Exactness, integrality, and log modifications

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1 Introduction

The exactness and the integrality of morphisms of log schemes, introduced by Kazuya Kato [3], have been widely accepted as practically useful concepts in log geometry. For instance, a log smooth morphism is underlain by a flat morphism if it is integral, while the underlying morphism of a log smooth morphism in general might be fairly pathological.

In this note we will prove that, under a few minor conditions, any morphism of fs log schemes can be modified by so-called log modifications to be an exact, and moreover an integral morphism. Our main theorem is the following (Theorem 3.13 and Theorem 3.16):

Theorem. *Let $f\colon X \to Y$ be a morphism of locally noetherian fs log schemes. Suppose the underlying structure of f and Y are quasi-compact.*

(1) *There exists a log-ideal $\mathcal{K}$ of Y such that the base change of f by the log blow-up of Y along $\mathcal{K}$ is exact.*

(2) *There exists a commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \longrightarrow & Y \end{array}$$

of fs log schemes such that the horizontal arrows are log modifications and $\widetilde{f}$ is integral.

The log blow-up is also the concept which was already studied by Kazuya Kato in his unpublished works (cf. [4][5]). Roughly speaking, a log blow-up is the composite of a usual blow-up centered at an ideal which comes from an ideal of the log strtucture (log-ideal) followed by normalization. The log modification is a certain class of universally surjective log étale morphisms, which of course includes log blow-up's.

Notice that the second part of the theorem above (especially for a log smooth morphism f) can be compared to a weak log-version of the famous flattening theorem by Raynaud and Gruson [9].

Notation and conventions. All sheaves are considered with respect to the étale topology. Log schemes are denoted simply by capital letters. For a log scheme X, we denote by X°, $\mathcal{O}_X$, $\mathcal{M}_X$, and $\overline{\mathcal{M}}_X$ the underlying scheme, the structure sheaf of X°, the log structure of X, and the characteristic ($= \mathcal{M}_X/\mathcal{O}_X^\times$) of X, respectively. For a commutative and unitary monoid P the associated group (resp. the subgroup of invertible elements) is denoted by P^{gp} (resp. $P^\times$).

2 Exact morphism and integral morphism

In this section we recall the definition of exactness and integrality, and basic properties of them, especially in relation to charts; we also recall the notion of neat and good charts which is useful in the next section.

Definition 2.1

(1) A homomorphism $Q \to P$ of monoids is said to be *exact* if the diagram

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ Q^{\mathrm{gp}} & \longrightarrow & P^{\mathrm{gp}} \end{array}$$

is Cartesian in the category of sets.

(2) A morphism $f: X \to Y$ of log schemes is said to be *exact at* $x \in X$ if the homomorphism $\overline{\mathcal{M}}_{Y,\overline{y}} \to \overline{\mathcal{M}}_{X,\overline{x}}$ is exact, where $y = f(x)$. (*Here and hereafter* $\bar{\ }$ *means the separable closure of points.*) The morphism f is said to be *exact* if it is exact at any point of X.

If a homomorphism $h: Q \to P$ of monoids is exact and $Q^{\times} = 1$, then h is injective. In particular, if a morphism $f: X \to Y$ is exact at $x \in X$, then $\overline{\mathcal{M}}_{Y,\overline{y}} \to \overline{\mathcal{M}}_{X,\overline{x}}$ is injective.

Let P be a monoid and A a ring. We denote by $\mathrm{S}(P)_A$ the log scheme underlain by the scheme $\operatorname{Spec} A[P]$ endowed with the log structure induced by $P \to A[P]$. We write, in particular, $\mathrm{S}(P) = \mathrm{S}(P)_{\mathbf{Z}}$. The log structure of $\mathrm{S}(P)$ is denoted by $\mathcal{M}_P$. (Caution: Even if $Q \to P$ is exact, $\mathrm{S}(P) \to \mathrm{S}(Q)$ is not necessarily exact.)

Let X be a log scheme. A chart of X modeled on a monoid P is a strict morphism $\lambda: X \to \mathrm{S}(P)$ of log schemes. The chart λ is said to be good at a point $x \in X$ if $P \to \overline{\mathcal{M}}_{X,\overline{x}}$ is an isomorphism. Good charts always exist at any point, provided X is fs.

Let $f: X \to Y$ be a morphism of log schemes. A chart of f modeled on a homomorphism $h: Q \to P$ of monoids is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{S}(P) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{S}(h)} \\ Y & \longrightarrow & \mathrm{S}(Q) \end{array}$$

with the horizontal arrows strict. This chart of f is said to be *neat* at $x \in X$ if h^{gp} is injective, $P^{\mathrm{gp}}/Q^{\mathrm{gp}} \to \overline{\mathcal{M}}_{X/Y,\overline{x}}$ is an isomorphism, and $P \to \overline{\mathcal{M}}_{X,\overline{x}}$ is exact. This chart of f is said to be *good* at $x \in X$ if the charts $X \to \mathrm{S}(P)$ and $Y \to \mathrm{S}(Q)$ are good at x and $f(x)$, respectively.

Lemma 2.2 (A. Ogus [8, Theorem 2.13]) *Let $f: X \to Y$ be a morphism of fine log schemes and $x \in X$ a point. Suppose there exists a chart $Y \to \mathrm{S}(Q)$ of Y modeled on a fine monoid Q. Then f has fppf locally a neat chart at x modeled on a homomorphism of fine monoids extending the given chart $Y \to \mathrm{S}(Q)$. Furthermore, if the order of the torsion part of $\overline{\mathcal{M}}^{\mathrm{gp}}_{X/Y,\overline{x}}$ is invertible in the residue field $k(\overline{x})$, then the neat chart as above can be found étale locally.*

Corollary 2.3 *If $f: X \to Y$ is a morphism of fs log schemes exact at $x \in X$, then f allows fppf locally a good chart at x.*

Proof. Take étale locally a good chart $Y \to \mathrm{S}(Q)$ at $y = f(x)$ and extend it fppf locally to a neat chart $(X \to \mathrm{S}(P), Y \to \mathrm{S}(Q), Q \to P)$ at x. Since $\overline{\mathcal{M}}_{Y,\overline{y}} \to \overline{\mathcal{M}}_{X,\overline{x}}$ is injective, the chart $X \to \mathrm{S}(P)$ is automatically good at x. □

Definition 2.4 A homomorphism $h: Q \to P$ of integral monoids is said to be *integral* if for any integral monoid Q' and a homomorphism $Q' \to Q$, the push-out $P \oplus_Q Q'$ in the category of monoids is integral.

It is known that the integrality of $h: Q \to P$ is equivalent to the following condition: If $a_1, a_2 \in Q$, $b_1, b_2 \in P$ and $h(a_1)b_1 = h(a_2)b_2$, there exist $a_3, a_4 \in Q$ and $b \in P$ such that $b_1 = h(a_3)b$, $b_2 = h(a_4)b$, and $a_1a_3 = a_2a_4$ (cf. [3, (4.1)]). It is also known that $h: Q \to P$ is integral and injective if and only if the map $\mathbf{Z}[h]: \mathbf{Z}[Q] \to \mathbf{Z}[P]$ is flat.

Lemma 2.5

(1) *Let $h: Q \to P$ be an integral homomorphism of integral monoids such that $h^{-1}(P^\times) = Q^\times$. Then h is exact.*

(2) *Consider the commutative diagram*

$$\begin{array}{ccc} P' & \xrightarrow{a} & P \\ {\scriptstyle h'}\uparrow & & \uparrow{\scriptstyle h} \\ Q' & \longrightarrow & Q \end{array}$$

of integral monoids such that the horizontal arrow are surjective. Then:

(a) *If h' is integral and there exists a submonoid $P'' \subseteq P'$ such that $P'/P'' \cong P$, then h is integral.*

(b) *If h is integral and the horizontal arrows are exact, then h' is integral.*

(*Here the quotient of a monoid M by a submonoid N is defined by the coset space $M/\sim$ with the induced monoid structure, where $a \sim b$ if and only if there exist $c, d \in N$ such that $ac = bd$.*)

Proof. Straightforward. □

Definition 2.6 A morphism $f: X \to Y$ of integral log schemes is said to be *integral at $x \in X$* if $\overline{\mathcal{M}}_{Y,\overline{y}} \to \overline{\mathcal{M}}_{X,\overline{x}}$ is integral, where $y = f(x)$. We say f is *integral* if it is integral at any point of X.

Note that the integrality of $\overline{\mathcal{M}}_{Y,\overline{y}} \to \overline{\mathcal{M}}_{X,\overline{x}}$ is equivalent to that of $\mathcal{M}_{Y,\overline{y}} \to \mathcal{M}_{X,\overline{x}}$. By Lemma 2.5 (1), integral morphisms are exact. If $h: Q \to P$ is integral, then by Lemma 2.5 (2) (a), the morphism $\mathrm{S}(h): \mathrm{S}(P) \to \mathrm{S}(Q)$ is integral.

In what follows, all push-outs are taken in the category of fs monoids, unless otherwise explicitely specified. Accordingly, all fiber products of fs log schemes are taken in the category fs log schemes, unless otherwise specified.

Let $P \leftarrow Q \to Q'$ be homomorphisms of fs monoids and $P^{\mathrm{gp}} \leftarrow Q^{\mathrm{gp}} \to Q'^{\mathrm{gp}}$ the induced morphisms, which gives rise to the push-out $P^{\mathrm{gp}} \oplus_{Q^{\mathrm{gp}}} Q'^{\mathrm{gp}}$ in the category of finitely generated abelian groups.

Lemma 2.7 *The push-out of $P \leftarrow Q \to Q'$ in the category of fs monoids is given by the smallest fs submonoid in $P^{\mathrm{gp}} \oplus_{Q^{\mathrm{gp}}} Q'^{\mathrm{gp}}$ containing those images of P and*

Q', *or what amounts to the same, the saturation of the submonoid in* $P^{\mathrm{gp}} \oplus_{Q^{\mathrm{gp}}} Q'^{\mathrm{gp}}$ *generated by those images of* P *and* Q'.

Proof. It is enough to prove that the push-out in the category of fine monoids is the submonoid in $P^{\mathrm{gp}} \oplus_{Q^{\mathrm{gp}}} Q'^{\mathrm{gp}}$ generated by the images of P and Q', which is easy to verify. $\square$

Corollary 2.8 *For a homomorphism* $Q \to P$ *of fs monoids such that* $Q^{\mathrm{gp}} \cong P^{\mathrm{gp}}$, *the push-out of* $P \leftarrow Q \to P$ *is given by* P.

We denote the push-out of $P \leftarrow Q \to Q'$ by $P_{Q'}$ if the monoid Q is evident in the context.

Lemma 2.9 *For an exact homomorphism* $Q \to P$ *of fs monoids and a homomorphism* $Q \to Q'$ *of fs monoids, the induced homomorphism* $Q' \to P_{Q'}$ *is exact.*

Proof. It is straightfoward to see that the induced homomorphism $Q' \to P'$ is exact, where P' is the submonoid in $P^{\mathrm{gp}} \oplus_{Q^{\mathrm{gp}}} Q'^{\mathrm{gp}}$ generated by the images of P and Q'. The monoid $P_{Q'}$ is the saturation of P'. Since Q' is saturated, we see that $Q' \to P_{Q'}$ is also exact. $\square$

Corollary 2.10 *Exactness is stable under base change.*

3 Log blow-up and log modification

Definition 3.1 Let Q be a monoid. A subset $K \subseteq Q$ (possibly empty) is said to be an ideal of Q if $ax \in K$ for any $a \in Q$ and $x \in K$.

Any ideal of a finitely generated monoid is finitely generated as an ideal. Products and intersections of ideals are obviously ideals. The correspondence $K \mapsto \overline{K} := \pi(K) \cdot \overline{Q}$ gives rise to the natural bijection between the set of all ideals in Q and the set of all ideals in $\overline{Q}$, where $\pi: Q \to \overline{Q} = Q/Q^{\times}$ is the natural projection.

Notation 3.2 Let Q be an fs monoid. For a subset $E \subseteq Q^{\mathrm{gp}}$ we denote by $Q\langle E\rangle$ the smallest fs submonoid in Q^{gp} which contains Q and E.

Definition 3.3 Let X be a log scheme. A *log-ideal* of X is a sheaf of ideals $\mathcal{K}$ of $\mathcal{M}_X$.

Notation 3.4

(1) For an ideal $K \subseteq Q$ we denote by $\widetilde{K}$ the log-ideal of $\mathrm{S}(Q)$ generated by the image of K.

(2) Let $f: X \to Y$ be a morphism of log schemes and $\mathcal{K}$ a log-ideal of Y. Then we denote by $f^{\bullet}\mathcal{K}$ the log-ideal of X generated by the image of $f^{-1}\mathcal{K}$.

Clearly, for $f: X \to Y$ and $g: Y \to Z$ and a log-ideal $\mathcal{K}$ of Z, we have $(g \circ f)^{\bullet}\mathcal{K} = f^{\bullet}g^{\bullet}\mathcal{K}$.

A log scheme X is called *locally noetherian* if the underlying scheme X° is locally noetherian.

Lemma 3.5 (A. Ogus [8, Proposition 2.6]) *Let* X *be a locally noetherian fine log scheme and* $\mathcal{K}$ *a log-ideal. Then the following conditions are equivalent:*

(1) *For any $x, y \in X$ such that x belongs to the topological closure of $\{y\}$, the natural map $\mathcal{K}_{\overline{x}} \to \mathcal{K}_{\overline{y}}$ is surjective.*
(2) *For any point $x \in X$ and any étale neighborhood $\iota: U \to X$ of x with chart $\lambda: U \to \mathrm{S}(P)$ set $K = \beta_{\overline{x}}^{-1}(\mathcal{K}_{\overline{x}})$, where $\beta: P \to \mathcal{M}_X|_U$. Then replacing U by a sufficiently small étale neighborhood of x, we have $\iota^{\bullet}\mathcal{K} = \lambda^{\bullet}\widetilde{K}$.*

Definition 3.6 Let X be a locally noetherian fine log scheme. Then a log-ideal $\mathcal{K}$ of X is said to be *coherent* if it satisfies the conditions in Lemma 3.5.

Definition 3.7 A coherent log-ideal $\mathcal{K}$ of X is said to be *locally principal* if it is étale locally generated by a single global section.

Definition 3.8 Let X be a locally noetherian fs log scheme and $\mathcal{K}$ a coherent log-ideal of X. Then the *log blow-up* $\rho_{\mathcal{K}}: \mathrm{B}_{\mathcal{K}}(X) \to X$ of X along $\mathcal{K}$ is a morphism of locally noetherian fs log schemes satisfying the following conditions:
(1) The log ideal $\rho_{\mathcal{K}}^{\bullet}\mathcal{K}$ of $\mathrm{B}_{\mathcal{K}}(X)$ is locally principal.
(2) For any morphism $f: X' \to X$ of locally noetherian fs log schemes such that $f^{\bullet}\mathcal{K}$ is locally principal, there exists a unique morphism $\widetilde{f}: X' \to \mathrm{B}_{\mathcal{K}}(X)$ such that $f = \rho_{\mathcal{K}} \circ \widetilde{f}$.

The log blow-up along $\mathcal{K}$ is unique up to isomorphisms if it exists. We are going to construct $\rho_{\mathcal{K}}: \mathrm{B}_{\mathcal{K}}(X) \to X$. We should first start with the case $X = \mathrm{S}(P)$ for an fs monoid P.

The following construction is due to Kazuya Kato [5]: Let $K \subseteq P$ be an ideal. Then the log blow-up $\mathrm{B}_K(P) \to \mathrm{S}(P)$ of $\mathrm{S}(P)$ along K is constructed as follows: Let $I(K)$ be the ideal of $\mathbf{Z}[P]$ generated by K. Consider the natural morphism $\operatorname{Proj} \oplus_n I(K)^n \to \operatorname{Spec} \mathbf{Z}[P]$. $\operatorname{Proj} \oplus_n I(K)^n$ has the affine open covering

$$\operatorname{Proj} \oplus_n I(K)^n = \bigcup_{p \in K} \operatorname{Spec} \mathbf{Z}[P\langle p^{-1}K\rangle].$$

(It is enough for p to run over the generators of K.) The canonical log structures given by $P\langle p^{-1}K\rangle \to \mathbf{Z}[P\langle p^{-1}K\rangle]$ obviously glue to a fine log structure on $\operatorname{Proj} \oplus_n I(K)^n$. The fs log scheme $\mathrm{B}_K(P)$ is defined by the saturation of this resulting log scheme. The morphisms $\mathrm{S}(P\langle p^{-1}K\rangle) \to \mathrm{S}(P)$ give rise to the desired morphism $\mathrm{B}_K(P) \to \mathrm{S}(P)$, which turns out to be the log blow-up in this case.

Let X be an fs log scheme underlain by a locally noetherian scheme and $\mathcal{K}$ a coherent log-ideal. Suppose X has a chart $\lambda: X \to \mathrm{S}(P)$ modeled on an fs monoid P, and that the log-ideal $\mathcal{K}$ is generated by global sections. Take a point $x \in X$ and set $K = \beta_{\overline{x}}^{-1}(\mathcal{K}_{\overline{x}})$, where $\beta: P \to \mathcal{M}_X$. Since $\mathcal{K}$ is generated by global sections, the ideal K does not depend on the choice of x. In this situation we define the log blow-up $\mathrm{B}_{\mathcal{K}}(X) \to X$ by

$$\mathrm{B}_{\mathcal{K}}(X) := X \times_{\mathrm{S}(P)} \mathrm{B}_K(P) \longrightarrow X.$$

It can be checked without so much pain that this morphism satisfies the conditions in Definition 3.8. In particular, it does not depend on the choice of the chart λ.

In general we take an étale covering $\bigcup_{i \in I} U_i \to X$ such that each U_i has allows a chart $U_i \to \mathrm{S}(P_i)$ and $\mathcal{K}|_{U_i}$ is generated by global sections. Then from what we have seen above the local log blow-up's $\mathrm{B}_{\mathcal{K}|_{U_i}}(U_i) \to U_i$ glue to a morphism of log schemes $\mathrm{B}_{\mathcal{K}}(X) \to X$ which is nothing but the log blow-up of X along the coherent log-ideal $\mathcal{K}$. Note that log blow-up's are log étale.

The following lemmas are easy to see by the universality of log blow-up's:

Lemma 3.9 *Let X be a locally noetherian fs log scheme and $\mathcal{K}$ a log-ideal of X. Then we have $\mathrm{B}_{\mathcal{K}}(X) \times_X \mathrm{B}_{\mathcal{K}}(X) \cong \mathrm{B}_{\mathcal{K}}(X)$.*

Lemma 3.10 *Let $\mathcal{K}$ and $\mathcal{K}'$ be coherent log-ideals of X. Then we have a canonical isomorphisms*

$$\mathrm{B}_{\mathcal{K}*\mathcal{K}'}(X) \cong \mathrm{B}_{\rho^{\bullet}_{\mathcal{K}}\mathcal{K}'}(\mathrm{B}_{\mathcal{K}}(X)) \cong \mathrm{B}_{\rho^{\bullet}_{\mathcal{K}'}\mathcal{K}}(\mathrm{B}_{\mathcal{K}'}(X))$$

over X, where $\mathcal{K} * \mathcal{K}'$ denote the log-ideal of X generated by $\mathcal{K}$ and $\mathcal{K}'$.

Example 3.11 It is worth while describing log blow-up's in the toric case: Let Σ be a fan of $N \cong \mathbf{Z}^r$ and T_Σ the assiciated toric variety over a ring A. For any cone $\sigma \in \Sigma$, the affine patch $T_\sigma = \operatorname{Spec} A[\sigma^\vee \cap M]$ (where $M = \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$) has the canonical log structure which gives rise to $\mathrm{S}(\sigma^\vee \cap M)_A$. These patches glue to an fs log scheme which is also denoted by T_Σ. Let $T = \operatorname{Spec} A[M]$ be the torus included in T_Σ as a Zariski open set and $i\colon T \hookrightarrow T_\Sigma$ the open immersion. Then T is exactly the locus on which the log structure is trivial. Furthermore, the log structure $\mathcal{M}_{T_\Sigma}$ on T_Σ is isomorphic to

$$\mathcal{O}_{T_\Sigma} \cap i_*\mathcal{O}_T^\times \hookrightarrow \mathcal{O}_{T_\Sigma}.$$

Log blow-up's of toric varieties coincides with those described in [6, I, §2]: For a log-ideal $\mathcal{K}$ of T_Σ, the ideal $\mathcal{I}(\mathcal{K})$ of $\mathcal{O}_{T_\Sigma}$ generated by $\mathcal{K}$ is invariant under the action of T. Conversely, to a T-invariant ideal of $\mathcal{O}_{T_\Sigma}$ we can associate the log ideal of T_Σ by taking the intersection with $\mathcal{M}_{T_\Sigma}$. This correspondence gives rise to the bijection between the set of all log-ideals of T_Σ and that of all T-invariant ideals of $\mathcal{O}_{T_\Sigma}$. If $\mathcal{I}$ is a T-invariant ideal of $\mathcal{O}_{T_\Sigma}$, the log blow-up along $\mathcal{I}$ is given by a certain subdivision $\widetilde{\Sigma} \to \Sigma$ of the fan Σ; the fan $\widetilde{\Sigma}$ is the subdivision consisting of cones on which the associated piecewise linear function $\mathrm{ord}\mathcal{I}$.

Any log blow-up of a toric variety therefore comes from subdivision of the fan. But a morphism of toric varieties by subdivision is not necessarily a log blow-up.

Lemma 3.12 *Let $Q \to P$ be a homomorphism of fs monoids such that $Q^{\mathrm{gp}} \cong P^{\mathrm{gp}}$. Then there exists an ideal $K \subseteq Q$ and a strict open immersion $\mathrm{S}(P) \hookrightarrow \mathrm{B}_K(Q)$ such that*

$$\begin{array}{ccc} \mathrm{S}(P) & = & \mathrm{S}(P) \\ \big\downarrow & & \big\downarrow \\ \mathrm{S}(Q) & \longleftarrow & \mathrm{B}_K(Q) \end{array}$$

is a Cartesian diagram of fs log schemes.

Proof. Write $T = \mathrm{S}(P)$ and $S = \mathrm{S}(Q)$. We can set $P = Q\langle \frac{a_1}{b}, \ldots, \frac{a_r}{b} \rangle$ in $Q^{\mathrm{gp}} \cong P^{\mathrm{gp}}$, where $b, a_1, \ldots, a_r \in Q$. Let $K = (b, a_1, \ldots, a_r)$ be the ideal of Q. Then the log blow-up $\mathrm{B}_K(Q)$ has the covering

$$\mathrm{B}_K(Q) = \bigcup_{i=0,\ldots,r} V_i,$$

where

$$V_0 = \mathrm{S}(Q\langle \frac{a_1}{b}, \dots, \frac{a_r}{b}\rangle),$$

which is isomorphic to T, and

$$V_j = \mathrm{S}(Q\langle \frac{b}{a_j}, \frac{a_1}{a_j}, \dots, \frac{a_r}{a_j}\rangle)$$

for $j > 0$. By Lemma 2.7 we have $T \times_S V_0 \cong T$ and $T \times_S V_j \cong \mathrm{S}(P\langle \frac{b}{a_j}\rangle)$ for $j > 0$. Since we can check $T \times_S (V_0 \cap V_j) = T \times_S V_j$, we have $T \times_S \mathrm{B}_K(Q) \cong T$. □

Theorem 3.13 *Let $f: X \to Y$ be a morphism of locally noetherian fs log schemes. Suppose f° and Y° are quasi-compact. Then there exists a log-ideal $\mathcal{K}$ of Y such that the base change $X_{\mathrm{B}_{\mathcal{K}}(Y)} \to \mathrm{B}_{\mathcal{K}}(Y)$ by the log blow-up of Y along $\mathcal{K}$ is exact.*

Proof. First we assume that $\overline{\mathcal{M}}^{\mathrm{gp}}_{X/Y}$ is constant and that there exists a chart $Y \to \mathrm{S}(Q)$. Then by Lemma 2.2 there exists an fppf covering $\{X_i \to X\}$ of X consisting of finitely many members such that each $X_i \to Y$ allows a chart $(\lambda_i: X_i \to \mathrm{S}(P_i), \mu: Y \to \mathrm{S}(Q), h_i: Q \to P_i)$ such that h_i is injective and $\overline{\mathcal{M}}^{\mathrm{gp}}_{X/Y} \cong P_i^{\mathrm{gp}}/Q^{\mathrm{gp}}$. Set $Q_i = P_i \cap Q^{\mathrm{gp}}$. Then $Q \to Q_i$ yields $Q^{\mathrm{gp}} \cong Q_i^{\mathrm{gp}}$. Set $Y_i = Y \times_{\mathrm{S}(Q)} \mathrm{S}(Q_i)$ for each i. By Lemma 3.12 we have an ideal K_i of Q such that $\mathrm{S}(Q_i) \hookrightarrow \mathrm{B}_{K_i}(Q)$ gives the base change. Set $\mathcal{K}_i = \mu^\bullet \widetilde{K}_i$ and consider the log blow-up $\mathrm{B}_{\mathcal{K}_i}(Y) \to Y$. Then we have $X_i \times_Y \mathrm{B}_{\mathcal{K}_i}(Y) \cong X_i$, since $Y_i \times_Y \mathrm{B}_{\mathcal{K}_i}(Y) \cong Y_i$. Since $Y_i \to \mathrm{B}_{\mathcal{K}_i}(Y)$ is strict, we have $\overline{\mathcal{M}}^{\mathrm{gp}}_{X_i/\mathrm{B}_{\mathcal{K}_i}(Y)} \cong \overline{\mathcal{M}}^{\mathrm{gp}}_{X_i/Y}$, which is constant and is isomorphic to $P_i^{\mathrm{gp}}/Q_i^{\mathrm{gp}}$. Hence it follows from the exactness of $Q_i \to P_i$ that $X_i \to Y_i$ is exact, and hence $X_i \to \mathrm{B}_{\mathcal{K}_i}(Y)$ is exact. Let $\mathcal{K}$ be the log-ideal of Y generated by $\mathcal{K}_i$'s. Since the exactness is stable under base changes, the base change $X_{\mathrm{B}_{\mathcal{K}}(Y)} \to \mathrm{B}_{\mathcal{K}}(Y)$ restricted on each X_i is exact, and hence the whole $X_{\mathrm{B}_{\mathcal{K}}(Y)} \to \mathrm{B}_{\mathcal{K}}(Y)$ is exact. Hence we have proved the theorem in this particular case.

In general we may still assume Y allows a chart $Y \to \mathrm{S}(Q)$. The sheaf $\overline{\mathcal{M}}^{\mathrm{gp}}_{X/Y}$ is constructible. We look at each locally closed set of X° on which $\overline{\mathcal{M}}^{\mathrm{gp}}_{X/Y}$ is constant, and take a log blow-up to make f exact on this locus. Finally, we gather up the resulting log-ideal of Y and then take sufficiently large log blow-up. □

Definition 3.14 A morphism $\rho: \widetilde{X} \to X$ of locally noetherian fs log schemes is called a *log modification* if for any point $x \in X$ there exists an étale neighborhood $U \to X$ of x, and a log blow-up $Z \to \widetilde{X}_U$ such that the composite $Z \to \widetilde{X}_U \to U$ is also a log blow-up.

Note that if X is quasi-compact the morphism $\rho: \widetilde{X} \to X$ is a log modification if and only if there exists a log blow-up $Z \to \widetilde{X}$ such that the composite $Z \to \widetilde{X} \to X$ is also a log blow-up.

The above definition of log modifications is motivated by the following facts:

- A morphism $\rho: \widetilde{X} \to X$ of quasi-projective varieties over a field is a modification (i.e. proper and birational morphism) if and only if there exists a blow-up $Z \to \widetilde{X}$ such that the composite $Z \to X$ is a blow-up; this follows easily from [1, Theorem 7.17] and Chow's lemma.
- Any morphism of toric varieties coming from subdivision of fans are log modifications due to De Concini-Procesi's Theorem (cf. [7, §1.7]).

Note that log modifications are log étale.

Lemma 3.15

(1) *Log modifications are universally surjective. (A morphism $X \to Y$ of fs log schemes is said to be universally surjective if for any morphism $Y' \to Y$ of fs log schemes the underlying morphism of the base change $X_{Y'} \to Y'$ is surjective.)*

(2) *If $X \to Y$ and $Y \to Z$ are log modifications, then the composite $X \to Z$ is a log modification.*

(3) *If $\widetilde{X} \to X$ is a log modification and $Y \to X$ is a morphism, then the base change $\widetilde{X}_Y \to Y$ is a log modification.*

Proof. First we prove (1). It is enough to prove that log blow-up's are universally surjective. By the universality of log blow-up's it suffices to show that any blow-up is surjective. By an easy observation we reduce to verify whether $\mathrm{B}_K(P) \to \mathrm{S}(P)$ is surjective for a monoid P and an ideal K. But this is trivial since the underlying morphism of $\mathrm{B}_K(P) \to \mathrm{S}(P)$ is a normalized blow-up in the usual sense. To prove (2) we may assume there exists a log blow-up $\mathrm{B}(Z) \to Z$ which factors through Y by a log blow-up. We can take a log blow-up $\mathrm{B}(Y) \to Y$ which factors through X by a log blow-up, since Y is easily seen to be quasi-compact. Take the log blow-up $\mathrm{B}(\mathrm{B}(Z)) \to \mathrm{B}(Z)$ along the pull-back of the blown-up center of $\mathrm{B}(Y) \to Y$. Then we have the unique morphism $\mathrm{B}(\mathrm{B}(Z)) \to \mathrm{B}(Y)$, and then (2) is proved. (3) is straightforward. □

Theorem 3.16 *Let $f\colon X \to Y$ be a morphism of locally noetherian fs log schemes. Suppose f° and Y° are quasi-compact. Then there exists a commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \longrightarrow & Y \end{array}$$

of fs log schemes such that the horizontal arrows are log modifications and $\widetilde{f}$ is integral.

Proof. By Theorem 3.13 we may assume f is exact, since base changes by log blow-up's are log modifications. Since we have assumed Y° is quasi-compact, we may assume Y allows a chart $\mu\colon Y \to \mathrm{S}(Q)$ which is good at a point. By Corollary 2.3 we can take fppf locally on X a chart $(\lambda\colon X \to \mathrm{S}(P), \mu\colon Y \to \mathrm{S}(Q), h\colon Q \to P)$ of f which is good at a point. Then $\mathrm{S}(h)\colon \mathrm{S}(P) \to \mathrm{S}(Q)$ is a morphism of toric varieties which comes from a morphism of fans $\Sigma' \to \Sigma$ (both of which are defined by single cones) underlain by $\varphi\colon N' \to N$.

For any ray $\rho \in \Sigma'_1$ the image of ρ in N is either a ray or a point. If it is a ray, then take the primitive base $n_1 \in N$ of $\varphi(\rho)$, and extend it to a $\mathbf{Z}$-basis $n_1, \ldots, n_r$ of N. The $r+1$-rays spanned by $n_1, \ldots, n_r, -(n_1 + \cdots + n_r)$ give the projective space $\mathbf{P}^r_{\mathbf{Z}}$, and the very ample divisor $\mathcal{O}(1)$ gives rise to the support function, which we denote by s_ρ. In case ρ is mapped to the point we set $s_\rho = 0$. Then

$$s = \sum_{\rho \in \Sigma'_1} s_\rho$$

gives a piecewise linear function on $N \otimes_{\mathbf{Z}} \mathbf{R}$ which yields a log-ideal $\mathcal{K}$ on $\mathrm{S}(Q)$ (cf. [6, I, Theorem 9]). Set $\mathcal{K}' = \mathrm{S}(h)^\bullet \mathcal{K}$. The induced morphism $\mathrm{B}_{\mathcal{K}'}(P) \to \mathrm{B}_{\mathcal{K}}(Q)$ is also an equivariant morphism of toric varieties which comes from $\widetilde{\Sigma}' \to \widetilde{\Sigma}$, where $\widetilde{\Sigma}$ (resp. $\widetilde{\Sigma}'$) is the subdivision of Σ (resp. Σ') consisting of biggest cones on which s (resp. $\varphi^*_{\mathbf{R}}s$) is linear. Since any cones in $\widetilde{\Sigma}'$ is mapped surjectively onto some cone in $\widetilde{\Sigma}$, that the morphism of associated toric varieties $\mathrm{B}_{K'}(P) \to \mathrm{B}_K(Q)$ has constant fiber dimension. Taking more auxiliary log blow-up's of $\mathrm{B}_K(Q)$ and $\mathrm{B}_{K'}(P)$, we may assume $\mathrm{B}_K(Q)$ is non-singular. In this case the morphism $\mathrm{B}_{K'}(P) \to \mathrm{B}_K(Q)$ is flat (since toric varieties are Cohen-Macaulay), or what amounts to the same, integral (as a morphism of fs log schemes) by [3, (4,1)].

We gather up these resulting local log-ideals on Y to make the log-ideal $\mathcal{K}$. We set $\mathcal{K}' = f^{\bullet}\mathcal{K}$ and then consider $\mathrm{B}_{\mathcal{K}'}(X) \to \mathrm{B}_{\mathcal{K}}(Y)$. Then by what we have seen above it is fppf locally integral, and hence is integral. □

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