A Characterization of the Uniform Transmission Property for Pseudodifferential Operators

Elmar Schrohe

Max-Planck-Arbeitsgruppe
"Partielle Differentialgleichungen und Komplexe Analysis"
Universität Potsdam
Am Neuen Palais 10
D-14415 Potsdam

Germany

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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Elmar Schrohe

Max-Planck-Arbeitsgruppe
"Partielle Differentialgleichungen und Komplexe Analysis"
Universität Potsdam
Am Neuen Palais 10
D - 14415 POTSDAM

and

Fachbereich Mathematik
Johannes Gutenberg-Universität
Saarstr. 21
D - 55099 MAINZ

Abstract

It is shown that the pseudodifferential operators with the (uniform two-sided) transmission property can be characterized by the behavior of their iterated commutators with multipliers and vector fields tangential to the boundary on ordinary and wedge Sobolev spaces. This extends previous work by Grubb and Hörmander.
Together with a corresponding characterization of the singular Green operators the above result will be used to prove that the algebra of elements of order and type zero in Boutet de Monvel's calculus is a submultiplicative Fréchet algebra.

Key Words: Boundary value problems, Boutet de Monvel's calculus, transmission property, wedge Sobolev spaces.

AMS Subject Classification: 35 S 15, 47 D 25, 46 E 35

Introduction

Boutet de Monvel's calculus makes boundary value problems accessible to pseudodifferential methods. It is a highly efficient tool for studying Fredholm criteria and index theory (Boutet de Monvel [2], Grubb [11], Schrohe [24], Rempel, Schulze [22], [30]), but also topics like functional calculus for boundary problems, Navier Stokes equations, and elasto-dynamics (Grubb [11], [12], Schrohe [24], Ebin and Simanca [6]).
On the other hand, this calculus is technically highly refined. In its usual presentation, [22], [11], the various operators are described by rather delicate estimates on their symbols or symbol kernels.
In the predecessor of this paper it could be shown that the singular Green operators, usually one of the more difficult parts in the theory, had a fairly simple and natural description in terms of the boundedness of their commutators with multipliers and vector fields tangential to the boundary on the wedge Sobolev spaces introduced by Schulze; in fact, the result was motivated by a theorem of Schulze in [29].
In theorem 3.1 I am proving that also the pseudodifferential operators with the transmission property fit into the framework of wedge Sobolev spaces. Again, I am considering commutators with multipliers and vector fields tangential to the boundary.
Extending a result of Grubb and Hörmander [14] on the local transmission property, I first show in theorem 2.1 that a pseudodifferential operator P has the (uniform two-sided) transmission property on $\mathbf{R}^n_+$ provided its commutators with tangential multipliers and vector fields are bounded on the usual Sobolev spaces over $\mathbf{R}^n_+$.
This yields the characterization of the pseudodifferential operators with the transmission property. For one thing, the boundedness of the tangential commutators on the wedge Sobolev spaces *a forteriori* implies the transmission property; on the other hand, it turns out that pseudodifferential operators with the transmission property are indeed bounded on wedge Sobolev spaces.
The difference to the singular Green operators is that these are smoothing in the normal direction, while the pseudodifferential operators are order preserving.
Characterizations by commutators have a long standing tradition in the theory of pseudodifferential operators, cf. for instance the work of H.O. Cordes and his associates, see [4] for references.
For a large class of pseudodifferential operators, R. Beals gave a characterization in terms of the boundedness of their commutators with multipliers and vector fields on Sobolev spaces. Adapted to the usual symbol classes his result reads as follows:

Theorem (Beals 1977). *Let $m \in \mathbf{R}, 0 \le \delta \le \rho \le 1, \delta < 1$, and let $P : \mathcal{S}(\mathbf{R}^n) \longrightarrow \mathcal{S}'(\mathbf{R}^n)$ be a continuous operator. Then P is a pseudodifferential operator with a symbol in $S^m_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n)$ if and only if for all $s \in \mathbf{R}$, and all multi-indices $\alpha, \beta \in \mathbf{N}^n_0$, the iterated commutators $\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)P$ have bounded extensions*

$$\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)P : H^s(\mathbf{R}^n) \longrightarrow H^{s-m+\rho|\alpha|-\delta|\beta|}(\mathbf{R}^n).$$

Cordes has obtained a similar characterization of $S^0_{0,0}$ by different methods [3]; a new proof of Beals' result was given by Ueberberg [32].
Obviously, theorems 1.13 and 3.1 are as close to Beals' as one might hope to come in this more complicated situation.

Beals' characterization has a number of interesting consequences, among them the spectral invariance of the algebra of pseudodifferential operators of order zero in $\mathcal{L}(L^2(\mathbf{R}^n))$, [1] theorem 3.2, or the stability of the spectrum with respect to changes of the space the operators are acting on, cf. Leopold and Schrohe [19], [20], [23].
In addition, Gramsch, Ueberberg, and Wagner showed in [10] that a characterization of an algebra via commutators and order shifts on Sobolev spaces allows to introduce topology on this algebra which makes it topologically an intersection of Banach algebras, a 'submultiplicative' Fréchet algebra.
Starting from this result, it will be shown in a subsequent paper, [9], joint work with B. Gramsch, that these characterizations of the singular Green operators and the pseudodifferential operators with the transmission property, respectively, imply the submultiplicativity of the algebra of elements of order and type zero in Boutet de Monvel's calculus.
Submultiplicativity is of particular interest in connection with the results of N. C. Philipps on K-theory for submultiplicative Fréchet algebras, [21], and the results of Gramsch on non-abelian cohomology and Oka principle in submultiplicative Ψ^*-algebras, [8], cf. [7] (the Ψ^*-property for the algebra of elements of order and type zero was established in [24], in the classical case by Schulze in [28]).

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1 Notation. Pseudodifferential Operators and the Transmission Property. Singular Green Operators and Wedge Sobolev Spaces

We will start with a short review of the essential notation on symbol classes and the transmission property.

1.1 Definition.
(a) For $m \in \mathbf{R}, S^m_{1,0} = S^m_{1,0}(\mathbf{R}^k \times \mathbf{R}^n)$ denotes the set of all smooth functions p on $\mathbf{R}^k \times \mathbf{R}^n, k, n \in \mathbf{N}$, satisfying the estimates

$$|D^\alpha_\xi D^\beta_x p(x,\xi)| \le C_{\alpha\beta} \langle\xi\rangle^{m-|\alpha|} \tag{1}$$

for all $x \in \mathbf{R}^k, \xi \in \mathbf{R}^n$. Here, $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. The choice of best constants in (1) gives the Fréchet topology for $S^m_{1,0}$.
Occasionally, we shall also write $p^{(\alpha)}_{(\beta)}(x,\xi)$ instead of $\partial^\alpha_\xi D^\beta_x p(x,\xi)$.
In general, the symbols will take values in matrices over $\mathbf{C}$, but for the purposes here it will be sufficient to deal with scalar functions.
(b) A symbol $p \in S^m_{1,0}$ defines a pseudodifferential operator $\operatorname{Op} p$ or $p(x,D)$ by

$$[p(x,D)u](x) = [\operatorname{Op} p\, u](x) = (2\pi)^{-n/2} \int e^{ix\xi} p(x,\xi)\hat{u}(\xi)d\xi, \tag{2}$$

where u is a rapidly decreasing function and the hat denotes the Fourier transform.
(c) For $s \in \mathbf{R}, H^s(\mathbf{R}^n)$ denotes the usual Sobolev space on $\mathbf{R}^n$, cf. [18], ch. 3, definition 2.1. For $s, t \in \mathbf{R}$, let

$$H^{s,t}(\mathbf{R}^n) = \{\langle x\rangle^{-t} u : u \in H^s(\mathbf{R}^n_x)\}.$$

$H^{s,t}(\mathbf{R}^n, E), E$ a Hilbert space, denotes the vector-value analog.

1.2 Notation on the half-space. We will write $\mathbf{R}^n_+ = \{(x_1, ..., x_n) : x_n > 0\}$ and $x = (x', x_n), \xi = (\xi', \xi_n)$ with $x' = (x_1, \ldots, x_{n-1}), \xi' = (\xi_1, \ldots, \xi_{n-1})$.
(a) For a function or distribution f on $\mathbf{R}^n$ let $\mathrm{r}^+ f$ denote its restriction to $\mathbf{R}^n_+$; for a function g on $\mathbf{R}^n_+$ denote by $\mathrm{e}^+ g$ its extension to $\mathbf{R}^n$ by zero. Similarly define r^- and e^-.
(b) Let $\mathcal{S}(\mathbf{R}^n_+) = \{\mathrm{r}^+ f : f \in \mathcal{S}(\mathbf{R}^n)\}$, and $H^{s,t}(\mathbf{R}^n_+) = \{\mathrm{r}^+ f : f \in H^{s,t}(\mathbf{R}^n)\}, s, t \in \mathbf{R}$. $H^{s,t}_0(\mathbf{R}^n_+)$ is the closure of $C^\infty_0(\mathbf{R}^n_+)$ in the topology of $H^{s,t}(\mathbf{R}^n)$.
(c) Let $H = H^+ \oplus H^-_0 \oplus H'$, where

$$H^+ = \{(\mathrm{e}^+ f)^\wedge : f \in \mathcal{S}(\mathbf{R}_+)\}, H^-_0 = \{(\mathrm{e}^- f)^\wedge : f \in \mathcal{S}(\mathbf{R}_-)\},$$

and H' denotes the space of all polynomials. For $d \in \mathbf{N}_0$ denote by H_d the subspace of H consisting of all functions $f(t)$ that are $\mathrm{O}(\langle t \rangle^{d-1})$.

1.3 Green operators and Singular Green Operators. A *Green operator of order and type zero* in Boutet de Monvel's calculus on $\mathbf{R}^n_+$ is an operator of the form

$$A = \begin{bmatrix} P_+ + \mathrm{G} & \mathrm{K} \\ \mathrm{T} & \mathrm{S} \end{bmatrix} : \begin{matrix} C^\infty_0(\bar{\mathbf{R}}^n_+) \\ \oplus \\ C^\infty_0(\mathbf{R}^{n-1}) \end{matrix} \longrightarrow \begin{matrix} C^\infty(\bar{\mathbf{R}}^n_+) \\ \oplus \\ C^\infty(\mathbf{R}^{n-1}) \end{matrix},$$

where P is a pseudodifferential operator with the transmission property of order zero, see 1.4, $P_+ = \mathrm{r}^+ P \mathrm{e}^+$, and G is a singular Green operator of order and type zero, i.e. with a symbol kernel in $\tilde{\mathcal{B}}^{-1,0}$, the precise definition being given in 1.5. (For the description of the singular Green operators I am using symbol *kernels* rather than the singular Green *symbols*, because it makes things slightly easier.)
K is a Poisson operator, T a trace operator, and S is a pseudodifferential operator with a symbol in $S^0_{1,0}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1})$.

The most interesting part within this setting, however, is the algebra

$$\mathcal{A} = \{A : A = P_+ + G\}$$

of the elements in the upper left corner, and I will from now on focus on it, omitting the details about Poisson and trace operators. They may be found in [2], [11], or [22].

There are currently several definitions of the transmission property available. The one I will be using is the following

1.4 Definition. A symbol $p \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ has the transmission property if for every $k \in \mathbf{N}_0$,

$$\partial^k_{x_n} p(x', x_n, \xi', \langle \xi' \rangle \xi_n)|_{x_n=0} \in S^m_{1,0}(\mathbf{R}^{n-1}_{x'} \times \mathbf{R}^{n-1}_{\xi'}) \hat{\otimes}_\pi H_{d,\xi_n},$$

where $d = \max\{\mathrm{entier}(m) + 1, 0\}$, cf. [22].

There will be a detailed discussion of various forms of the transmission property, below. Before, however, I would like to give the definition of the singular Green operators and their symbol kernels.

1.5 Definition. Let $\mu \in \mathbf{R}$. The class $\tilde{\mathcal{B}}^{\mu,0}$ consists of all smooth functions g on $\mathbf{R}^{n-1}_{x'} \times \mathbf{R}^{n-1}_{\xi'} \times \mathbf{R}_{+x_n} \times \mathbf{R}_{+y_n}$ (symbol kernels) satisfying the estimates

$$\|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_{\xi'}^{\alpha} D_{x'}^{\beta} g(x',\xi',x_n,y_n)\|_{L^2(\mathbf{R}_{+x_n}\times\mathbf{R}_{+y_n})} = \mathrm{O}(\langle\xi'\rangle^{\mu+1-k+k'-m+m'-|\alpha|}) \qquad (1)$$

for every fixed choice of $k, k', m, m', \alpha, \beta$.
Such a symbol kernel g induces the singular Green operator $\mathrm{Op}_G\, g$ by

$$[\mathrm{Op}_G\, g(f)](x) = (2\pi)^{\frac{n-1}{2}} \iint_0^\infty e^{ix'\xi'} g(x',\xi',x_n,y_n)(\mathcal{F}_{x'\to\xi'}f)(\xi',y_n)\,dy_n\,d\xi', \qquad (2)$$

$f \in \mathcal{S}(\mathbf{R}^n_+)$; g is called the symbol kernel of $\mathrm{Op}_G\, g$.
For fixed x', ξ' let the operator $g(x',\xi',D_n)$ be defined on $\mathcal{S}(\mathbf{R}_+)$ by

$$[g(x',\xi',D_n)f](x_n) = \int_0^\infty g(x',\xi',x_n,y_n)f(y_n)\,dy_n;$$

then

$$\mathrm{Op}_G\, g = \mathrm{Op}'g(x',\xi',D_n),$$

where Op' denotes the usual pseudodifferential action with respect to the x', ξ'–variables for operator-valued symbols.

Let us now have a closer look at the transmission property.

1.6 Remark. In [14], Grubb and Hörmander use the following notation.
(a) An operator P has the *transmission property* for a manifold with boundary, provided it maps functions that are smooth up to the boundary to functions that are again smooth up to the boundary. This obviously is a weaker condition than that in 1.4.
(b) A symbol $p \in S^m_{1,0}(\mathbf{R}^{n-1}_{x'} \times \mathbf{R}_{\xi_n})$ has the *uniform transmission property with respect to* $\mathbf{R}^n_+$ provided that for every $N \in \mathbf{N}_0$, and all multi-indices $\gamma \in \mathbf{N}^n$,

$$x_n^N D_{x'}^{\gamma'} D_{x_n}^{\gamma_n} \mathcal{F}^{-1}_{\xi_n\to x_n} p(x',\xi_n) \text{ is bounded on } \mathbf{R}^n_+.$$

Moreover, $p \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ has the uniform transmission property, provided that for all multi-indices $\alpha, \beta \in \mathbf{N}^n$, $p^{(\alpha)}_{(\beta)}(x',0,0,\xi_n)$ has the uniform transmission property (it suffices to ask this for $\alpha_n = 0, \beta' = 0$).
(a) and (b) are one-sided notions of the transmission property: they only check the behavior of the functions as $x_n \to 0^+$.
(c) In [12] def. 1.7, Grubb has introduced the $\mathcal{H}$-condition:
A symbol $p \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n), m \in \mathbf{Z}$, satisfies the $\mathcal{H}$-condition, if for all multi-indices α, β,

$$D_\xi^\alpha D_x^\beta p(x',0,0,\xi_n) \in S(\mathbf{R}^{n-1}, H_{m+1}).$$

Adapted to the above definition of H, $S(\mathbf{R}^{n-1}, H_{m+1})$ is the space of all functions $f = f(x',t) \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R})$ such that for all β' the standard H-estimates given for $k, r, N \in \mathbf{N}_0$ by

$$\left| D_t^k t^r \left[D_{x'}^{\beta'} f(x',t) - \sum_{m-N<j\le m} s_{j,\beta}(x')t^j \right] \right| \le C(x')\,|t|^{m-k+r-N} \text{ for } |t| \ge 1 \qquad (1)$$

are finite with a continuous function C on $\mathbf{R}^{n-1}$.
She shows in [12] theorem 1.9 that this is equivalent to a version of the transmission property in 1.4 based on the non-uniform symbol classes.
In contrast to (a) and (b), (c), like 1.4, is a two-sided notion: it also takes into account what happens beyond $\{x_n = 0\}$.

The following proposition gives a connection between these notions.

1.7 Proposition. *Let $p \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n), m \in \mathbf{Z}$. Then the following are equivalent*
(i) *For all multi-indices $\alpha, \beta \in \mathbf{N}_0^n$*

$$D_\xi^\alpha D_x^\beta p(x', 0, 0, \pm\xi_n) \in S^{m-|\alpha|}_{1,0,utr}(\mathbf{R}^{n-1} \times \mathbf{R})$$

in the sense of [14], *cf.* 1.6(b).
(ii) *For all $j \in \mathbf{N}_0$, all multi-indices $\alpha' \in \mathbf{N}_0^{n-1}$, and all fixed ξ',*

$$(x', \xi_n) \mapsto D_{\xi'}^{\alpha'} D_{x_n}^j p(x', 0, \xi', \pm\xi_n) \in S^{m-|\alpha'|}_{1,0,utr}(\mathbf{R}^{n-1} \times \mathbf{R}).$$

(iii) *For all $j \in \mathbf{N}_0$, all multi-indices $\alpha' \in \mathbf{N}_0^{n-1}$, and all fixed ξ',*

$$\mathrm{r}^+ \mathcal{F}^{-1}_{\xi_n \to x_n} D_{\xi'}^{\alpha'} D_{x_n}^j p(x', 0, \xi', +\xi_n) \in \mathcal{S}(\mathbf{R}_+, C_b^\infty(\mathbf{R}^{n-1}_{x'})), \text{ and} \qquad (1)$$
$$\mathrm{r}^+ \mathcal{F}^{-1}_{\xi_n \to x_n} D_{\xi'}^{\alpha'} D_{x_n}^j p(x', 0, \xi', -\xi_n) \in \mathcal{S}(\mathbf{R}_+, C_b^\infty(\mathbf{R}^{n-1}_{x'})). \qquad (2)$$

Here, C_b^∞ denotes the smooth functions with bounded derivatives of all orders.
(iv)*For all $j \in \mathbf{N}_0$, all multi-indices $\alpha' \in \mathbf{N}_0^{n-1}$, and all fixed ξ',*

$$D_{\xi'}^{\alpha'} D_{x_n}^j p(x', 0, \xi', \xi_n) \in H_d(\mathbf{R}_{\xi_n}, C_b^\infty(\mathbf{R}^{n-1}_{x'})) = H_d(\mathbf{R}_{\xi_n}) \hat{\otimes}_\pi C_b^\infty(\mathbf{R}^{n-1})$$

with $d = \max\{m - |\alpha'| + 1, 0\}$.
(v) *p satisfies Grubb's $\mathcal{H}$-condition uniformly in x', i.e. the C at the right hand side in* 1.6(1) *is independent of x'.*
(vi) *p has the transmission property in the sense of* 1.4.

In the language of Grubb and Kokholm [15], p has the uniform two-sided transmission property. The signs "+" and "-" reflect the two directions of approaching $\{x_n = 0\}$.

Proof. (i) $\Rightarrow$ (ii) is shown in [14], p. 6.
(ii) $\Rightarrow$ (i). Since $D_{x'}^{\beta'} D_{\xi_n}^{\alpha_n}$ maps $S^{m-|\alpha'|}_{1,0,utr}(\mathbf{R}^{n-1} \times \mathbf{R})$ into $S^{m-|\alpha|}_{1,0,utr}(\mathbf{R}^{n-1} \times \mathbf{R})$, we may simply take $\xi' = 0$.
(ii) $\Leftrightarrow$ (iii). This is a reformulation of the definition in [14]. We have

$$a(x', \xi_n) \in S^m_{1,0,utr}(\mathbf{R}^{n-1} \times \mathbf{R}) \quad \text{iff} \quad \sup_{x', x_n} |\mathrm{r}^+ x_n^N D_{x_n}^j D_{x'}^{\beta'} (\mathcal{F}^{-1}_{\xi_n \to x_n} a)(x', x_n)| < \infty,$$

and this is just the semi-norm system defining the topology of $\mathcal{S}(\mathbf{R}_+, C_b^\infty(\mathbf{R}^{n-1}))$.
(iii) $\Rightarrow$ (iv). $\mathcal{S}(\mathbf{R}_+)$ is a nuclear space and therefore

$$\mathcal{S}(\mathbf{R}_+, C_b^\infty(\mathbf{R}^{n-1})) = \mathcal{S}(\mathbf{R}_+) \hat{\otimes}_\pi C_b^\infty(\mathbf{R}^{n-1}).$$

On the other hand,

$$H_d(\mathbf{R}) = \mathcal{F}\left[\mathrm{e}^+ \mathcal{S}(\mathbf{R}_+) \oplus \mathrm{e}^- \mathcal{S}(\mathbf{R}_-) \oplus \mathbf{C}_d[\delta]\right],$$

where $\mathbf{C}_d[\delta]$) denotes linear combinations of derivatives of δ of degree $< d$. The inverse Fourier transform furnishes a topological isomorphism from $H_d(\mathbf{R})$ to $e^+\mathcal{S}(\mathbf{R}_+) \oplus e^-\mathcal{S}(\mathbf{R}_-) \oplus \mathbf{C}_d[\xi_n]$.
Since all derivatives of p have polynomial growth,

$$\mathcal{F}^{-1}_{\xi_n \to x_n} D^{\alpha'}_{\xi'} D^j_{x_n} p(x', 0, \xi', \xi_n) \in \mathcal{S}'(\mathbf{R}_{\xi_n})$$

for fixed x', ξ'.
By assumption, the singular support of this distribution is in $\{x_n = 0\}$. There, it is a linear combination of derivatives of Dirac's δ of finite order. Both, the coefficients and the order may depend on x' and ξ'.
It is easy to see, however, that the order is $\leq d-1$ and that the coefficients are C^∞_b in x', ξ' being fixed: The Fourier transform of this linear combination of derivatives of δ is the corresponding polynomial in ξ_n. It can be written as the difference of $D^{\alpha'}_{\xi'} D^j_{x_n} p(x', 0, \xi', \xi_n)$, which is $O(\langle \xi_n \rangle^{m-|\alpha'|})$, and the Fourier transforms of the functions in (1) and (2) which are $O(\langle \xi_n \rangle^{-1})$. Therefore the maximal order or, equivalently, the maximal degree of this polynomial is $\leq d-1$, and all coefficients are bounded functions of x'. Derivatives of the coefficients with respect to x' correspond to derivatives of p, and to those we may apply the same argument.
Hence

$$D^{\alpha'}_{\xi'} D^j_{x_n} p(x', 0, \xi', \xi_n) \in \left\{e^+\mathcal{S}(\mathbf{R}_+) \oplus e^-\mathcal{S}(\mathbf{R}_-) \oplus \mathbf{C}_d[\xi_n]\right\} \hat{\otimes}_\pi C^\infty_b(\mathbf{R}^{n-1}_{x'})$$

and we obtain the assertion.
(iv) $\Rightarrow$ (iii) is immediate.
(iv) $\Leftrightarrow$ (v) is just a reformulation of Grubb's definition [12] def. 1.7, uniformly in x', since we may always apply derivatives with respect to ξ_n and x'. Notice that Grubb's space $S(\mathbf{R}^{n-1}, H_d)$ is $H_d(\mathbf{R}) \hat{\otimes}_\pi C^\infty(\mathbf{R}^{n-1})$.
(v) $\Leftrightarrow$ (vi) This is just the calculation made for the proof of Grubb's theorem [12] 1.9, uniformly in x' : While (vi) seems to be the stronger condition, (v) implies that there exist $s_{j,\alpha,\beta} \in S^{m-j-|\alpha|}_{1,0}$ such that

$$\left| \xi_n^r D^\alpha_\xi D^\beta_x p(x', 0, \xi) - \sum_{0 \leq j+r \leq m-|\alpha|+r} s_{j,\alpha,\beta}(x', \xi') \xi_n^{j+r} \right| \leq C_{\alpha,\beta,r} \langle \xi' \rangle^{m+1-|\alpha|+r} \langle \xi \rangle^{-1}, \quad (3)$$

and this is another characterization of the uniform transmission property, cf. [11] definition 2.2.7, [22] section 2.2.2.1, proposition 3.

Finally two more important concepts.

1.8 The ad–notation. For multi-indices $\alpha, \beta \in \mathbf{N}^n_0$ and an operator T acting on functions or distributions on $\mathbf{R}^n$, let

$$\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)T = \mathrm{ad}^{\alpha_1}(-ix_1) \cdots \mathrm{ad}^{\alpha_n}(-ix_n)\mathrm{ad}^{\beta_1}(D_{x_1}) \cdots \mathrm{ad}^{\beta_n}(D_{x_n})T.$$

Here, $\mathrm{ad}^0(-ix_j)T = T$, and $\mathrm{ad}^k(-ix_j)T = [-ix_j, \mathrm{ad}^{k-1}(-ix_j)T], k = 1, 2, \ldots$; the iterated commutators $\mathrm{ad}^{\beta_j}(D_{x_j})T$ are defined correspondingly. Of course, we are assuming for the moment that all compositions involved make sense.

The following lemma is simple but very useful; it follows easily from 1.1(2):

1.9 Lemma. *If $P = \operatorname{Op} p$ is the pseudodifferential operator with the symbol p, then $\operatorname{ad}^\alpha(-ix)\operatorname{ad}^\beta(D_x)P$ is the pseudodifferential operator with the symbol $\partial_\xi^\alpha D_x^\beta p$.*

Wedge Sobolev spaces were introduced by Schulze, cf. [27], section 3.1.

1.10 Definition. (a) For $f \in L^2(\mathbf{R}_+), \lambda > 0$, let

$$(\kappa_\lambda f)(t) = \lambda^{\frac{1}{2}} f(\lambda t). \tag{1}$$

This defines a unitary map

$$\kappa_\lambda : L^2(\mathbf{R}_+) \longrightarrow L^2(\mathbf{R}_+)$$

with respect to the sesquilinear form

$$\langle f, g\rangle = \int_{\mathbf{R}_+} f(t)\bar{g}(t)dt.$$

(b) More generally, let E be a Banach space and suppose that $\{\kappa_\lambda : \lambda \in \mathbf{R}_+\}$ is a strongly continuous group of operators on E, i.e. $\lambda \mapsto \kappa_\lambda \in C(\mathbf{R}_+, \mathcal{L}_\sigma(E))$, and $\kappa_\lambda\kappa_\rho = \kappa_{\lambda\rho}$.
The *wedge Sobolev space* modelled on E, $\mathcal{W}^s(\mathbf{R}^q, E), s \in \mathbf{R}, q \in \mathbf{N}_0$ is defined as the completion of $\mathcal{S}(\mathbf{R}^q, E) = \mathcal{S}(\mathbf{R}^q)\hat{\otimes}_\pi E$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbf{R}^q,E)} = \left(\int \langle\eta\rangle^{2s} \|\kappa_{\langle\eta\rangle^{-1}}\mathcal{F}_{y\to\eta}u(\eta)\|_E^2 d\eta\right)^{\frac{1}{2}}.$$

Here, $\mathcal{F}_{y\to\eta}u$ denotes the Fourier transform of the E-valued function or distribution u,

$$\mathcal{F}_{y\to\eta}u(\eta) = (2\pi)^{-q/2}\int e^{-iy\eta}u(y)dy.$$

(c) For $s, t \in \mathbf{R}$, let

$$\mathcal{W}^{s,t}(\mathbf{R}^q, E) = \{\langle y\rangle^{-t} u : u \in \mathcal{W}^s(\mathbf{R}^q, E)\}.$$

In general, the wedge Sobolev space will depend on the choice of the group action on E. Here, however, we will only deal with the usual weighted Sobolev spaces on $\mathbf{R}_+$, cf. 1.2(b), and we will always use the group defined by (1).
(d) Let $\{E_k : k \in \mathbf{N}\}$ be a sequence of Banach spaces with $E_{k+1} \hookrightarrow E_k, E = \text{proj-lim } E_k$, and suppose that the group action coincides on all spaces. Then

$$\mathcal{W}^{s,t}(\mathbf{R}^q, E) = \text{proj-lim } \mathcal{W}^{s,t}(\mathbf{R}^q, E_k).$$

Vice versa, if $E_k \hookrightarrow E_{k+1}, E = \text{ind-lim } E_k$, and the group action is the same for all spaces, then

$$\mathcal{W}^{s,t}(\mathbf{R}^q, E) = \text{ind-lim } \mathcal{W}^{s,t}(\mathbf{R}^q, E_k).$$

1.11 Remark.
(a) $\mathcal{S}(\mathbf{R}^n_+) = \text{proj-lim}_{\,s,t\to\infty} H^{s,t}(\mathbf{R}^n_+)$.
(b) $\mathcal{S}'(\mathbf{R}^n_+) = \text{ind-lim}_{\,s,t\to\infty} H_0^{-s,-t}(\mathbf{R}^n_+)$.
(c) $\mathcal{W}^s(\mathbf{R}^q, H^s(\mathbf{R}_+)) = H^s(\mathbf{R}^{q+1}_+), s \geq 0$,
(d) $\mathcal{W}^s(\mathbf{R}^q, H^s_0(\mathbf{R}_+)) = H^s_0(\mathbf{R}^{q+1}_+), s \leq 0$.
For (c) and (d), cf. [27], section 3.1.1, (17) and (18).

1.12 Definition and Lemma, cf. [27], section 3.1.2, proposition 10. Let $E = H^{\sigma,\tau}(\mathbf{R}_+)$ for some choice of $\sigma, \tau \in \mathbf{R}$, and let $E' = H_0^{-\sigma,-\tau}(\mathbf{R}_+)$ denote its dual with respect to the extension of the sesquilinear form

$$(u,v)_{E,E'} = \int u(x)\bar{v}(x)dx$$

defined for $u \in C_0^\infty(\bar{\mathbf{R}}_+), v \in C_0^\infty(\mathbf{R}^n)$.
We obtain a natural duality $\mathcal{W}^{s,t}(\mathbf{R}^q, E), \mathcal{W}^{-s,-t}(\mathbf{R}^q, E')$ and a non-degenerate sesquilinear form by

$$\begin{aligned} \langle f, g\rangle_{\mathcal{W}^{s,t}(\mathbf{R}^q,E),\mathcal{W}^{-s,-t}(\mathbf{R}^q,E')} &= \int (\kappa_{\langle\eta\rangle^{-1}}\mathcal{F}_{y\to\eta}f, \kappa_{\langle\eta\rangle^{-1}}\mathcal{F}_{y\to\eta}g)_{E,E'}d\eta \qquad (1)\\ &= \int (\mathcal{F}_{y\to\eta}f, \mathcal{F}_{y\to\eta}g)_{E,E'}d\eta, \end{aligned}$$

noting that the group $\{\kappa_\lambda : \lambda \in \mathbf{R}_+\}$ of 1.10 (a) is unitary with respect to $(\cdot,\cdot)_{E,E'}$ i.e.

$$(\kappa_\lambda u, \kappa_\lambda v)_{E,E'} = (u,v)_{E,E'}.$$

(1) extends the usual Sobolev space sesquilinear form on $\mathbf{R}_+^{q+1}$.

In [26], the following characterization of the singular Green operators in Boutet de Monvel's calculus in terms of the behavior of their iterated commutators has been proven. It was motivated by a result of B.–W. Schulze, [29] theorem 3.1.

1.13 Theorem. *Let $G : \mathcal{S}(\mathbf{R}_+^n) \longrightarrow \mathcal{S}'(\mathbf{R}_+^n)$ be a continuous linear operator. Then the following is equivalent:*
(i) *$G = \mathrm{Op}_G g$ for some $g \in \tilde{\mathcal{B}}^{-1,0}$.*
(ii) *For all multi-indices $\alpha, \beta \in \mathbf{N}_0^{n-1}$, all $s, t \in \mathbf{R}$, the operator $\mathrm{ad}^\alpha(-ix')\mathrm{ad}^\beta(D_{x'})G$ has a continuous extension*

$$\mathrm{ad}^\alpha(-ix')\mathrm{ad}^\beta(D_{x'})G : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, \mathcal{S}'(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s-|\alpha|,t}(\mathbf{R}^{n-1}, \mathcal{S}(\mathbf{R}_+)). \qquad (1)$$

(iii) *G has the mapping properties in (1) for $t = 0$.*

1.14 Definition. Call an operator with the properties in 1.13 a singular Green operator of order and type zero.

2 Boundedness of Pseudodifferential Operators on Sobolev Spaces and the Uniform Transmission Property

The uniform (one-sided) transmission property can be characterized by the boundedness of commutators on Sobolev spaces over $\mathbf{R}_+^n$. This is the contents of the following theorem, the main result in this section. The proof relies on an argument given by Grubb and Hörmander [14] in order to characterize the weaker form of the transmission property presented in 1.6.

2.1 Theorem. *Let $m \in \mathbf{Z}, p \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n), P = \operatorname{Op} p, P_+ = r^+ P e^+$, and assume that for all multi-indices $\alpha', \beta' \in \mathbf{N}_0^{n-1}$ and all $s \geq 0$*

$$\operatorname{ad}^{\alpha'}(-ix')\operatorname{ad}^{\beta'}(D_{x'})P_+ : H^s(\mathbf{R}^n_+) \longrightarrow H^{s-m-|\alpha'|}(\mathbf{R}^n_+) \tag{1}$$

is bounded.
Then p has the uniform transmission property with respect to $\mathbf{R}^n_+$, i.e. for all $\alpha, \beta \in \mathbf{N}_0^n$

$$p^{(\alpha)}_{(\beta)}(x', 0, 0, \xi_n) \in S^{m-|\alpha|}_{1,0,utr}(\mathbf{R}^{n-1} \times \mathbf{R}). \tag{2}$$

2.2 Reduced hypothesis. We may replace 2.1(1) by the following weaker assumption: *For every choice of $\alpha \in \mathbf{N}_0^n$ there is a $K \in \mathbf{N}$, an $\tilde{m} \in \mathbf{Z}$, and an $s_0 >> 0$ such that*

$$\operatorname{ad}^{\alpha}(-ix)P_+ : H^s_{[K]}(\mathbf{R}^n_+) \longrightarrow H^{s-\tilde{m}}(\mathbf{R}^n_+) \tag{1}$$

is continuous for all $s \geq s_0$.

Here, $H^s_{[K]}(\mathbf{R}^n_+)$ denotes all functions in $H^s(\mathbf{R}^n_+)$ which have a zero of order $\geq K$ at $x_n = 0$. Property 2.2(1) is indeed weaker than 2.1(1), although we have $\alpha \in \mathbf{N}^n$: For our purposes we may replace P by the operator $M_\phi P M_\phi$, where M_ϕ denotes multiplication with a smooth function $\phi = \phi(x_n)$, vanishing outside $\{|x_n| < 1\}$ and $\equiv 1$ near zero. Therefore multiplication with x_n from the left or right equals multiplication with a bounded smooth function.

The proof will be given in a series of steps.

2.3 Remark. From [14] we know already that $p^{(\alpha)}_{(\beta)}(x', 0, 0, \xi_n) \in S^{m-|\alpha|}_{1,0,tr}(\mathbf{R}^{n-1} \times \mathbf{R})$, where tr stands for the transmission property in the sense of 1.6(a).

2.4 Reduction. It suffices to show 2.1(2) for $\alpha = \beta = 0$: For $K > \beta_n$ and $u \in H^s_{[K]}(\mathbf{R}^n_+)$

$$[\operatorname{Op} p^{(\alpha)}_{(\beta)}]_+ u = [\operatorname{ad}^{\alpha}(-ix)\operatorname{ad}^{\beta}(D_x)P]_+ u = \operatorname{ad}^{\beta}(D_x)[\operatorname{ad}^{\alpha}(-ix)P]_+ u. \tag{1}$$

So $[\operatorname{Op} p^{(\alpha)}_{(\beta)}]_+$ also satisfies 2.2(1) with $\tilde{m}$ replaced by $\tilde{m} + |\beta|$, and the task is the same.

2.5 Reduction. We only have to show that

$$\mathcal{F}^{-1}_{\xi_n \to x_n} p(x', 0, 0, \xi_n) \text{ is bounded on } \mathbf{R}^n_+. \tag{1}$$

Proof. Basically, we would have to show that for arbitrary $N \in \mathbf{N}$, β' and γ,

$$x_n^N D^{\beta'}_{x'} D^{\gamma}_{x_n} \mathcal{F}^{-1}_{\xi_n \to x_n} p(x', 0, 0, \xi_n) \text{ is bounded on } \mathbf{R}^n_+. \tag{2}$$

Expression (2) however equals $\mathcal{F}^{-1}_{\xi_n \to x_n}(-D_{\xi_n})^N \xi_n^\gamma D^{\beta'}_{x'} p(x', 0, 0, \xi_n)$, and

$$q(x, \xi) = (-D_{\xi_n})^N \xi_n^\gamma D^{\beta'}_{x'} p \in S^{m-N+\gamma}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n).$$

Moreover, for $u \in H^s_{[K]}$, K large, $[\operatorname{Op} q]_+ u = [\operatorname{ad}^N(x_n)\operatorname{ad}^{\beta'}(D_{x'})P]_+ \circ D^\gamma_{x_n} u$ has property 2.2(1), and we are in the same situation as before.

2.6 Observation. 2.5(1) is easily established when $m < -1$, for then

$$\mathcal{F}^{-1}_{\xi_n \to x_n} p(x', 0, 0, \xi_n) = \frac{1}{2\pi} \int e^{ix_n\xi_n} p(x', 0, 0, \xi_n) d\xi_n$$

is a convergent integral; the integrand is $O(\langle \xi_n \rangle^m)$.

2.7 Reduction. Let us therefore assume that we have already shown the boundedness of $\mathcal{F}^{-1}_{\xi_n \to x_n} q(x', 0, 0, \xi_n)$ on $\mathbf{R}^n_+$ for all $q \in S^{\mu}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, where $[\operatorname{Op} q]_+$ has property 2.2(1) and $m > \mu \in \mathbf{Z}$.

2.8 Definition. Choose $0 \neq v \in C_0^\infty(\mathbf{R}^{n-1})$. For $t \in \mathbf{R}^{n-1}$ let $v_t(x') = v(x' - t)$. Moreover, let M_{v_t} denote the operator of multiplication with v_t. By p_t denote the symbol of $PM_{v_t} =: P_t$.

2.9 Reduction. It is sufficient to show that

$$|\mathcal{F}^{-1}_{\xi_n \to x_n} p_t(x', 0, 0, \xi_n)| \leq C, \text{ independent of } t. \tag{1}$$

Proof. P_t has the double symbol $q_t(x, y, \xi) = p(x, \xi) v_t(y')$. The calculus gives

$$p_t(x, \xi) = v_t(x') p(x, \xi) + \sum_{0 < |\alpha'| < N} \frac{1}{\alpha'!} \partial^{\alpha'}_{\xi'} p(x, \xi) D^{\alpha'}_{x'} v_t(x') + r_{Nt}(x', 0, 0, \xi_n),$$

with $r_{Nt}(x, \xi) \in S^{m-N}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, uniformly in t, since $v_t \in S^0_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, uniformly in t. Therefore,

$$v_t(x') \mathcal{F}^{-1}_{\xi_n \to x_n} p(x', 0, 0, \xi_n) = \mathcal{F}^{-1}_{\xi_n \to x_n} p_t(x', 0, 0, \xi_n)$$

$$- \sum_{0 < |\alpha'| < N} \frac{1}{\alpha'!} \partial^{\alpha'}_{\xi'} \mathcal{F}^{-1}_{\xi_n \to x_n} p(x', 0, 0, \xi_n) D^{\alpha'}_{x'} v_t(x') - \mathcal{F}^{-1}_{\xi_n \to x_n} r_{Nt}(x, \xi),$$

$$=: f_1 + f_2 + f_3.$$

By assumption, f_1 is uniformly bounded in t. Since $\partial^{\alpha'}_{\xi'} p$ is of order lower than m, and since $[\operatorname{Op} \partial^{\alpha'}_{\xi'} p]_+ = [\operatorname{ad}^{\alpha'}(-ix')P]_+$ has property 2.2(1), f_2 is bounded by reduction 2.7. Finally, f_3 is uniformly bounded for $m - N < -1$ by the argument in 2.6. Together, we get 2.5(1).

2.10 Definition. For $\sigma \in \mathcal{S}(\mathbf{R})$, vanishing of order $\geq K$ in zero, define $u_t = v_t \otimes \sigma$, $b_t = \mathcal{F}_{x_n \to \xi_n} P e^+ (u_t)_+$ with $(u_t)_+ = r^+ u_t$ as in [14].

2.11 Lemma. $b_t \in S^{m-1}_{1,0,utr}(\mathbf{R}^{n-1} \times \mathbf{R})$, *uniformly in* t.

Proof. We know that $b_t \in S^{m-1}_{1,0}(\mathbf{R}^{n-1} \times \mathbf{R})$ uniformly, cf. [14],(1.4). So we have to consider

$$x_n^N D_x^\beta \mathcal{F}^{-1}_{\xi_n \to x_n} b_t(x', \xi_n) = x_n^N D_x^\beta P e^+ (u_t)_+,$$

for given N, β. It is bounded for the following reasons

(i) we may neglect x_n^N, since $p(x,\xi)=0$ for $|x_n|\geq 1$, and

(ii) For arbitrary s, $(u_t)_+ \in H^s_{[K]}(\mathbf{R}^n_+)$, uniformly in t, since the H^s-norm is translation invariant. Hence $Pe^+(u_t)_+ \in H^{s-\tilde{m}}(\mathbf{R}^n_+)$, uniformly in t, and this implies the boundedness of the derivatives of order β for sufficiently large s.

From now on follow closely the proof of theorem 1.5 in [14]

2.12 Definition. Fix $\sigma=\sigma_k \in C_0^\infty(\mathbf{R})$ equal to $x_n^{k-1}e^{x_n}/(k-1)!$ for $|x_n|<1$, and let $b_{tk}=\mathcal{F}_{x_n\to\xi_n}Pe^+(v_t\otimes\sigma_k)_+, k=1,2,\ldots$.

2.13 Lemma. *For fixed k, $\{b_{tk}: t\in\mathbf{R}^{n-1}\}$ is a bounded subset of $S^{m-k}_{1,0,utr}(\mathbf{R}^{n-1}\times\mathbf{R})$.*

Proof. By the argument in [14], $b_{tk}\in S^{m-k}_{1,0,utr}(\mathbf{R}^{n-1}\times\mathbf{R})$ for each t. Why uniformly? Obviously,

$$\mathcal{F}_{x_n\to\xi_n}e^+(v_t\otimes\sigma_k)_+ = v_t\otimes\mathcal{F}_{x_n\to\xi_n}e^+(\sigma_k)_+.$$

From the argument of [14], specialized to $n=1$, we know that $\mathcal{F}_{x_n\to\xi_n}e^+(\sigma_k)_+\in S^{-k}_{1,0}(\mathbf{R}_{\xi_n})$. Since $v_t\in S^0_{1,0}(\mathbf{R}^{n-1}\times\mathbf{R})$, uniformly in t, we have

$$v_t\otimes\mathcal{F}_{x_n\to\xi_n}e^+(\sigma_k)_+\in S^{-k}_{1,0}(\mathbf{R}^{n-1}_{x'}\times\mathbf{R}_{\xi_n}),$$

uniformly, thus $b_{tk}\in S^{m-k}_{1,0}(\mathbf{R}^{n-1}\times\mathbf{R})$, uniformly. Already in Lemma 2.11 we saw that the derivatives of $\mathcal{F}^{-1}_{\xi_n\to x_n}b_{tk}$ are uniformly bounded, so we obtain the assertion.

2.14 Rewriting b_{tk}. Noting that $b_{tk}=\mathcal{F}_{x_n\to\xi_n}Pe^+(v_t\otimes\sigma_k)_+=\mathcal{F}_{x_n\to\xi_n}P_te^+(1\otimes\sigma_k)_+$, write for a positive integer $M>|m|+1$

$$(2\pi)^{\frac{1}{2}}(i\xi_n-1)^k b_{tk}(x',\xi_n)=$$

$$=(i\xi_n-1)^k\sum_{j=0}^{M}\frac{1}{j!}(-iD_{x_n}D_{\xi_n})^j(p_t(x,\xi)(i\xi_n-1)^{-k})|_{x_n=0,\xi'=0}+r_{tk}(x',\xi_n),$$

where $r_{tk}\in S^{m-M}_{1,0}$, uniformly in t. Like in [14] rewrite this as

$$=\sum_{j=0}^{M}\sum_{\nu=0}^{j}(iD_{x_n})^j(-D_{\xi_n})^{j-\nu}p_t(x',0,0,\xi_n)k\cdots(k+\nu-1)(i\xi_n-1)^{-\nu}/\nu!(j-\nu)! \quad (1)$$
$$+r_{tk}(x',\xi_n).$$

2.15 Conclusion. Denote the sum 2.14(1) by $q(x',\xi_n,k)$. It is a polynomial in k of degree $\leq M$. Choose $M+1$ different large k's: $k_1,\ldots,k_{M+1}$; then for $\kappa\in\mathbf{C}$, Lagrange's interpolation formula gives

$$\begin{aligned} q(x',\xi_n,\kappa) &= \sum_{j=1}^{M+1}q(x',\xi_n,k_j)L_j(\kappa)\\ &= \sum_{j=1}^{M+1}[(2\pi)^{\frac{1}{2}}(i\xi_n-1)^{k_j}b_{t,k_j}(x',\xi_n)-r_{tk_j}(x',\xi_n)]L_j(\kappa)\\ &\in S^m_{1,0,utr}(\mathbf{R}^{n-1}\times\mathbf{R})+S^{m-M}_{1,0}(\mathbf{R}^{n-1}\times\mathbf{R}),\end{aligned}$$

uniformly in t. Here, L_j are the Lagrange interpolation polynomials as in [14]. Letting $\kappa = 0$ we obtain

$$\sum_{j=0}^{M}(-iD_{x_n}D_{\xi_n})^j p_t(x',0,0,\xi_n)/j! \in S^m_{1,0,utr}(\mathbf{R}^{n-1}\times\mathbf{R}) + S^{m-M}_{1,0}(\mathbf{R}^{n-1}\times\mathbf{R}),$$

uniformly in t. The same argument applies to $(-iD_{x_n}D_{\xi_n})^l p_t(x',0,0,\xi_n), l = 0,\dots M$: by making K larger, we also have 2.2(1) for $[\mathrm{Op}\,(-iD_{x_n}D_{\xi_n})^l p_t(x,\xi)]_+, l = 0,\dots,M$. So

$$\sum_{j=0}^{M}(-iD_{x_n}D_{\xi_n})^{j+l} p_t(x',0,0,\xi_n)/j! \in S^m_{1,0,utr}(\mathbf{R}^{n-1}\times\mathbf{R}) + S^{m-M}_{1,0}(\mathbf{R}^{n-1}\times\mathbf{R}),$$

uniformly in t. Multiply by $(-1)l/l!$, and sum for $l = 0,\dots M$. Then

$$\sum_{j=0}^{M}\sum_{l=0}^{M}\frac{(-1)^l}{j!l!}(-iD_{x_n}D_{\xi_n})^{j+l} p_t(x',0,0,\xi_n) \in S^m_{1,0,utr}(\mathbf{R}^{n-1}\times\mathbf{R}) + S^{m-M}_{1,0}(\mathbf{R}^{n-1}\times\mathbf{R}), \quad (1)$$

uniformly in t. The left hand side of (1), however, equals

$$p_t(x',0,0,\xi_n) + \sum_{0\le j,l\le M,\, j+l>M}\frac{(-1)^l}{j!l!}(-iD_{x_n}D_{\xi_n})^{j+l} p_t(x',0,0,\xi_n) \quad (2)$$

The second summand in (2) belongs to $S^{m-M}_{1,0}(\mathbf{R}^{n-1}\times\mathbf{R})$, uniformly, hence

$$p_t(x',0,0,\xi_n) \in S^m_{1,0,utr}(\mathbf{R}^{n-1}\times\mathbf{R}) + S^{m-M}_{1,0}(\mathbf{R}^{n-1}\times\mathbf{R}),$$

uniformly. Remembering that $m - M < -1$, we conclude that

$$\mathcal{F}^{-1}_{\xi_n\to x_n} p_t(x',0,0,\xi_n)$$

is bounded on $\mathbf{R}^n_+$, uniformly in t. This is what we had to show by reduction 2.9.

3 Characterization of the Pseudodifferential Operators with the Transmission Property

In this section, we shall prove the following theorem.

3.1 Theorem. *Let $P : \mathcal{S}(\mathbf{R}^n) \longrightarrow \mathcal{S}'(\mathbf{R}^n)$ be a continuous operator. Then the following assertions are equivalent.*

(i) $P = \mathrm{Op}\,p$ *for some* $p \in S^0_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)$ *with the transmission property of* 1.4.

(ii) *P has the following properties*

(α) *for all multi-indices $\alpha, \beta \in \mathbf{N}^n$ and all $s \in \mathbf{R}$, $\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)P$ has a bounded extension*

$$\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)P : H^s(\mathbf{R}^n) \longrightarrow H^{s+|\alpha|}(\mathbf{R}^n).$$

(β) *for all multi-indices* $\alpha', \beta' \in \mathbf{N}^{n-1}$, *all* $s, t, \sigma, \tau \in \mathbf{R}$, $\mathrm{ad}^{\alpha'}(-ix')\mathrm{ad}^{\beta'}(D_{x'})P_+$ *has a bounded extension*

$$\mathrm{ad}^{\alpha'}(-ix')\mathrm{ad}^{\beta'}(D_{x'})P_+ : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, H^{\sigma,\tau}_{\{0\}}(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s+|\alpha'|,t}(\mathbf{R}^{n-1}, H^{\sigma+|\alpha'|,\tau}_{\{0\}}(\mathbf{R}_+)).$$

(γ) *The properties in* (β) *also hold for the formal adjoint* ${P_+}^* = {P^*}_+$ *of* P.

Here, $H^{\tilde\sigma,\tilde\tau}_{\{0\}}(\mathbf{R}_+)$ *denotes the space* $H^{\tilde\sigma,\tilde\tau}(\mathbf{R}_+)$ *for* $\sigma \geq 0$ *and the space* $H^{\tilde\sigma,\tilde\tau}_0(\mathbf{R}_+)$ *for* $\sigma \leq 0$.

(iii) P *has the mapping properties in* (ii) *for the unweighted wedge Sobolev spaces, i.e. for* $t = 0$.

3.2 Remark. We know from Beals' theorem that condition (ii.α) ensures that P is a pseudodifferential operator with a symbol in $S^0_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$.
Of course, we might start with this assumption and omit (ii.α). The present form has been chosen in view of [9], where I want the complete characterization by commutators.

Proof of theorem 3.1. Let us first check that the conditions in (iii) imply (i): By 3.2 we only have to make sure that the symbol of P has the transmission property. By 1.7 this is equivalent to showing that for all α, β, $D^\alpha_\xi D^\beta_x p(x', 0, 0, \pm\xi_n)$ has the uniform transmission property with respect to $\mathbf{R}^n_+$; in other words, p has the uniform two-sided transmission property.
Using reflection and the pseudodifferential calculus, this in turn requires that both, P and P^* have the uniform transmission property with respect to $\mathbf{R}^n_+$, cf. [14] corollary 1.8. In theorem 2.1, the uniform transmission property has been characterized in terms of the boundedness of

$$\mathrm{ad}^{\alpha'}(-ix')\mathrm{ad}^{\beta'}(D_{x'})P_+ : H^s(\mathbf{R}^n_+) \longrightarrow H^{s-m-|\alpha'|}(\mathbf{R}^n_+)$$

for all multi-indices $\alpha', \beta' \in \mathbf{N}^{n-1}$ and all $s \geq 0$.
Applying remark 1.11 we know that $\mathcal{W}^s(\mathbf{R}^{n-1}, H^s(\mathbf{R}_+)) = H^s(\mathbf{R}^n_+)$; therefore the properties (iii) are sufficient.

For the proof of the converse direction we only have to show that the commutators have the mapping properties in (ii.β) and (ii.γ). This part will be split up in a series of lemmas. In the following, I shall write $q \in S^m_{1,0,tr}(\mathbf{R}^n \times \mathbf{R}^n)$ to indicate that q has the transmission property of 1.4. We start with a technical result.

3.3 Reduction of the order. Cf. also [13]. Let $\chi \in \mathcal{S}(\mathbf{R})$ with $\mathrm{supp}\mathcal{F}^{-1}\chi \subseteq \mathbf{R}_-$, $\chi(0) = 1$, and $a \gg 0$. For $\mu \in \mathbf{Z}$ let

$$\begin{aligned} r^\mu_-(\xi) &= \left(\chi\left(\frac{\xi_n}{a\langle\xi'\rangle}\right)\langle\xi'\rangle - i\xi_n\right)^\mu, \\ r^\mu_+(\xi) &= \bar{r}^\mu_-(\xi). \end{aligned}$$

The definition makes sense, since $r_\pm(\xi)/(\langle\xi'\rangle \pm i\xi_n) = 1 + r(\xi)$, where $|r| = O(a^{-1})$.
For sufficiently large a,

$$[R^\mu_+]_+ := [\mathrm{Op}\, r^\mu_+]_+ : \mathcal{W}^s(\mathbf{R}^{n-1}, H^{\sigma,\tau}_0(\mathbf{R}_+)) \to \mathcal{W}^{s-\mu}(\mathbf{R}^{n-1}, H^{\sigma-\mu,\tau}_0(\mathbf{R}_+))$$

is a bounded operator for every choice of $s, \sigma, \tau \in \mathbf{R}$. The operator

$$[R_-^\mu]_+ := [\mathrm{Op}\, r_-^\mu]_+ : \mathcal{W}^s(\mathbf{R}^{n-1}, H^{\sigma,\tau}(\mathbf{R}_+)) \to \mathcal{W}^{s-\mu}(\mathbf{R}^{n-1}, H^{\sigma-\mu,\tau}(\mathbf{R}_+))$$

is bounded, provided $\sigma > -\frac{1}{2}$. Here, I consider e^+ a trivial action on $H_0^{\sigma,\tau}(\mathbf{R}_+)$. The first operator is an isomorphism, $([R_+^\mu]_+)^{-1} = [R_+^{-\mu}]_+$; if additionally $\sigma > \mu - \frac{1}{2}$, then also the second is an isomorphism, and $([R_-^\mu]_+)^{-1} = [R_-^\mu]_+$.

Proof. Since $r_\pm^\mu$ are x−independent symbols, it is sufficient to show that they are operator-valued symbols on the wedge spaces, cf. [27] section 3.2.1, lemma 7, i.e.

$$\|\kappa_{\langle\xi'\rangle^{-1}} D_{\xi'}^\alpha [\mathrm{Op}_{x_n} r_-^\mu(\xi)]_+ \kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(H^{\sigma,\tau}(\mathbf{R}_+), H^{\sigma-\mu,\tau}(\mathbf{R}_+))} = \mathrm{O}(\langle\xi'\rangle^{\mu-|\alpha|}), \tag{1}$$

and

$$\|\kappa_{\langle\xi'\rangle^{-1}} D_{\xi'}^\alpha [\mathrm{Op}_{x_n} r_+^\mu(\xi)]_+ \kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(H_0^{\sigma,\tau}(\mathbf{R}_+), H_0^{\sigma-\mu,\tau}(\mathbf{R}_+))} = \mathrm{O}(\langle\xi'\rangle^{\mu-|\alpha|}). \tag{2}$$

First let $\alpha = 0$ and notice that

$$\kappa_{\langle\xi'\rangle^{-1}} D_{\xi'}^\alpha [\mathrm{Op}_{x_n} r_\pm^\mu(\xi)]_+ \kappa_{\langle\xi'\rangle} = [\mathrm{Op}_{x_n} r_\pm^\mu(\xi', \langle\xi'\rangle\, \xi_n)]_+ = \langle\xi'\rangle^\mu\, [\mathrm{Op}_{x_n} r_\pm^\mu(0,\xi_n)]_+. \tag{3}$$

Let us show (1). For abbreviation write $R = \mathrm{Op}_{x_n} r_-^\mu(0,\xi_n)$. For $\tau = 0$, the fact that $r_-^\mu(0,\xi_n)$ is a symbol with the transmission property implies that

$$R_+ : H^{\sigma,0}(\mathbf{R}_+) \to H^{\sigma-\mu,0}(\mathbf{R}_+)$$

is continuous, provided $\sigma > -\frac{1}{2}$. So let τ be arbitrary. By interpolation, we may assume that $\tau \in 2\mathbf{Z}$. Clearly,

$$R_+ : H^{\sigma,\tau}(\mathbf{R}_+) \to H^{\sigma-\mu,\tau}(\mathbf{R}_+) \tag{4}$$

is bounded iff

$$\langle x_n\rangle^\tau\, R_+\, \langle x_n\rangle^{-\tau} : H^{\sigma,0}(\mathbf{R}_+) \to H^{\sigma,0}(\mathbf{R}_+)$$

is. On the other hand, either $\langle x_n\rangle^\tau$ or $\langle x_n\rangle^{-\tau}$ is a polynomial. Without loss of generality assume the former is. Using the identity

$$x_n R_+ = (x_n R)_+ = (R x_n)_+ + [x_n, R]_+ = R_+ x_n + \mathrm{Op}_{x_n}(-D_{\xi_n} r_-^\mu)(0,\xi_n)$$

we may move factors x_n^k to the other side. Since $x_n^k\, \langle x_n\rangle^{-\tau}$ – as a multiplication operator – is bounded on $H^{\sigma,0}(\mathbf{R}_+)$, the boundedness of (4) follows from the fact that $(-D_{\xi_n} r_-^\mu)(0,\xi_n)$ also has the transmission property and therefore maps $H^{\sigma,0}(\mathbf{R}_+)$ to $H^{\sigma-\mu,0}(\mathbf{R}_+)$. This shows the case $\alpha = 0$ because of the factor $\langle\xi'\rangle$ in (3).
Let $1 \le j \le n-1$. Then

$$\partial_{\xi_j} r_\pm^\mu(\xi) = \mu\; r_\pm^{\mu-1}(\xi) \frac{\xi_j}{\langle\xi'\rangle} \left[\chi'(\frac{\xi_n}{a\,\langle\xi'\rangle})\frac{\xi_n}{a\,\langle\xi'\rangle} + \chi(\frac{\xi_n}{a\,\langle\xi'\rangle})\right]. \tag{5}$$

Therefore

$$\begin{aligned}
\kappa_{\langle\xi'\rangle^{-1}} [\mathrm{Op}_{x_n} \partial_{\xi_j} r_\pm^\mu]_+ \kappa_{\langle\xi'\rangle} &= \mathrm{Op}_{x_n} \left[\mu\; r^{\mu-1}(\xi', \langle\xi'\rangle\, \xi_n) \frac{\xi_j}{\langle\xi'\rangle} \left(\chi'(\frac{\xi_n}{a})\frac{\xi_n}{a} + \chi(\frac{\xi_n}{a})\right)\right]_+ \qquad (6) \\
&= \mu\, \langle\xi'\rangle^{\mu-1}\, \frac{\xi_j}{\langle\xi'\rangle} \left[\mathrm{Op}_{x_n} r^{\mu-1}(0,\xi_n) \left(\chi'(\frac{\xi_n}{a})\frac{\xi_n}{a} + \chi(\frac{\xi_n}{a})\right)\right]_+ \\
&=: \mu\, \langle\xi'\rangle^{\mu-1}\, \frac{\xi_j}{\langle\xi'\rangle} \tilde{R}_+
\end{aligned}$$

The functions $\chi'(\frac{\xi_n}{a})\frac{\xi_n}{a}$ and $\chi(\frac{\xi_n}{a})$ are rapidly decreasing, so have the transmission property. The same argument as before then shows that

$$\tilde{R}_+ : H^{\sigma,\tau}(\mathbf{R}_+) \to H^{\sigma-\mu+1,\tau}(\mathbf{R}_+)$$

is bounded. Together with the factor $\langle\xi'\rangle^{\mu-1}$, this yields the assertion for $|\alpha| = 1$.
The computations in (5) and (6) show the behavior of the higher order derivatives. The argument for r_+^μ is the same.
Now for the statement that these operators define isomorphisms. As a function of $\xi_n, r_-^{\pm\mu}$ belongs to H^- while $r_+^{\pm\mu}$ is the sum of a polynomial and a function in H^+. For $u \in C_0^\infty(\mathbf{R}_+)$ we therefore have $\mathrm{r}^+\mathrm{Op}_{x_n} r_+^{\pm\mu} u = 0$ on $\bar{\mathbf{R}}_-$. For $u \in \mathcal{S}(\mathbf{R})$ we have $\mathrm{r}^+\mathrm{Op}_{x_n} r^{\pm\mu}(f - \mathrm{e}^+\mathrm{r}^+ f) = 0$. Hence

$$[\mathrm{Op}_{x_n} r_\pm^\mu]_+ [\mathrm{Op}_{x_n} r_\pm^\mu]_+ = \mathrm{Op}_{x_n} r_\pm^\mu \mathrm{Op}_{x_n} r_\pm^\mu]_+ = id. \tag{7}$$

Here, id denotes the identity on $H^{\sigma,\tau}(\mathbf{R}_+)$ for the sign " $-$ " and the identity on $H_0^{\sigma,\tau}(\mathbf{R}_+)$ for the sign " $+$ ".

3.4 Corollary. Let $s, t, \sigma, \tau \in \mathbf{R}$. Then

$$[R_+^\mu]_+ = [\mathrm{Op}\, r_+^\mu]_+ : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, H_0^{\sigma,\tau}(\mathbf{R}_+)) \to \mathcal{W}^{s-\mu,t}(\mathbf{R}^{n-1}, H_0^{\sigma-\mu,\tau}(\mathbf{R}_+))$$

is a topological isomorphism, and so is

$$[R_-^\mu]_+ = [\mathrm{Op}\, r_-^\mu]_+ : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, H^{\sigma,\tau}(\mathbf{R}_+)) \to \mathcal{W}^{s-\mu,t}(\mathbf{R}^{n-1}, H^{\sigma-\mu,\tau}(\mathbf{R}_+)),$$

provided $\sigma > -\frac{1}{2}, \sigma - \mu > -\frac{1}{2}$.

Proof. Consider r_-^μ. By interpolation,it is sufficient to assume $t \in 2\mathbf{Z}$ and to show the boundedness of

$$\langle x'\rangle^t [R_-^\mu]_+ \langle x'\rangle^{-t} : \mathcal{W}^s(\mathbf{R}^{n-1}, H^{\sigma,\tau}(\mathbf{R}_+)) \to \mathcal{W}^{s-\mu}(\mathbf{R}^{n-1}, H^{\sigma-\mu,\tau}(\mathbf{R}_+)).$$

Now either $\langle x'\rangle^t$ or $\langle x'\rangle^{-t}$ is a polynomial. For $1 \le j \le n-1$ we can use the commutator identity

$$[x_j, (R_-^\mu)_+] = ([x_j, R_-^\mu])_+ = (\mathrm{Op}\,(-D_{\xi_j} r_-^\mu))_+,$$

like in the proof of 3.3.

3.5 Lemma. *In order to show that the iterated commutators in theorem* 3.1 *have the mapping properties in* (ii.β) *and* (ii.γ) *it is sufficient to establish the following:*
(i) *For all* $p \in S^0_{1,0,tr}(\mathbf{R}^n \times \mathbf{R}^n)$ *and* $M \in \mathbf{N}_0$,

$$[\mathrm{Op}\, p]_+ : \mathcal{W}^0(\mathbf{R}^{n-1}, H^{(M,0)}(\mathbf{R}_+)) \longrightarrow \mathcal{W}^0(\mathbf{R}^{n-1}, H^{(M,0)}(\mathbf{R}_+))$$

is bounded, and
(ii) *for all* $p \in S^0_{1,0,tr}(\mathbf{R}^n \times \mathbf{R}^n)$ *and* $M \in \mathbf{N}_0$,

$$[\mathrm{Op}\, p]_+ : \mathcal{W}^0(\mathbf{R}^{n-1}, H^{(0,M)}(\mathbf{R}_+)) \longrightarrow \mathcal{W}^0(\mathbf{R}^{n-1}, H^{(0,M)}(\mathbf{R}_+))$$

is bounded.

Proof. *Step* 1. Property 3.1(i) implies property 3.1(ii.α), since

$$\mathrm{ad}^{\alpha}(-ix)\mathrm{ad}^{\beta}(D_x)P = \mathrm{Op}\,(\partial_\xi^\alpha D_x^\beta p).$$

Step 2. It is now enough to prove 3.1(ii.β), since the symbol of the adjoint also belongs to $S^0_{1,0,tr}(\mathbf{R}^n \times \mathbf{R}^n)$.

Step 3. We may assume that $\alpha' = \beta' = 0$ in 3.1(ii.β):

$$\mathrm{ad}^{\alpha'}(-ix')\mathrm{ad}^{\beta'}(D_{x'})P_+ = \left[\mathrm{Op}\,(\partial_{\xi'}^{\alpha'} D_{x'}^{\beta'} p)\right]_+,$$

and $q = \partial_{\xi'}^{\alpha'} D_{x'}^{\beta'} p \in S^{-|\alpha'|}_{1,0,tr}$. For $\sigma \geq 0$, lemma 3.4 shows that

$$[\mathrm{Op}\,q]_+ : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, H^{\sigma,\tau}(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s+|\alpha'|,t}(\mathbf{R}^{n-1}, H^{\sigma+|\alpha'|,\tau}(\mathbf{R}_+)).$$

is bounded iff $[R_-^{|\alpha'|}]_+\,[\mathrm{Op}\,q]_+$ is bounded on $\mathcal{W}^{s,t}(\mathbf{R}^{n-1}, H^{\sigma,\tau}(\mathbf{R}_+))$.
The fact that $r_-^{|\alpha'|}(\xi)$ belongs to H^- as a function of ξ_n implies that $[R_-^{|\alpha'|}]_+[\mathrm{Op}\,q]_+ = [R_-^{|\alpha'|}\mathrm{Op}\,q]_+$. Since $R_-^{|\alpha'|}\mathrm{Op}\,q$ is a pseudodifferential operator of order zero with the transmission property we obtain the assertion for $\sigma \geq 0$.
If $\sigma < 0$ and $\sigma + |\alpha'| \leq 0$, then by 3.4 the boundedness of

$$[\mathrm{Op}\,q]_+ : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, H_0^{\sigma,\tau}(\mathbf{R}_+)) \to \mathcal{W}^{s+|\alpha'|,t}(\mathbf{R}^{n-1}, H_0^{\sigma+|\alpha'|,\tau}(\mathbf{R}_+))$$

is equivalent to the boundedness of $[R_+^{|\alpha'|}]_+[\mathrm{Op}\,q]_+$ on $\mathcal{W}^{s,t}(\mathbf{R}^{n-1}, H_0^{\sigma,\tau}(\mathbf{R}_+))$. Now $[R_+^{|\alpha'|}]_+$ $[\mathrm{Op}\,q]_+ = [R_+^{|\alpha'|}\mathrm{Op}\,q]_+ + G$, where G is a singular Green operator of order and type zero, cf. definition 1.14. As before, $R_+^{|\alpha'|}\mathrm{Op}\,q$ is a pseudodifferential operator with the transmission property. Since G is bounded on $\mathcal{W}^{s,t}(\mathbf{R}^{n-1}, H_0^{\sigma,\tau}(\mathbf{R}_+))$ by 1.13, we obtain the assertion also for this case.
Finally let $\sigma \in \mathbf{Z}$ with $\sigma < 0 < \sigma + |\alpha'| =: \tilde{\sigma}$. Using that $H_0^{0,\tau}(\mathbf{R}_+) \cong H^{0,\tau}(\mathbf{R}_+)$ it follows from 3.4 that the composition

$$[R_+^{-\sigma}]_+[R_-^{\tilde{\sigma},\tau}]_+ : \mathcal{W}^{s+|\alpha'|,t}(\mathbf{R}^{n-1}, H^{\tilde{\sigma}}(\mathbf{R}_+)) \to \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, H_0^{\sigma,\tau}(\mathbf{R}_+))$$

is an isomorphism. But $[R_+^{-\sigma}]_+[R_-^{\tilde{\sigma},\tau}]_+[\mathrm{Op}\,q]_+ = [R_+^{-\sigma}R_-^{\tilde{\sigma},\tau}\mathrm{Op}\,q]_+ + G'$ with a singular Green operator G' of order and type zero. This completes the proof of step 3.

Step 4. We may assume that $s = t = 0$: Let E be one of the weighted Sobolev spaces on $\mathbf{R}_+$. By the interpolation results for wedge Sobolev spaces established by Hirschmann [16], theorem 6.4, we may assume that $s, t \in 2\mathbf{Z}$. The operator

$$[\mathrm{Op}\,p]_+ : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, E) \longrightarrow \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, E) \tag{1}$$

is bounded iff

$$[\mathrm{Op}\,\langle\xi'\rangle^s]\,\langle x'\rangle^t[\mathrm{Op}\,p]_+\langle x'\rangle^{-t}\,[\mathrm{Op}\,\langle\xi'\rangle^{-}s] : \mathcal{W}^0(\mathbf{R}^{n-1}, E) \longrightarrow \mathcal{W}^0(\mathbf{R}^{n-1}, E) \tag{2}$$

is bounded. Now, $\langle x'\rangle^t[\mathrm{Op}\,p]_+\langle x'\rangle^{-t} = [\langle x'\rangle^t\mathrm{Op}\,p\,\langle x'\rangle^{-t}]_+$, and the operator inside the brackets belongs to $S^0_{1,0,tr}$, since we may use commutator identities like in the proof of 3.4 to move factors x_j^k from one side to the other.

Since $s \in 2\mathbf{Z}$, either $\mathrm{Op}\,\langle\xi'\rangle^{s}$ or $\mathrm{Op}\,\langle\xi'\rangle^{-s}$ is a *differential* operator with constant coefficients. For $1 \le j \le n-1$,

$$\partial_{x_j} P_+ = P_+ \partial_{x_j} - [P, \partial_{x_j}]_+.$$

The commutator is a pseudodifferential operator with a symbol in $S^0_{1,0,tr}$. So we can move the differential operator to the other side. On the left hand side we obtain a finite sum of operators with symbols of order zero with the transmission property.

Step 5. For $s = 0$ it is sufficent to prove the boundedness of $[\mathrm{Op}\,p]_+$ on $\mathcal{W}^0(\mathbf{R}^{n-1}, E)$ for the spaces

$$E = H^{(M,0)}(\mathbf{R}_+) \text{ or } E = H^{(0,M)}(\mathbf{R}_+) \tag{3}$$

with $M \ge 0$. The reason is the following.

(i) $H_0^{(-M,0)}(\mathbf{R}_+) = \left[H^{(M,0)}(\mathbf{R}_+)\right]'$ and

(ii) $H^{(0,-M)}(\mathbf{R}_+) = \left[H^{(0,M)}(\mathbf{R}_+)\right]'$,

where the duality is with respect to the standard $L^2(\mathbf{R}_+)$ sesquilinear form.

(iii) Once we have established the boundedness for the spaces of the type (3), we may switch to the x_n-adjoint and obtain boundedness of $[\mathrm{Op}\,p]_+$ for the spaces

$$E = H_0^{(-M,0)}(\mathbf{R}_+) \text{ or } E = H^{(0,-M)}(\mathbf{R}_+) \tag{4}$$

noting that the x_n-adjoint also belongs to $S^0_{1,0,tr}$.

(iv) Each space

$$E = H_0^{(-\sigma,\tau)}(\mathbf{R}_+) \text{ or } E = H^{(\sigma,\tau)}(\mathbf{R}_+)$$

$\sigma \ge 0, \tau \in \mathbf{R}$, can be obtained by interpolation from the spaces of the four types in (3) and (4).

3.6 Two computations. Let $k \in \mathbf{N}_0$. Then $\langle x_n\rangle^{2k} = \sum_{j=0}^{k} \binom{k}{j} x_n^{2j}$, and

(a)

$$\begin{aligned}
\|f\|^2_{\mathcal{W}^0(\mathbf{R}^{n-1}, H^{(0,k)}(\mathbf{R}_+))} &= \int \|\mathcal{F}_{y\to\eta} f(\eta, \langle\eta\rangle^{-1} y_n)\,\langle\eta\rangle^{-\frac{1}{2}}\,\langle y_n\rangle^{k}\|^2_{L^2(\mathbf{R}_+)} d\eta \\
&= \int\int_0^\infty |\mathcal{F}_{y\to\eta} f(\eta, \langle\eta\rangle^{-1} y_n)|^2\,\langle\eta\rangle^{-1} \sum_{j=0}^{k} \binom{k}{j} y_n^{2j}\, dy_n\, d\eta \\
&= \sum_{j=0}^{k} \binom{k}{j} \int\int_0^\infty |\mathcal{F}_{y\to\eta} f(\eta, x_n)|^2\,\langle\eta\rangle^{2j}\, x_n^{2j}\, dx_n\, d\eta \\
&= \sum_{j=0}^{k} \binom{k}{j} \|x_n^j f\|^2_{H^j(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+))}.
\end{aligned}$$

(b)

$$\|f\|^2_{\mathcal{W}^0(\mathbf{R}^{n-1}, H^{(k,0)}(\mathbf{R}_+))} = \int \|\mathcal{F}_{y\to\eta} f(\eta, \langle\eta\rangle^{-1} y_n)\,\langle\eta\rangle^{-\frac{1}{2}}\|^2_{H^k(\mathbf{R}_+)} d\eta$$

$$= \int \sum_{j=0}^{k} \int_0^\infty |\partial_{y_n}^j (\mathcal{F}_{y\to\eta}) f(\eta, \langle\eta\rangle^{-1} y_n)|^2 \langle\eta\rangle^{-1} dy_n d\eta$$

$$= \sum_{j=0}^{k} \int \int_0^\infty \langle\eta\rangle^{-2j} |(\partial_{y_n}^j \mathcal{F}_{y\to\eta}) f(\eta, x_n) x_n^j|^2 dx_n d\eta$$

$$= \sum_{j=0}^{k} \|\partial_{x_n}^j f\|^2_{H^{-j}(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+))}.$$

3.7 Lemma. *Let $p \in S^0_{1,0,tr}$. Then for every $M \in \mathbf{N}_0$,*

$$[\operatorname{Op} p]_+ : \mathcal{W}^0(\mathbf{R}^{n-1}, H^{(0,M)}(\mathbf{R}_+)) \longrightarrow \mathcal{W}^0(\mathbf{R}^{n-1}, H^{(0,M)}(\mathbf{R}_+))$$

is bounded.

Proof. We shall use the description of the norms of 3.6(a). First note that for $l, j \in \mathbf{N}_0, l \le j, D_{\xi_n}^{j-l} p \in S^{l-j}_{1,0,tr}$, and hence

$$[\operatorname{Op}(D_{\xi_n}^{j-l} p)]_+ : H^l(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+)) \longrightarrow H^j(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+))$$

is bounded by an extension of the usual boundedness result of Calderón and Vaillancourt to the case of symbols with values in $\mathcal{L}(E, F)$, E, F Hilbert spaces. Therefore we conclude

$$\begin{aligned}
&\|[\operatorname{Op} p]_+ f\|^2_{\mathcal{W}^0(\mathbf{R}^{n-1}, H^{(0,k)}(\mathbf{R}_+))} \\
= &\sum_{j=0}^{k} \binom{k}{j} \|x_n^j [\operatorname{Op} p]_+ f\|^2_{H^j(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+))} \\
= &\sum_{j=0}^{k} \binom{k}{j} \left\| \sum_{l=0}^{j} \binom{j}{l} [\operatorname{Op}(-D)_{\xi_n}^{j-l} p]_+ (x_n^l f) \right\|^2_{H^j(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+))} \\
\le &\sum_{j=0}^{k} \sum_{l=0}^{j} \binom{k}{j} \binom{j}{l} \|[\operatorname{Op} D_{\xi_n}^{j-l} p]_+\|_{\mathcal{L}(H^l(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+)), H^j(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+)))} \\
&\cdot \|(x_n^l f)\|^2_{H^l(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+))} \\
\le &C \|f\|^2_{\mathcal{W}^0(\mathbf{R}^{n-1}, H^{(0,k)}(\mathbf{R}_+))}
\end{aligned}$$

3.8 Lemma. *Let $f \in \mathcal{S}(\mathbf{R}^n_+)$. Then for $k = 0, 1, 2, \ldots$*

$$\gamma_k : f \mapsto \partial_{x_n}^k f(x', 0)$$

defines a bounded map from $\mathcal{W}^0(\mathbf{R}^{n-1}, H^{(k+1,0)}(\mathbf{R}_+))$ to $H^{-\frac{1}{2}-k}(\mathbf{R}^{n-1})$.

Proof. This is essentially a consequence of the estimate

$$|g(0)|^2 = -\int_0^\infty \partial_{x_n} |g(x_n)|^2 dx_n \le 2\|\partial_{x_n} g\|_{L^2(\mathbf{R}_+)} \|g\|_{L^2(\mathbf{R}_+)},$$

valid for $g \in \mathcal{S}(\mathbf{R}_+)$. It implies that

$$\begin{aligned}
&\|\gamma_k f\|^2_{H^{-\frac{1}{2}-k}} \\
= &\int \langle\eta\rangle^{-1-2k} |(\mathcal{F}_{y\to\eta}\partial^k_{x_n} f)(\eta,0)|^2 d\eta \\
\le &\int \langle\eta\rangle^{-1-2k} \|(\mathcal{F}_{y\to\eta}\partial^k_{x_n} f)(\eta,x_n)\|_{L^2}\ \|(\mathcal{F}_{y\to\eta}\partial^{k+1}_{x_n} f)(\eta,x_n)\|_{L^2} d\eta \\
\le &\left(\int \langle\eta\rangle^{-2k} \|(\mathcal{F}_{y\to\eta}\partial^k_{x_n} f)(\eta,x_n)\|^2_{L^2} d\eta\right)^{\frac{1}{2}} \left(\int \langle\eta\rangle^{-2k-2} \|(\mathcal{F}_{y\to\eta}\partial^{k+1}_{x_n} f)(\eta,x_n)\|^2_{L^2} d\eta\right)^{\frac{1}{2}} \\
\le &\|\partial^k_{x_n} f\|_{H^{-k}(\mathbf{R}^{n-1},L^2(\mathbf{R}_+))} \|\partial^{k+1}_{x_n} f\|_{H^{-k-1}(\mathbf{R}^{n-1},L^2(\mathbf{R}_+))} \\
\le &\|f\|^2_{\mathcal{W}^0(\mathbf{R}^{n-1},H^{(k+1,0)}(\mathbf{R}_+))}.
\end{aligned}$$

3.9 Lemma. *Let* $p \in S^0_{1,0,tr}, M \in \mathbf{N}_0$. *Then*

$$P_+ = [\mathrm{Op}\, p]_+ : \mathcal{W}^0(\mathbf{R}^{n-1}, H^{(M,0)}(\mathbf{R}_+)) \longrightarrow \mathcal{W}^0(\mathbf{R}^{n-1}, H^{(M,0)}(\mathbf{R}_+))$$

is bounded.

Proof. By 3.6 we have to estimate

$$\|\partial^k_{x_n} P_+ f\|_{H^{-k}(\mathbf{R}^{n-1},L^2(\mathbf{R}_+))} : 0 \le k \le M$$

in terms of

$$\|\partial^k_{x_n} f\|_{H^{-k}(\mathbf{R}^{n-1},L^2(\mathbf{R}_+))} : 0 \le k \le M.$$

By induction, the equation

$$\partial_{x_n} P_+ f = ([\partial_{x_n}, P])_+ + P_+\partial_{x_n} f + \mathrm{r}^+ P\,(\gamma_0 f \otimes \delta))$$

on $\mathbf{R}^n_+$ shows that for suitable constants $c^{[k]}_j, d^{[k]}_{j,l,m}$,

$$\partial^k_{x_n} P_+ f = \sum_{j=0}^{k} c^{[k]}_j [\mathrm{Op}\,(\partial^j_{x_n} p]_+(\partial^{k-j}_{x_n} f) + \sum_{j+l+m=k-1} d^{[k]}_{j,l,m} \mathrm{r}^+ \mathrm{Op}\,(\partial^j_{x_n} p)(\gamma_l \otimes \delta^m)$$

The terms of the first sum are easy to handle, for $\partial^{k-j}_{x_n} f \in H^{-k}(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+))$, and there $[\mathrm{Op}\,\partial^j_{x_n} p]_+$ is bounded, again by an operator-valued version of Calderón and Vaillancourt's theorem.
Now for the second sum. The operator γ_l yields a bounded map from $\mathcal{W}^0(\mathbf{R}^{n-1}, H^{(M,0)}(\mathbf{R}_+))$ to $H^{-\frac{1}{2}-l}(\mathbf{R}^{n-1})$ by lemma 3.8. For $v \in \mathcal{S}(\mathbf{R}^{n-1})$, the operator

$$K_{jm} : v \mapsto \mathrm{r}^+ \mathrm{Op}\,(\partial^j_{x_n} p)(v \otimes \delta^{(m)})$$

is a Poisson operator of order m, cf. [22] section 2.3.2.3, or [11] 2.7.5 (of order $m+1$ in Grubb's notation). The space $H^{-k}(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+))$ on the other hand is just the space Grubb calls $\mathrm{H}^{0,-k}(\mathbf{R}^n_+)$, [11], $(A.41)$.
So by [11] theorem 2.5.1 (or rather a uniform version of it) the fact that $j+m+l=k-1$ implies that

$$\|K_{jm} v\|_{H^{-k}(\mathbf{R}^{n-1},L^2(\mathbf{R}_+))} \le C\|v\|_{H^{\frac{1}{2}+m-k}(\mathbf{R}^{n-1})} \le C\|v\|_{H^{-\frac{1}{2}-l}(\mathbf{R}^{n-1})}$$

Hence $\|\partial^k_{x_n} P_+ f\|_{H^{-k}(\mathbf{R}^{n-1},L^2(\mathbf{R}_+))}$ is bounded in terms of $\|f\|_{\mathcal{W}^0(\mathbf{R}^{n-1},H^{(M,0)}(\mathbf{R}_+))}$.

3.10 Conclusion. Together, 3.5, 3.7 and 3.9 complete the proof of theorem 3.1.

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