Rationality of moduli spaces of parabolic bundles

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ABSTRACT. The moduli space of parabolic bundles with fixed determinant over a smooth curve of positive genus is proved to be rational whenever one of the multiplicities associated to the quasi-parabolic structure is equal to one. It follows that the moduli space of nonparabolic bundles is in general stably rational, and rational in many cases.

1. INTRODUCTION

Let X be a smooth complex curve of genus $g \geq 1$, L a line bundle of degree d over X, and $\mathcal{M}_{r,L}$ the moduli space of semistable bundles E of rank r with determinant L.

Conjecture 1.1. *$\mathcal{M}_{r,L}$ is rational, i.e. it is birational to a projective space.*

Despite many positive results [12], this is still an open problem, even for $(r,d) = 1$.

In this paper, we study a closely related problem, namely the birational classification of moduli spaces of parabolic bundles over X. These moduli spaces occur naturally

(i) as unitary representation spaces of Fuchsian groups [10],
(ii) as moduli spaces of Yang-Mills connections on X with an orbifold metric [5], and
(iii) as moduli spaces of certain semistable bundles over an elliptic surface [3].

The theory developed in [7] and extended here shows that their birational type depends only on the *quasi-parabolic* structure. The methods of [12] then prove, in many cases, that these moduli spaces are rational. The weaker result, Theorem 6.1, uses only Newstead's theorem, while the stronger one, Theorem 6.2, requires an adaptation of his inductive argument.

A direct consequence is that $\mathcal{M}_{r,L}$ is stably rational, which had been proved by Ballico in the case $(r,d) = 1$ [2]. We then show why this and our bound on the level implies Conjecture 1.1 under the assumptions that $(r,d) = 1$ and either $(g,d) = 1$ or $(g, r-d) = 1$.

A number of useful facts are established along the way. One key point is Proposition 3.2, which gives a simple criterion for the existence of a universal bundle of stable parabolic bundles. We also extend the theory developed in [7] in several important ways (Theorems

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4.1, 4.2, and 5.3); the first two are standard but necessary for our purposes and the third is completely new. Its proof requires the idea of shifting a parabolic sheaf [17], which also provides a framework for the Hecke correspondence. All of these results play a crucial role in the proofs of Theorems 6.1 and 6.2.

A brief word about the organization of this paper: §2 introduces the notation used in the following sections, §3 discusses the existence of universal families, §4 summarizes and extends the theory of [7], §5 describes shifting and the Hecke correspondence, and §6 contains the proofs of the main results and their corollaries.

Before we begin, we would like to acknowledge a certain debt to the work of Newstead, upon which a number of our arguments depend, and without which this paper would be inconceivable.

2. Notation

Let X be a smooth curve of genus $g \geq 1$ and D a reduced divisor in X. If E is a $\mathbb{C}^r$ bundle over X, then a *parabolic structure* on E with respect to D is just a collection of weighted flags in the fibers of E over each $p \in D$ of the form

$$E_p = F_1(p) \supset F_2(p) \supset \cdots \supset F_{s_p}(p) \supset 0, \tag{1}$$

$$0 \leq \alpha_1(p) < \alpha_2(p) < \cdots < \alpha_{s_p}(p) < 1. \tag{2}$$

Holomorphic bundles E with parabolic structures are called *parabolic bundles*, and we use the notation E_* to indicate the bundle (or, equivalently, locally-free sheaf) E together with a choice of parabolic structure. A morphism $\phi : E_* \longrightarrow E'_*$ of parabolic bundles is a bundle map satisfying $\phi(F_i(p)) \subset F'_{j+1}(p)$ whenever $\alpha_i(p) > \alpha'_j(p)$ for all $p \in D$. We use the tensor product notation $H^0(E_*^\vee \otimes E'_*)$ for these morphisms, where $E_*^\vee$ denotes the dual parabolic bundle (cf. [17]).

A *quasi-parabolic* structure on E is what is left after the weights are forgotten, it is determined topologically by the *multiplicities* $m(p) = (m_1(p), \dots, m_{s_p}(p))$, defined for each $p \in D$ by $m_i(p) = \dim F_i(p) - \dim F_{i+1}(p)$.

A subbundle E' inherits a parabolic structure from one on E in a canonical way: The flag in E'_p is gotten by intersecting with the flag in E_p and the weights are determined by choosing maximal weights with the property that the inclusion map from E' to E is parabolic (p. 213, [10]). Parabolic structures on quotients E'' of E have a similar description (loc. cit.).

A parabolic bundle E_* is called *stable* if every proper holomorphic subbundle E' satisfies $\mu(E'_*) < \mu(E_*)$, where

$$\mu(E_*) = \text{pardeg } E_*/r = \deg E/r + \sum_{p \in D} \sum_{i=1}^{s_p} m_i(p)\alpha_i(p)/r.$$

The parabolic bundle E_* is called *semistable* if $\mu(E'_*) \leq \mu(E_*)$ for each subbundle E'_*. The construction of the moduli space $\mathcal{M}_\alpha$ of semistable parabolic bundles, as a normal, projective variety, is given in [10]. The subspace $\mathcal{M}^s_\alpha$ of stable bundles is smooth, in particular, if every semistable bundle is stable, then $\mathcal{M}_\alpha$ is smooth.

Let $\Delta^r = \{(a_1, \ldots, a_r) \mid 0 \leq a_1 \leq \cdots \leq a_r < 1\}$ and define $W = \{\alpha : D \longrightarrow \Delta^r\}$. Points in W determine both the weights and the multiplicities. Conversely, given a weight α in the sense of (2), the associated point in W is gotten by repeating each $\alpha_i(p)$ according to it multiplicity $m_i(p)$. We abuse notation slightly by referring to points in W as weights. This gives an obvious notion of when a weight is *compatible* with a choice of multiplicities, and for a given m, we define the open face of weights compatible with m to be

$$V_m = \{\alpha \in W \mid \alpha_{i-1}(p) = \alpha_i(p) \Leftrightarrow \sum_{k=1}^{j} m_k(p) < i \leq \sum_{k=1}^{j+1} m_k(p)\}.$$

A weight in the interior of W specifies full flags at each $p \in D$. For every other choice of m, V_m is contained in the boundary of W. Now W is a simplicial set, and the face relations give a natural ordering on $\{V_m\}$ and we write $V_m > V_{m'}$ if $V_{m'}$ is a proper face contained in the closure of V_m. This agrees with the natural ordering on m gotten by successive refinement.

Weights for which $\mathcal{M}_\alpha$ is not necessarily smooth satisfy $\mu(E'_*) = \mu(E_*)$ for some proper subbundle E'. Letting E'' be the quotient, then the short exact sequence of parabolic bundles $E'_* \xrightarrow{\iota} E_* \xrightarrow{\pi} E''_*$ determines a partition of (d, r, m) in the obvious way: $(d', d''), (r', r'')$ and (m', m'') are the degrees, ranks, and multiplicities of (E', E''). (We define m' and m'' here slightly unconventionally, namely

$$\begin{aligned} m'_i(p) &= \dim(F_i(p) \cap \iota(E'_p)) - \dim(F_{i+1}(p) \cap \iota(E'_p)), \\ m''_i(p) &= \dim(\pi(F_i(p)) \cap E''_p) - \dim(\pi(F_{i+1}(p)) \cap E''_p), \end{aligned}$$

for $p \in D$ and $1 \leq i \leq s_p$.) Notice that $r', r'' > 0$ and $m'_i(p), m''_i(p) \geq 0$. Write $\xi = (d', r', m')$. For fixed ξ, the set of weights compatible with m for which $\mu(E'_*) = \mu(E_*)$ is the hyperplane H_ξ in V_m given by the equation

$$\sum_{p \in D} \sum_{i=1}^{s_p} (m_i(p) - m'_i(p))\alpha_i(p) = rd' - r'd. \tag{3}$$

There are only finitely many hyperplanes; the above equation puts a bound on d' and all other quantities are already bounded. The hyperplanes induce a chamber structure on V_m, a *chamber* being a connected component of $V_m \setminus \cup_\xi H_\xi$. Weights contained in a chamber are called *generic*.

3. Families of parabolic bundles

In this section, we establish the existence of a universal family of stable parabolic bundles parametrized by $\mathcal{M}^s_\alpha$ whenever V_m contains a generic weight. This condition is shown to be equivalent to requiring that the multiplicities $\{m_i(p)\}$ and the degree d form a set which is relatively prime. The conclusion, Proposition 3.2, is that $\mathcal{M}_\alpha$ is fine in this case. This is consistent with the known results:

(i) $\mathcal{M}_{r,L}$ is fine if $(r,d)=1$ (Theorem 5.12, [13]),

(ii) $\mathcal{M}_\alpha$ is fine if $D=p$ and $m(p)=(r-1,1)$ (Théorème 32, [14]).

Definition 3.1. *Given a bundle $U \to S \times X$, we adopt the notation $U_s = U|_{\{s\}\times X}$. We also use π_S for the projection map $S \times X \to S$.*

(i) *A family of stable parabolic bundles parametrized by a variety S is a bundle U over $S \times X$ so that U_s is a stable parabolic bundle of a fixed type (i.e. fixed d, m and α) for all $s \in S$.*

(ii) *Two families U and U' parametrized by S are equivalent, written $U' \sim U$, if there exists a line bundle L over S so that $U' \cong U \otimes \pi_S^* L$.*

It follows from the construction of Mehta and Seshadri that $\mathcal{M}_\alpha$ is a coarse moduli space (see Remark 4.6 of [10], or [9]). This means that for any family of stable parabolic bundles U parametrized by S, there is a unique morphism $\psi_U : S \to \mathcal{M}_\alpha^s$ so that $\psi_U(s) = [U_s]$. A *universal* family $\mathcal{U}^\alpha$ is one parametrized by $\mathcal{M}_\alpha^s$ so that for any family U parametrized by S, we have $U \sim (\psi_U \times 1_X)^* \mathcal{U}^\alpha$. If there exists a universal family, then $\mathcal{M}_\alpha^s$ is a *fine* moduli space. (See [13] for a more thorough explanation of these matters.) Such a universal family, if it exists, is clearly only determined up to equivalence.

It suffices to find a family $\mathcal{U}^\alpha$ over $\mathcal{M}_\alpha^s$ such that $\mathcal{U}_e^\alpha \cong E_*$ for $e = [E_*] \in \mathcal{M}_\alpha$, for then it follows that any two families U and U' parametrized by S are equivalent if and only if $\psi_U = \psi_{U'}$. For if $U \sim U'$, then $U_s \cong U'_s$ for all $s \in S$, which shows that $\psi_U(s) = \psi_{U'}(s)$. Conversely, if $\psi_U = \psi_{U'}$, then $U_s \cong U'_s$ for all $s \in S$. Since U_s and U'_s are both stable, $H^0(U_s^\vee \otimes U'_s) \cong \mathbb{C}$ and $(R^0\pi_S)(U^\vee \otimes U')$ is a locally free sheaf of rank 1 over S whose corresponding line bundle gives $U \sim U'$.

To describe the universal family $\mathcal{U}^\alpha$, we need to review the construction of $\mathcal{M}_\alpha$ (see [10] and [9]). Let Q be the Hilbert scheme of coherent sheaves over X which are quotients of $\mathcal{O}_X^{\oplus N}$ with fixed Hilbert polynomial (that of $E(k)$ for $k \gg g$), where $N = h^0(E)$. Let U be the universal family on $Q \times X$. Define R to be the subscheme of Q of points $r \in Q$ so that U_r is a locally free sheaf which is generated by its global sections and $h^1(U_r) = 0$. Let $\widetilde{R}$ be the total space of the universal flag bundle over R with flag type that of the quasi-parabolic structure. Then $\widetilde{R}$ has the local universal property for parabolic bundles (p. 16, [9]).

The subsets $\widetilde{R}^s$ ($\widetilde{R}^{ss}$) corresponding to the stable (semistable) parabolic bundles are invariant under the natural action of $\mathrm{GL}(N) = \mathrm{Aut}(\mathcal{O}_X^{\oplus N})$, and $\mathcal{M}_\alpha$ is a good quotient of $\widetilde{R}^{ss}$ (with linearization induced by the weights α), and $\mathcal{M}_\alpha^s$ is the geometric quotient of $\widetilde{R}^s$.

The center of $\mathrm{GL}(N)$ acts trivially on R and $\widetilde{R}$, but nontrivially on the locally universal bundles. In fact, $\lambda(\mathrm{id})$ acts on $\widetilde{U}$ by scalar multiplication by λ in the fibers (this follows from p. 138, [13]). Given a line bundle L over $\widetilde{R}^{ss}$ with a natural lift of the $\mathrm{GL}(N)$ action such that $\lambda(\mathrm{id})$ acts by multiplication by λ, then using $\widetilde{U}^{ss}$ to denote $\widetilde{U}|_{\widetilde{R}^{ss}\times X}$, the quotient of $\widetilde{U}^{ss}\otimes\pi_X^*L^{-1}$ gives a universal family.

Proposition 3.2. *Such a line bundle L exists if either of the two equivalent conditions hold:*

(i) *The set $\{d, m_i(p) \mid s\in D, 1\le i\le s_p\}$ is relatively prime.*

(ii) *The face V_m containing α contains a generic weight.*

If either of these conditions are satisfied, then the moduli space $\mathcal{M}_\alpha$ is fine.

The idea of the proof is to find line bundles L_k for each $k\in\{d, m_i(p)\}$ over $\widetilde{R}^{ss}$ with natural actions of $\mathrm{GL}(N)$ such that $\lambda(\mathrm{id})$ acts by scalar multiplication by λ^k. Then (i) gives the existence of $k_1,\dots,k_l\in\{d,m_i(p)\}$ and integers $a_1,\dots,a_l$ so that $a_1k_1+\cdots a_lk_l=1$. The required line bundle is then the tensor product $L=L_{k_1}^{a_1}\otimes\cdots\otimes L_{k_l}^{a_l}$. At the end of the proof, we will show that (i) and (ii) are equivalent.

We start with a lemma.

Lemma 3.3. *If E_* is parabolic semistable and H_* is a parabolic line bundle of degree h, then*

$$h^1(H_*^\vee\otimes E_*)\neq 0 \quad\Rightarrow\quad d\le r(2g-2+h)+r^2n. \tag{4}$$

Proof. Serre duality for parabolic bundles (Proposition 3.7 of [17]) implies that

$$h^1(H_*^\vee\otimes E_*)\le h^0(E_*^\vee\otimes H_*\otimes K(D)).$$

(If we had used $h^0(E_*^\vee\otimes\widehat{H}_*\otimes K(D))$, the circumflex over H_* indicating *strongly* parabolic morphisms, we would get the usual statement of Serre duality with equality, see [17, 8].) Suppose that $\phi: E\longrightarrow H\otimes K(D)$ is a non-zero map and let E' be the subbundle generated by $\operatorname{Ker}\phi$. Then

$$\deg E'\ge\deg E-\deg H\otimes K(D)=d-h-(2g-2+n).$$

Considering E'_* with its canonical parabolic structure as a subbundle of rank $r-1$, the inequality (4) follows easily from this, semistability of E_*, and the inequalities $\operatorname{pardeg}E'_*\ge\deg E'$ and $\operatorname{pardeg}E_*\ge\deg E+rn$. □

Proof of Proposition. Write the weights α without repetition. Choose $\ell: D\longrightarrow\mathbb{Z}$ with $1\le\ell_p\le s_p+1$ and set $\beta(p)=\alpha_{\ell_p}(p)$. (Take $\beta(p)>\alpha_{s_p}$ if $\ell_p=s_p+1$.) For $h\in\mathbb{Z}$, define

$$\chi(\ell,h)=d+r(1-g-h)-\sum_{p\in D}\sum_{i=1}^{\ell_p-1}m_i(p).$$

Let H_* be the parabolic line bundle with $\deg H=h<d/r-rn-(2g-2)$ and with weights $\beta(p)$ at $p\in D$. It follows from the lemma that if E_* is semistable, then $h^1(H_*^\vee\otimes E_*)=0$. Thus $h^0(H_*^\vee\otimes E_*)=\chi(\ell,h)$ by Riemann-Roch. Hence $(R^0\pi_{\widetilde{R}^{ss}})(\widetilde{U}^{ss}\otimes\pi_X^*H_*)$ is a locally

free sheaf of rank $\chi(\ell,h)$ over $\widetilde{R}^{ss}$. Let $L(\ell,h)$ be the determinant of the corresponding bundle. By construction, the $\mathrm{GL}(N)$ action on $\widetilde{U}$ induces one on this bundle (and hence on $L(\ell,h)$); $\lambda(\mathrm{id})$ acts by scalar multiplication by λ on the bundle and by $\lambda^{\chi(\ell,h)}$ on $L(\ell,h)$. It is now a simple exercise in high school algebra to see that we can choose h,h' and ℓ,ℓ' so that $\lambda(\mathrm{id})$ acts on $L(\ell,h)\otimes L(\ell',h')$ by λ^k for any $k\in\{d,m_i(p)\}$.

This proves the conclusion of the proposition assuming (i), and now we show that conditions (i) and (ii) are equivalent. Suppose first that (i) does not hold. Consider E_* as a quasi-parabolic bundle without holomorphic structure, which will be specified later. Since the set $\{d,m_i(p)\}$ is not relatively prime, there exists a prime number q so that q divides d and each $m_i(p)$. Note that q also divides $r=\sum_{i=1}^{s_p}m_i(p)$. Set $d'=d/q, r'=r/q$ and $m_i'(p)=m_i(p)/q$. Consider now the quasi-parabolic bundle E_*' with degree d', rank r', and multiplicities m'. Any choice of weights α on E_* induces (the same!) weights on E_*', and it follows that since $g\geq 1$, there is some holomorphic structure for which E_*' is semistable. Define the holomorphic structure on E_* by

$$E_*=E_*'\oplus\overset{q}{\cdots}\oplus E_*'.$$

It follows that E_* is semistable but not stable for *any* choice of compatible weights. This implies that V_m does not contain a generic weight.

Suppose conversely that V_m does not contain a generic weight. Since V_m is affine, $V_m\subset H_\xi$ for some $\xi=(r',d',m')$ Using (3), we conclude that for all $\alpha\in V_m$,

$$\sum_{p\in D}\sum_{i=1}^{s_p}(rm_i'(p)-r'm_i(p))\alpha_i(p)=rd'-r'd.$$

(Here, we are still thinking of α without repetition.) We can vary each $\alpha_i(p)$ continuously by some small amount, and it follows that

$$rm_i'(p)-r'm_i(p)=0=rd'-r'd$$

for all i and p. Since $r'<r$, there exists a prime q such that q^k divides r but not r'. Hence q divides d and each $m_i(p)$. Thus the set $\{d,m_i(p)\}$ is not relatively prime. □

4. The variation and degeneration theorems

In this section, we describe and extend the theory of [7]. This allows us to compare the moduli spaces of parabolic bundles $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ when

(i) $\alpha,\beta\in V_m$ are generic weights in adjacent chambers,

(ii) $\alpha\in V_l$ and $\beta\in V_m$ are generic weights with $V_l>V_m$.

Cases (i) and (ii) correspond to Theorem 3.1 and Proposition 3.4 of [7]. We present slightly stronger versions of those results tailored for our purposes here.

Starting with (i), suppose that $\alpha,\beta\in V_m$ are generic weights separated by a single hyperplane H_ξ. Choose $\gamma\in H_\xi$ on the straight line connecting α to β. Then $\mathcal{M}_\gamma$ is stratified by the Jordan-Hölder type of the underlying bundle, and since γ lies on only one hyperplane, there are exactly two strata: the stable bundles $\mathcal{M}^s_\gamma$ and the strictly semistable bundles Σ_γ. Writing $\xi = (r', d', m')$ for the partition, then it is not hard to see that $\Sigma_\gamma \cong \mathcal{M}_{\gamma'} \times \mathcal{M}_{\gamma''}$, with the obvious definitions for γ' and γ'' coming from the partition ξ.

Theorem 4.1. *There are natural algebraic maps ϕ_α and ϕ_β*

$$\begin{array}{ccc} \mathcal{M}_\alpha & & \mathcal{M}_\beta \\ & \phi_\alpha \searrow \quad \swarrow \phi_\beta & \\ & \mathcal{M}_\gamma & \end{array}$$

which are generized blow-downs along projectivizations of vector bundles over Σ_γ, where the projective fiber dimensions e_α and e_β satisfy $e_\alpha + e_\beta + 1 = \operatorname{codim} \Sigma_\gamma$.

The proof is the same as in [7], the only difference being the actual computation of the numbers e_α and e_β, which we discuss now. We assume that $E_* \sim_S E'_* \oplus E''_*$, where $[E_*] \in \Sigma_\gamma$ and $\sim_S$ denotes Sesahdri equivalence (i.e. isomorphic Jordan-Hölder form). The topological type of the parabolic bundles E'_* and E''_* does not change as $[E_*]$ varies within Σ_γ. We use $(r', r''), (d', d'')$ and (m', m'') to denote the ranks, degrees, and multiplicities of (E'_*, E''_*), written as in §2. The moduli spaces $\mathcal{M}_\alpha, \mathcal{M}_\beta$, and $\mathcal{M}_\gamma$ have dimension

$$(g-1)r^2 + 1 + \frac{1}{2}\sum_{p\in D} r^2 - \sum_{i=1}^{s_p} m_i(p)^2.$$

Using a similar formula for $\Sigma_\gamma = \mathcal{M}^{\gamma'} \times \mathcal{M}^{\gamma''}$, we find that

$$\operatorname{codim} \Sigma_\gamma = r'r''(2g-1) - 1 + \sum_{p\in D}\sum_{i=1}^{s_p} m'_i(p) m''_i(p).$$

Now we claim that

$$h^0(E''^{\vee}_* \otimes E'_*) = 0 = h^0(E'^{\vee}_* \otimes E''_*).$$

This is true for any $\alpha' \in V_m$, as one of these equations is true for α, the other for β, but H^0 is constant as the weights are varied within V_m. Let $\mathcal{U}'$ and $\mathcal{U}''$ be the families parametrized by Σ_γ gotten by pulling back the universal families $\mathcal{U}^{\gamma'}$ and $\mathcal{U}^{\gamma''}$, whose existence follows from Proposition 3.2. Then the vector bundles referred to in the theorem are

$$(R^1\pi_{\Sigma_\gamma})(\mathcal{U}''^{\vee} \otimes \mathcal{U}') \text{ and } (R^1\pi_{\Sigma_\gamma})(\mathcal{U}'^{\vee} \otimes \mathcal{U}'').$$

The projectivizations of these bundles have dimensions

$$e_\alpha = h^1(E''^{\vee}_* \otimes E'_*) - 1 = r''d' - r'd'' + r'r''(g-1) + \chi(\mathcal{Q}) - 1, \tag{5}$$

$$e_\beta = h^1(E'^{\vee}_* \otimes E''_*) - 1 = r'd'' - r''d' + r'r''(g-1) + \chi(\mathcal{Q}') - 1, \tag{6}$$

where $\mathcal{Q}$ and $\mathcal{Q}'$ are skyscraper sheaves supported on D obtained as the quotients

$$\mathfrak{ParHom}(E''_*, E'_*) \longrightarrow \mathfrak{Hom}(E'', E') \longrightarrow \mathcal{Q},$$

$$\mathfrak{ParHom}(E'_*, E''_*) \longrightarrow \mathfrak{Hom}(E', E'') \longrightarrow \mathcal{Q}'.$$

It is a nice exercise to see

$$\chi(\mathcal{Q}) + \chi(\mathcal{Q}') = \sum_{p\in D} \left(r'r'' - \sum_{(i,j)\in S_e(p)} m_i'(p) m_j''(p) \right),$$

where $S_e(p) = \{(i,j) \mid \gamma_i'(p) = \gamma_j''(p)\}$. This shows $e_\alpha + e_\beta + 1 = \operatorname{codim} \Sigma_\gamma$.

Now suppose that $\alpha \in V_l$ and $\beta \in V_m$, that $V_l > V_m$, and that α and β lie in incident chambers.

Theorem 4.2. *There exists a map $\psi : \mathcal{M}_\alpha \longrightarrow \mathcal{M}_\beta$ whose restriction to $\psi^{-1}(\mathcal{M}_\beta^s)$ is a morphism with fiber a flag variety in case α is generic.*

The existence of a map ψ was proved in [5], where it was also shown that the fibers over $\mathcal{M}_\beta^s$ are flag varieties. What remains is to explain why the restriction is a morphism when α is generic. This however follows quite easily from the existence of the universal family over $\mathcal{M}_\alpha$. If $[E_*] \in \psi^{-1}(\mathcal{M}_\beta^s)$, then, by definition, E_* is stable and remains stable when viewed as parabolic bundle with weights β. Applying this to the restriction of the universal family $\mathcal{U}^\alpha$ to $\psi^{-1}(\mathcal{M}_\beta^s) \times X$, we obtain a family of stable bundles (with weights β and multiplicities m) parameterized by $\psi^{-1}(\mathcal{M}_\beta^s)$. Because $\mathcal{M}_\beta$ is a coarse moduli space, we get a morphism from $\psi^{-1}(\mathcal{M}_\beta^s)$ to $\mathcal{M}_\beta$, which obviously coincides with ψ.

Remark. In the special case where both α and β are generic, this implies that ψ is a fibration in the Zariski topology with fiber a flag variety.

5. Shifting and the Hecke correspondence

In this section, we introduce the notion of a shifted parabolic bundle, which is the result of changing the weights, multiplicities, and degree of E_* in a prescribed way. In some sense, shifting is a symmetry of a larger weight space, one which includes bundles of different degrees. Two applications of shifting are discussed at the end.

Shifting is most naturally described in terms of parabolic sheaves. If $\mathcal{E}$ is a locally free sheaf on X, then a *parabolic structure* on $\mathcal{E}$ consists of a weighted filtration of the form

$$\mathcal{E} = \mathcal{E}_{\alpha_1} \supset \mathcal{E}_{\alpha_2} \supset \cdots \supset \mathcal{E}_{\alpha_l} \supset \mathcal{E}_{\alpha_{l+1}} = \mathcal{E}(-D), \tag{7}$$

$$0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_l < \alpha_{l+1} = 1. \tag{8}$$

We can define $\mathcal{E}_x$ for $x \in [0,1]$ by setting $\mathcal{E}_x = \mathcal{E}_{\alpha_i}$ if $\alpha_{i-1} < x \le \alpha_i$, and then extend to $x \in \mathbb{R}$ by setting $\mathcal{E}_{x+1} = \mathcal{E}_x(-D)$. We call the resulting filtered sheaf $\mathcal{E}_*$ a parabolic sheaf and $\mathcal{E} = \mathcal{E}_0$ the underlying sheaf.

We can define parabolic subsheaves, degree, and stability for these objects, and there is a categorical equivalence between locally free parabolic sheaves and parabolic bundles. We describe this in case $D = p$, the general case being quite similar ([15], [6]).

Suppose that E_* is a parabolic bundle given by flags and weights in the fibers as in (1) and (2). Define $\mathcal{E}_*$ by setting

$$\mathcal{E}_x = \ker(E \to E_p/F_i),$$

for $\alpha_{i-1} < x < \alpha_i$. Thus $\mathcal{E}_*$ is a parabolic sheaf. Conversely, given a parabolic sheaf $\mathcal{E}_*$, the quotient $\mathcal{E}_0/\mathcal{E}_1 = \mathcal{E}/\mathcal{E}(-p)$ is a skyscraper sheaf with support p and fiber that of $\mathcal{E}$. Defining a flag in this fiber by setting $F_i = (\mathcal{E}_{\alpha_i}/\mathcal{E}_1)_p$ and associating the weight α_i, we obtain a parabolic bundle in the sense of (1) and (2).

The category of parabolic sheaves is developed in [15], where one finds for example the definitions of tensor products $\mathcal{E}_* \otimes \mathcal{E}'_*$ and duals $\mathcal{E}^\vee_*$. We use this notation freely in the calculations of §6 involving sheaf cohomology and point out that $H^i(\mathcal{E}_*) = H^i(\mathcal{E})$.

Definition 5.1. *Given a parabolic sheaf $\mathcal{E}_*$ and $\eta \in \mathbb{R}$, define the shifted parabolic sheaf $\mathcal{E}_*[\eta]_*$ by setting $\mathcal{E}_*[\eta]_x = \mathcal{E}_{x+\eta}$.*

Remark. The above operation can be refined in case $D = p_1 + \cdots + p_n$. If $\eta = (\eta_1, \ldots, \eta_n)$, then one can shift $\mathcal{E}_*$ by η_i at each $p_i \in D$ ([15], [5]).

It is not difficult to verify that $\mathcal{E}_*[\eta]_*$ is (semi)stable if and only if $\mathcal{E}_*$ is (semi)stable, and it follows that this defines an isomorphism between the associated moduli spaces of parabolic bundles.

We can easily describe the parabolic structure on the shifted bundle $\mathcal{E}'_* = \mathcal{E}_*[\eta]_*$ in case $0 < \eta \le 1$ and $D = p$. Let E'_* denote the parabolic bundle associated to $\mathcal{E}'_*$. If i is the integer with $\alpha_i < \eta \le \alpha_{i+1}$, then the weights of E'_* are given by

$$\alpha'_j = \begin{cases} \alpha_{j+i} - \eta & \text{for } j = 1, \ldots, r-i, \\ 1 + \alpha_{j-r+i} - \eta & \text{for } j = r-i+1, \ldots, r. \end{cases} \tag{9}$$

The quasi-parabolic structure of E'_* has multiplicities m' given by a cyclic permutation of m, i.e. $m' = (m_{i+1}, \ldots, m_s, m_1, \ldots, m_i)$. Although $\mathcal{E}'$ is a subsheaf of $\mathcal{E}$, E' is *not* a subbundle of E, so one must appeal to sheaf theory in order to define the flag in E'_p. This is a simple exercise in tracing through the equivalence between locally free parabolic sheafs and parabolic bundles given above.

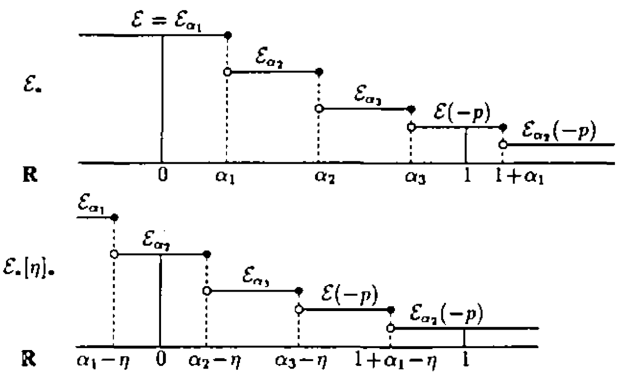

FIGURE 1. The parabolic sheaf $\mathcal{E}_*$ shifted by η with $\alpha_1 < \eta < \alpha_2$.

There are two interesting applications of shifting we discuss now. The first is the Hecke correspondence. Using $\mathcal{M}_{r,d}$ to denote the moduli space of semistable bundles of rank r and degree d, the Hecke correspondence gives a means of comparing $\mathcal{M}_{r,d}$ and $\mathcal{M}_{r,d'}$ through the use of parabolic bundles. For $r = 2$, this was observed in a remark at the end of [8].

To start, define $\epsilon_+(d,r), \epsilon_-(d,r)$, and $\epsilon(d,r)$ for $d, r \in \mathbb{Z}$ with $r > 0$ by

$$\begin{aligned} \epsilon_\pm(d,r) &= \inf\{\pm(\tfrac{d}{r} - \tfrac{d'}{r'}) \mid d', r' \in \mathbb{Z},\ 1 \leq r' < r, \text{ and } \pm(\tfrac{d}{r} - \tfrac{d'}{r'}) > 0\} \\ \epsilon(d,r) &= \min\{\epsilon_\pm(d,k) \mid k = 1, \dots, r\}. \end{aligned}$$

It is easy to see that $\epsilon_\pm(d,k) > 0$ for all k, thus $\epsilon(d,r) > 0$ as well.

Suppose that E is a bundle over X of degree d and rank r and suppose further that E' is a proper subbundle. If $\mu(E') < \mu(E)$, then $\mu(E) - \mu(E') \geq \epsilon_+(d,r)$. Similarly, if $\mu(E') > \mu(E)$, then $\mu(E') - \mu(E) \geq \epsilon_-(d,r)$.

Proposition 5.2. *Suppose that E_* satisfies $\sum_{p\in D}\sum_{i=1}^{s_p} m_i(p)\alpha_i(p) < \epsilon(d,r)/2$.*

(i) *If E is stable as a regular bundle, then E_* is parabolic stable.*

(ii) *If E_* is parabolic stable, then E is semistable as a regular bundle.*

Proof. (i) If E'_* is a proper parabolic subbundle of E_*, then

$$\mu(E'_*) \leq \mu(E') + \epsilon(d,r)/2 < \mu(E') + \epsilon_+(d,r) \leq \mu(E) < \mu(E_*),$$

thus E_* is parabolic stable.

(ii) If E' is a subbundle of E, then

$$\mu(E') \leq \mu(E'_*) < \mu(E_*) < \mu(E) + \epsilon(d,r)/2 < \mu(E) + \epsilon_-(d,r),$$

hence $\mu(E') \leq \mu(E)$ and E is semistable. □

We thus get a map $\mathcal{M}_\alpha \longrightarrow \mathcal{M}_{r,d}$ which is actually a special case of the map of Theorem 4.2. By choosing the weights and quasi-parabolic structure correctly, we can fit $\mathcal{M}_{r,d}$ and $\mathcal{M}_{r,d-1}$ into a chain diagram of maps as follows. Let $D = p$ and $m = (1, r-1)$. Choose weights $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 + (r-1)\alpha_2 < \epsilon(r,d)/2$. Choose η with $\alpha_1 < \eta < \alpha_2$, and let E'_* denote the parabolic bundle E_* shifted by η. Notice that E'_* has degree $d-1$, weights $\alpha' = (\alpha_2 - \eta, 1 - \eta + \alpha_1)$, and multiplicities $m' = (r-1, 1)$. Choose $\beta' \in V_{m'}$ generic with $(r-1)\beta'_1 + \beta'_2 < \epsilon(r,d)/2$. Connect α' to β' in $V_{m'}$ by a line passing through a finite number of hyperplanes $H_{\xi^1}, \dots, H_{\xi^n}$. Choose weights α^i in the intermediate chambers and $\gamma^i \in H_{\xi^i}$ for $i = 1, \dots, n$ with $\alpha^n = \beta'$. Theorem 4.1 applies each time we cross a hyperplane, while the above proposition gives us maps from $\mathcal{M}_\alpha$ to $\mathcal{M}_{r,d}$ and from $\mathcal{M}_{\beta'}$ to $\mathcal{M}_{r,d-1}$, which, when restricted to the preimages of $\mathcal{M}^s_{r,d}$ and $\mathcal{M}^s_{r,d-1}$, are $\mathbb{P}^{r-1}$ bundles by Theorem 4.2. This is summarized in the following diagram.

$$\begin{array}{ccccccc} \mathcal{M}_\alpha \cong \mathcal{M}_{\alpha'} & & & \mathcal{M}_{\alpha^1} & & & \mathcal{M}_{\beta'} \\ \downarrow & \searrow & \swarrow & & \searrow & \swarrow & \downarrow \\ \mathcal{M}_{r,d} & & \mathcal{M}_{\gamma^1} & & \cdots & & \mathcal{M}_{r,d-1} \end{array}$$

The second application of shifting is to extend the results of [5] to a case which is natural from the point of view of representations of Fuchsian groups but less natural from the point of view of parabolic bundles. Assume for simplicity that $\mu(E_*) = 0$ and $D = p$. Thus, $\deg E = -k$ for some $0 \leq k < r$, and the relevant weight space is

$$W_k = \{(\alpha_1, \ldots, \alpha_r) \in \Delta^r \mid \alpha_1 + \cdots + \alpha_r = k\}.$$

Consider the union $\widetilde{W} = \bigcup_{k=0}^{r-1} W_k$, where we identify

$$\partial_0 W_k = \{\gamma \in W_k \mid \gamma_1 = 0\}$$

with its companion set

$$\partial_1 \overline{W}_{k+1} = \{\overline{\gamma} \in \overline{W}_{k+1} \mid \overline{\gamma}_r = 1\}$$

via the identification

$$\partial_0 W_k \ni \gamma = (0, \gamma_2, \ldots, \gamma_n) \sim (\gamma_2, \ldots, \gamma_n, 1) = \overline{\gamma} \in \partial_1 \overline{W}_{k+1}. \tag{10}$$

Then $\widetilde{W}$ is the Weyl chamber for $\mathrm{SU}(r)$, and from this point of view $\partial_0 W_k$ is an interior hyperplane of $\widetilde{W}$. (Notice that $\partial_0 W_k$ really does satisfy the condition (3) for being a hyperplane.)

Theorem 4.1 does not carry over to this case immediately because points in W_k and W_{k+1} are weights on parabolic bundles of different degrees. Given a quasi-parabolic bundle of degree $-k$, what is needed is a canonical procedure to construct a quasi-parabolic bundle of degree $-(k+1)$. This is precisely what is provided by the shifting operation. Thought of in terms of $\widetilde{W}$, the following theorem extends Theorem 4.1 to the case where $H_\xi = \partial_0 W$.

We use the notation $\mathcal{M}_\alpha(k, m)$ for the moduli space when E_* has degree $-k$, multiplicities m, and weights α.

Theorem 5.3. *Suppose that $\gamma \in \partial_0 W_k \cap V_m$ and that $\alpha \in W_k \cap V_m$ is a generic weight near to γ. Choose $\eta \in \mathbb{R}$ with $0 < \eta < \gamma_{m_1+1}$. Define $\overline{\gamma} \in \partial_1 \overline{W}_{k+1}$ as in (10). Let E'_* be E_* shifted by η, and denote the multiplicities of E'_* by m'. Set $k' = -\deg E' = k + m_1$. Let $\beta \in W_{k'} \cap V_{m'}$ be generic near $\overline{\gamma}$. Then there are projective algebraic maps ϕ_α and ϕ_β*

$$\begin{array}{ccc} \mathcal{M}_\alpha(m,k) & & \mathcal{M}_\beta(m',k') \\ & {\scriptstyle\phi_\alpha}\searrow \quad \swarrow{\scriptstyle\phi_\beta} & \\ & \mathcal{M}_\gamma(m,k) & \end{array}$$

satisfying the conclusion of Theorem 4.1.

Proof. By the choice of α, β and η, we see that $\alpha_{m_1} < \eta < \alpha_{m_1+1}$, $\eta < \beta_1$ and $\eta < \gamma_{m_1+1}$. Consequently, the shifting operation defines the following isomorphisms:

$$\begin{aligned} \mathcal{M}_\alpha(m,k) &\cong \mathcal{M}_{\alpha'}(m',k'), \\ \mathcal{M}_\beta(m',k') &\cong \mathcal{M}_{\beta'}(m',k'), \\ \mathcal{M}_\gamma(m,k) &\cong \mathcal{M}_{\gamma'}(m',k'), \end{aligned}$$

where $\alpha', \beta', \gamma' \in V_{m'}$ are defined from α, β, γ as in (9). Now Theorem 4.1 applies to the shifted moduli spaces to prove the theorem. One can calculate e_α and e_β by applying formulas (5) and (6) to α', β' and γ'. □

Remark. Theorem 5.3 solves a problem mentioned at the end of [5] and has a nice application to the knot invariants introduced in [4].

6. Rationality of moduli spaces of parabolic bundles

Let L be a holomorphic line bundle over a curve X of genus $g \geq 1$. Denote by

(i) $\mathcal{M}_{r,L}$ the moduli space of semistable bundles E of rank r with $\det E = L$, and by

(ii) $\mathcal{M}_{\alpha,L}$ the moduli space of parabolic bundles E_* with weights α and $\det E = L$.

Theorem 4.1 holds for the moduli spaces with fixed determinant with no essential difference, and one concludes that if V_m contains a generic weight, then the birational type of $\mathcal{M}_{\alpha,L}$ is independent of $\alpha \in V_m$.

Theorem 4.2 also holds for the fixed determinant moduli spaces, and if α and β are generic, then the fibration

$$\mathcal{M}_{\alpha,L} \longrightarrow \mathcal{M}_{\beta,L}$$

has fiber a flag variety (which is rational). Hence $\mathcal{M}_{\alpha,L}$ is rational whenever $\mathcal{M}_{\beta,L}$ is. The goal is then to prove rationality with the coarsest choice of multiplicities m. At one extreme, we have the trivial flag, whose moduli space is exactly $\mathcal{M}_{r,L}$. Proposition 2 of [10] implies that $\mathcal{M}_{r,L}$ is rational if $\deg L = \pm 1 \mod (r)$, and so then Proposition 4.2 implies that $\mathcal{M}_{\alpha,L}$ is also rational for any $\alpha \in V_m$ if $\deg L = \pm 1 \mod (r)$.

Theorem 6.1. *If $m(p) = (1, \dots, 1)$ for some $p \in D$, then the moduli space $\mathcal{M}_{\alpha,L}$ is rational.*

Proof. First, use Theorem 4.2 to reduce to the case $D = p$ by forgetting all the other flag structures. If E'_* denotes the bundle obtained by shifting E_* by some η with $\alpha_1 < \eta < \alpha_2$, then $\det E' = L' = L(-p)$. It follows that shifting by η defines an isomorphism from $\mathcal{M}_{\alpha,L}$ to $\mathcal{M}_{\alpha',L'}$. Repeated application of shifting puts us in the case $\deg L = 1 \mod (r)$, and then Newstead's theorem and Theorem 4.2 imply that $\mathcal{M}_{\alpha,L}$ is rational. □

The above argument works in slightly more generality. We can always shift our bundle to be any of the $\mathcal{E}_x$ appearing in the filtration (7) and illustrated in Figure 1. Thus, whenever one of these terms in the filtration is of a degree to which Newstead's theorem applies, the corresponding moduli space of parabolic bundles is rational.

The next theorem is a considerable strengthening of the previous one.

Theorem 6.2. *If $m_i(p) = 1$ for some $p \in D$ and some $1 \leq i \leq s_p$, then $\mathcal{M}_{\alpha,L}$ is rational.*

Before delving into the proof of this theorem, we mention some interesting consequences. Recall that a variety V is called *stably rational* if $V \times \mathbb{P}^k$ is rational for some k. If V is stably rational, then the *level* of V is the smallest integer k with this property.

Corollary 6.3. *For any r and L, $\mathcal{M}_{r,L}$ is stably rational with level $k \leq r-1$.*

Proof. Theorems 4.2 and 6.2 imply $\mathcal{M}^s_{r,L} \times \mathbb{P}^{r-1}$ is rational. □

Ballico proved stable rationality of $\mathcal{M}_{r,L}$ for $(r,d) = 1$ using a different approach [2].

We now apply this last result to Conjecture 1.1.

Corollary 6.4. *Suppose $g > 1$ and $(r,d) = 1$. By tensoring with a line bundle, we can assume that $0 < d < r$. If either $(g,d) = 1$ or $(g, r-d) = 1$, then $\mathcal{M}_{r,L}$ is rational.*

Proof. Suppose first that $(g, r-d) = 1$. Let L be a line bundle of degree $r(g-1)+d$. Then Newstead's construction applies and proves that $\mathcal{M}_{r,L}$ is birational to $\mathcal{M}_{r-d,L} \times \mathbb{P}^\chi$, where $\chi = (g-1)(r^2 - (r-d)^2)$. But the above corollary implies that $\mathcal{M}_{r-d,L}$ is stably rational with level $k \leq r-d-1 \leq \chi$, hence $\mathcal{M}_{r,L}$ is rational.

The case $(g,d) = 1$ follows by the same argument after applying duality, which interchanges (r,d) and $(r, r-d)$. □

Remark. Conjecture 1.1 was previously known [12] in the following three cases:

(i) $d = \pm 1 \mod (r)$,
(ii) $(r,d) = 1$ and g a prime power, and
(iii) $(r,d) = 1$ and the two smallest distinct primes factors of g have sum greater than r.

Conjecture 6.4 applies in each case. More importantly, it applies in many cases not covered by (i), (ii) or (iii). In fact, for a given r and d with $(r,d) = 1$, one can easily list those g for which the conjecture remains open. For example, if $r = 110$ and $d = 43$, then Corollary 6.4 applies as long as g is not a multiple of $d \cdot (r-d) = 43 \cdot 67 = 2881$.

Proof of Theorem. Set $d = \deg L$. The theorem is clearly true for $r = 1$ and follows from Theorem 6.1 for $r = 2$, so assume $r > 2$. Notice that by tensoring with a line bundle, we can suppose

$$r(g-1) < d \leq rg.$$

By Theorem 4.2, we can again assume that $D = p$, and by shifting and another application of Theorem 4.2, if necessary, we can arrange it so that $m(p) = (r-1, 1)$. Write

$$\alpha = \alpha(p) = (\overbrace{\alpha_1, \cdots, \alpha_1}^{r-1}, \alpha_2).$$

Proposition 3.2 implies that V_m contains a generic weight and that $\mathcal{M}_{\alpha,L}$ parameterizes a universal family $\mathcal{U}^\alpha$. By Theorem 4.1, the birational type of $\mathcal{M}_{\alpha,L}$ is independent of choice of compatible weights, so we can assume that the weights are small enough to satisfy the hypothesis of Proposition 5.2 (this comes up at various technical points in the argument, e.g. the proof of Claim 6.5).

Consider the following two cases.

CASE I: $d = rg$. Choose η with $\alpha_1 < \eta < \alpha_2$, and let $E'_* = E_*[\eta]_*$. Denote the weights of E'_* by α' as in (9). If $\det E = L$, then $\det E' = L' = L(-(r-1)p)$ has degree $d' = d - (r-1)$. Since $d' = 1 \mod (r)$, Proposition 2 of [12] implies that $\mathcal{M}_{r,L'}$ is rational, and Proposition 4.2 then implies that $\mathcal{M}_{\alpha',L'}$ is also rational. Rationality of $\mathcal{M}_{\alpha,L}$ now follows from the isomorphism of the moduli spaces $\mathcal{M}_{\alpha,L} \cong \mathcal{M}_{\alpha',L'}$ defined by shifting by η.

CASE II: $r(g-1) < d < rg$. The idea is to use induction to construct a nonempty, Zariski-open subset $\mathcal{M}$ of affine space of dimension $(r^2-1)(g-1) + r - 1$ $(= \dim \mathcal{M}_{\alpha,L})$ and a family of stable parabolic bundles $\mathcal{U}$ parametrized by $\mathcal{M}$ with $\det \mathcal{U}_\xi = L$ for all $\xi \in \mathcal{M}$. The universal property of $\mathcal{U}^\alpha$ then gives a map $\psi_{\mathcal{U}} : \mathcal{M} \longrightarrow \mathcal{M}_{\alpha,L}$. If, in addition, we have $\mathcal{U}_{\xi_1} \cong \mathcal{U}_{\xi_2} \Leftrightarrow \xi_1 = \xi_2$, then $\psi_{\mathcal{U}}$ is injective and rationality of $\mathcal{M}_{\alpha,L}$ follows from that of $\mathcal{M}$ and the dimension condition.

Set $r' = rg - d$, $r'' = r - r'$ and $\alpha' = (\overbrace{\alpha_1, \cdots, \alpha_1}^{r'-1}, \alpha_2)$. Assume that both α and α' are generic. Let $\mathcal{U}^{\alpha'}$ be the universal family parametrized by $\mathcal{M}_{\alpha',L}$ and $I_* = \mathcal{O}_X[\alpha_1]_*$ be the trivial parabolic line bundle with weight α_1. If $e' = [E'_*] \in \mathcal{M}_{\alpha',L}$, then because $E'^{\vee}_* \otimes I_*$ is a stable parabolic bundle of negative parabolic degree, $h^0(E'^{\vee}_* \otimes I_*) = 0$ and

$$n \stackrel{\text{def}}{=} h^1(E'^{\vee}_* \otimes I_*) = (2r' + r'')(g-1) + r'' + 1 \tag{11}$$

is independent of e'. Since $\mathcal{U}^{\alpha'}_{e'} \cong E'_*$, it follows that

$$(R^1\pi_{\mathcal{M}_{\alpha',L}})(\mathcal{U}^{\alpha'^{\vee}} \otimes \pi_X^*(I_*))$$

is locally free. The associated vector bundle $V \stackrel{\pi}{\longrightarrow} \mathcal{M}_{\alpha',L}$ has rank n and fiber over e' naturally isomorphic to $H^1(E'^{\vee}_* \otimes I_*)$.

Let $\mathcal{U}' = (\pi \times 1_X)^*\mathcal{U}^{\alpha'}$ and $\mathcal{I} = \pi_X^* I_*$ be pullback bundles over $V \times X$. These are families of parabolic bundles parametrized by V. There is an extension

$$0 \longrightarrow \mathcal{I}^{\oplus r''} \longrightarrow \mathcal{U} \longrightarrow \mathcal{U}' \longrightarrow 0 \tag{12}$$

of bundles over $V^{\oplus r''} \times X$, such that, for $\xi \in V_{e'}^{\oplus r''}$, $\mathcal{U}_\xi$ is the parabolic bundle E_*^ξ described as the short exact sequence

$$0 \longrightarrow I_*^{\oplus r''} \longrightarrow E_*^\xi \longrightarrow E'_* \longrightarrow 0 \tag{13}$$

corresponding to the extension class $\xi \in H^1(E'^\vee_* \otimes I_*^{\oplus r''})$.

Using stability of E'_* and triviality of $I_*^{\oplus r''}$, it follows that

$$\mathrm{Aut}(E'_*) \times \mathrm{Aut}(I_*^{\oplus r''}) \cong \mathbb{C}^* \times \mathrm{GL}(r'', \mathbb{C}).$$

This group acts naturally as fiber-preserving maps on the bundle $V^{\oplus r''}$ since

$$V_{e'}^{\oplus r''} \cong H^1(E'^\vee_* \otimes I_*^{\oplus r''}) = H^1(E'^\vee_* \otimes I_*)^{\oplus r''},$$

and two extension classes ξ_1 and ξ_2 in the same orbit have associated bundles E^{ξ_1} and E^{ξ_2} which are isomorphic. We can ignore the $\mathbb{C}^*$ action here because $(z, 1) \cdot \xi = (1, z) \cdot \xi$ for $z \in \mathbb{C}^*$ and $\xi \in V^{\oplus r''}$.

Using the inductive hypothesis and local triviality of V, we can choose a nonempty Zariski-open subset $\mathcal{M}'$ of $\mathcal{M}_{\alpha', L}$ isomorphic to a Zariski-open subset of affine space of dimension $(r'^2 - 1)(g - 1) + r' - 1$ such that $V|_{\mathcal{M}'} \cong \mathcal{M}' \times H^1(E'^\vee_* \otimes I_*)$ (E'_* is fixed). Lemma 2 of [12] applies here and produces a Zariski-open subspace $\mathcal{M}' \times W$ of $V^{\oplus r''}|_{\mathcal{M}'}$ invariant under the group action, and affine subspace $A \subset W$ so that every orbit in W intersects A precisely once. In fact, A can be chosen as a Zariski open subset of the Grassmannian $G(r'', n)$. In any case, it should be clear that A has dimension $r''(n - r'')$. Using equation (11) and the fact that $r' + r'' = r$, we see that $\mathcal{M}' \times A$ is a Zariski-open subset of affine space of dimension

$$\begin{aligned} \dim \mathcal{M}' \times A &= (r'^2 - 1)(g - 1) + r' - 1 + r''(n - r'') \\ &= (r'^2 - 1)(g - 1) + r' - 1 + r''((2r' + r'')(g - 1) + 1) \\ &= (r^2 - 1)(g - 1) + r - 1. \end{aligned}$$

Let $\mathcal{M}$ be the subset of $V^{\oplus r''}$ defined by

$$\mathcal{M} = \{\xi \in \mathcal{M}' \times A \mid H^1(\mathcal{U}_\xi) = 0\},$$

and consider the bundle $\mathcal{U}$ restricted to $\mathcal{M}$, which we continue to denote $\mathcal{U}$. For $\xi \in V^{\oplus r''}$, let $E_*^\xi = \mathcal{U}_\xi$. Clearly E_*^ξ is a parabolic bundle with weights α and determinant L, thus $\mathcal{M}$ parameterizes a family of parabolic bundles. By the upper semi-continuity theorem, $\mathcal{M}$ is Zariski-open in $\mathcal{M}' \times A$.

We claim that $\mathcal{M}$ is nonempty. Fix $e' = [E'_*] \in \mathcal{M}'$ and consider the set

$$N = \{\xi \in H^1(E'^\vee_* \otimes I_*^{\oplus r''}) \mid h^1(E_*^\xi) = 0\}.$$

If $N \cap A \neq \emptyset$, then $\mathcal{M}$ is nonempty. Clearly, N is invariant under the action of $\mathrm{GL}(r'', \mathbb{C})$, so it is enough to show $N \cap W \neq \emptyset$. There is a natural map

$$\delta : H^1(E'^{\vee}_* \otimes I_*^{\oplus r''}) \times H^0(E'_*) \longrightarrow H^1(I_*^{\oplus r''})$$

with $\delta_\xi = \delta(\xi, \cdot) : H^0(E'_*) \longrightarrow H^1(I_*^{\oplus r''})$ the coboundary map of the long exact sequence in homology of (13). Now $H^0(E'_*) = H^0(E')$, and since $\alpha_1 + (r'-1)\alpha_2 < \epsilon(r,d)/2$, by Proposition 5.2, E' is semistable as a non-parabolic bundle. Serre duality implies that $h^1(E') = h^0(E'^{\vee} \otimes K)$, and we compute

$$\begin{aligned}\deg(E'^{\vee} \otimes K) &= -d + r'(1-g) \\ &\leq (r+r')(1-g) - r'',\end{aligned}$$

which is negative since $r'' \geq 1$ and $g \geq 1$. This implies that $h^1(E'_*) = 0$, and Riemann-Roch implies that $h^0(E'_*) = r''g$. Because $h^1(I_*^{\oplus r''}) = r''g$, we see that

$$\xi \in N \iff H^1(E_*^\xi) = 0 \iff \delta_\xi \text{ is an isomorphism.}$$

But δ is obviously onto and $\dim(\ker \delta) = r''n$. The set N has complement

$$N^c = \{\xi \in H^1(E'^{\vee}_* \otimes I_*^{\oplus r''}) \mid \delta(\xi, s) = 0 \text{ for some } 0 \neq s \in H^0(I_*^{\oplus r''})\}.$$

But $\delta(\xi, s) = 0 \Rightarrow \delta(\xi, zs) = 0$ for all $z \in \mathbb{C}$, which shows that the map $\ker \delta \longrightarrow N^c$ has fibers of dimension ≥ 1. Hence $\dim N^c \leq \dim(\ker \delta) - 1 < r''n$, and we see that N is nonempty and Zariski-open. Thus $N \cap W \neq \emptyset$ and it follows that $\mathcal{M}$ is nonempty.

We now prove that $\mathcal{M}$ parameterizes a family of stable parabolic bundles, using again the inequality $(r-1)\alpha_1 + \alpha_2 < \epsilon(r,d)/2$ and Proposition 5.2.

Claim 6.5. (i) *E_*^ξ is stable for all $\xi \in \mathcal{M}$.*
(ii) $E_*^{\xi_1} \cong E_*^{\xi_2} \iff \mathrm{GL}(r'', \mathbb{C}) \cdot \xi_1 = \mathrm{GL}(r'', \mathbb{C}) \cdot \xi_2$ *for all* $\xi_1, \xi_2 \in \mathcal{M}$.

Proof. (i) Suppose to the contrary that E_*^ξ is not parabolic stable for some $\xi \in \mathcal{M}$. Let G_* be a rank s parabolic subbundle of E_*^ξ with $\mu(G_*) > \mu(E_*)$. Then $\mu(G) \geq \mu(E^\xi)$, since otherwise

$$\mu(G_*) < \mu(G) + \epsilon(d,r)/2 < \mu(E^\xi) < \mu(E_*^\xi).$$

As in the argument of Lemma 6 of Newstead, the map $G \longrightarrow E'$ has a factorization as $G \to G^1 \to G^2 \to E'$ and the arguments there give the following inequalities:

$$\deg(G^2) \geq \deg(G) \geq \frac{sd}{r}, \tag{14}$$

$$\mathrm{rank}(G^2) \leq \mathrm{rank}(G) - h^0(G) \leq \frac{sr'}{r}. \tag{15}$$

These imply that $\mu(G^2) - \mu(E') \geq 0$. But E'_* is parabolic stable, so by Proposition 5.2, E' is semistable and $\mu(G^2) = \mu(E')$. Thus, we must have equalities in equations (14) and

(15), in particular $\mu(G) = \mu(E^\xi)$. But since $\mu(G_*) > \mu(E_*^\xi)$, we see that G_* must inherit the weight α_2, which implies that G_*^2 also inherits α_2, and it now follows that

$$\mu(G_*^2) - \mu(E_*') = \frac{(s_2-1)\alpha_1 + \alpha_2}{s_2} - \frac{(r'-1)\alpha_1 + \alpha_2}{r'} > 0,$$

where $s_2 = \operatorname{rank} G^2 < r'$. This contradicts the parabolic stability of E_*' and completes the proof of part (i).

(ii) Since $\Leftarrow$ is true independent of the vanishing of H^1, we only prove $\Rightarrow$. Suppose $E_*^{\xi_1} \cong E_*^{\xi_2}$ and set $\pi_X(E_*^{\xi_i}) = e_i' = [E_*^{i'}] \in \mathcal{M}_{\alpha',L}$. Notice that $h^1(E_*^{\xi_i}) = 0$, and so $h^0(E_*^{\xi_i}) = \chi(E_*^{\xi_i}) = r''$. It follows that every holomorphic section of $E_*^{\xi_i}$ has its image contained in $I_*^{\oplus r''}$. Hence any isomorphism $\varphi : E_*^{\xi_1} \longrightarrow E_*^{\xi_2}$ defines a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_*^{\oplus r''} & \longrightarrow & E_*^{\xi_1} & \longrightarrow & E_*^{1'} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \varphi''} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \varphi'} & & \\ 0 & \longrightarrow & I_*^{\oplus r''} & \longrightarrow & E_*^{\xi_2} & \longrightarrow & E_*^{2'} & \longrightarrow & 0 \end{array}$$

where both φ' and φ'' are isomorphisms, and so $\xi_2 = (\varphi' \times \varphi'') \cdot \xi_1$. □

Part (i) of the claim and the universal property of $\mathcal{U}^\alpha$ gives a map $\mathcal{M} \xrightarrow{\psi} \mathcal{M}_{\alpha,L}$, which is injective by part (ii). Since $\mathcal{M}$ is nonempty, $\dim \mathcal{M} = \dim \mathcal{M}_{\alpha,L}$, so rationality of $\mathcal{M}_{\alpha,L}$ follows from that of $\mathcal{M}$. This concludes the proof in Case II. □

Remark. We had originally hoped to prove rationality of $\mathcal{M}_{\alpha,L}$ with the weaker hypothesis that α is generic, but the argument does not hold in this generality. For consider the case $D = p$. By tensoring with a line bundle and shifting, we can assume that

$$r(g-1) < d \leq r(g-1) + m_1.$$

Hence, the subbundle split off in the induction is again a sum of parabolic line bundles with the same weights. The difficulty is in proving that the quotient E_*' has *generic* weights α'.

Proposition 3.2 implies that E_*' admits a generic weight if and only if the set $\{d, m_i'(p)\}$ is relatively prime. The statement

$$(d, m_1, \dots, m_s) = 1 \Rightarrow (d, m_1', \dots, m_s') = 1,$$

which is what we would need to prove here, is unfortunately false (notice that $m_1' = m_1 - d + r(g-1)$ and $m_i' = m_i$ otherwise).

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