(v) = (\pi \circ j_2)(\eta)(v) \quad \text{for all } \eta \in \mathcal{H}_v^{sph} \Big\}.$$

We prove now that the subspace $(I_\chi^{\deg})^{spz}$ inside a degenerate principal series $(\pi, I_\chi^{\deg})$ for G includes the *spherical vectors* in $I_\chi^{\deg}$ (that is, the $Sp_{2n}(\mathfrak{o})$-invariant vectors in $I_\chi^{\deg}$).

This is essentially a direct computation. The only not-quite-classical ingredient concerns the double coset space

$$P^{\text{Sieg}} \setminus G \,/\, j(Sp_n(F) \times Sp_n(F)).$$

By now it is well-known (e.g., see [7]) that under this action of $P^{\text{Sieg}} \times (Sp_n(F) \times Sp_n(F))$ on G there are finitely-many orbits, with a unique open dense orbit

$$\Omega = P^{\text{Sieg}} \cdot \xi \cdot j(Sp_n(F) \times Sp_n(F)) \qquad (\text{with } \xi = \begin{pmatrix} I_n & & & \\ & I_n & & \\ & I_n & I_n & \\ I_n & & & I_n \end{pmatrix}).$$

The isotropy group of $P\xi$ under the action of $Sp_n(F) \times Sp_n(F)$ on $P^{\text{Sieg}} \backslash G$ is readily seen to be a diagonal copy of $H = Sp_n(F)$, namely

$$\Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} d & c \\ b & a \end{pmatrix} \ :\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H = Sp_n(F) \Big\}.$$

Thus,

$$\Omega = P^{\text{Sieg}} \cdot \xi \cdot j_i(H) \qquad (\text{for } i = 1, 2).$$

Write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\natural} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

and observe that

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\natural})^{-1} = \begin{pmatrix} a^\top & -c^\top \\ -b^\top & d^\top \end{pmatrix}.$$

Thus,

$$\chi(\xi \cdot j(h \times h^\natural) \cdot \xi^{-1}) = 1 \qquad (\text{for all } h \in H = Sp_n(F)).$$

for these (unramified) characters χ. Further we need a particular element w of the spherical Weyl group given by (1.14). This has the property that conjugation by w interchanges the two copies of $H = Sp_n(F)$ inside G, namely

$$w\, j_1(h)\, w^{-1} = j_2(h) \qquad (\text{for all } h \in H).$$

Further, notice that w *normalizes* our choice of representative ξ, and moreover lies in $P^{\text{Sieg}} \cap Sp_{2n}(\mathfrak{o})$.

Let f be a spherical vector in I_χ^{deg}, that is, $f(gk) = f(g)$ for all $k \in Sp_{2n}(\mathfrak{o})$ and $g \in G$. For η in the spherical Hecke algebra $\mathcal{H}_v^{sph}$ on $H = Sp_n(F)$, by definition,

$$(\pi \circ j_1)(\eta) f(g) = \int_H \eta(h)\, f(g\, j_1(h)) dh.$$

Since $(\pi \circ j_1)(\eta) f$ will in any case be a left χ-equivariant continuous function on G, it suffices to take g in the dense orbit Ω. For that matter, of course we can take g in a set of representatives for $P^{\text{Sieg}} \backslash \Omega$, namely $\xi j_1(H)$. Let $g = j_1(h_o)$ with $h_o \in H$. By the isotropy subgroup relation,

$$(\pi \circ j_1)(\eta) f(g) = \int_H \eta(h)\, f(\xi\, j_1(h_o)\, j_1(h))\, dh = \int_H \eta(h)\, f(\xi\, j_2(h_o^\natural)^{-1}\, j_1(h))\, dh.$$

Since $w \in Sp_{2n}(\mathfrak{o})$, and since $w\xi w^{-1} = \xi$,

$$f(\xi\, j_2(h_o^\natural)^{-1}\, j_1(h)) = f(\xi\, j_2(h_o^\natural)^{-1}\, j_1(h) w^{-1})$$
$$= f(w^{-1} w \xi w^{-1}\, w j_2(h_o^\natural)^{-1}\, w^{-1}\, w j_1(h) w^{-1}) = f(\xi\, j_1(h_o^\natural)^{-1}\, j_2(h)).$$

Thus,

$$(\pi \circ j_1)(\eta) f(\xi\, j_1(h_o)) = (\pi \circ j_2)(\eta) f(\xi\, j_1(h_o^\natural)^{-1})).$$

If we could justify taking h_o such that $(h_o^\natural)^{-1} = h_o$ then we would have the desired result. To do so, we use the right $Sp_{2n}(\mathfrak{o})$-invariance of f and the two-sided $Sp_n(\mathfrak{o})$-invariance of η. Since η is *left* $Sp_n(\mathfrak{o})$-invariant, in the integral we can replace h by kh with any $k \in Sp_n(\mathfrak{o})$, thus effectively right-multiplying h_o by k without changing the value of the integral.

On the other hand, now we take advantage of the fact that we are using Sp_n and not GSp_n. Since f is right $Sp_{2n}(\mathfrak{o})$-invariant, the argument to f can be right-multiplied by $i_2(k)$ for $k \in Sp_n(\mathfrak{o})$, and thus compute

$$f(\xi\, j_1(h_o h)) = f(\xi\, j_1(h_o h)\, j_2(k)) = f(\xi\, j_2(k)\, j_1(h_o h)) = f(\xi\, j_1(k^\natural)^{-1}\, j_1(h_o h)).$$

That is, we can left-multiply h_o by $Sp_n(\mathfrak{o})$ without changing the value of the integral. By the p-adic Cartan decomposition for $Sp_n(F)$, there are *diagonal* representatives h_o for

$$Sp_n(\mathfrak{o}) \setminus Sp_n(F) \,/\, Sp_n(\mathfrak{o}).$$

Happily, diagonal elements h_o of $Sp_n(F)$ have the desired property

$$(h_o^\natural)^{-1} = h_o$$

This finishes the proof. □

3. Ikeda lifts and representation theory

In this section we recall the definition and construction of Ikeda lifts, then prove Corollary 1.4.

For positive integers n and κ, let M_κ^n be the space of Siegel modular forms for the Siegel modular group $\Gamma_n = Sp_n(\mathbb{Z})$ on the Siegel upper half-space $\mathfrak{H}_n$. The action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp_n(\mathbb{R})$ on $\mathfrak{H}_n$ is

$$\gamma(z) \;=\; (az+b)(cz+d)^{-1}.$$

The Petersson slash operator $|_k$ on complex-valued functions f on $\mathfrak{H}_n$ is

$$\big(f|_\kappa \gamma\big)(z) \;=\; \det{(cz+d)^{-\kappa}} \;\; f(\gamma(z)).$$

Thus, f is a Siegel modular form of weight κ when f is holomorphic, satisfies the functional equation $f|_\kappa\gamma = f$ for all $\gamma \in \Gamma_n$. For elliptic modular forms there is the usual condition on growth at infinity, but by the Koecher principle no growth condition is needed for $n > 1$. Siegel modular forms f have Fourier expansions

$$f(z) \;=\; \sum_T A(T)\; e^{2\pi i\, \mathrm{tr}(Tz)}, \tag{3.1}$$

where T runs over half-integral positive semi-definite n-by-n matrices. The space S_κ^n of cusp forms is the collection of modular forms with $A(T) = 0$ for T not positive-definite. Let $\mathcal{H}_n$ be the Hecke algebra associated to the Hecke pair $(\Gamma_n, Sp_n(\mathbb{Q}))$. It is well known that this algebra is commutative and decomposes into local components $\mathcal{H}_n = \otimes_p \mathcal{H}_{n,p}$, where $\mathcal{H}_{n,p}$ is the Hecke algebra associated to the Hecke pair $(\Gamma_n, Sp_n(\mathbb{Z}[\frac{1}{p}]))$. For $T = \sum_j a_j\, \Gamma_n \gamma_j \in \mathcal{H}_n$, the corresponding Hecke operator on the space of Siegel modular forms is $Tf = \sum_j a_j\, f|_\kappa \gamma_j$. Let $F \in S_\kappa^n$ be a Hecke eigenform with Satake parameters

$\{\beta_1^{\pm 1}, \ldots, \beta_n^{\pm 1}\}$, in the notation of Ikeda [15], page 472. The standard L-function of F is

$$L(s, F, \mathrm{st}) = \zeta(s) \prod_p \left(\prod_{i=1}^n (1 - \beta_i \, p^{-s}) \, (1 - \beta_i^{-1} \, p^{-s}) \right)^{-1}. \tag{3.2}$$

For a primitive newform $f \in S_{2\kappa}$ (i.e., $a(1) = 1$ and f is a Hecke eigenform) with Fourier expansion $f(z) = \sum_{t=1}^{\infty} a(t) \, e^{2\pi i \, tz}$ the standard GL_2 L-function is

$$\begin{aligned} L(s, f) &= \prod_p \left(1 - a(p) \, X + p^{2\kappa - 1} X^2\right)^{-1} \\ &= \prod_p \left((1 - \alpha_p p^{\frac{2\kappa-1}{2}} \, p^{-s}) \, (1 - \alpha_p^{-1} p^{\frac{2\kappa-1}{2}} \, p^{-s}) \right)^{-1}. \end{aligned} \tag{3.3}$$

For all $n \in \mathbb{N}$ with $n + \kappa \equiv 0 \pmod 2$ Ikeda [14] explicitly constructed a Hecke eigenform $I^{2n}(f) \in S_{\kappa+n}^{2n}$ with Satake parameters

$$\left\{ \left(\alpha_p \, p^{-(2n+1)/2}\right)^{\pm 1}, \left(\alpha_p \, p^{-(2n-1)/2}\right)^{\pm 1}, \ldots, \left(\alpha_p \, p^{(2n+1)/2}\right)^{\pm 1} \right\}. \tag{3.4}$$

Proof of Corallary 1.4:
Let $F_o \in \mathbb{I}_{\kappa}^{2n} \subseteq S_{\kappa+n}^{2n}$ be an Ikeda lift of an elliptic modular form $f \in S_{2\kappa}$. We can assume F_o and f Hecke eigenforms, and

$$L(s, f) = \prod_p \left((1 - \alpha_p p^{\frac{2\kappa-1}{2}} \, p^{-s}) \, (1 - \alpha_p^{-1} p^{\frac{2\kappa-1}{2}} \, p^{-s}) \right)^{-1}.$$

By strong approximation, F_o yields a cuspidal automorphic function F on $Sp_{2n}(\mathbb{A}_{\mathbb{Q}})$. Since F_o is a Hecke eigenform, and is holomorphic, F generates an irreducible cuspidal automorphic representation π which factors over places as $\pi \approx \otimes_v' \pi_v$ with π_v spherical at finite places v. Since F_o is an Ikeda lift, the Satake parameters of π_v are given by (3.4). Thus, at finite places v attached to primes p, π_v is a degenerate principal series representation

$$\pi_v \approx \mathrm{Ind}_{P_v^{\mathrm{Sieg}}}^{G_v} \omega_v \circ \det. \tag{3.5}$$

Here P_v^{Sieg} is the v-adic points of the Siegel parabolic, G_v is $Sp_{2n}(\mathbb{Q}_v)$, and ω_v is the unramified character of $\mathbb{Q}_v^{\times}$ determined by $\omega_v(p) = \alpha_v$. Next we apply Theorem 1.2 to this local representation since we have at all finite places degenerate principal series and a spherical vector. It is now a routine excercise to translate this back into the claim of the theorem, giving the desired result.

4. Proof of Theorem 1.1 and Theorem 1.5

In this section we prove two converse theorems. Theorem 1.1 is stated in classical terms and Theorem 1.5 is a result in representation theory. Let $F \in S_\kappa^2$ be a Siegel modular form of degree 2. We assume F to be a Hecke eigenform. In this paper and [14], [15], this means F is an eigenform of Hecke operators with respect to the Hecke pair $\mathcal{H}_2\left(\Gamma_2, Sp_2(\mathbb{Q})\right)$. We do not assume that F is also an eigenform with respect to the Hecke algebra $\mathcal{H}_2^+$ related to the Hecke pair $\mathcal{H}_2^+ = (\Gamma_2, G^+Sp_2(\mathbb{Q}))$, where

$$G^+Sp_n(\mathbb{Q}) = \Big\{\gamma \in Gl_{2n}(\mathbb{Q}) | \, \gamma^T J\gamma = \mu(\gamma)\, J, \text{ with } \mu(\gamma) \in \mathbb{Q} \text{ and } \mu(\gamma) > 0\Big\}.$$

If we would assume the stronger condition F to be an eigenform with respect to the algebra $\mathcal{H}_2^+$, then it is well-known that with respect to the spinor L-function, which contains more information than the standard L-function $L(F, s, \mathrm{st})$, the following converse theorem is true. F is a Saito-Kurokawa lift if and only if the spinor L-function $L(F, s)$ of F is $\zeta(s+\kappa-1)\,\zeta(s+\kappa-2)\,L(f, s)$, where $f \in S_{2k-2}$ is an eigenform with respect to $\mathcal{H}_1^+$. From [14], an Ikeda lift $F = I^2(f) \in S_\kappa^2$ of an eigenform $f \in S_{2\kappa-2}$ has standard L-function

$$L(s, F, \mathrm{st}) \;=\; \zeta(s)\; L(s+\kappa-1, f)\, L(s+\kappa-2, f). \tag{4.1}$$

By considering the Fourier coefficients of $I^2(f)$ Ikeda has shown that these fullfill the Maass relations, and, hence, his lift is the Saito-Kurokawa lift of f. This result does *not* imply that a Hecke eigenfunction $F \in S_\kappa^2$ with standard L-function of the form (4.1) is necessarily a lift. Nevertheless, it is true that F is a lift. This is the content of Theorem 1.1.

Proof. [*Theorem 1.1*] Let $G \in S_k^2$ be a Hecke eigenform with standard L-function given as in (1.2). Then the p-component of the automorphic representation π generated by G is a degenerate principal series. Hence locally at finite places, π satisfies the Hecke duality relation. This implies that G satisfies $T^\uparrow(G) = T^\downarrow(G)$ for all Hecke operators T related to the Hecke pair $(SL_2(\mathbb{Z}), SL_2(\mathbb{Q}))$. Applying Theorem 4.1 in [13] leads to the desired result. □

We have shown that degenerate principal series satisfy the Hecke duality relation (at finite places), but we do not know whether the converse holds. In fact, the converse seems unlikely. Nevertheless, in any case, we have a global version of this, at least in the case of Siegel modular forms of degree 2, as follows. Let $\pi = \otimes'\pi_v$ be a cuspidal automorphic representation attached to a Hecke eigenform $F \in S_\kappa^2$, with π_v a degenerate principal series for almost all finite v. We have already proven that degenerate principal series satisfy the local Hecke duality relation, so, for almost all p, F satisfies the Hecke duality relation $T^\uparrow(F) = T^\downarrow(F)$ for $T \in \mathcal{H}_{2,p}$. Then, from [13], Theorem 4.1

and the arguments in the proof, F is a Saito-Kurokawa lift. Hence F is an eigenform of the larger Hecke algebra $\mathcal{H}_2^+$. This proves Theorem 1.5.

5. Applications

Integral representation of L-functions of cuspforms by integration against Eisenstein series in effect presents these L-functions as *decomposition coefficients* in spectral decomposition of restrictions of automorphic forms. Although often only implicit, this mechanism is an important recurring theme in the theory of automorphic forms, with both analytic and arithmetic applications: from among the enormous range of range of literature from the early [27], [25], one may mention [16], [17], [28], [29], (and many other later papers of Shimura), also [4], [5], and [10], [6], and the partial survey of examples of the Rankin-Selberg method in [3]. The most typical scenario involves part of a spectral decomposition of the restriction of an Eisenstein series to a smaller reductive group. Less often, such an integral representation involves part of a spectral decomposition of the restriction of a *cuspform*, as in [1], and [11], for example. Some analytical aspects of spectral decompositions of cuspforms on low-rank groups appear in [26], [2], and [19].

Recently [15] introduced *Miyawaki lifts*, projections of pullbacks of cusp forms in $\mathbb{I}_\kappa^{2n}$, which are lifts of newforms, with respect to the Petersson scalar product $\langle\,,\,\rangle$ to $S_{\kappa+n}^r$ $(r \leq n)$. We briefly recall those results. Let us fix a partition $m+r$ of the positive integer n with $m, r \in \mathbb{N}_0$. Let $F \in S_{\kappa+n}^{2n}$ and $\Phi \in S_{\kappa+n}^r$ any non-trival cusp forms. Attached to the partition $m+r$ we have the imbedding $j : \mathfrak{H}_{2m+r} \times \mathfrak{H}_r \longrightarrow \mathfrak{H}_{2n}$ given by $j(z,w) = \left(\begin{smallmatrix} z & 0 \\ 0 & w \end{smallmatrix}\right)$. It is well known that $F \circ j \in S_{\kappa+n}^{2m+r} \otimes S_{\kappa+n}^r$. Hence the function $\mathcal{F}_{F,\Phi}$ defined in the introduction (1.10) is well defined. Let $\langle\,,\,\rangle$ be the Petersson scalar product on $S_{\kappa+n}^r$ then $\mathcal{F}_{F,\Phi}(z) = \langle F \circ j(z,*), \Phi\rangle$. Hecke eigenforms are always normalized in such a way that all the Fourier coefficients are totally real algebraic numbers. Now we can state the main result of Ikeda in [15]: Let $F = I^{2n}(f) \in S_{\kappa+n}^{2n}$ be the Ikeda lift of a normalized newform $f \in S_{2\kappa}$ and $\Phi \in S_{\kappa+n}^r$ a normalized Hecke eigenform. Assume $\mathcal{F}_{F,\Phi} \neq 0$, then $\mathcal{F}_{F,\Phi}$ is a Hecke eigenform with standard L-function

$$L(s,\Phi,\mathrm{st}) \prod_{i=1}^{2n} L(s+\kappa+n-i, f). \tag{5.1}$$

In [15], Ikeda recalls two conjecture of Miyawaki. Conjecture 2.1 predicts the exact form of the standard zeta function of the unique Hecke eigenform of weight 12 and degree 3. More generally, conjecture 2.2 states that for two given newforms $f \in S_{2\kappa-4}$ and $g \in S_\kappa$, there exists a Hecke eigenform in S_κ^3 such that the standard L-function is given by the data of f and g in a certain way. Ikeda's method to construct such forms is highly dependent on the non-vanishing of the Miyawaki lift. With this non-vanishing, he could prove both conjectures. He was able to prove conjecture 2.1, and gave several examples toward conjecture 2.2.

Hence, the question about non-vanishing of the Miyawaki lift becomes crucial. At this point we want to remark that even in the case $n = r = 1$, where it is known that for every newform $\Phi \in S_{\kappa+1}$ a newform f exists such that $\mathcal{F}_{I^2(f),\Phi} \neq 0$, it is an open question whether to every f a Φ exists such that $\mathcal{F}_{I^2(f),\Phi} \neq 0$. The pullback $F \circ j$ of a modular form F can be viewed as the 0-th Taylor coefficient. Hence, to guarantee non-vanishing, one has to involve the higher Taylor coefficients, too. This can described via invariant differential operators ([13]). In the case of Siegel modular forms of degree 2 this is $1-1$. Hence, it is natural to generalize the definition of the Miyawaki lift introduced by Ikeda. We alter the notation to highlight the fact that Miyawaki lifts posses certain duality properties inherited by the Hecke duality relations.

Let $n = m + r$ and let $k + n$ be even. Assume $F' \in S^{2m+r}_{\kappa+n} \otimes S^{r}_{\kappa+n}$ and $\Phi \in S^{r}_{\kappa+n}$, then we define

$$\widehat{\Phi}_{F'}(z) = \Big\langle F'(z, *), \Phi \Big\rangle. \tag{5.2}$$

It is obvious that $\widehat{\Phi}_{F'} \in S^{2m+r}_{\kappa+n}$. A straightforword way to obtain functions F' of the type just described is via differential operators $\mathcal{D}_{2\nu}$ mentioned earlier. The main point is the following. Assume that for every $\nu \in \mathbb{N}_0$ and every even positive integer k we have non-trivial linear maps

$$\mathcal{D}_{2\nu} : S^{2n}_{\kappa} \longrightarrow \mathrm{Sym}^2 S^{n}_{\kappa+2\nu}. \tag{5.3}$$

with the additional property $(\mathcal{D}_{2\nu}F)\,|_{\kappa+2\nu} j_i(\gamma) = \mathcal{D}_{2\nu}\left(F|_{\kappa} j_i(\gamma)\right)$ for all $\gamma \in Sp_n(\mathbb{R})$ and $i = 1, 2$. Such maps exist (and are unique) at least for $\kappa > 2n$. We use the following normalization. For $\nu = 0$ let $\mathcal{D}_0 F = F \circ j$. And in the case of Siegel modular forms of degree 2, we refer to [13], where

$$\mathcal{D}_{2\nu}F(\tau,\tilde{\tau}) = \left(p_{k,2\nu}\left(\frac{1}{2\pi\, i}\frac{\partial}{\partial z}, \left(\frac{1}{2\pi\, i}\right)^2 \frac{\partial}{\partial \tau}\frac{\partial}{\partial \tilde{\tau}} \right)\right) F \Bigg|_{z=0}. \tag{5.4}$$

Here $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tilde{\tau} \end{smallmatrix}\right) \in \mathfrak{H}_2$ and $p_{a,b}$ is the the Gegenbauer polyonomial as defined in [13]. Now we apply these operators to Ikeda lifts. Let $n = r$ and $\mathcal{D}_{2\nu} : S^{2n}_{\kappa+r} \longrightarrow \mathrm{Sym}^2 S^{n}_{\kappa+r+2\nu}$. Let $(\Phi_j)_j$ be a normalized basis of Hecke eigenforms of $S_{\kappa+r+2\nu}$. Then we have

$$\mathcal{D}_{2\nu}F = \sum_{i,j} \alpha_{i,j}\, \Phi_i \otimes \Phi_j \ \ (\alpha_{i,j} \in \mathbb{C}). \tag{5.5}$$

If we assume that $F \in \mathbb{I}^{2r}_{\kappa}$ then $T^{\uparrow}F = T^{\downarrow}F$ for all Hecke operators $T \in \mathcal{H}_r$. Since these operators commute with our differential operators

$$(T \otimes I_r)\,(\mathcal{D}_{2\nu}F) = (I_r \otimes T)\,(\mathcal{D}_{2\nu}F).$$

Here T acts as usual on the related components of the tensor product with weight $k+r+2\nu$. If $T(G_i) = \lambda_i(T)\, G_i$ then $\alpha_{i,j}\, \lambda_i(T) = \alpha_{i,j}\, \lambda_j(T)$ for all $T \in \mathcal{H}_r$. Hence let $\alpha_{i,j} \neq 0$, then G_i and G_j are in the same eigenspace. Next consider $\widehat{G}_{\mathcal{D}_{2\nu}F}$. We have

$$\widehat{G}_{\mathcal{D}_{2\nu}F} = \sum_{i,j} \alpha_{i,j} G_i \otimes \langle G_j, G\rangle. \tag{5.6}$$

Let λ_G be the $\mathbb{C}$-algebra homomorphism of $\mathcal{H}_r$ induced by the eigenvalues of G, then $\widehat{G}_{\mathcal{D}_{2\nu}F} \in S^r_{\kappa+r+2\nu}(\lambda_G)$. In the case of $r=1$ of F an Ikeda lift of degree 2 it follows directly from the multiplicity one theorem for SL_2 (see [24]) that

$$\widehat{G}_{\mathcal{D}_{2\nu}F} = \beta(\nu)\; G$$

It follows form [13] that there is $G_\nu \in S_{\kappa+1+2\nu}$ for $\nu = 0,1$ such that $\beta(0)$ or $\beta(1)$ is non-zero. This gives a preliminary answer to the question of Ikeda about non-vanishing. Conversely, for every Hecke eigenform $G \in S_{\kappa+1}$ there exists a Hecke eigenform $F \in \mathbb{I}^2_\kappa$ such that $\widehat{G}_{\mathcal{D}_0 F} = \beta(0)\; G$ and $\beta(0) \neq 0$. Moreover let $F \in \mathbb{I}^4_\kappa$ and $G_\nu \in \mathbb{I}^2_{\kappa+2\nu+1}$ be two Hecke eigenforms lifted from elliptic cuspforms $f \in S_{2\kappa}$ and $g \in S_{2\kappa+2+4\nu}$. Since we have proven that the standard zeta function of the Ikeda lift of degree 2 uniquely determines the function itself, $\widehat{G}_{\mathcal{D}_{2\nu}F} = \beta(\nu)\; G$. Hence Theorem 1.6 and Corollary 1.7 are proven.

References

[1] A. N. Andrianov: *Symmetric squares of zeta functions of Siegel modular forms of genus* 2 Trudy Mat. Inst. Steklov **142** (1976), 22-45.

[2] J. Bernstein and A. Reznikov *Analytic continuation of representations and estimates of automorphic forms*, Ann. of Math. **150**, 1999, 329-352.

[3] D. Bump, *The Rankin-Selberg method: an introduction and survey*, in Automorphic representations, L-functions, and applications: progress and prospects, Ohio State Univ. Math. Res. Inst. Publ. **11**, 41-73.

[4] P. Garrett: *Pullbacks of Eisenstein series; applications.* Automorphic forms of several variables (Katata, 1983), 114–137, Progr. Math., **46** Birkhäuser Boston, Boston, MA, 1984.

[5] P. Garrett: Decomposition of Eisenstein series: triple product L-functions. Ann. Math. **125** (1987), 209-235.

[6] P. Garrett: *Euler products of global integrals*, in vol. 2 of *Automorphic Forms, Automorphic Representations, and Arithmetic* ed. R. Doran, Z.-L. Dou, G. Gilbert, Proc. Symp. Pure Math. 66 (1999), AMS, 35-102.

[7] P. Garrett: *On the arithmetic of Siegel-Hilbert cuspforms: Petersson inner products and Fourir coefficients* Invent. math. **107** (1992), 453-481.

[8] P. Garrett, M. Harris: *Special values of triple product L-functions.* Amer. J. Math. **115** (1993), 159-238.

[9] S. Gelbart: *Automorphic Forms on Adele Groups* Princeton Univ. Press, (1975).

[10] S. Gelbart, I.I. Piatetski-Shapiro, S. Rallis, *Explicit Construction of L-functions*, SLN 1254, Springer, NY, 1987.

[11] B. Heim: *Über Poincare Reihen und Restriktionsabbildungen.* Abh. Math. Sem. Univ. Hamburg, **68** (1998), 79-89.

[12] B. Heim: *Pullbacks of Eisenstein series, Hecke-Jacobi theory and automorphic L-functions.* In: Automorphic Froms, Automorphic Representations and Arithmetic. Proceedings of Symposia of Pure Mathematics **66**, part 2 (1999).

[13] B. Heim: *On the Spezialschar of Maass. Preprint, submitted* 2006

[14] T. Ikeda: *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$. Ann. of Math. **154** no. 3 (2001), 641-681.

[15] T. Ikeda: *Pullback of the lifting of elliptic cusp forms and Miyawakis conjecture.* Duke Math. Journal, **131** no. 3 (2006), 469-497.

[16] H. Klingen, *Über den arithmetischen Charakter der Fourierkoeffizienten von Modulformen*, Math. Ann. **147** (1962), 176-188.

[17] H. Klingen, *Über einen Zusammenhang swischen Siegelschen und Hermitschen Modulfunktionen*, Abh. Math. Sem. Univ. Hamburg **27** (1964), 1-12.

[18] W. Kohnen, H. Kojima: *A Maass space in higher degree.* Compos. Math, **141** no. 2 (2005), 313-322.

[19] B. Krotz, R. Stanton, *Holomorphic extensions of representations, I. Automorphic functions*, Ann. of Math. (2) **159** (2004), 641-724.

[20] N. Kurokawa: *Examples of eigenvalues of Hecke operators on Siegel cuspforms of degree two.* Inventiones Math. **49** (1978), 149-165.

[21] H. Maass: *Über eine Spezialschar von Modulformen zweiten Grades I.* Invent. Math. **52** (1979), 95-104.

[22] H. Maass: *Über eine Spezialschar von Modulformen zweiten Grades II.* Invent. Math. **53** (1979), 249-253.

[23] H. Maass: *Über eine Spezialschar von Modulformen zweiten Grades III.* Invent. Math. **53** (1979), 255-265.

[24] D. Ramakrishnan: *Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2).* Ann. math. **152** (2000), 45-111.

[25] R. Rankin, *Contributions to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetical functions, I, and II*, Proc. Camb. Phil. Soc. **35** (1939), 351-356 and 357-372.

[26] P. Sarnak, *Integrals of products of eigenfunctions*, Int. Math. Res. Notices, no. 6 (1994), 251-260.

[27] A. Selberg, *Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist* Arch. Math.Naturvid. **43** (1940), 47-50.

[28] G. Shimura, *On the Fourier coefficients of modular forms of several variables*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **17** (1975), 261-268.

[29] G. Shimura, *On some arithmetic properties of modular forms of one and several variables*, Ann. of Math. **102** (1975), no. 3, 491-515. Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **17** (1975), 261-268.

[30] D. Zagier: *Sur la conjecture de Saito-Kurokawa (d'après H. Maass.* Sém. Delange-Pisot-Poitou 1979/1980, Progress in Math. **12** (1980), 371 -394.

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