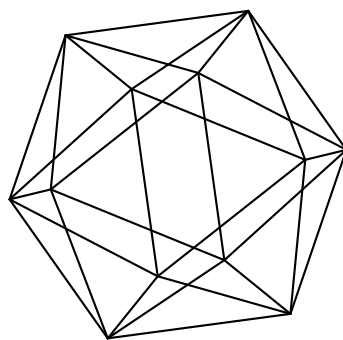


# Max-Planck-Institut für Mathematik Bonn

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the Yang-Mills flow on Kähler manifolds

by

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# ANALYTIC CYCLES, BOTT-CHERN FORMS, AND SINGULAR SETS FOR THE YANG-MILLS FLOW ON KÄHLER MANIFOLDS

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ABSTRACT. It is shown that the singular set for the Yang-Mills flow on unstable holomorphic vector bundles over compact Kähler manifolds is completely determined by the Harder-Narasimhan-Seshadri filtration of the initial holomorphic bundle. We assign a multiplicity to irreducible top dimensional components of the singular set of a holomorphic bundle with a filtration by saturated subsheaves. We derive a singular Bott-Chern formula relating the second Chern form of a smooth metric on the bundle to the Chern current of an admissible metric on the associated graded sheaf. This is used to show that the multiplicities of the top dimensional bubbling locus defined via the Yang-Mills density agree with the corresponding multiplicities for the Harder-Narasimhan-Seshadri filtration. The set theoretic equality of singular sets is a consequence.

## 1. INTRODUCTION

The purpose of this paper is the exact determination of the bubbling locus for the limit of unstable integrable connections on a hermitian vector bundle over a compact Kähler manifold  $(X, \omega)$  along the Yang-Mills flow. Roughly speaking, our theorem states that the set of points where curvature concentration occurs coincides with a subvariety canonically determined by a certain filtration of the initial holomorphic bundle by saturated subsheaves. This result builds on work of several authors on both the analytic and algebraic sides of this picture, and so below we present a brief description of some of this background.

From the analytic point of view, the original compactness result that informs all subsequent discussion is that of Uhlenbeck [38] which, combined with the result of the unpublished preprint [40] implies that a sequence  $A_i$  of integrable unitary connections on a hermitian vector bundle  $E \rightarrow X$  with uniformly bounded Hermitian-Einstein tensors  $\sqrt{-1}\Lambda_\omega F_{A_i}$  has a subsequence that weakly converges locally, modulo gauge transformations and in a certain Sobolev norm outside a set  $Z^{an}$  of Hausdorff (real) codimension at least 4, to a unitary connection  $A_\infty$  (see Theorem 2.5 below). This played an important role in the fundamental work of Uhlenbeck and Yau [41]. We call  $Z^{an}$  the *analytic singular set* (or *bubbling set*). If  $A_\infty$  is Yang-Mills, then by the removable singularities theorem it extends to a unitary connection on a hermitian vector bundle  $E_\infty$  defined off a set of (real) codimension at least 6. We call this extension an *Uhlenbeck limit* and note that  $E_\infty$  may be topologically distinct from  $E$  on its set of definition. The corresponding holomorphic bundle also extends as a reflexive coherent analytic sheaf  $\mathcal{E}_\infty \rightarrow X$ .

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The long time existence of the Yang-Mills flow on integrable connections over Kähler manifolds was proved by Donaldson [13]. Hong and Tian have shown, using a combination of blow-up analysis of the sequence near the singular set and geometric measure theory techniques, that in fact the convergence can be taken to be  $C^\infty$  away from the bubbling set and that the bubbling set itself is a holomorphic subvariety. This second fact relies on a structure theorem of King [25] or a generalized version [20] due to Harvey and Shiffman, for closed, positive, rectifiable currents on complex manifolds. More precisely, in [36] Tian gives a decomposition  $Z^{an} = Z_b^{an} \cup \text{sing } A_\infty$  into a rectifiable piece  $Z_b^{an}$  and a set  $\text{sing } A_\infty$  over which the connection  $A_\infty$  cannot be extended, and having zero  $(2n - 4)$ -dimensional Hausdorff measure, where  $n$  is the complex dimension of  $X$ . The results in [36] together with the main result of [25] or [20] imply that  $Z_b^{an}$  is a subvariety of pure complex codimension 2. Then a result contained in [37] shows that  $\text{sing } A_\infty$  is also an analytic subvariety of codimension at least 3. Moreover, weak convergence of the measures defined by the Yang-Mills densities  $|F_{A_i}|^2 d\text{vol}_\omega$  leads to the definition of a natural density function  $\Theta$  supported on  $Z^{an}$ . In [36] Tian shows that  $\Theta$  assigns an integer weight to each irreducible codimension 2 component  $Z$  of  $Z_b^{an}$ . We call this weight the *analytic multiplicity*  $m_Z^{an}$  of the component  $Z$ .

On the algebraic side, associated to a holomorphic vector bundle  $\mathcal{E} \rightarrow X$  we have the Harder-Narasimhan-Seshadri (HNS) filtration and its associated graded sheaf  $\text{Gr}(\mathcal{E})$ , which is locally free away from a complex analytic subvariety  $Z^{alg}$  of codimension  $\geq 2$ . The sheaf  $\text{Gr}(\mathcal{E})$  is uniquely determined up to isomorphism by  $\mathcal{E}$  and the Kähler class  $[\omega]$ , and therefore so is  $Z^{alg}$ , which we call the *algebraic singular set* (the terminology, which has taken hold, is a bit inaccurate in the sense that we do *not* assume that  $X$  be a projective algebraic manifold). The reflexive sheaf  $\text{Gr}(\mathcal{E})^{**}$  is locally free outside a subvariety of codimension  $\geq 3$ . The restriction of the torsion sheaf  $\text{Gr}(\mathcal{E})^{**}/\text{Gr}(\mathcal{E})$  has a generic rank on each irreducible codimension 2 component  $Z$  of  $Z^{alg}$ , and we call this rank the *algebraic multiplicity*  $m_Z^{alg}$  of the component  $Z$ .

A hermitian metric on the locally free part of a torsion-free sheaf on  $X$  is called *admissible* if its Chern connection has square integrable curvature and bounded Hermitian-Einstein tensor (see Section 2.1). By the main result of Bando-Siu [3], the sheaf  $\text{Gr}(\mathcal{E})^{**}$  carries an admissible Hermitian-Einstein metric whose Chern connection is the direct sum of the Hermitian-Yang-Mills connections on its stable summands, and this is unique up to gauge. The main result linking the two pictures described above is the fact that in the case of Uhlenbeck limits along the Yang-Mills flow,  $\mathcal{E}_\infty$  is actually isomorphic to  $\text{Gr}(\mathcal{E})^{**}$ . In particular, the limiting connection  $A_\infty$  is gauge equivalent to the Bando-Siu connection and is independent of the choice of subsequence. This result is due to Daskalopoulos [9] and Råde [30] for  $\dim_{\mathbb{C}} X = 1$  (where there are no singularities), Daskalopoulos-Wentworth [10] for  $\dim_{\mathbb{C}} X = 2$ , and Jacob [22, 23, 24] and Sibley [32] in higher dimensions. A priori,  $Z^{an}$  also depends on a choice of subsequence along the Yang-Mills flow, whereas  $Z^{alg}$  is uniquely determined as previously mentioned. In  $\dim_{\mathbb{C}} X \geq 3$ , the identification of the limiting structure  $A_\infty$  just mentioned does not address the relationship between these two singular sets; to do so is the aim of this paper. We now formulate our main theorem as follows.

**Theorem 1.1.** *Let  $\mathcal{E} \rightarrow X$  be a hermitian holomorphic vector bundle over a compact Kähler manifold with Chern connection  $A_0$ , and let  $Z^{alg}$  denote the algebraic singular set associated to the Harder-Narasimhan-Seshadri filtration of  $\mathcal{E}$ . Let  $A_t$ ,  $0 \leq t < +\infty$ , denote the Yang-Mills flow with initial condition  $A_0$ . Then:*

- (1) *For any sequence  $t_i \rightarrow \infty$  defining an Uhlenbeck limit  $A_{t_i} \rightarrow A_\infty$  with bubbling set  $Z^{an}$ , then  $Z^{an} = Z^{alg}$  as sets.*
- (2) *Modulo unitary gauge transformations,  $A_t \rightarrow A_\infty$  smoothly away from  $Z^{alg}$  as  $t \rightarrow \infty$  to the admissible Yang-Mills connection  $A_\infty$  on a reflexive sheaf isomorphic to  $\text{Gr}(\mathcal{E})^{**}$ .*
- (3) *For any irreducible component  $Z \subset Z^{alg}$  of complex codimension 2, then  $Z \subset Z_b^{an}$  and the analytic and algebraic multiplicities of  $Z$  are equal.*

**Remark 1.2.** (i) Theorem 1.1 generalizes to higher dimensions the result of [11] in the case  $\dim_{\mathbb{C}} X = 2$ .  
(ii) Item (2) follows from (1) by combining the work of Hong-Tian (for the smooth convergence) and Jacob, Sibley (for the identification of the limit).  
(iii) Under certain technical assumptions on the growth of norms of the second fundamental forms associated to the HNS filtration near  $Z^{an}$ , Collins and Jacob [8] show that (1) holds. The proof of (1) in this paper relies on the structure theorems of Tian and King/Harvey-Shiffman, and the equality of multiplicities from item (3).

A key step in the proof of Theorem 1.1 is a singular version of the usual Bott-Chern formula which is of independent interest. Suppose  $\mathcal{E} \rightarrow X$  is a holomorphic bundle with a filtration by saturated subsheaves and associated graded sheaf  $\text{Gr}(\mathcal{E})$ . Given hermitian structures, then since  $\mathcal{E}$  and  $\text{Gr}(\mathcal{E})$  are topologically isomorphic away from the singular set  $Z^{alg}$  the Bott-Chern formula relates representatives of the second Chern characters  $ch_2$  in terms of Chern connections as an equation of smooth forms outside this set. If the hermitian metric on  $\text{Gr}(\mathcal{E})$  is admissible, we can extend this equality over the singular set as an equation of currents, at the cost of introducing on one side of the equation the current defined by the analytic cycle associated to the irreducible codimension 2 components  $\{Z_j^{alg}\}$  of  $Z^{alg}$  with the multiplicities  $m_j^{alg}$  defined above. The result is the following

**Theorem 1.3.** *Let  $\mathcal{E} \rightarrow X$  be a holomorphic vector bundle with a filtration by saturated subsheaves and hermitian metric  $h_0$ . If  $h$  is an admissible metric on  $\text{Gr}(\mathcal{E})$ , then the following equation of closed currents holds:*

$$(1.1) \quad ch_2(\text{Gr}(\mathcal{E}), h) - ch_2(\mathcal{E}, h_0) = \sum_j m_j^{alg} Z_j^{alg} + dd^c \Psi$$

where  $\Psi = \Psi(h, h_0)$  is a  $(1, 1)$ -current on  $X$ , smooth away from  $Z^{alg}$ . Here,  $Z_j^{alg}$  is regarded as a  $(2, 2)$ -current by integration over its set of smooth points, and  $ch_2(\text{Gr}(\mathcal{E}), h)$  denotes the extension of the Chern form (3.1) as a current on  $X$ .

The organization of this paper is as follows. In Section 2 we recall the definition of the Yang-Mills functional and its negative gradient flow, along with the statement of the main result of [3]. We review the HNS filtration and give a precise definition of the associated algebraic multiplicity. We also recall the version of Uhlenbeck compactness that applies to integrable connections with bounded Hermitian-Einstein tensors. We describe the work of [36] and [21] in a bit more detail and elaborate the notion of analytic multiplicity.

Section 3 is devoted to the proof of Theorem 1.3. It will be shown that (1.1) is essentially a consequence of the cohomological statement that the second Chern character of the torsion sheaf  $\mathrm{Gr}(\mathcal{E})^{**}/\mathrm{Gr}(\mathcal{E})$  is equal to the analytic cycle appearing on the right hand side of (1.1). This latter fact is probably well-known to algebraic geometers, and in the projective case it can be obtained from the Grothendieck-Riemann-Roch theorem of Baum-Fulton-MacPherson [4, 5]. In the setting of arbitrary compact complex manifolds the desired identity, Proposition 3.1, follows from the generalization of BFM due to Levy [27]. The statement in that reference is written in terms of analytic and topological  $K$ -theory, and much of Section 3.2 is therefore devoted to recalling this formalism and using it to obtain the statement about the second Chern character mentioned above. In order to go further, we also need to prove that the second Chern current  $ch_2(\mathrm{Gr}(\mathcal{E}), h)$  for an admissible metric  $h$  is closed and represents  $ch_2(\mathrm{Gr}(\mathcal{E})^{**})$  in cohomology (Proposition 3.3). The proof relies on the monotonicity formula and  $L^p$ -estimates derived by Uhlenbeck in the aforementioned paper [40], as well as an argument of Tian [36] which was used in the case of admissible Yang-Mills connections. We should point out that other versions of singular Bott-Chern currents exist in the literature (e.g. the work of Bismut-Gillet-Soulé [6]).

In Section 4 we prove a slicing lemma showing that the analytic multiplicity may be computed by restricting to a (real) 4-dimensional slice through a generic smooth point of an irreducible component of the analytic singular set (Lemma 4.1). Since a parallel result holds for the currents of integration appearing in the Bott-Chern formula, we can use this and Theorem 1.3 to compare algebraic and analytic multiplicities. Combined with an argument similar to that used in [11], this leads to a proof of the main theorem.

## 2. PRELIMINARIES

**2.1. Stability, Hermitian-Einstein metrics, and the Yang-Mills flow.** Unless otherwise stated,  $X$  will be a compact Kähler manifold of complex dimension  $n$  with Kähler form  $\omega$ . Let  $\Lambda_\omega$  denote the formal adjoint of the Lefschetz operator given by wedging with  $\omega$ . Let  $\mathcal{E} \rightarrow X$  be a hermitian holomorphic vector bundle with metric  $h$ , Chern connection  $A = (\mathcal{E}, h)$ , and curvature  $F_A$ . Then  $\sqrt{-1}\Lambda_\omega F_A$  is a hermitian endomorphism of the underlying hermitian bundle  $(E, h)$ , and it is called the *Hermitian-Einstein tensor*. The equality

$$(2.1) \quad \sqrt{-1}\Lambda_\omega F_A = \mu \mathbf{I}_E$$



where  $\mu$  is constant may be viewed as an equation for the metric  $h$ . A solution to (2.1) is called a *Hermitian-Einstein metric*, and the corresponding Chern connection is called *Hermitian-Yang-Mills*.

If  $\mathcal{E} \rightarrow X$  is a torsion-free sheaf, then its  $\omega$ -slope is defined

$$\mu_\omega(\mathcal{E}) = \frac{1}{\text{rank } \mathcal{E}} \int_X c_1(\mathcal{E}) \wedge \omega^{n-1}.$$

Then  $\mathcal{E}$  is called *stable* (resp. *semistable*), if  $\mu_\omega(\mathcal{F}) < \mu_\omega(\mathcal{E})$  (resp.  $\leq$ ) for all coherent subsheaves  $\mathcal{F} \subset \mathcal{E}$  with  $0 < \text{rank } \mathcal{F} < \text{rank } \mathcal{E}$ . The term *polystable* refers to a sheaf which splits holomorphically into a direct sum of stable sheaves, all of the same slope. The Donaldson-Uhlenbeck-Yau theorem [13, 14, 41] states that a holomorphic bundle  $\mathcal{E}$  admits a Hermitian-Einstein metric if and only if the bundle is polystable. If the volume of  $X$  is normalized to be  $2\pi/(n-1)!$ , then the constant in (2.1) is  $\mu = \mu_\omega(E)$ .

A hermitian metric  $h$  on the locally free part of a torsion-free sheaf  $\mathcal{E}$  is called  $\omega$ -*admissible* if  $|\Lambda_\omega F_A|_h \in L^\infty(X)$  and  $|F_A|_h \in L^2(X, \omega)$ , where  $A$  is the Chern connection  $(\mathcal{E}, h)$ . The Hermitian-Einstein condition (2.1) has the same meaning in this context. We will sometimes refer to  $A$  as an *admissible connection*. The notion of admissibility was introduced by Bando and Siu who also proved the Hitchin-Kobayashi correspondence in this context.

**Theorem 2.1** (Bando-Siu [3]). *Let  $\mathcal{E} \rightarrow X$  be a torsion-free coherent sheaf with reflexivization  $\mathcal{E}^{**}$ . Then:*

- (i) *there exists an admissible metric on  $\mathcal{E}$ ;*
- (ii) *any admissible metric on  $\mathcal{E}$  extends to a metric on the locally free part of  $\mathcal{E}^{**}$  which is in  $L^p_{2,loc}$  for all  $p$ ;*
- (iii) *there exists an admissible Hermitian-Einstein metric on  $\mathcal{E}^{**}$  if and only if  $\mathcal{E}^{**}$  is polystable.*

The *Yang-Mills flow* of unitary connections on a hermitian bundle  $(E, h)$  is given by the equations:

$$(2.2) \quad \frac{\partial A_t}{\partial t} = -d_{A_t}^* F_{A_t}, \quad A(0) = A_0$$

Donaldson [13] shows that if  $X$  is Kähler and  $A_0$  is integrable, then a solution to (2.2) exists (modulo gauge transformations) for all  $0 \leq t < +\infty$ . Eq. (2.2) is formally the negative gradient flow for the *Yang-Mills functional*:

$$YM(A) = \int_X |F_A|^2 d\text{vol}_\omega.$$

Critical points of this functional are called *Yang-Mills connections* and satisfy  $d_A^* F_A = 0$ . By the Kähler identities, if  $E$  admits an integrable Yang-Mills connection  $A$ , then it decomposes holomorphically and isometrically into a direct sum of the (constant rank) eigenbundles of  $\sqrt{-1}\Lambda_\omega F_A$ , and the induced connections are Hermitian-Yang-Mills. Similarly, an admissible Yang-Mills connection on a reflexive sheaf gives a direct sum decomposition into reflexive sheaves admitting admissible Hermitian-Yang-Mills connections.

**2.2. The HNS filtration and the algebraic singular set.** Let  $\mathcal{E} \rightarrow X$  be a holomorphic bundle. We say that  $\mathcal{E}$  is *filtered by saturated subsheaves* if there are coherent subsheaves

$$(2.3) \quad 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$$

with torsion-free quotients  $\mathcal{Q}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ . The associated graded sheaf is

$$\mathrm{Gr}(\mathcal{E}) = \bigoplus_{i=1}^{\ell} \mathcal{Q}_i$$

We define the *algebraic singular set* of a filtration to be  $Z^{\mathrm{alg}} = \mathrm{sing} \mathrm{Gr}(\mathcal{E})$ , i.e. the complement of the open set where  $\mathrm{Gr}(\mathcal{E})$  is locally free. Since the  $\mathcal{Q}_i$  are torsion-free, the singular set is an analytic subvariety of codimension at least 2. Note that we have

$$(2.4) \quad Z^{\mathrm{alg}} = \mathrm{supp}(\mathrm{Gr}(\mathcal{E})^{**}/\mathrm{Gr}(\mathcal{E})) \cup \mathrm{sing} \mathrm{Gr}(\mathcal{E})^{**}$$

since if  $x \notin \mathrm{supp}(\mathrm{Gr}(\mathcal{E})^{**}/\mathrm{Gr}(\mathcal{E}))$ , and  $x \notin \mathrm{sing} \mathrm{Gr}(\mathcal{E})^{**}$ , then  $\mathrm{Gr}(\mathcal{E})$  must be locally free at  $x$ . We also record the simple

**Lemma 2.2.** *Each  $\mathcal{E}_i$  is reflexive. Moreover,  $\mathcal{E}_i$  is locally free on  $X - Z^{\mathrm{alg}}$ .*

*Proof.* Consider the last quotient

$$0 \longrightarrow \mathcal{E}_{\ell-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q}_\ell \longrightarrow 0,$$

Since  $\mathcal{E}$  is reflexive and  $\mathcal{Q}_\ell$  torsion-free,  $\mathcal{E}_{\ell-1}$  is reflexive (cf. [26, Prop. V.5.22]). The result for  $\mathcal{E}_i$  follows by repeatedly applying this argument. For the second statement, start with the first quotient  $\mathcal{Q}_1 = \mathcal{E}_1$ . Then there is an exact sequence:

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{Q}_2 \longrightarrow 0,$$

If both  $\mathcal{E}_1$  and  $\mathcal{Q}_2$  are locally free, then this sequence splits at the level of stalks, and so  $\mathcal{E}_2$  is also locally free. Iterating this argument proves the claim.  $\square$

The following a priori structure of  $Z^{\mathrm{alg}}$  will also be important.

**Proposition 2.3.** *On the complement of  $\mathrm{sing} \mathrm{Gr}(\mathcal{E})^{**}$ ,  $Z^{\mathrm{alg}}$  has pure codimension 2.*

*Proof.* Choose a coordinate ball  $B_\sigma(x) \subset X - \mathrm{sing} \mathrm{Gr}(\mathcal{E})^{**}$ , and assume that  $\mathrm{codim}(B_\sigma(x) \cap Z^{\mathrm{alg}}) \geq 3$ . Set  $U = B_\sigma(x) - Z^{\mathrm{alg}}$ . The first step in the filtration is:

$$(2.5) \quad 0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \xrightarrow{q_2} \mathcal{Q}_2 \longrightarrow 0$$

By Lemma 2.2,  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are reflexive. Furthermore, since  $B_\sigma(x)$  misses  $\mathrm{sing} \mathrm{Gr}(\mathcal{E})^{**}$ ,  $\mathcal{E}_1 = \mathcal{E}_1^{**}$  and  $\mathcal{Q}_2^{**}$  are locally free. Since  $\mathcal{Q}_2^* = (\mathcal{Q}_2^{**})^*$  is also locally free, tensoring (2.5) by  $\mathcal{Q}_2^*$  leaves the sequence exact. Consider the resulting exact sequence in cohomology:

$$(2.6) \quad H^0(U, \mathcal{E}_2 \otimes \mathcal{Q}_2^*) \longrightarrow H^0(U, \mathcal{Q}_2 \otimes \mathcal{Q}_2^*) \longrightarrow H^1(U, \mathcal{E}_1 \otimes \mathcal{Q}_2^*)$$

Since  $\mathcal{E}_1 \otimes \mathcal{Q}_2^*$  is locally free on  $B_\sigma(x)$  and  $\mathrm{codim}(B_\sigma(x) \cap Z^{\mathrm{alg}}) \geq 3$ , it follows from Scheja's theorem [31, Sec. 3, Satz 3] that  $H^1(U, \mathcal{E}_1 \otimes \mathcal{Q}_2^*) \simeq H^1(B_\sigma(x), \mathcal{E}_1 \otimes \mathcal{Q}_2^*) = \{0\}$ . In particular, the image of

the identity  $\mathbf{I}_{\mathcal{Q}_2} \in H^0(U, \text{Hom}(\mathcal{Q}_2, \mathcal{Q}_2)) = H^0(U, \mathcal{Q}_2 \otimes \mathcal{Q}_2^*)$  by the coboundary map, is trivial. By exactness of (2.6) this means there is  $\varphi : \mathcal{Q}_2 \rightarrow \mathcal{E}_2$  on  $U$  satisfying  $q_2 \circ \varphi = \mathbf{I}_{\mathcal{Q}_2}$ . By normality,  $\varphi$  extends to a map  $\tilde{\varphi} : \mathcal{Q}_2^{**} \rightarrow \mathcal{E}_2$  on  $B_\sigma(x)$ . If  $\tilde{q}_2 : \mathcal{E}_2 \rightarrow \mathcal{Q}_2^{**}$  is the map obtained by composing  $q_2$  with the inclusion  $\mathcal{Q}_2 \hookrightarrow \mathcal{Q}_2^{**}$ , then clearly  $\tilde{q}_2 \circ \tilde{\varphi} = \mathbf{I}_{\mathcal{Q}_2^{**}}$ . In particular,  $\tilde{q}_2$  is surjective, and hence  $\mathcal{Q}_2 = \mathcal{Q}_2^{**}$ . So there is no contribution to  $Z^{alg}$  from this term in the filtration. Moreover, since  $\mathcal{E}_1$  and  $\mathcal{Q}_2$  are locally free, eq. (2.5) implies as in the previous lemma that  $\mathcal{E}_2$  is locally free on  $B_\sigma(x)$ . Now consider the next step in the filtration:  $0 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \xrightarrow{q_3} \mathcal{Q}_3 \rightarrow 0$ . Again,  $\mathcal{E}_2$  and  $\mathcal{Q}_3^{**}$  are locally free on  $B_\sigma(x)$ , and the argument proceeds as above. Continuing in this way, we conclude that  $B_\sigma(x) \cap Z^{alg} = \emptyset$ . The statement in the proposition follows.  $\square$

The main example of interest in this paper is the following. Recall that for any reflexive sheaf  $\mathcal{E}$  on a Kähler manifold  $X$  there is a canonical filtration of  $\mathcal{E}$  by saturated subsheaves  $\mathcal{E}_i$  whose successive quotients are torsion-free, semistable. The slopes  $\mu_i = \mu(\mathcal{Q}_i)$  satisfy  $\mu_1 > \mu_2 > \dots > \mu_\ell$ . This filtration is called the *Harder-Narasimhan filtration* of  $\mathcal{E}$ . Moreover, there is a further filtration of the quotients by subsheaves so that the successive quotients are stable. We call this the *Harder-Narasimhan-Seshadri (HNS) filtration*. The associated graded sheaf  $\text{Gr}(\mathcal{E})$  is a direct sum of stable torsion-free sheaves. It depends on the choice of Kähler class  $[\omega]$  but is otherwise canonically associated to  $\mathcal{E}$  up to permutation of isomorphic factors. The  $(\text{rank } \mathcal{E})$ -vector  $\vec{\mu}$  obtained by repeating each  $\mu_1, \dots, \mu_\ell$ ,  $\text{rank } \mathcal{Q}_i$  times, is called the *Harder-Narasimhan type* of  $\mathcal{E}$ .

Two more remarks:

- Strictly speaking, the HNS construction gives rise to a *double* filtration; however, this fact presents no difficulties, and for simplicity we shall treat the HNS filtration like the general case. For more details the reader may consult [10], [32], or the book [26].
- We sometimes use the same notation  $\text{Gr}(\mathcal{E})$  for the associated graded of a general filtration as well as for the HNS filtration of  $\mathcal{E}$ . The context will hopefully make clear which is meant.

**2.3. Multiplicities associated to the support of a sheaf.** If  $\mathcal{F}$  is a coherent sheaf on a complex manifold  $X$ , then the support  $Z = \text{supp } \mathcal{F}$  is a closed, complex analytic subvariety of  $X$ . Moreover,  $\text{supp } \mathcal{F}$  is the vanishing set  $V(\mathcal{J}_{\mathcal{F}})$ , where  $\mathcal{J}_{\mathcal{F}} \subset \mathcal{O}_X$  is the annihilator ideal sheaf whose presheaf on an open set  $U$  is the subset of functions  $\mathcal{O}_X(U)$  that annihilate all local sections in  $\mathcal{F}(U)$ . This ideal gives  $\text{supp } \mathcal{F}$  the structure of a complex analytic subspace with structure sheaf  $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}_{\mathcal{F}}$ . Now  $Z$  has a decomposition  $Z = \cup_j Z_j$  into irreducible components  $Z_j$ , with structure sheaves  $\mathcal{O}_{Z_j}$ . If necessary, we take the reduced structures, so that each  $Z_j$  is a reduced and irreducible complex subspace of  $X$  with ideal  $\mathcal{J}_j$  (i.e.  $\text{supp } \mathcal{O}_X/\mathcal{J}_j = Z_j$ ). The fact that  $Z_j$  is irreducible means that the complex manifold  $Z_j - \text{sing } Z_j$  is connected.

Note that for the inclusion  $\iota : Z_j \hookrightarrow X$ , the restriction  $\iota^*\mathcal{F}$  of  $\mathcal{F}$  to  $Z_j$  is a coherent sheaf of  $\mathcal{O}_{Z_j}$  modules. In this way we may regard  $\mathcal{F}$  as a sheaf on  $Z_j$ . The fibres of  $\mathcal{F}$  on  $Z$  are the finite

dimensional  $\mathbb{C}$ -vector spaces

$$\mathcal{F}(z) = \mathcal{F}_z / \mathfrak{m}_z \mathcal{F}_z = \mathcal{F}_z \otimes_{\mathcal{O}_{Z_j, z}} \mathbb{C},$$

where  $\mathfrak{m}_z$  is the maximal ideal in the local ring  $\mathcal{O}_{Z_j, z}$ , and the rank,  $\text{rank}_z \mathcal{F}$  at a point is the dimension of this vector space.

It is not difficult to see (cf. [17, p. 91]) that  $i^* \mathcal{F}$  is locally free at a point  $z_0 \in Z_j$  if and only if the function  $z \mapsto \text{rank}_z \mathcal{F}$  is constant for  $z \in Z_j$  near  $z_0$ . Therefore away from the set  $\text{sing } Z_j \cup \text{sing } i^* \mathcal{F}$  of points where  $i^* \mathcal{F}$  fails to be locally free and  $Z_j$  is singular, this function is constant (since the set of smooth points of  $Z_j$  is connected). The set  $\text{sing } Z_j \cup \text{sing } i^* \mathcal{F}$  is a proper subvariety of  $Z_j$ , and nowhere dense in  $Z_j$  (since  $Z_j$  is reduced). In particular, this subvariety has dimension less than  $Z_j$ , and therefore the generic rank of  $i^* \mathcal{F}$  on  $Z_j$  is well-defined. Another way of saying this is that if  $Z_j$  has codimension  $k$ ,  $z \in Z_j$  is a generic smooth point, and  $\Sigma$  is a (locally defined) complex submanifold of  $X$  of dimension  $k$  intersecting  $Z_j$  transversely at  $z$ , then the  $\mathbb{C}$ -vector space  $(\mathcal{O}_\Sigma)_z / (\mathcal{J}_j|_\Sigma)_z$  is finite dimensional, and its dimension is generically constant and equal to the rank of  $i^* \mathcal{F}$ .

**Definition 2.4.** Given a coherent sheaf  $\mathcal{F}$  on  $X$  and an irreducible component  $i : Z \hookrightarrow \text{supp } \mathcal{F}$ , define the *multiplicity*  $m_Z$  of  $\mathcal{F}$  along  $Z$  to be the rank of  $i^* \mathcal{F}$ .

For any  $k$  we can define an  $(n - k)$ -cycle associated to  $\mathcal{F}$  by:

$$(2.7) \quad [\mathcal{F}]_k = \sum_{\substack{\text{irred. } Z \subset \text{supp } \mathcal{F} \\ \text{codim } Z = k}} m_Z [Z]$$

Of particular interest in this paper is the associated graded sheaf  $\mathcal{F} = \text{Gr}(\mathcal{E})$  of a locally free sheaf  $\mathcal{E}$  with a filtration by saturated subsheaves. The quotient by the inclusion  $\text{Gr}(\mathcal{E}) \hookrightarrow \text{Gr}(\mathcal{E})^{**}$  yields a torsion sheaf which has support in codimension 2. The irreducible components  $\{Z_j^{\text{alg}}\}$  of  $Z^{\text{alg}}$  with codimension = 2 have associated algebraic multiplicities from Definition 2.4. We will denote these by  $m_j^{\text{alg}}$  and refer to them as the *algebraic multiplicities* of the filtration.

Note that in the case  $\dim_{\mathbb{C}} X = 2$ ,  $\text{Gr}(\mathcal{E})^{**}$  is locally free and  $\text{Gr}(\mathcal{E})^{**} / \text{Gr}(\mathcal{E})$  is supported at points. The structure sheaf of singular point  $z$  is  $\mathcal{O}_{X, z} / \mathfrak{m}_z = \mathbb{C}$ , so the fibre at this point is just the stalk, and the multiplicity is the  $\mathbb{C}$ -dimension of the stalk. This was the definition of  $m_z^{\text{alg}}$  used in [11].

**2.4. Uhlenbeck limits and the analytic singular set.** We briefly recall the Uhlenbeck compactness theorem. It is a combination of the results of [38] and [40] (see also Theorem 5.2 of [41]).

**Theorem 2.5** (Uhlenbeck). *Let  $X$  be a compact Kähler manifold of complex dimension  $n$  and  $(E, h)$  a hermitian vector bundle on  $X$ . Let  $A_i$  be a sequence of integrable unitary connections on  $E$  with  $|\Lambda_\omega F_{A_i}|$  uniformly bounded and  $\|d_{A_i}^* F_{A_i}\|_{L^2} \rightarrow 0$ . Fix  $p > 2n$ . Then there is:*

- a subsequence (still denoted  $A_i$ )

- a closed subset  $Z^{an} \subset X$  of finite  $(2n - 4)$ -Hausdorff measure;
- a hermitian vector bundle  $(E_\infty, h_\infty)$  defined on  $X - Z^{an}$  and  $L^p_{2,loc}$ -isometric to  $(E, h)$ ;
- an admissible Yang-Mills connection  $A_\infty$  on  $E_\infty$ ;

such that up to unitary gauge equivalence  $A_i$  converges weakly in  $L^p_{1,loc}(X - Z^{an})$  to  $A_\infty$ .

We call the resulting connection  $A_\infty$  an *Uhlenbeck limit* and the set  $Z^{an}$  the *analytic singular set*. A priori both depend on the choice of subsequence in the statement of the theorem.

**Remark 2.6.** By [3], the holomorphic structure  $\bar{\partial}_{A_\infty}$  on  $E_\infty$  extends as a reflexive sheaf  $\mathcal{E}_\infty \rightarrow X$ , and the metrics extend smoothly to  $X - \text{sing } \mathcal{E}_\infty$ . As mentioned at the end of Section 2.1, since the limiting connection is Yang-Mills,  $\mathcal{E}_\infty$  decomposes holomorphically and isometrically as a direct sum of stable reflexive sheaves with admissible Hermitian-Einstein metrics.

The following result identifies Uhlenbeck limits of certain sequences of isomorphic bundles.

**Theorem 2.7** ([10, 32]). *Let  $A_i$  be a sequence of connections in a complex gauge orbit of a holomorphic bundle  $\mathcal{E}$  with Harder-Narasimhan type  $\bar{\mu}$  and satisfying the hypotheses of Theorem 2.5. Assume further that functionals  $\text{HYM}_\alpha(A_i) \rightarrow \text{HYM}_\alpha(\bar{\mu})$  for  $\alpha \in [1, \infty)$  in a set that includes 2 and has a limit point. Then any Uhlenbeck limit of  $A_i$  defines a reflexive sheaf  $\mathcal{E}_\infty$  which is isomorphic to  $\text{Gr}(\mathcal{E})^{**}$ .*

Here, the functionals  $\text{HYM}_\alpha$  are generalizations of the Yang-Mills energy that were introduced in [1]. All the hypotheses of Theorem 2.7 are in particular satisfied if  $A_i$  is a sequence  $A_{t_i}$ ,  $t_i \rightarrow \infty$ , along the Yang-Mills flow. In this case, Hong and Tian [21] prove that the convergence is in fact  $C^\infty$  away from  $Z^{an}$ .

We now give a more precise definition of the analytic singular set. For a sequence of connections  $A_i$  satisfying the hypotheses of Theorem 2.5 and with Uhlenbeck limit  $A_\infty$ , define (cf. [36, eq. (3.1.4)]),

$$Z^{an} = \bigcap_{\sigma_0 \geq \sigma > 0} \left\{ x \in X : \liminf_{i \rightarrow \infty} \sigma^{4-2n} \int_{B_\sigma(x)} |F_{A_i}|^2 d\text{vol}_\omega \geq \varepsilon_0 \right\}.$$

The numbers  $\varepsilon_0, \sigma_0$ , are those that appear in the  $\varepsilon$ -regularity theorem [38, 40] and depend only on the geometry of  $X$  (see also Section 3.3 below). The density function is defined by taking a weak limit of the Yang-Mills measure:

$$|F_{A_i}|^2(x) d\text{vol}_\omega \longrightarrow |F_{A_\infty}|^2(x) d\text{vol}_\omega + \Theta(x) \mathcal{H}^{2n-4} \llcorner Z_b^{an}$$

For almost all  $x \in Z^{an}$  with respect to  $(2n - 4)$ -Hausdorff measure  $\mathcal{H}^{2n-4}$ ,

$$\Theta(x) = \lim_{\sigma \rightarrow 0} \lim_{i \rightarrow \infty} \sigma^{4-2n} \int_{B_\sigma(x)} |F_{A_i}|^2 d\text{vol}_\omega$$

The closed subset of  $Z^{an}$  defined by

$$Z_b^{an} = \overline{\{x \in X \mid \Theta(x) > 0, \lim_{\sigma \rightarrow 0} \sigma^{4-2n} \int_{B_\sigma(x)} |F_{A_\infty}|^2 d\text{vol}_\omega = 0\}}$$

is called the *blow-up locus* of the sequence  $A_i$ .

By the removable singularities theorem of Tao and Tian [34] (see also Theorem 5.1 of [37]), there is a gauge transformation  $g$  on  $X - Z^{an}$ , such that  $g(A_\infty)$  extends smoothly over the blow-up locus. Moreover, it can be shown that  $\mathcal{H}^{2n-4}(\overline{Z^{an} - Z_b^{an}}) = 0$ . Therefore, we can express

$$(2.8) \quad Z^{an} = Z_b^{an} \cup \text{sing } A_\infty$$

and  $\text{sing } A_\infty$  is the  $\mathcal{H}^{2n-4}$ -measure zero set where  $A_\infty$  is singular.

If the sequence  $\{A_i\}$  happens to be a sequence of Hermitian-Yang-Mills connections, or if it is a sequence along the Yang-Mills flow, then one can say much more about the blow-up locus. Namely, we have the following theorem, proven in the two different cases in [36] and [21] respectively.

**Theorem 2.8** (Tian, Hong-Tian). *If  $A_i$  is either a sequence of Hermitian-Yang-Mills connections, or a sequence  $A_i = A_{t_i}$  of connections along the Yang-Mills flow with  $t_i \rightarrow \infty$ , and  $A_i$  has Uhlenbeck limit  $A_\infty$ , then its blow-up locus  $Z_b^{an}$  is a holomorphic subvariety of pure codimension 2. Furthermore, the density  $\Theta$  is constant along each of the irreducible codimension 2 components  $Z_j^{an} \subset Z^{an}$  and there exist positive integers  $m_j^{an}$  such that for any smooth  $(2n - 4)$ -form  $\Omega$ , we have*

$$\frac{1}{8\pi^2} \lim_{i \rightarrow \infty} \int_X \Omega \wedge \text{Tr}(F_{A_i} \wedge F_{A_i}) = \frac{1}{8\pi^2} \int_X \Omega \wedge \text{Tr}(F_{A_\infty} \wedge F_{A_\infty}) + \sum_j m_j^{an} \int_{Z_j^{an}} \Omega.$$

It follows that  $Z^{an}$  is an analytic subvariety. We will call the numbers  $m_j^{an}$ , the *analytic multiplicities*. For more details see [36], [37], and [21].

### 3. A SINGULAR BOTT-CHERN FORMULA

**3.1. Statement of results.** Throughout this section we consider holomorphic vector bundles  $\mathcal{E}$  on a compact Kähler manifold  $(X, \omega)$  with a *general* filtration (2.3) by saturated subsheaves. As above, let  $\text{Gr}(\mathcal{E})$  denote the graded sheaf associated to the filtration, and  $\text{Gr}(\mathcal{E})^{**}$  its sheaf theoretic double dual. Then  $Z^{alg}$  will refer to the algebraic singular set defined in Section 2.2. The codimension 2 components  $Z_j^{alg} \subset Z^{alg}$  have multiplicities  $m_j^{alg}$  as in Definition 2.4.

There are two key steps in the proof of Theorem 1.3. The first is the cohomological statement of the following result which will be proved in Section 3.2.

**Proposition 3.1.** *Let  $X$  be a compact, complex manifold, and  $\mathcal{T} \rightarrow X$  a torsion sheaf. Assume that  $\text{supp } \mathcal{T}$  has codimension  $p$ . Denote by  $Z_j \subset \text{supp } \mathcal{T}$  the irreducible components of codimension exactly  $p$  and by  $[Z_j]$  their corresponding homology classes. Then for all  $k < p$ ,  $ch_k(\mathcal{T}) = 0$ , and  $ch_p(\mathcal{T}) = \text{PD}(\sum_j m_{Z_j} [Z_j])$  in  $H^{2p}(X, \mathbb{Q})$ .*

In other words, the  $p$ -th component of the Chern character of a torsion sheaf with support in codimension  $p$  is the the cohomology class of the cycle  $[\text{supp } \mathcal{T}]_p$ , and all lower components vanish. This result is probably well known, but we have not been able to find a proof in the literature. We therefore give a proof here which will use the subsequent discussion of the Grothendieck-Riemann-Roch theorem for complex spaces.

Note that on a non-projective compact, complex manifold, a coherent sheaf does not in general have a global resolution by locally free coherent analytic sheaves. See the appendix of [42] for an explicit counter-example. Therefore, we cannot define Chern classes directly in this way. However, after tensoring with the sheaf  $\mathcal{A}_X$  of germs of real analytic functions on  $X$  there is a resolution by real analytic vector bundles (cf. [16]).

**Definition 3.2.** Let  $X$  be a compact, complex manifold and  $\mathcal{F} \rightarrow X$  a coherent sheaf. Choose a global resolution:  $0 \rightarrow E_r \rightarrow E_{r-1} \rightarrow \cdots \rightarrow E_0 \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X \rightarrow 0$ , where the  $E_i$  are real analytic complex vector bundles on  $X$ . Then define

$$ch_p(\mathcal{F}) = \sum_{i=0}^r (-1)^i ch_p(E_i)$$

One can show that this definition does not depend on the choice of global resolution (cf. [7]). The Chern characters appearing in Proposition 3.1 are defined in this way. For other approaches to Chern classes of coherent sheaves, see [18, 19].

For a smooth hermitian metric  $h_0$  on  $\mathcal{E}$ , let  $A_0 = (\mathcal{E}, h_0)$  denote the Chern connection. Then by Chern-Weil theory the smooth, closed  $(2, 2)$ -form:  $ch_2(\mathcal{E}, h_0) = -(1/8\pi^2) \text{Tr}(F_{A_0} \wedge F_{A_0})$ , represents the Chern character  $ch_2(\mathcal{E})$  in cohomology. For an *admissible* metric  $h$  on  $\text{Gr}(\mathcal{E})$ , let

$$(3.1) \quad ch_2(\text{Gr}(\mathcal{E}), h) = -\frac{1}{8\pi^2} \text{Tr}(F_A \wedge F_A)$$

where  $A$  is the Chern connection of  $(\text{Gr}(\mathcal{E}), h)$  on its locally free locus  $X - Z^{alg}$ . Note that since  $F_A \in L^2$ , eq. (3.1) defines a  $(2, 2)$ -current on  $X$  by setting

$$(3.2) \quad ch_2(\text{Gr}(\mathcal{E}), h)(\Omega) = -\frac{1}{8\pi^2} \int_X \Omega \wedge \text{Tr}(F_A \wedge F_A)$$

for any smooth  $(2n-4)$ -form  $\Omega$ . The second step in the proof of Theorem 1.3 is the following result which will be proved in Section 3.3 below.

**Proposition 3.3.** *Let  $\mathcal{E}$  and  $\text{Gr}(\mathcal{E})$  be as in the statement of Theorem 1.3. Then for any admissible metric  $h$  on  $\text{Gr}(\mathcal{E})$ , the smooth form  $ch_2(\text{Gr}(\mathcal{E}), h)$  on  $X - Z^{alg}$  defined by (3.1), extends as a closed  $(2, 2)$ -current on  $X$ . Moreover, this current represents  $ch_2(\text{Gr}(\mathcal{E})^{**})$  in rational cohomology.*

Assuming the two results above, we have the

*Proof of Theorem 1.3.* Consider the exact sequence

$$0 \longrightarrow \text{Gr}(\mathcal{E}) \longrightarrow \text{Gr}(\mathcal{E})^{**} \longrightarrow \text{Gr}(\mathcal{E})^{**}/\text{Gr}(\mathcal{E}) \longrightarrow 0.$$

Then by the additivity of  $ch_2$  over exact sequences we have

$$ch_2(\text{Gr}(\mathcal{E})^{**}/\text{Gr}(\mathcal{E})) = ch_2(\text{Gr}(\mathcal{E})^{**}) - ch_2(\text{Gr}(\mathcal{E})) = ch_2(\text{Gr}(\mathcal{E})^{**}) - ch_2(\mathcal{E}),$$

where we have used that  $ch_2(\text{Gr}(\mathcal{E})) = ch_2(\mathcal{E})$ , by the definition of  $ch_2$  and additivity. Applying Proposition 3.3 to the right hand side, and Proposition 3.1 to the left hand side of the equation

above we obtain

$$\frac{1}{8\pi^2} (\mathrm{Tr}(F_{A_0} \wedge F_{A_0}) - \mathrm{Tr}(F_A \wedge F_A)) = \sum_j m_j^{\mathrm{alg}} Z_j^{\mathrm{alg}} + dd^c \Psi$$

for some  $(1, 1)$ -current  $\Psi$  by the  $dd^c$ -lemma for currents. By elliptic regularity,  $\Psi$  may be taken to be smooth away from  $\mathrm{sing} \mathrm{Gr}(\mathcal{E}) = Z^{\mathrm{alg}}$ .  $\square$

We also note that Theorems 1.3 and 2.8 combine to give the following

**Corollary 3.4.** *The currents  $\sum_j m_j^{\mathrm{an}} Z_j^{\mathrm{an}}$  and  $\sum_k m_k^{\mathrm{alg}} Z_k^{\mathrm{alg}}$  are cohomologous.*

**3.2. Levy's Grothendieck-Riemann-Roch Theorem and the cycle  $ch_p(\mathcal{J})$ .** To prove Proposition 3.1 we recall a very general version of the Riemann-Roch theorem for complex spaces. One may think of this theorem as translating algebraic (holomorphic) data into topological data. It is expressed in terms of  $K$ -theory, so we recall some basic definitions. For the subsequent discussion  $X$  will denote a compact, complex space. Note that for such a space there is a topological embedding  $X \hookrightarrow \mathbb{C}^N$ .

Let  $K_0^{\mathrm{hol}}(X)$  denote the Grothendieck group of the category of coherent analytic sheaves on  $X$ , that is, the free abelian group generated by isomorphism classes of coherent sheaves modulo the relation given by exact sequences. We will write  $K_0^{\mathrm{top}}(X)$  for the homology  $K$ -theory of the topological space underlying  $X$ . This is the homology theory corresponding to the better known topological  $K$ -theory  $K_{\mathrm{top}}^0(X)$  given by the Grothendieck group of the category of topological vector bundles. The group  $K_0^{\mathrm{top}}(X)$  may be defined in this case by choosing an embedding  $X \hookrightarrow \mathbb{C}^N$  and declaring  $K_0^{\mathrm{top}}(X) = K_{\mathrm{top}}^0(\mathbb{C}^N, \mathbb{C}^N - X)$ , where the group on the right hand side is the usual relative  $K$ -theory (see for example [5]). For a proper map  $f : X \rightarrow Y$ , we can also define a pushforward map  $f_* : K_0^{\mathrm{top}}(X) \rightarrow K_0^{\mathrm{top}}(Y)$  (see [5]), by factoring  $f$  as an inclusion composed with a projection.

With all of this understood, the version of the Grothendieck-Riemann-Roch theorem proven by Levy [27] states that there is a natural transformation of functors  $\alpha$  from  $K_0^{\mathrm{hol}}$  to  $K_0^{\mathrm{top}}$ . Explicitly, this means that for any two compact complex spaces  $X$  and  $Y$ , there are maps

$$\alpha_X : K_0^{\mathrm{hol}}(X) \rightarrow K_0^{\mathrm{top}}(X), \quad \alpha_Y : K_0^{\mathrm{hol}}(Y) \rightarrow K_0^{\mathrm{top}}(Y)$$

such that for any proper morphism  $f : X \rightarrow Y$  the following diagram commutes:

$$\begin{array}{ccc} K_0^{\mathrm{hol}}(X) & \xrightarrow{\alpha_X} & K_0^{\mathrm{top}}(X) \\ f_! \downarrow & & \downarrow f_* \\ K_0^{\mathrm{hol}}(Y) & \xrightarrow{\alpha_Y} & K_0^{\mathrm{top}}(Y) \end{array}$$

Here  $f_!$  is Grothendieck's direct image homomorphism given by

$$f_!([\mathcal{F}]) = \sum_i (-1)^i [R^i f_* \mathcal{F}],$$

and  $f_*$  is the pushforward map in  $K$ -theory. We also have the usual Chern character

$$ch^* : K_{\mathrm{top}}^0(\mathbb{C}^N, \mathbb{C}^N - X) \rightarrow H^{2*}(\mathbb{C}^N, \mathbb{C}^N - X, \mathbb{Q}).$$



We may furthermore define an homology Chern character:

$$ch_* : K_0^{\text{top}}(X) \longrightarrow H_{2*}(X, \mathbb{Q}),$$

by taking a topological embedding  $X \hookrightarrow \mathbb{C}^N$  and then composing the maps:

$$K_0^{\text{top}}(X) = K_{\text{top}}^0(\mathbb{C}^N, \mathbb{C}^N - X) \xrightarrow{ch^*} H^{2*}(\mathbb{C}^N, \mathbb{C}^N - X, \mathbb{Q}) \cong H_{2*}(X, \mathbb{Q}),$$

where the last isomorphism is Lefschetz duality.

If  $X$  is nonsingular, so that Poincaré duality gives an isomorphism  $K_0^{\text{top}}(X) \cong K_{\text{top}}^0(X)$ , then the relationship between the homology Chern character and the ordinary Chern character  $ch^* : K_{\text{top}}^0(X) \longrightarrow H^{2*}(X, \mathbb{Q})$  is given by:

$$ch_*(\text{PD } \eta) = \text{PD}[ch^*(\eta) \cdot Td(X)].$$

The homology Chern character is also a natural transformation with respect to the pushforward maps. By composition there is a natural transformation of functors

$$\tau = ch_* \circ \alpha : K_0^{\text{hol}}(X) \rightarrow H_{2*}(X, \mathbb{Q})$$

that satisfies the corresponding naturality property, i.e. for any proper map of complex spaces  $f : X \rightarrow Y$ :  $\tau \circ f_! = f_* \circ \tau$ . Here  $f_!$  is as before, and  $f_*$  is the usual induced map in homology.

For a coherent sheaf  $\mathcal{F}$ ,  $\tau(\mathcal{F})$  is called the *homology Todd class* of  $\mathcal{F}$  and satisfies a number of important properties.

(1) If  $\mathcal{F}$  is locally free and  $X$  is smooth, then  $\tau(\mathcal{F})$  is the Poincaré dual to

$$ch(\mathcal{F}) \cdot Td(X) \in H^{2*}(X, \mathbb{Q}),$$

and in particular  $\tau(\mathcal{O}_X) = \text{PD}(Td(X))$ .

(2) For an embedding  $\iota : X \hookrightarrow Y$ , we have  $\tau(\iota_* \mathcal{F}) = \iota_*(\tau(\mathcal{F}))$ .

(3) If the support  $\text{supp } \mathcal{F}$  has dimension  $k$ , then the components of  $\tau(\mathcal{F})$  in (real) dimensions  $r > 2k$  vanish.

(4) For any exact sequence of sheaves:  $0 \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{H} \rightarrow 0$ , we have  $\tau(\mathcal{F}) = \tau(\mathcal{G}) + \tau(\mathcal{H})$ .

Property (4) follows from [27, Lemma 3.4(b)] together with the additivity of the homology Chern character. For  $\mathcal{F}$  locally free, property (1) is a restatement of [27, Lemma 3.4(c)] as follows. The statement there is that for  $\mathcal{F}$  locally free,  $\alpha(\mathcal{F})$  is Poincaré dual to the topological vector bundle corresponding to  $\mathcal{F}$  in  $K_{\text{top}}^0(X)$ , then (1) follows by applying  $ch_*$ . Property (2) follows from the naturality of  $\tau$  and the fact that the higher direct images of an embedding vanish. Property 3 follows from dimensional considerations. Indeed, since  $\text{supp } \mathcal{F}$  is an analytic subvariety, it can be triangulated, and if it has (real) dimension  $2k$  the homology in higher dimensions will be zero. Hence,  $\tau$  evaluated on the restriction of  $\mathcal{F}$  to  $\text{supp } \mathcal{F}$  has zero components in dimension  $> 2k$ . This combined with property (2) applied to the embedding  $\iota : \text{supp } \mathcal{F} \hookrightarrow X$  proves (3).

Now we have the following important consequence of these properties.

**Proposition 3.5.** *For a coherent sheaf  $\mathcal{F}$  on a complex manifold, the component of  $\tau(\mathcal{F})$  in degree  $2 \dim \text{supp } \mathcal{F}$  is the homology class of the cycle*

$$\sum_j \left( \text{rank}_{\mathcal{O}_{Z_j}} \mathcal{F}|_{Z_j} \right) [Z_j] = \sum_j m_{Z_j} [Z_j]$$

where the  $Z_j$  are the irreducible components of  $\text{supp } \mathcal{F}$  of top dimension.

*Proof.* First, we prove that for a reduced and irreducible, closed complex subspace  $Z$  of  $X$ , and a coherent sheaf  $\mathcal{F}$  on  $Z$ ,  $\tau(\mathcal{F})_{2 \dim Z} = (\text{rank } \mathcal{F})[Z]$ , where  $[Z]$  denotes the generator of  $H_{2 \dim Z}(Z, \mathbb{Q})$ . Note that if  $Z$  is smooth and  $\mathcal{F}$  is locally free, the statement follows directly from property (1). By Hironaka's theorem we may resolve the singularities of  $Z$  by a sequence of blow-ups along smooth centers. By Hironaka's flattening theorem, we can perform a further sequence of blow-ups along smooth centers to obtain a complex manifold  $\widehat{Z}$ , and a map  $\pi : \widehat{Z} \rightarrow Z$  such that

$$\pi : \widehat{Z} - \pi^{-1}(\text{sing } Z \cup \text{sing } \mathcal{F}) \rightarrow Z - (\text{sing } Z \cup \text{sing } \mathcal{F})$$

is a biholomorphism, and  $\widehat{\mathcal{F}} = \pi^* \mathcal{F} / \text{tor}(\pi^* \mathcal{F})$  is locally free. Then  $\tau(\widehat{\mathcal{F}})_{2 \dim \widehat{Z}} = (\text{rank } \widehat{\mathcal{F}})[\widehat{Z}] = (\text{rank } \mathcal{F})[\widehat{Z}]$ . The fundamental class of an analytic subvariety is always equal to the pushforward of the fundamental class of a resolution of singularities (see [42, Ch. 11.1.4]), so we have that  $\pi_*(\tau(\widehat{\mathcal{F}})_{2 \dim \widehat{Z}}) = (\text{rank } \mathcal{F})[Z]$ . By naturality, this is also  $\tau(\pi_! \widehat{\mathcal{F}})_{2 \dim Z}$ . Since  $\pi$  is, in particular, a proper map, the stalks of the higher direct image sheaves are given by  $(R^i \pi_* \widehat{\mathcal{F}})_z = H^i(\pi^{-1}(z), \widehat{\mathcal{F}})$ , and so  $R^i \pi_* \widehat{\mathcal{F}}$  is supported on a proper subvariety for  $i > 0$ . Therefore,  $\tau(R^i \pi_* \widehat{\mathcal{F}})_{2 \dim Z} = 0$  for  $i > 0$  by property (3). It follows that  $\tau(\pi_* \widehat{\mathcal{F}})_{2 \dim Z} = (\text{rank } \mathcal{F})[Z]$ . In fact, since  $\text{tor}(\pi^* \mathcal{F})$  is supported on a divisor, by properties (1), (3), and (4), and the above argument applied to the higher direct images of  $\pi^* \mathcal{F}$ , we have

$$(3.3) \quad (\text{rank } \mathcal{F})[Z] = \pi_*(\tau(\widehat{\mathcal{F}})_{2 \dim \widehat{Z}}) = \pi_*(\tau(\pi^* \mathcal{F})_{2 \dim Z}) = \tau(\pi_* \pi^* \mathcal{F})_{2 \dim Z}.$$

On the other hand the natural map  $\mathcal{F} \xrightarrow{\alpha} \pi_* \pi^* \mathcal{F}$  is an isomorphism away from the proper subvariety  $\text{sing } Z \cup \text{sing } \mathcal{F}$ . Therefore, the sheaves  $\ker(\alpha)$  and the quotient  $Q$  of  $\pi_* \pi^* \mathcal{F}$  by  $\mathcal{F} / \ker(\alpha)$  are supported on  $\text{sing } Z \cup \text{sing } \mathcal{F}$ . Considering the exact sequence:

$$0 \longrightarrow \mathcal{F} / \ker(\alpha) \longrightarrow \pi_* \pi^* \mathcal{F} \longrightarrow Q \longrightarrow 0,$$

we see by (4) that

$$(3.4) \quad \tau(\pi_* \pi^* \mathcal{F}) = \tau(\mathcal{F} / \ker(\alpha)) + \tau(Q) = \tau(\mathcal{F}) - \tau(\ker(\alpha)) + \tau(Q)$$

Since  $\ker(\alpha)$  and  $Q$  are supported on  $Z_{\text{sing}}$ , (3) implies  $\tau(\ker(\alpha))_{2 \dim Z} = \tau(Q)_{2 \dim Z} = 0$ . Therefore, taking the top dimensional component in (3.4) we obtain:

$$\tau(\mathcal{F})_{2 \dim Z} = \tau(\pi_* \pi^* \mathcal{F})_{2 \dim Z},$$

so by (3.3),  $\tau(\mathcal{F})_{2 \dim Z} = (\text{rank } \mathcal{F})[Z]$ .

Now we prove the proposition. If  $\text{supp } \mathcal{F} = Z \xrightarrow{\iota} X$  is irreducible so that  $\iota^* \mathcal{F}$  has constant rank on  $Z$ , then by what we have just proven  $\tau(\mathcal{F}|_Z)_{2 \dim Z} = (\text{rank } \mathcal{F}|_Z)[Z]$ . By naturality, and the fact that  $\iota_* \iota^* \mathcal{F} = \mathcal{F}$ , we have  $\tau(\mathcal{F})_{2 \dim Z} = (\text{rank } \mathcal{F}|_Z)[Z]$ . If  $Z$  has several irreducible components

$Z_1, \dots, Z_l$ , of dimension  $\dim Z$  then we may similarly consider the embeddings  $\iota_j : Z_j \hookrightarrow X$ . Then  $\text{rank } \mathcal{F}|_{Z_j} = \text{rank}((\iota_j)_* \mathcal{F}|_{Z_j})$ , and this latter sheaf is supported on  $Z_j$ , so that  $\tau((\iota_j)_* \mathcal{F}|_{Z_j})_{2 \dim Z} = (\text{rank } \mathcal{F}|_{Z_j})[Z_j]$ . Moreover, the natural map  $\mathcal{F} \rightarrow \bigoplus_j (\iota_j)_* \mathcal{F}|_{Z_j}$  is an injection, and the quotient is supported on the pairwise intersections of the  $Z_j$  (and the irreducible components of lower dimension), and so has zero  $\tau$  in dimension  $2 \dim Z$ . Therefore by properties (3) and (4) we have

$$\tau(\mathcal{F})_{2 \dim Z} = \tau\left(\bigoplus_j (\iota_j)_* \mathcal{F}|_{Z_j}\right)_{2 \dim Z} = \sum_j (\text{rank } \mathcal{F}|_{Z_j})[Z_j].$$

□

**Remark 3.6.** If  $X$  and  $Y$  are compact, complex manifolds, Atiyah and Hirzebruch [2] prove the topological Grothendieck-Riemann-Roch theorem. That is, for any continuous map  $f : X \rightarrow Y$ , the diagram:

$$\begin{array}{ccc} K_{\text{top}}^0(X) & \xrightarrow{ch^*} & H^{2*}(X, \mathbb{Q}) \\ f_* \downarrow & & \downarrow f_* \\ K_{\text{top}}^0(Y) & \xrightarrow{ch^*} & H^{2*}(Y, \mathbb{Q}) \end{array}$$

commutes up to multiplication by Todd classes on both sides, where  $f_*$  on both sides of the diagram is the Gysin map given by the induced map in homology and Poincaré duality. Combining this with Levy's theorem, it follows that for a proper holomorphic map  $f : X \rightarrow Y$  and any class  $\eta \in K_0^{\text{hol}}(X)$ ,

$$f_*(ch(\eta) \cdot Td(X)) = ch(f_!(\eta)) \cdot Td(Y)$$

in  $H^{2*}(Y, \mathbb{Q})$ . This is the exact analogue of the formula proven in [29], where the identity is in the Hodge ring rather than rational cohomology.

**Remark 3.7.** Define  $K_{\text{hol}}^0(X)$  to be the Grothendieck group of holomorphic vector bundles on a complex manifold  $X$ . Let  $\mathcal{F}$  be a coherent analytic sheaf on  $X$  defining a class  $\eta \in K_0^{\text{hol}}(X)$ . By the lemma in Fulton [15], there is a complex manifold  $\widehat{X}$  with a proper morphism  $\pi : \widehat{X} \rightarrow X$  an element  $\zeta \in K_{\text{hol}}^0(\widehat{X})$  such that  $\pi_!(\text{PD } \zeta) = \eta$ , (where  $\text{PD } \zeta$  is given by the class  $\zeta \otimes [\mathcal{O}_{\widehat{X}}]$ , the cap product with the fundamental class in  $K_0^{\text{hol}}(\widehat{X})$ ). Then by property (1), naturality of  $\tau$ , and the previous remark:

$$\begin{aligned} \tau(\mathcal{F}) &= \tau(\eta) = \pi_*(\tau(\text{PD } \zeta)) = \pi_*[(ch^* \text{PD } \zeta \cdot Td(\widehat{X})) \cap [\widehat{X}]] \\ &= [ch(\pi_!(\text{PD } \zeta)) \cdot Td(X)] \cap [X] = ch(\mathcal{F}) \cdot Td(X) \cap [X]. \end{aligned}$$

Therefore, in fact property (1) holds for arbitrary coherent sheaves.

Finally, we have the

*Proof of Proposition 3.1.* If  $\mathcal{T}$  is a torsion sheaf on a complex manifold  $X$ , then  $Z = \text{supp } \mathcal{T}$  is a complex analytic subvariety of  $X$  (via the annihilator ideal sheaf). Suppose  $\text{codim } Z = p$ . Then

for any  $k < p$ , by property (3) we have  $\tau(\mathcal{T})_{\dim 2n-2k} = 0$ . Applying Remark 3.7 we have:

$$\tau(\mathcal{T})_{\dim 2n-2} = \text{PD} \left[ c_1(\mathcal{T}) + \frac{\text{rank } \mathcal{T}}{2} c_1(X) \right] = \text{PD}[c_1(\mathcal{T})]$$

since  $\mathcal{T}$  is torsion and therefore has generic rank 0. If  $p \geq 2$ , the left hand side vanishes. By induction we therefore see that for any  $k < p$ , we have:

$$0 = \tau(\mathcal{T})_{\dim 2n-2k} = \text{PD} \left[ \sum_{i \leq k} ch_i(\mathcal{T}) \cdot Td_{k-i}(X) \right] = \text{PD}[ch_k(\mathcal{T})].$$

By Proposition 3.5,  $\tau(\mathcal{T})_{\dim 2n-2p} = \sum_j m_{Z_j} [Z_j]$ , and by Remark 3.7 it is also  $[ch(\mathcal{T}) \cdot Td(X)]_p \cap [X]$ . Since all components of the Chern character of degree less than  $p$  are zero, this is equal to  $\text{PD}[ch_p(\mathcal{T})]$ , so we have:  $ch_p(\mathcal{T}) = \text{PD}(\sum_j m_{Z_j} [Z_j])$ .  $\square$

**3.3. Proof of Proposition 3.3.** In this section we show that the second Chern character form of an admissible connection on a reflexive sheaf actually represents the cohomology class of the second Chern character, at least when the sheaf satisfies a certain technical, topological assumption. The proof of this result follows the general argument in [36]. However, instead of an admissible Yang-Mills connection and the corresponding monotonicity formula of Price which are the context of [36], we have the Chern connection of an admissible metric in the sense of Section 2.1. The uniform bound on the Hermitian-Einstein tensor means we may instead use the monotonicity formula and  $L^p$ -estimates derived for integrable connections on Kähler manifolds in [40]. For completeness, we provide details of the proof below.

First let us review the two key results of [40] that we will need. Let  $E \rightarrow X$  be a hermitian bundle with an integrable connection  $A$ , and let  $\sup_X |\Lambda_\omega F_A| \leq H_0$ . For constants  $C_1, C_2$ , and  $x \in X, \sigma > 0$ , define

$$(3.5) \quad e_A(x, \sigma) = C_1 \sigma^4 H_0^2 + (1 + C_2 \sigma^2)^{2n-2} \sigma^{4-2n} \int_{B_\sigma(x)} |F_A|^2 dvol_\omega$$

Then for appropriately chosen  $C_1, C_2, \sigma_0 > 0$  (depending only on the geometry of  $X$ ), it follows from [40, Thm. 3.5] that  $e_A(x, \sigma)$  is monotone increasing, i.e. for all  $0 < \sigma \leq \rho \leq \sigma_0$ ,

$$(3.6) \quad e_A(x, \sigma) \leq e_A(x, \rho)$$

Next, fix  $p > 2n$ . Then there are constants  $\varepsilon_0 > 0, C > 0$  (depending only on  $p$  and the geometry of  $X$ ) such that if  $4\sigma \leq \sigma_0$  and  $e_A(x, 4\sigma) < \varepsilon_0$ , then:

$$(3.7) \quad \left( \sigma^{2p-2n} \int_{B_\sigma(x)} |F_A|^p dvol_\omega \right)^{1/p} \leq C \left\{ \left( \sigma^{4-2n} \int_{B_{4\sigma}(x)} |F_A|^2 dvol_\omega \right)^{1/2} + \sigma^2 H_0 \right\}$$

(see [40, Thm. 2.6]). Under these assumptions, it follows from [38, Sec. 2] that one can find a gauge transformation  $g$  so that

$$(3.8) \quad \sup_{B_{\sigma/2}(x)} |g(A)| \leq \frac{C}{\sigma} \left( \sigma^{2p-2n} \int_{B_\sigma(x)} |F_A|^p dvol_\omega \right)^{1/p}$$

With these preliminaries we give the

*Proof of Proposition 3.3.* Let  $A$  denote the Chern connection of  $(\text{Gr}(\mathcal{E}), h)$ . We wish to show the current defined in (3.2) is closed, i.e.  $ch_2(\text{Gr}(\mathcal{E}), h)(\Omega) = 0$  for any  $\Omega = d\phi$ . The argument closely follows the proof in [36, Prop. 2.3.1]. By using a partition of unity we may assume  $\phi$  is compactly supported in a coordinate ball  $U$ . Note that  $\text{Gr}(\mathcal{E})$  is smoothly isomorphic to the underlying vector bundle  $E$  of  $\mathcal{E}$  on  $X - Z^{alg}$ . Hence, we may assume that  $A$  is an integrable connection on a trivial bundle on  $U$ , smooth away from  $Z$ , with  $F_A \in L^2$  and  $|\Lambda_\omega F_A|$  uniformly bounded by  $H_0$ . For  $0 < \sigma < \sigma_0$ ,  $0 < \varepsilon < \varepsilon_0$ , let

$$E_{\sigma, \varepsilon} = \{x \in U : e_A(x, 4\sigma) \geq \varepsilon\}$$

where  $e_A(x, \sigma)$  is defined in (3.5). Denote  $\sigma$ -neighborhoods of subsets of  $U$  by  $\mathcal{N}_\sigma$ .

Choose a covering  $\{B_{2\sigma}(x_k), B_{2\sigma}(y_k)\}$  of  $Z \cup E_{\sigma, \varepsilon}$ , with  $x_k \in Z$ ,  $y_k \in E_{\sigma, \varepsilon}$ , and such that the balls  $B_\sigma(x_k)$ ,  $B_\sigma(y_k)$ , are all disjoint. If  $x \notin \bigcup_k B_{6\sigma}(x_k) \cup B_{2\sigma}(y_k)$ , then  $B_{4\sigma}(x) \subset X - Z$ , and  $e_A(x, 4\sigma) < \varepsilon$ . In particular, for any  $x \notin \mathcal{N}_{8\sigma}(Z) \cup \mathcal{N}_{4\sigma}(E_{\sigma, \varepsilon})$ , there is a gauge transformation  $g$  such that (3.8) holds. As in [36, pp. 217-218], we can piece together the gauge transformations to obtain a global Chern-Simons form away from  $\mathcal{N}_{8\sigma}(Z) \cup \mathcal{N}_{4\sigma}(E_{\sigma, \varepsilon})$ :

$$CS(A) = \text{tr}(A \wedge F_A + (1/3)A \wedge A \wedge A)$$

for  $A$  in this gauge, with  $dCS(A) = \text{tr}(F_A \wedge F_A)$ . Now for  $x \in B_{16\sigma}(x_k) - B_{8\sigma}(x_k)$ , and using (3.7) and (3.8),

$$\begin{aligned} |CS(A)(x)| &\leq |A(x)||F_A(x)| + (1/3)|A(x)|^3 \\ &\leq \frac{1}{2\sigma}|A(x)|^2 + \frac{\sigma}{2}|F_A(x)|^2 + \frac{1}{3}|A(x)|^3 \\ &\leq \frac{C}{\sigma^3}(\sigma^{2p-2n} \int_{B_\sigma(x)} |F_A|^p d\text{vol}_\omega)^{2/p} + \frac{\sigma}{2}|F_A(x)|^2 \\ &\leq C\sigma^{1-2n} \int_{B_{4\sigma}(x)} |F_A|^2 d\text{vol}_\omega + C\sigma H_0^2 + \frac{\sigma}{2}|F_A(x)|^2 \\ (3.9) \quad &\leq C\sigma^{1-2n} \int_{B_{20\sigma}(x_k)} |F_A|^2 d\text{vol}_\omega + C\sigma H_0^2 + \frac{\sigma}{2}|F_A(x)|^2 \end{aligned}$$

Similarly, for  $y \in B_{8\sigma}(y_k) - B_{4\sigma}(y_k)$ ,

$$(3.10) \quad |CS(A)(y)| \leq C\sigma^{1-2n} \int_{B_{12\sigma}(y_k)} |F_A|^2 d\text{vol}_\omega + C\sigma H_0^2 + \frac{\sigma}{2}|F_A(y)|^2$$

(we have assumed  $\varepsilon \leq 1$ ). Now choose a smooth cut-off function  $\eta$ ,  $\eta(t) \equiv 0$  for  $t \leq 1$ ,  $\eta(t) \equiv 1$  for  $t \geq 2$ . It follows that

$$\begin{aligned}
\left| \int_X d\phi \wedge \text{Tr}(F_A \wedge F_A) \right| &= \lim_{\sigma \rightarrow 0} \left| \int_X \eta(\text{dist}(x, Z)/8\sigma) \eta(\text{dist}(x, E_{\sigma, \varepsilon})/4\sigma) d\phi \wedge dCS(A) \right| \\
&\leq \lim_{\sigma \rightarrow 0} \left\{ \int_{8\sigma \leq \text{dist}(x, Z) \leq 16\sigma} \frac{1}{8\sigma} |d\phi| |CS(A)| d\text{vol}_\omega(x) \right. \\
&\quad \left. + \int_{4\sigma \leq \text{dist}(x, E_{\sigma, \varepsilon}) \leq 8\sigma} \frac{1}{4\sigma} |d\phi| |CS(A)| d\text{vol}_\omega(x) \right\} \\
&\leq C \sup |d\phi| \lim_{\sigma \rightarrow 0} \sum_k \left\{ \int_{B_{20\sigma}(x_k)} (|F_A|^2 + CH_0^2) d\text{vol}_\omega \right. \\
&\quad \left. + \int_{B_{12\sigma}(y_k)} (|F_A|^2 + CH_0^2) d\text{vol}_\omega \right\} \\
(3.11) \quad &\leq C \sup |d\phi| \lim_{\sigma \rightarrow 0} \int_{\mathcal{N}_{20\sigma}(Z) \cup \mathcal{N}_{12\sigma}(E_{\sigma, \varepsilon})} (|F_A|^2 + CH_0^2) d\text{vol}_\omega
\end{aligned}$$

since the number of  $i, j, k, l$  such that the balls  $B_{20\sigma}(x_i)$ ,  $B_{20\sigma}(x_j)$ ,  $B_{12\sigma}(y_k)$ ,  $B_{12\sigma}(y_l)$ , intersect is bounded independently of  $\sigma$ .

**Claim .** For  $0 < \sigma' \leq \sigma$ , we have the following inclusions:

$$\begin{aligned}
\mathcal{N}_{20\sigma'}(Z) \cup \mathcal{N}_{12\sigma'}(E_{\sigma', \varepsilon}) &\subset \mathcal{N}_{20\sigma}(Z) \cup \mathcal{N}_{12\sigma}(E_{\sigma, \varepsilon}) \\
\bigcap_{\sigma > 0} \{ \mathcal{N}_{20\sigma}(Z) \cup \mathcal{N}_{12\sigma}(E_{\sigma, \varepsilon}) \} &\subset Z
\end{aligned}$$

Indeed, if  $y \in \mathcal{N}_{12\sigma'}(E_{\sigma', \varepsilon})$  and  $y \notin \mathcal{N}_{20\sigma}(Z)$ , then there is  $x \in E_{\sigma', \varepsilon}$  such that  $d(x, y) < 12\sigma' \leq 12\sigma$ , and if  $z \in Z$ , then  $20\sigma \leq d(y, z) \leq d(x, y) + d(x, z) < d(x, z) + 12\sigma$ , so  $B_{4\sigma}(x) \subset X - Z$ . Now (3.6) applies, and so  $\varepsilon \leq e_A(x, 4\sigma') \leq e_A(x, 4\sigma)$ . It follows that  $x \in E_{\sigma, \varepsilon}$  and  $y \in \mathcal{N}_{12\sigma}(E_{\sigma, \varepsilon})$ . This proves the first statement in the claim. The second statement follows from the fact that  $A$  is smooth away from  $Z^{alg}$ ; hence,  $\lim_{\sigma \rightarrow 0} e_A(x, \sigma) = 0$  for  $x \notin Z^{alg}$ . Now by the claim and the dominated convergence theorem, the limit on the right hand side of (3.11) vanishes, and closedness of  $ch_2(\text{Gr}(\mathcal{E}), h)$  follows.

Since  $ch_2(\text{Gr}(\mathcal{E}), h)$  is a closed current it defines a cohomology class. By Poincaré duality, to check that indeed  $[ch_2(\text{Gr}(\mathcal{E}), h)] = ch_2(\text{Gr}(\mathcal{E})^{**})$ , it suffices to show that for any smooth, closed  $(2n - 4)$ -form  $\Omega$  whose cohomology class is dual to a 4-dimensional rational homology class  $[\Sigma]$ ,

$$(3.12) \quad ch_2(\text{Gr}(\mathcal{E})^{**})[\Sigma] = ch_2(\text{Gr}(\mathcal{E}), h)(\Omega)$$

Since a multiple of a rational homology class is represented by an embedded manifold, and since  $\text{Gr}(\mathcal{E})^{**}$  is locally free away from a set of (real) codimension  $\geq 6$ , by a transversality argument we may assume (after passing to an integer multiple) that the homology class  $[\Sigma]$  is represented by a smoothly embedded submanifold  $\Sigma \subset X - \text{sing Gr}(\mathcal{E})^{**}$ . By the Thom isomorphism we may then choose the form  $\Omega$  to be compactly supported in  $X - \text{sing Gr}(\mathcal{E})^{**}$ . Find a global resolution

$$(3.13) \quad 0 \longrightarrow E_r \longrightarrow E_{r-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow \text{Gr}(\mathcal{E})^{**} \otimes_{\mathcal{O}_X} \mathcal{A}_X \longrightarrow 0$$

where the  $E_i$  are real analytic complex vector bundles on  $X$ , and fix smooth connections  $\nabla_i$  on  $E_i$ . Then by Definition 3.2,

$$(3.14) \quad ch_2(\mathrm{Gr}(\mathcal{E})^{**}) = \left[ -\frac{1}{8\pi^2} \sum_{i=0}^r (-1)^i \mathrm{tr}(F_{\nabla_i} \wedge F_{\nabla_i}) \right]$$

If we choose a sequence of smooth hermitian metrics  $h_j^{**}$  on the locally free part of  $\mathrm{Gr}(\mathcal{E})^{**}$  with Chern connections  $A_j^{**}$ , then since (3.13) is an exact sequence of analytic vector bundles away from  $\mathrm{sing} \mathrm{Gr}(\mathcal{E})^{**}$ , there are smooth forms  $\Psi_j$  such that

$$(3.15) \quad -\frac{1}{8\pi^2} \mathrm{tr}(F_{A_j^{**}} \wedge F_{A_j^{**}}) + \frac{1}{8\pi^2} \sum_{i=0}^r (-1)^i \mathrm{tr}(F_{\nabla_i} \wedge F_{\nabla_i}) = d\Psi_j$$

on  $X - \mathrm{sing} \mathrm{Gr}(\mathcal{E})^{**}$ . Finally, by Theorem 2.1 (ii), we may arrange that  $h_j^{**} \rightarrow h$  in  $L_{2,loc}^p$  for some  $p > 2n$ . Then  $h_j^{**}$  and  $(h_j^{**})^{-1}$  are uniformly bounded on compact subsets of  $X - \mathrm{sing} \mathrm{Gr}(\mathcal{E})^{**}$ , and it follows that  $F_{A_j^{**}} \rightarrow F_A$  in  $L_{loc}^2$ . Then for  $\Omega$  as above, we obtain from (3.14) and (3.15) that

$$\begin{aligned} ch_2(\mathrm{Gr}(\mathcal{E})^{**})[\Sigma] &= -\frac{1}{8\pi^2} \int_X \Omega \wedge \sum_{i=0}^r (-1)^i \mathrm{tr}(F_{\nabla_i} \wedge F_{\nabla_i}) \\ &= -\int_X \Omega \wedge \left( \frac{1}{8\pi^2} \mathrm{tr}(F_{A_j^{**}} \wedge F_{A_j^{**}}) + d\Psi_j \right) \\ &= -\frac{1}{8\pi^2} \int_X \Omega \wedge \mathrm{tr}(F_{A_j^{**}} \wedge F_{A_j^{**}}) \quad \text{for all } j \\ &= -\frac{1}{8\pi^2} \lim_{j \rightarrow \infty} \int_X \Omega \wedge \mathrm{tr}(F_{A_j^{**}} \wedge F_{A_j^{**}}) \\ &= -\frac{1}{8\pi^2} \int_X \Omega \wedge \mathrm{tr}(F_A \wedge F_A) \end{aligned}$$

Hence, from the definition (3.2), eq. (3.12) holds, and this completes the proof of the proposition.  $\square$

#### 4. COMPARISON OF SINGULAR SETS

**4.1. A slicing lemma.** Let  $z$  be a smooth point of a codimension 2 subvariety  $Z \subset X$ . We say that  $\Sigma$  is a *transverse slice* to  $Z$  at  $z$  if  $\Sigma \cap Z = \{z\}$  and  $\Sigma$  is the intersection of a linear subspace  $\mathbb{C}^2 \hookrightarrow \mathbb{C}^n$  in some coordinate ball centered at  $z$  that is transverse to  $T_z Z$  at the origin. Suppose that  $T$  is a smooth, closed  $(2, 2)$  form and that we have an equation

$$T = mZ + dd^c \Psi$$

where  $\Psi$  is a  $(1, 1)$ -current,  $mZ$  is the current of integration over the nonsingular points of  $Z$  with multiplicity  $m$ , and the equation is taken in the sense of distributions. Then for a transverse slice,

$$(4.1) \quad m = \int_{\Sigma} T - \int_{\partial \Sigma} d^c \Psi$$

Indeed, if we choose local coordinates so that a neighborhood of  $z$  is biholomorphic to a polydisk  $\Delta \in \mathbb{C}^n$ , and  $Z \cap \Delta$  is given by the coordinate plane  $z_1 = z_2 = 0$ , then by King's formula (cf. [12, Ch. III, 8.18]) we have an equation of currents on  $\Delta$ :

$$T = dd^c [\Psi + mudd^c u]$$

where  $u(z) = (1/2) \log(|z_1|^2 + |z_2|^2)$  (here,  $-\pi i dd^c = \partial \bar{\partial}$ ). Write  $T = dd^c \alpha$  for a smooth form  $\alpha$  on  $\Delta$ . By the regularity theorem and the Poincaré lemma, we can find a smooth form  $\beta$  such that

$$d^c [\Psi - \alpha + mudd^c u] = d\beta$$

on  $\Delta$ . Let  $\Delta_\varepsilon = \{z \in \Delta : |z| \leq \varepsilon\}$ . It follows that:

$$\begin{aligned} \int_\Sigma T &= \lim_{\varepsilon \rightarrow 0} \int_{\Sigma \cap \Delta_\varepsilon} T = \lim_{\varepsilon \rightarrow 0} \int_{\Sigma \cap \Delta_\varepsilon} dd^c \Psi = \int_{\partial \Sigma} d^c \Psi - \lim_{\varepsilon \rightarrow 0} \int_{\Sigma \cap \partial \Delta_\varepsilon} d^c \Psi \\ &= \int_{\partial \Sigma} d^c \Psi - \lim_{\varepsilon \rightarrow 0} \int_{\Sigma \cap \partial \Delta_\varepsilon} d^c \alpha + m \lim_{\varepsilon \rightarrow 0} \int_{\Sigma \cap \partial \Delta_\varepsilon} d^c (udd^c u) \\ &= \int_{\partial \Sigma} d^c \Psi + m \end{aligned}$$

by direct computation. The next result shows that the analytic multiplicities may also be calculated by restricting to transverse slices.

**Lemma 4.1.** *Let  $A_i$  be as in Theorem 2.8 and  $Z \subset Z_b^{an}$  an irreducible component of the blow-up set. For a transverse slice  $\Sigma$  at a smooth point  $z \in Z$ , we have:*

$$m_Z^{an} = \lim_{i \rightarrow \infty} \frac{1}{8\pi^2} \int_\Sigma \{ \text{Tr}(F_{A_i} \wedge F_{A_i}) - \text{Tr}(F_{A_\infty} \wedge F_{A_\infty}) \}.$$

*Proof.* Assume without loss of generality that  $\Sigma \subset B_\sigma(z)$ , where the ball is chosen so that  $A_i \rightarrow A_\infty$  smoothly and uniformly on compact subsets of  $B_{2\sigma}(z) - Z$ . We furthermore assume the exponential map  $\exp_z$  at  $z$  defines a diffeomorphism onto  $B_{2\sigma}(z)$ , and that  $Z \cap B_{2\sigma}(z)$  is a submanifold. For  $\lambda > 0$ , let  $A_{i,\lambda}$  be the connection on  $T_z X$  obtained by pulling back  $A_i$  by the exponential map, followed by the rescaling  $v \mapsto \lambda v$  (cf. [36, Sec. 3]). Then by definition of the blow-up connection and the uniqueness of tangent cones, there is a sequence  $\lambda_i \downarrow 0$  such that

$$m_Z^{an} = \lim_{i \rightarrow \infty} \frac{1}{8\pi^2} \int_{V^\perp \cap \mathbf{B}_1(0)} \left\{ \text{tr}(F_{A_{i,\lambda_i}} \wedge F_{A_{i,\lambda_i}}) - \text{tr}(F_{A_\infty, \lambda_i} \wedge F_{A_\infty, \lambda_i}) \right\}$$

(see [36, eq. (4.2.7)] and [21, eq. (5.5)]). The notation  $V^\perp$  denotes the orthogonal complement of  $V = T_z Z \subset T_z X$ , and  $\mathbf{B}_1(0)$  is the unit ball about the origin. Let  $S_{\lambda_i} = \exp_z(\lambda_i(V^\perp \cap \mathbf{B}_1(0)))$ . For sufficiently large  $i$  (i.e.  $0 < \lambda_i$  small), we may assume  $S_{\lambda_i} \subset B_\sigma(z)$ . At this point we choose smooth maps  $u_i : V^\perp \cap \mathbf{B}_1(0) \times [0, 1] \rightarrow B_{2\sigma}(z)$ , such that

- $u_i(V^\perp \cap \mathbf{B}_1(0), 0) = \Sigma$ ;
- $u_i(V^\perp \cap \mathbf{B}_1(0), 1) = S_{\lambda_i}$ ;
- $u_i(v, t) \in B_{2\sigma}(z) - Z$  for all  $|v| = 1$  and all  $t \in [0, 1]$ .

To be precise, the  $u_i$  can be constructed as follows. Without loss of generality assume:



- (i)  $Z \cap B_{2\sigma}(0) = \{\exp_z(g(w), w) : w \in V \cap \mathbf{B}_1(0)\}$ . Here,  $g(0) = 0$ , so we have  $|g(w)| \leq C_1|w|$  for some constant  $C_1$  and all  $w \in V \cap \mathbf{B}_1(0)$ .
- (ii)  $\Sigma = \{\exp_z(v, f(v)) : v \in V^\perp \cap \mathbf{B}_1(0)\}$ . We may assume this form for any transverse slice by the implicit function theorem. Since  $f(0) = 0$ , we may assume, after possibly shrinking the slice, that  $|f(v)| \leq C_2$  for all  $v \in V^\perp \cap \mathbf{B}_1(0)$ , where  $C_1 C_2 \leq 1/2$ .
- (iii)  $S_{\lambda_i} = \{\exp_z(\lambda_i v, 0) : v \in V^\perp \cap \mathbf{B}_1(0)\}$ .

Now for  $v \in V^\perp \cap \mathbf{B}_1(0)$ ,  $t \in [0, 1]$ , set

$$u_i(v, t) = \exp_z(((1-t) + t\lambda_i)v, (1-t)f(v))$$

Then by (ii) and (iii), the image of  $u_i(\cdot, 0)$  is  $\Sigma$ , and the image of  $u_i(\cdot, 1)$  is  $S_{\lambda_i}$ . Moreover, we may assume that  $u(v, t) \in B_{2\sigma}(x)$  by changing the radius of  $\mathbf{B}_1(0)$ . Finally, note that by (ii),  $|(1-t)f(v)| \leq (1-t)C_2$ , whereas  $|((1-t) + t\lambda_i)v| = (1-t) + t\lambda_i$  for  $|v| = 1$ . If  $u_i(v, t) \in Z$ , then by (i) we would have  $(1-t) + t\lambda_i \leq C_1 C_2 (1-t) \leq (1/2)(1-t)$ , which is impossible since  $\lambda_i > 0$ . Hence,  $u_i(v, t) \notin Z$  for  $|v| = 1$ . With this understood, we obtain

$$\begin{aligned}
0 &= \frac{1}{8\pi^2} \int_{V^\perp \cap \mathbf{B}_1(0) \times [0,1]} d\{u_i^* \operatorname{tr}(F_{A_i} \wedge F_{A_i}) - u_i^* \operatorname{tr}(F_{A_\infty} \wedge F_{A_\infty})\} \\
&= \frac{1}{8\pi^2} \int_{V^\perp \cap \mathbf{B}_1(0)} \left\{ \operatorname{tr}(F_{A_i, \lambda_i} \wedge F_{A_i, \lambda_i}) - \operatorname{tr}(F_{A_\infty, \lambda_i} \wedge F_{A_\infty, \lambda_i}) \right\} \\
(4.2) \quad &\quad - \frac{1}{8\pi^2} \int_{\Sigma} \{ \operatorname{tr}(F_{A_i} \wedge F_{A_i}) - \operatorname{tr}(F_{A_\infty} \wedge F_{A_\infty}) \} \\
&\quad + \frac{1}{8\pi^2} \int_{V^\perp \cap \partial \mathbf{B}_1(0) \times [0,1]} u_i^* \{ \operatorname{tr}(F_{A_i} \wedge F_{A_i}) - \operatorname{tr}(F_{A_\infty} \wedge F_{A_\infty}) \}
\end{aligned}$$

Since  $A_i$  and  $A_\infty$  are connections on the same bundle away from  $Z$ , we may write  $d_{A_i} = d_{A_\infty} + a_i$ , and define the Chern-Simons transgression,

$$(4.3) \quad CS(A_i, A_\infty) = \operatorname{tr}(a_i \wedge d_{A_\infty}(a_i)) + (2/3)a_i \wedge a_i \wedge a_i + 2a_i \wedge F_{A_\infty}$$

Then

$$\begin{aligned}
&\frac{1}{8\pi^2} \int_{V^\perp \cap \partial \mathbf{B}_1(0) \times [0,1]} u_i^* \{ \operatorname{tr}(F_{A_i} \wedge F_{A_i}) - \operatorname{tr}(F_{A_\infty} \wedge F_{A_\infty}) \} \\
&= \frac{1}{8\pi^2} \int_{V^\perp \cap \partial \mathbf{B}_1(0) \times [0,1]} u_i^* (dCS(A_i, A_\infty)) \\
&= \frac{1}{8\pi^2} \int_{V^\perp \cap \partial \mathbf{B}_1(0)} CS(A_i, \lambda_i, A_\infty, \lambda_i) - \frac{1}{8\pi^2} \int_{\partial \Sigma} CS(A_i, A_\infty)
\end{aligned}$$

Since  $A_i \rightarrow A_\infty$  uniformly away from  $Z$  and  $A_i, \lambda_i, A_\infty, \lambda_i \rightarrow 0$  on compact subsets of  $V^\perp - \{0\}$  (cf. [21, p. 469]), this term vanishes as  $i \rightarrow \infty$ . We conclude from (4.2) that

$$\begin{aligned}
&\lim_{i \rightarrow \infty} \frac{1}{8\pi^2} \int_{V^\perp \cap \mathbf{B}_1(0)} \{ \operatorname{tr}(F_{A_i, \lambda_i} \wedge F_{A_i, \lambda_i}) - \operatorname{tr}(F_{A_\infty, \lambda_i} \wedge F_{A_\infty, \lambda_i}) \} \\
&= \lim_{i \rightarrow \infty} \frac{1}{8\pi^2} \int_{\Sigma} \{ \operatorname{tr}(F_{A_i} \wedge F_{A_i}) - \operatorname{tr}(F_{A_\infty} \wedge F_{A_\infty}) \}
\end{aligned}$$

and the proof is complete.  $\square$

**4.2. Proof of Theorem 1.1.** Throughout this section we let  $A_i = A_{t_i}$ ,  $t_i \rightarrow \infty$  be a sequence along the Yang-Mills flow, and suppose  $A_i \rightarrow A_\infty$  is an Uhlenbeck limit with analytic singular set  $Z^{an}$ . Let  $Z^{alg}$  denote the algebraic singular set of the HNS filtration of the initial holomorphic bundle  $\mathcal{E}$ . Then we have the following

**Proposition 4.2.** *Let  $Z$  be an irreducible codimension 2 component of  $Z^{alg}$ . Then  $Z \subset Z_b^{an}$  and  $m_Z^{alg} = m_Z^{an}$ . Moreover,  $Z_b^{an} \subset Z^{alg}$ .*

To isolate contributions from individual components, we will first need an argument similar to the one used in [11, Lemma 6].

**Lemma 4.3.** *Let  $Z \subset Z^{alg}$  be an irreducible codimension 2 component. Then there exists a modification  $\pi : \widehat{X} \rightarrow X$  with center  $\mathbf{C}$  and exceptional set  $\mathbf{E} = \pi^{-1}(\mathbf{C})$ , and a filtration of  $\pi^*\mathcal{E}$  with associated graded sheaf  $\mathrm{Gr}(\pi^*\mathcal{E}) \rightarrow \widehat{X}$  and singular set  $\mathrm{sing} \mathrm{Gr}(\pi^*\mathcal{E})$ , all with the following properties:*

- (i)  $\mathrm{Gr}(\pi^*\mathcal{E}) \simeq \mathrm{Gr}(\mathcal{E})$  on  $\widehat{X} - \mathbf{E} = X - \mathbf{C}$ .
- (ii)  $\mathrm{codim}(Z \cap \mathbf{C}) \geq 3$ .
- (iii)  $\mathrm{codim}(\pi(\mathrm{sing} \mathrm{Gr}(\pi^*\mathcal{E})) - Z) \geq 3$ .

*Proof.* By Hironaka's theorem, we may find a resolution  $\widehat{X}_1 \rightarrow X$  of the singularities of  $Z^{alg}$ . Note that the center of this modification has codimension  $\geq 3$  in  $X$ . Let  $W \subset Z^{alg}$  be a codimension 2 irreducible component other than  $Z$ , and let  $\widehat{W}_1$  denote the strict transform of  $W$  in  $\widehat{X}_1$ . By assumption,  $\widehat{W}_1$  is smooth. We are going to define a sequence of monoidal transformations

$$\widehat{X}_n \longrightarrow \widehat{X}_{n-1} \longrightarrow \cdots \longrightarrow \widehat{X}_1 \longrightarrow X$$

First, let  $\pi_2 : \widehat{X}_2 \rightarrow \widehat{X}_1$  be the blow-up of  $\widehat{X}_1$  along  $\widehat{W}_1$ , and consider the induced filtration of  $\pi_2^*\mathcal{E}$  by saturated subsheaves with associated graded  $\mathrm{Gr}(\pi_2^*\mathcal{E})$ . After a possible further desingularization in codimension 3, we may assume without loss of generality that  $\mathrm{sing} \mathrm{Gr}(\pi_2^*\mathcal{E})$  is smooth in  $\widehat{X}_2$ . Moreover, any codimension 2 component of  $\mathrm{sing} \mathrm{Gr}(\pi_2^*\mathcal{E})$  that contains the generic  $\mathbb{P}^1$ -fiber of the exceptional divisor of  $\widehat{X}_2 \rightarrow \widehat{X}_1$  projects in  $X$  to a proper subvariety of  $W$ ; hence, up to a codimension 3 set in  $X$ , we may ignore these components. Let  $\widehat{W}_2$  denote the union of the (other) codimension 2 components of  $\mathrm{sing} \mathrm{Gr}(\pi_2^*\mathcal{E})$  in the exceptional set of  $\widehat{X}_2 \rightarrow \widehat{X}_1$ . Again,  $\widehat{W}_2$  is smooth by assumption. Define  $\widehat{X}_3$  to be the blow-up of  $\widehat{X}_2$  along  $\widehat{W}_2$ . Repeat this process in the same manner to obtain recursively  $\widehat{X}_k$ , for  $k$  greater than 3. We now claim that after a finite number of steps  $n$ , this process stabilizes:  $\widehat{W}_n$  is empty and  $\widehat{X}_{n+1} = \widehat{X}_n$ . In other words, the part of  $\mathrm{sing} \mathrm{Gr}(\pi_n^*\mathcal{E})$  in the exceptional set of  $\widehat{X}_n \rightarrow \widehat{X}_1$  projects in  $X$  to a proper subvariety of  $W$ . Note that this then implies  $\mathrm{codim}(\pi_n(\mathrm{sing} \mathrm{Gr}(\pi_n^*\mathcal{E})) - Z) \geq 3$ . To prove the claim it clearly suffices to consider the case of a single step filtration:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q}^{**} \longrightarrow \mathcal{T} \longrightarrow 0$$

where  $\mathcal{S}$  and  $\mathcal{E}$  are locally free at generic points of  $W$ . Let  $\mathcal{J}$  be the sheaf of ideals generated by the determinants of  $\text{rank } \mathcal{S} \times \text{rank } \mathcal{S}$  minors of the map  $\mathcal{S} \rightarrow \mathcal{E}$  with respect to local trivializations of both bundles near a point  $p \in W$ . The vanishing set of  $\mathcal{J}$  is  $W$  by definition. If  $\Sigma$  is a transverse slice to  $W$  at the point  $p$ , let  $\mathcal{J}_\Sigma$  denote the ideal sheaf in  $\mathcal{O}_\Sigma$  generated by the restriction of the generators of  $\mathcal{J}$  to  $\Sigma$ . Note that  $\mathcal{O}_\Sigma/\mathcal{J}_\Sigma$  is supported precisely at  $p$ . By [28, Thm. 14.14, Thm. 14.13, and Thm. 14.10], and the fact that the stalk  $\mathcal{O}_{\Sigma,p}$  is a Cohen-Macaulay local ring, there are germs  $f_1, f_2 \in \mathcal{J}_{\Sigma,p}$  so that  $\dim_{\mathbb{C}}(\mathcal{O}_{\Sigma,p}/\langle f_1, f_2 \rangle) = e(\mathcal{J}_{\Sigma,p})$ , the Hilbert-Samuel multiplicity of the ideal  $\mathcal{J}_{\Sigma,p}$ , and this latter number is constant for slices through generic smooth points of  $W$ . Let  $D_1, D_2$  be the divisors associated to  $f_1$  and  $f_2$ , and let  $\widehat{D}_1, \widehat{D}_2$  be the strict transforms of  $D_1, D_2$  in the blow-up of  $\Sigma$  at  $p$ . Then the intersection multiplicity  $\langle \widehat{D}_1, \widehat{D}_2 \rangle$  is *strictly* less than  $\langle D_1, D_2 \rangle$  (cf. [33, p. 210, Corollary 3]), the difference depending on the order of vanishing of  $f_1$  and  $f_2$  at  $p$ . This means that after a fixed number of blow-ups, depending only on  $\langle D_1, D_2 \rangle$ ,  $\widehat{D}_1$  and  $\widehat{D}_2$  are disjoint. But since  $D_1$  and  $D_2$  intersect only at  $p$ , the intersection multiplicity  $\langle D_1, D_2 \rangle$  is equal to  $\dim_{\mathbb{C}}(\mathcal{O}_{\Sigma,p}/\langle f_1, f_2 \rangle)$  by definition. It follows that after a finite number of blow-ups  $\pi_n : \widehat{X}_n \rightarrow X$  as described above, the number depending only upon the Hilbert-Samuel multiplicity  $e(\mathcal{J}_{\Sigma,p})$  of a generic slice, the strict transforms of the divisors corresponding to (the extensions of)  $f_1$  and  $f_2$  in  $\mathcal{J}$  intersect at most in a set  $\widehat{Z}_W$  that projects to a proper subvariety of  $W$ . If  $\widehat{\mathcal{S}}$  is the saturation of  $\pi_n^*\mathcal{S}$  in  $\pi_n^*\mathcal{E}$ , then  $\Lambda^{\text{rank } \mathcal{S}} \widehat{\mathcal{S}}$  is the saturation of  $\pi_n^*\Lambda^{\text{rank } \mathcal{S}} \mathcal{S}$ , and so  $\widehat{\mathcal{S}}$  is a subbundle away from  $\widehat{Z}_W$ . This proves the claim. The lemma now follows by carrying out the procedure above on all codimension 2 components of  $Z^{\text{alg}}$  other than  $Z$ .  $\square$

*Proof of Proposition 4.2.* Choose an irreducible codimension 2 component  $Z \subset Z^{\text{alg}}$ . We wish to show that  $m_Z^{\text{alg}} = m_Z^{\text{an}}$ . Since  $m_Z^{\text{alg}} \neq 0$ , it will follow that  $Z \subset Z^{\text{an}}$ . In fact, since  $\text{sing } A_\infty$  in the decomposition (2.8) has codimension at least 3,  $Z \subset Z_b^{\text{an}}$ . We therefore proceed to prove the equality of multiplicities.

First, since  $X$  is Kähler the rational homology class of  $[Z]$  is nonzero. Therefore, there is a class in  $H_4(X, \mathbb{Q})$  whose intersection product with  $[Z]$  is non-trivial. Since an integral multiple of any class (not in top dimension) can be represented by an embedded submanifold (see [35]), we may in particular choose a closed, oriented 4-real dimensional submanifold  $\Sigma \subset X$  representing a class  $[\Sigma]$  in  $H_4(X, \mathbb{Q})$ , so that the intersection product  $[\Sigma] \cdot [Z] \neq 0$ . Furthermore, since  $\dim Z^{\text{alg}} + \dim \Sigma = \dim X$  we may choose  $\Sigma$  so that it meets  $Z^{\text{alg}}$  only in the smooth points of the codimension 2 components, and this transversely. Since the intersection multiplicity  $[\Sigma] \cdot [Z] \neq 0$ , we have  $\Sigma \cap Z = \{z_1, \dots, z_p\}$  for some finite (non-empty) set of points. Clearly, we can assume  $\Sigma$  is a (positive or negatively oriented) transverse slice at each point of intersection with  $Z^{\text{alg}}$ . Let  $\widehat{X}$  and  $\text{Gr}(\pi^*\mathcal{E})$  be as in Lemma 4.3. By transversality and part (iii) of the lemma, we may arrange so that the strict transform  $\widehat{\Sigma}$  of  $\Sigma$  is embedded and  $\widehat{\Sigma}$  intersects  $\text{sing } \text{Gr}(\pi^*\mathcal{E})$  only along  $\pi^{-1}(Z)$ . Choose  $\sigma > 0$  so that for each  $k = 1, \dots, p$ ,

- $B_{2\sigma}(z_k) \subset X - \mathbf{C}$  (by Lemma 4.3 (ii))
- $B_{2\sigma}(z_k) \cap Z^{\text{alg}} \subset Z - \text{sing } \text{Gr}(\mathcal{E})^{**}$

Let  $A_\infty$  be a smooth connection on  $\text{Gr}(\mathcal{E})^{**}$  over  $\cup_{k=1}^p B_{2\sigma}(z_k)$ , and fix a Kähler metric on  $\widehat{X}$ . By the construction in [3] (see Theorem 2.1 (i)), and noting Lemma 4.3 (i), we can extend  $A_\infty$  to a  $\hat{\omega}$ -admissible connection  $\widehat{A}_\infty$  on  $\text{Gr}(\pi^*\mathcal{E})^{**}$ . Given a smooth connection  $A$  on  $\mathcal{E}$ , let  $\pi^*A$  denote the pull-back connection on  $\pi^*\mathcal{E}$ . Then by Theorem 1.3 we have

$$(4.4) \quad \frac{1}{8\pi^2} \text{tr}(F_{\pi^*A} \wedge F_{\pi^*A}) - \frac{1}{8\pi^2} \text{tr}(F_{\widehat{A}_\infty} \wedge F_{\widehat{A}_\infty}) = \sum \hat{m}_j^{alg} \widehat{W}_j + dd^c \widehat{\Psi}$$

where the  $\widehat{W}_j$  are the codimension 2 components of  $\text{supp}(\text{Gr}(\pi^*\mathcal{E})^{**}/\text{Gr}(\pi^*\mathcal{E}))$ . Notice that by Lemma 4.3 (i) and the choice of  $\Sigma$ ,  $\text{Gr}(\pi^*\mathcal{E})$  is locally free in a neighborhood of  $\widehat{\Sigma} \cap (\cup_{k=1}^p B_{2\sigma}(z_k))^c$ , and there is one component,  $\widehat{W}_1$  say, such that  $\pi(\widehat{W}_1) = Z$ , while all other components  $\widehat{W}_j$  miss  $\widehat{\Sigma}$ . Moreover,  $\hat{m}_1^{alg} = m_Z^{alg}$ . Now the difference of the Chern forms for  $\widehat{A}_\infty$  and  $\pi^*A$  are related by a Chern-Simons class  $dCS(\pi^*A, \widehat{A}_\infty) = dd^c \widehat{\Psi}$  in a neighborhood of  $\widehat{\Sigma}$  away from  $\pi^{-1}(Z)$  (cf. (4.3)). We can then use this fact to obtain, by (4.1) and (4.4),

$$(4.5) \quad \begin{aligned} ([\Sigma] \cdot [Z])m_Z^{alg} &= \frac{1}{8\pi^2} \int_{\Sigma \cap (\cup_{k=1}^p B_{2\sigma}(z_k))} \{\text{tr}(F_A \wedge F_A) - \text{tr}(F_{A_\infty} \wedge F_{A_\infty})\} - \int_{\Sigma \cap \partial(\cup_{k=1}^p B_{2\sigma}(z_k))} d^c \widehat{\Psi} \\ &= \frac{1}{8\pi^2} \int_{\Sigma \cap (\cup_{k=1}^p B_{2\sigma}(z_k))} \{\text{tr}(F_A \wedge F_A) - \text{tr}(F_{A_\infty} \wedge F_{A_\infty})\} + \int_{\widehat{\Sigma} \cap (\cup_{k=1}^p B_{2\sigma}(z_k))^c} dd^c \widehat{\Psi} \\ &= \frac{1}{8\pi^2} \int_{\Sigma \cap (\cup_{k=1}^p B_{2\sigma}(z_k))} \{\text{tr}(F_A \wedge F_A) - \text{tr}(F_{A_\infty} \wedge F_{A_\infty})\} + \int_{\Sigma \cap (\cup_{k=1}^p B_{2\sigma}(z_k))^c} dCS(\pi^*A, \widehat{A}_\infty) \\ &= \frac{1}{8\pi^2} \int_{\Sigma \cap (\cup_{k=1}^p B_{2\sigma}(z_k))} \{\text{tr}(F_A \wedge F_A) - \text{tr}(F_{A_\infty} \wedge F_{A_\infty})\} - \int_{\Sigma \cap \partial(\cup_{k=1}^p B_{2\sigma}(z_k))} CS(A, A_\infty) \end{aligned}$$

Finally, apply the above to a sequence  $A_i$  of connections converging to  $A_\infty$  away from  $Z^{an}$  as in Theorem 2.8. Then  $CS(A_i, A_\infty) \rightarrow 0$  on  $\Sigma \cap \partial B_{2\sigma}(z_k)$  for each  $k$ ; hence the second term in (4.5) vanishes in the limit. By Lemma 4.1,

$$\lim_{i \rightarrow \infty} \frac{1}{8\pi^2} \int_{\Sigma \cap (\cup_{k=1}^p B_{2\sigma}(z_k))} \{\text{tr}(F_{A_i} \wedge F_{A_i}) - \text{tr}(F_{A_\infty} \wedge F_{A_\infty})\} = ([\Sigma] \cdot [Z])m_Z^{an}$$

By (4.5) we therefore obtain  $([\Sigma] \cdot [Z])m_Z^{alg} = ([\Sigma] \cdot [Z])m_Z^{an}$ , and since  $[\Sigma] \cdot [Z] \neq 0$  we conclude that  $m_Z^{alg} = m_Z^{an}$ . This is the first assertion in the statement of Proposition 4.2. It implies that the cycle

$$\sum_j m_j^{an} Z_j^{an} - \sum_j m_k^{alg} Z_k^{alg}$$

has nonnegative coefficients. But by Corollary 3.4, the corresponding current is also cohomologous to zero. Hence, all codimension 2 components of  $Z_b^{an}$  must in fact be contained in  $Z^{alg}$  with the same multiplicities. As mentioned previously, by the theorem of Tian and Harvey-Shiffman,  $Z_b^{an}$  has pure codimension 2, and so this proves the second statement.  $\square$

Proposition 4.2 proves part (3) of Theorem 1.1. For part (1), note that by Proposition 4.2, the irreducible codimension 2 components of  $Z^{alg}$  and  $Z^{an}$  coincide. We claim that  $\text{sing } A_\infty = \text{sing } \text{Gr}(\mathcal{E})^{**}$ . Indeed, if  $p \notin \text{sing } \text{Gr}(\mathcal{E})^{**}$ , then by definition  $\text{Gr}(\mathcal{E})^{**}$  is locally free in a neighborhood of  $p$ . It follows from [3, Theorem 2 (c)] that the direct sum of the admissible Hermitian-Einstein

metrics on the stable summands of  $\text{Gr}(\mathcal{E})^{**}$  is smooth at  $p$ , and hence,  $p \notin \text{sing } A_\infty$ . Conversely, if  $p \notin \text{sing } A_\infty$  then the direct sum of the admissible Hermitian-Einstein metrics extends to a smooth bundle over  $p$ . But since  $\text{Gr}(\mathcal{E})^{**}$  is reflexive and hence normal, this implies  $p \notin \text{sing } \text{Gr}(\mathcal{E})^{**}$ . Given these equalities, part (1) now follows from the decompositions (2.8) and (2.4) and Proposition 2.3. By Remark 1.2 (ii), part (2) of Theorem 1.1 also follows. The proof of Theorem 1.1 is complete.

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