On a Jordan decomposition of general operators

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Jordan Decomposition

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ABSTRACT

For an arbitrary bounded linear operator in a Banach space we construct a broader locally convex space such that the operator and possible rational functions of the operator are continuously extendible to this space and all possible root vectors of the operator belong to this space . We study the problems of completeness of the system of root vectors and - in the case of completeness - we obtain the Jordan decomposition of the operator .

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§ 0. Introduction.

The Jordan Theorem on the normal form of a matrix is certainly one of the central results of Linear Algebra . It asserts that root vectors of a linear operator in a finite dimensional complex linear space V form a complete system (the root vectors are nontrivial solutions of the equations $(A - \lambda I)^k x = 0$).

The problem of obtaining an infinite dimensional version of this result was discussed very much (see, e.g., [2,5,7]), in particular because of its obvious importance in many problems of Analysis and, first of all , in the Fourier method . There are several problems immediately arising here , connected with adequate generalizations of most important notions .

So , the following notions play the principal roles in the finite dimensional situation:

(i) Spec A – the set of all $\lambda \in \mathbf{C}$ such that the equation $(A - \lambda I)x = 0$ has nontrivial solutions;

(ii) Eigenvectors = nontrivial solutions of the above equation;

(iii) Spec^k A – the set of all $\lambda \in \mathbf{C}$ such that there exists $x \in V$, $(A - \lambda I)^{k+1}x = 0$, $(A - \lambda I)^k x \neq 0$. So , $\text{Spec}^0 A = \text{Spec } A$;

(iv) Root vectors = solutions of the preceding equations.

The notion of spectrum may be very well generalized to the infinite dimensional setting , but as for all other notions - there are well known difficulties connected with the possible non-existence of eigenvectors and , further on, of root vectors.

In fact the Jordan theorem has two layers . The first one is a description of the algebra generated by the operator A, and the second one deals with the phenomenon of multiplicity of spectrum . Here we shall restrict ourselves only to the first layer and study the structure of the algebra , generated by A , leaving aside difficult problems connected with the multiple spectrum - we hope to return to them in the next publications.

Our approach is connected with the theory of generalized eigenvectors of a self adjoint operator in a Hilbert space ([3, 1]) The idea of this approach is to extend (in a natural way) the action of the operator in question to a broader space and try to find the missing eigenvectors and root vectors there . We show that it is possible to do this - we construct such an extension and really find all possible root vectors in the constructed space. The following well known example gives the flavor of the approach - consider the space $L_2(\mathbb{R})$ and the operator $-id/dx$ there, its spectrum is the whole real line $\mathbb{R}$, there exist eigenvectors $e^{i\lambda x}$ of this operator, but they all lie not in the initial space $L_2(\mathbb{R})$ but in a broader space, say, the space of bounded functions . Nevertheless , we know that there exists a very nice theory of decomposition of functions in integrals over these eigenvectors.

The crucial difference between the finite and the infinite dimensional situations is that there are no natural ways to extend an operator in a finite dimensional space to a broader space, whereas every infinite dimensional operator acts simultaneously in many spaces and it is very difficult to say in advance which space is more natural for this operator , so we must not restrict ourselves to the initial space when trying to find such objects as eigenvectors and root vectors.

The paper is organized as follows . § 1 contains necessary preliminaries . § 2 contains a description of the first step in the main construction . § 3 contains a theory of expansion of vectors into "integrals" over generalized eigenvectors with respect to a generalized measure . § 4 is devoted to a geometric description of the space containing all possible root vectors . In § 5 we describe the above construction in analytic terms and introduce some notions necessary for a generalized Gelfand Transform . § 6 is devoted to the generalized Gelfand Transform and problems of completeness of generalized root vectors . In § 7 we apply the previous considerations to obtain a Jordan decomposition for a general operator .

Preliminary versions of these results were announced at various conferences since 1983 - Chernogolovka (1983) , Voronezh (1985-1991) , Halle (1988) , Novgorod (1989) , Oberwolfach (1990) , Sapporo (1990) , Jerusalem (1991) , Beer-Sheba (1992).

§ 1. Main Notions

We consider the following situation : V, V' is a pair of complex Banach spaces such that either V' is the Banach dual space for V or V is the Banach dual space for V'. Let $p(\cdot)$ and $p'(\cdot)$ denote the norms in the spaces V and V', respectively.

Consider a pair of bounded linear operators

$$A: V \to V \quad \text{and} \quad A': V' \to V'$$

such that the usual identity holds:

$$\langle Ax,x' \rangle = \langle x,A'x' \rangle \quad (\forall x \in V, \forall x' \in V')$$

The norms of the operators are defined as usually and they coincide:

$$\| A : V \to V \| = \sup \{ |\langle Ax,x' \rangle| : p(x) \le 1, p'(x') \le 1 \} =$$

$$= \sup \{ |\langle x,A'x' \rangle| : p(x) \le 1, p'(x') \le 1 \} = \| A' : V' \to V' \|$$

The spectra of the operators are also defined as usually and they also coincide

$$\text{Spec } A = \{ \lambda \in \mathbb{C}: (A - \lambda \mathbb{I}) \text{ has no bounded inverse } \} =$$

$$= \{ \lambda \in \mathbb{C}: (A' - \lambda \mathbb{I}) \text{ has no bounded inverse } \} = \text{Spec } A'.$$

Spec A is a compact subset of $\mathbb{C}$.

DEFINITION 1.1. Consider the set $\mathbf{C} \setminus \text{Spec } A$. This set is a disjoint union of its connected components U_i, $i \in I$. One of these components contains ∞ , let U_∞ denote this component (so , $\infty \in I$). Choose a point $\lambda_i \in U_i$ for every $i \in I$, let $\lambda_\infty = \infty$, let Λ denote the set of these points .Consider the functions

$$\chi_i(\mu) = 1/(\mu - \lambda_i), \; i \in I, \; i \ne \infty$$

$$\chi_\infty(\mu) = \mu$$

Let Rat(A) denote the set of all polynomials of $\chi_i(\mu)$, $i \in I$, or in other terms, Rat(A) is the set of all rational functions with poles in the set Λ .

Certainly, the set Rat(A) depends upon the choice of Λ but really we shall use only some completions of Rat(A) and this dependence will disappear . Nevertheless for technical reasons we consider such an object.

DEFINITION 1.2. $R(A) = \{ f(A) : f(\cdot) \in Rat(A) \}$.

We need an equivalent definition of the set Spec A in terms of the algebra R(A).

PROPOSITION 1.3. *$\lambda \in$ Spec A if and only if for any $f \in Rat(A)$*

$$\| f(A) \| \geq | f(\lambda) |.$$

PROOF. First we prove the following assertion, almost equivalent to the Proposition 1.3:

$\lambda \in$ Spec A if and only if $\| \mathbb{I} - (A - \lambda\mathbb{I})B \| \geq 1$ for every $B \in R(A)$.

If for some $B \in R(A)$ we have $\| \mathbb{I} - (A - \lambda\mathbb{I})B \| = c < 1$, then we construct the operator $[(A - \lambda\mathbb{I})B\ ]^{-1}$ as an obviously converging series:

$$[(A - \lambda\mathbb{I})B\ ]^{-1} = [\mathbb{I} - (\mathbb{I} - (A - \lambda\mathbb{I})B\ )]^{-1} = \sum_{k=0}^{\infty} [\mathbb{I} - (A - \lambda\mathbb{I})B\ ]^{k}$$

Now one can easily show that the operator $(A - \lambda\mathbb{I})$ is invertible, so $\lambda \notin$ Spec A. Conversely, if $\lambda \notin$ Spec A then there exists $i \in I$ such that $\lambda \in U_i$. Take $\lambda_i \in \Lambda$, $\lambda_i \in U_i$. It follows from the Runge Theorem (see, e.g., [6]), that the function $z \mapsto 1/(z - \lambda)$ can be approximated by polynomials of $1/(z - \lambda_i)$ uniformly on any simple closed curve Γ lying in U_i, such that λ and λ_i are situated inside Γ. Really, introduce a new variable $w = = 1/(z - \lambda_i)$. The question is reduced to the problem of a uniform approximation of the function

$\varphi(w) = w/(1 + (\lambda_1 - \lambda)w)$ by polynomials of w on a contour $\Gamma\sim$ such that the function is analytic in a neighborhood of the domain bounded by the contour and the classical Runge Theorem guarantees the possibility of such an approximation.

Then the formulae

$$(A - \lambda \mathbb{I})^{-1} = \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{z - \lambda} (A - z\mathbb{I})^{-1}\, dz$$

$$P((A - \lambda_1 \mathbb{I})^{-1}) = \frac{1}{2\pi i} \oint_{\Gamma} P\left(\frac{1}{z - \lambda_1} \right) (A - z\mathbb{I})^{-1} dz$$

show that $(A - \lambda \mathbb{I})^{-1}$ may be approximated in the operator norm by polynomials of $(A - \lambda_1 \mathbb{I})^{-1}$

$$P_n((A - \lambda_1 \mathbb{I})^{-1}) \xrightarrow{\|\cdot\|} (A - \lambda \mathbb{I})^{-1}$$

Then, taking $B_n = P_n((A - \lambda \mathbb{I})^{-1})$ - they obviously belong to R(A) - we obtain that $\| \mathbb{I} - (A - \lambda \mathbb{I})B_n \| \to 0$. So, our assertion is proved.

In order to prove the Proposition 1.3 itself we first must note that if $\lambda \notin \mathrm{Spec}\, A$ then there exists $g \in \mathrm{Rat}(A)$ such that

$$\| \mathbb{I} - (A - \lambda \mathbb{I})g(A) \| < 1$$

and the inequality $\| f(A) \| \geq | f(\lambda) |$ fails for

$$f(z) = 1 - (z - \lambda)g(z) \in \mathrm{Rat}(A).$$

If $\lambda \in \mathrm{Spec}\, A$ then define $\phi_\lambda(\mu)$ as follows:

$$\phi_\lambda(\mu) = \begin{cases} f(\mu) - f(\lambda) / \mu - \lambda &, \mu \neq \lambda \\ f'(\lambda) &, \mu = \lambda \end{cases}$$

One can easily see that if $f \in \mathrm{Rat}(A)$ then ϕ_λ is also in Rat(A) and $f(\mu) = f(\lambda) + (\mu - \lambda)\phi_\lambda(\mu)$. For every function $f \in \mathrm{Rat}(A)$ such that $f(\lambda) \neq 0$ we obtain that

$$\| f(A) \| = \| f(\lambda)\mathbb{I} + (A - \lambda \mathbb{I})\, \phi_\lambda(A) \| =$$

$$= | f(\lambda) | \, \| \, \mathbb{I} + (A - \lambda \mathbb{I}) \frac{1}{f(\lambda)} \; \phi_\lambda(A) \, \| \geq | f(\lambda) | .$$

Let J(0) denote the completion of R(A) in the operator norm topology. One can easily prove that J(0) does not depend upon the choice of Λ in the definition of Rat(A) . J(0) is always supposed to be equipped with the operator norm . J(0) is a commutative Banach algebra.

As it was said before, we consider here only the situation of the "simple spectrum" , so we assume that the following condition holds:

Bicyclicity Condition (BC) . There exist $\Delta \in V$, $\nabla \in V'$ such that $\{ B\Delta : B \in R(A) \}$ is $\sigma(V,V')$ - dense in V and $\{ B'\nabla : B \in R(A) \}$ is $\sigma(V',V)$-dense in V'.

§ 2. Main construction - the first step .

Rigging . Consider the following linear mappings

$$\tau_\Delta : J(0) \to V , \quad \tau_\Delta(B) = B\Delta$$

$$\tau^\nabla : J(0) \to V' , \quad \tau^\nabla(B) = B'\nabla$$

Both mappings are continuous provided all spaces are equipped with the natural normed topologies, but it is more important for us that the mappings are continuous if we equip V and V' with the weak topologies $\sigma(V,V')$ and $\sigma(V',V)$, respectively, because we are interested in the dual mappings

$$(\tau_\Delta)' : V' \to J(0)'$$

$$(\tau^\nabla)' : V \to J(0)'$$

where J(0)' is the Banach dual space for J(0). It follows from the

Bicyclicity Condition that τ_Δ and τ^∇ are injections with weakly dense ranges and this means that $(\tau_\Delta)'$ and $(\tau^\nabla)'$ are also injections with $\sigma($ $J(0)',J(0)$ $)$-dense ranges (to prove the injectivity of, say, τ_Δ , one may go as follows : suppose that

$$0 \neq C \in J(0) \quad \text{and} \quad \tau_\Delta(C) = C\Delta = 0,$$

then the $\sigma(V,V')$-closed set Ker C does not coincide with $\{0\}$,V and for any $B \in J(0)$

$$B \,\mathrm{Ker}\, C \subset \mathrm{Ker}\, C$$

As $\Delta \in \mathrm{Ker}\, C$ then the set $\{ B\Delta : B \in J(0) \}$ is not $\sigma(V,V')$-dense in V, and this contradicts the Bicyclicity Condition).

So we have obtained the following injections with weakly dense ranges

$$J(0) \xrightarrow{\tau_\Delta} V \xrightarrow{(\tau^\nabla)'} J(0)'$$

$$J(0)' \xleftarrow{(\tau_\Delta)'} V' \xleftarrow{\tau^\nabla} J(0)$$

Let V_+ (resp., V^+) denote the lineal Im τ_Δ (resp., Im τ^∇), equipped with the norm transferred from J(0) by the operator τ_Δ (resp., τ^∇). Let V_-(resp., V^-) denote the completion of V (resp., V′) with respect to the norm, transferred from the lineal Im $(\tau^\nabla)' \subset J(0)'$ by the operator $(\tau^\nabla)'^{-1}$ (resp., from the lineal Im $(\tau_\Delta)' \subset J(0)'$ by the operator $(\tau_\Delta)'^{-1}$). So we obtain the following weakly dense inclusions:

$$V_+ \subset V \subset V_-$$
$$V^- \supset V' \supset V^+$$

and $\tau_\Delta : J(0) \to V_+$, $\tau^\nabla : J(0) \to V^+$, $(\tau^\nabla)' : V_- \to J(0)'$, $(\tau_\Delta)' : V^- \to J(0)$ are isometries . The space V_- (resp. V^-) is the

Banach dual space for V^+ (resp.,V_+) and the duality between them is an extension of the initial duality between V and V' .

PROPOSITION 2.1.*For any B, C ∈ J(0) the following statements hold*

(i) $BV_+ \subset V_+$, $B'V^+ \subset V^+$, $\tau_\Delta(BC) = B\,\tau_\Delta(C)$, $\tau^\nabla(BC) = B'\tau^\nabla(C)$;

(ii) $\| B: V_+ \to V_+ \| = \| B: V \to V \| = \| B': V' \to V' \| = \| B': V^+ \to V^+ \|$

(iii) *the action of the operator B on V is* $\sigma(V_-,V^+)$*-continuously extendible to* V_- *and* $\| B : V_- \to V_- \| = \| B : V \to V \|$;

the action of the operator B' on V' is $\sigma(V^-,V_+)$*-continuously extendible to* V^- *and* $\| B': V^- \to V^- \| = \| B' : V' \to V' \|$.

PROOF - an easy checking.

COROLLARY 2.2. *For any B ∈ J(0)*

$$\mathrm{Spec}\, B = \mathrm{Spec}\{B: V \to V\} = \mathrm{Spec}\{ B: V_+ \to V_+ \} = \mathrm{Spec}\{ B: V_- \to V_- \} =$$
$$= \mathrm{Spec}\{ B': V' \to V' \} = \mathrm{Spec}\{ B': V^+ \to V^+ \} = \mathrm{Spec}\{ B': V^- \to V^- \}$$

So we have continuously extended all the operators B from J(0) to a broader space V_- without changing norms of the operators and therefore without changing the spectra. This new space is more natural for these operators - it is shown, in particular, by the following simple and well known theorem (in slightly different terms - see , e.g. [4]).

THEOREM 2.3. $\lambda \in$ *Spec A if and only if there exists* $e_\lambda \in V_-$ *such that* $e_\lambda \neq 0$ *and* $Ae_\lambda = \lambda e_\lambda$.

PROOF. $\lambda \in \mathrm{Spec}\, A \Leftrightarrow \{ (A - \lambda \mathbb{I})B : B \in J(0) \}$ is not dense in J(0)

$\Leftrightarrow \; \exists\, \varphi_\lambda \in J(0)' : \varphi_\lambda(\mathbb{1}) \neq 0 \,,\; \varphi_\lambda((A - \lambda\mathbb{1})B) = 0$ for every $B \in J(0)$.

Take $e_\lambda = ((\tau^\nabla)')^{-1}(\varphi_\lambda)$, $e_\lambda \neq 0$ as $(\tau^\nabla)'$ is an isometry , take any $x' \in V^+$, $x' = \tau^\nabla(B)$ for some $B \in J(0)$. Then

$$\langle (A - \lambda\mathbb{1})e_\lambda \,,\, x' \rangle = \langle (A - \lambda\mathbb{1})\,[(\tau^\nabla)']^{-1}(\varphi_\lambda) \,,\, x' \rangle =$$

$$= \langle [(\tau^\nabla)']^{-1}(\varphi_\lambda) \,,\, (A' - \lambda\mathbb{1})\tau^\nabla(B) \rangle =$$

$$= \langle [(\tau^\nabla)']^{-1}(\varphi_\lambda) \,,\, \tau^\nabla((A - \lambda\mathbb{1})B) \rangle =$$

$$= \varphi_\lambda((A - \lambda\mathbb{1})B) = 0,$$

so $(A - \lambda\mathbb{1})e_\lambda = 0$. ∎

Note that $\langle e_\lambda \,,\, \nabla \rangle = \langle [(\tau^\nabla)']^{-1}\varphi_\lambda \,,\, \nabla \rangle = \varphi_\lambda(\, (\tau^\nabla)^{-1}(\nabla)\,) =$
$= \varphi_\lambda(\, (\tau^\nabla)^{-1}(\tau^\nabla\mathbb{1})\,) = \varphi_\lambda(\mathbb{1}) \neq 0$, so we normalize e_λ by the condition

$$\langle e_\lambda \,,\, \nabla \rangle = 1 \quad , \quad \forall\, \lambda \in \text{Spec } A$$

Similarly , $\lambda \in \text{Spec } A$ if and only if there exists $e^\lambda \in V^-$ such that

$$A'e^\lambda = \lambda e^\lambda \,,\quad \langle \Delta \,,\, e^\lambda \rangle = 1 \quad , \quad \forall\, \lambda \in \text{Spec } A.$$

REMARK 2.4. The core of Theorem 2.3 is that the functionals $\varphi_\lambda \in J(0)'$, normalized by the condition $\varphi_\lambda(\mathbb{1}) = 1$, are multiplicative and they are eigenvectors for the coregular action of $J(0)$ on $J(0)'$ (the coregular action is the action conjugate to the regular action of $J(0)$ on itself).

§ 3. Eigenvector expansions .

Completeness of the system of generalized eigenvectors . We have obtained generalized eigenvectors $\{ e_\lambda , \lambda \in \text{Spec } A \}$, belonging to V_- , so now we can pose and study the problem of completeness of this system of vectors . We consider the completeness in the weakest possible topology - the weak topology $\sigma(V_-,V^+)$. This is naturally equivalent to the problem of the $\sigma(J(0)',J(0))$ - completeness of the system of elements $\{ \varphi_\lambda : \lambda \in \text{Spec } A \}$ in $J(0)'$.

Consider the mapping

$$R(A) \ni f(A) \longmapsto f \,\big|_{\text{Spec } A} \in C(\text{Spec } A)$$

The inequality from the Proposition 1.3

$$\sup_{\mu \in \text{Spec} A} | f(\mu) | \le \| f(A) \|$$

shows that the mapping is correctly defined and is continuously extendible to the Gelfand Homomorphism

$$\wedge : J(0) \to C(\text{Spec } A)$$

(It is well known that Spec A in this situation may be identified with the space $\mathfrak{M}$ (J(0)) of maximal ideals (= the space of multiplicative functionals) of the algebra J(0), the function $\hat{A}$: $\mathfrak{M}(J(0)) \to$ Spec A delivers the necessary identification, a version of this identification was already used in this paper: $\lambda \mapsto \varphi_\lambda$ is a one-to-one correspondence between Spec A and the set of multiplicative functionals on J(0)).

The Gelfand Homomorphism may be also described as follows: for every $B \in J(0)$ and every $\lambda \in \text{Spec } A$ $\hat{B}(\lambda) = \varphi_\lambda(B)$.

THEOREM 3.2. *The system* $\{ \varphi_\lambda : \lambda \in \text{Spec } A \}$ *is* $\sigma(J(0)',J(0))$-

-complete in $J(0)'$ if and only if the algebra $J(0)$ is semisimple (i.e., $\mathrm{Ker}\ \wedge = \{ 0 \}$).

PROOF. The system $\{ \varphi_\lambda : \lambda \in \mathrm{Spec}\ A \}$ is $\sigma(J(0)', J(0))$ - complete if and only if there does not exist $B \in J(0)$, such that $B \neq 0$ and $\varphi_\lambda(B) = 0$, $\forall\ \lambda \in \mathrm{Spec}\ A$, and this is equivalent to the condition $\mathrm{Ker}\ \wedge = \{0\}$. ∎

A refinement of the completeness condition . Let $\partial\mathrm{Spec}\ A$ denote the boundary of Spec A .

PROPOSITION 3.3. *The systems of generalized eigenvectors $\{ \varphi_\lambda : \lambda \in \mathrm{Spec}\ A \}$ and $\{ \varphi_\lambda : \lambda \in \partial\mathrm{Spec}\ A \}$ are $\sigma(J(0)', J(0))$ - complete or noncomplete simultaneously .*

PROOF. For any $B \in J(0)$ the function $\lambda \mapsto \varphi_\lambda(B)$ is analytic in the interior of Spec A , as a uniform limit of analytic functions (Really , there exist $f_n(A) \in R(A)$ such that $\| f_n(A) - B \| \to 0$, so

$$0 \leftarrow \| f_n(A) - B \| \geq \sup_{\lambda \in \mathrm{Spec} A} |\varphi_\lambda(f_n(A) - B)| = \sup_{\lambda \in \mathrm{Spec} A} |f_n(\lambda) - \varphi_\lambda(B)| \ ),$$

therefore

$$\sup_{\lambda \in \mathrm{Spec} A} | \varphi_\lambda(B) | = \sup_{\lambda \in \partial \mathrm{Spec} A} | \varphi_\lambda(B) | .$$

So , if the system $\{ \varphi_\lambda : \lambda \in \partial\mathrm{Spec}\ A \}$ is noncomplete then there exists $B \in J(0)$ such that $B \neq 0$ and $\varphi_\lambda(B)|_{\partial \mathrm{Spec} A} \equiv 0$ then $\varphi_\lambda(B)|_{\mathrm{Spec} A} \equiv 0$ and the system $\{ \varphi_\lambda : \lambda \in \mathrm{Spec}\ A \}$ is also noncomplete. ∎

Generalized eigenvector decomposition . Let us suppose that the system of generalized eigenvectors $\{\varphi_\lambda : \lambda \in \partial\mathrm{Spec}\ A\}$ is $\sigma(J(0)',J(0))$ - complete. This is equivalent to the semisimplicity of the algebra J(0) , i.e. to the injectivity of the Gelfand Homomorphism

$$\wedge : J(0) \to C(\partial\mathrm{Spec}\ A).$$

Consider the conjugate mapping

$$\wedge' : \mathrm{Meas}(\partial\mathrm{Spec}\ A) \to J(0)'$$

where Meas(∂SpecA) is the space of Borel measures on ∂Spec A – the Banach dual space for C(∂Spec A). This mapping $\wedge'$ has a $\sigma(J(0)',J(0))$– - dense range (because of the injectivity of $\wedge$) , therefore each element of $J(0)'$ may be approximated by a net of elements of the type $\hat{}'(\mu_\alpha)$, where $\{ \mu_\alpha \}$ is a net of measures . Therefore we consider elements of $J(0)'$ as $\hat{}'$-images of " generalized measures" on ∂Spec A (functionals on Im ^ , equipped with the norm transferred from J(0) by the mapping ^).

THEOREM 3.4. *Suppose the set* $\{\varphi_\lambda : \lambda \in \partial\mathrm{Spec}\ A\}$ *is* $\sigma(J(0)',J(0))$*–complete (= the algebra J(0) is semisimple). Then for every* $\varphi \in J(0)'$ *there exists a generalized measure* $d\mu_\varphi$ *of ∂Spec A (= a bounded functional on the subalgebra Im ^) such that for every* $B \in J(0)$

$$\varphi(B) = \int_{\partial\mathrm{Spec}\ A} \varphi_\lambda(B)d\mu_\varphi(\lambda) = \lim_\alpha \int_{\partial\mathrm{Spec}A} \varphi_\lambda(B)\ d\mu_{\alpha,\varphi}(\lambda)$$

($\{d\mu_{\alpha,\varphi}\}$ *is a net of measures on ∂Spec A) .*

PROOF. $\varphi(B) = \int_{\partial \mathrm{Spec} A} \hat{B}(\lambda)\, d\mu_\varphi(\lambda) = \int_{\partial \mathrm{Spec} A} \varphi_\lambda(B)\, d\mu_\varphi(\lambda)$ ∎

Now we return to the spaces V , $V_{\pm}$, $V^{\pm}$. As previously, we let e_λ , $\lambda \in$ Spec A , denote the generalized eigenvectors of A , $e_\lambda \in V_-$, and we let e^λ, $\lambda \in$ Spec A , denote the generalized eigenvectors of A', $e^\lambda \in V^-$. The normalization conditions are :

$$\langle e_\lambda, \nabla \rangle = 1 \quad , \quad \forall \lambda \in \text{Spec } A$$

$$\langle \Delta, e^\lambda \rangle = 1 \quad , \quad \forall \lambda \in \text{Spec } A$$

We assume again that the algebra J(O) is semisimple - this is equivalent to the $\sigma(V_-,V^+)$-completeness of the system $\{e_\lambda,\ \lambda \in \text{Spec } A\}$ or , equivalently , to the $\sigma(V^-,V_+)$-completeness of the system $\{ e^\lambda :\ \lambda \in \partial\text{Spec } A \}$.

We want to decompose elements of V into "integrals" of e_λ over ∂Spec A . Generally speaking , this is impossible for all elements of V , but this is possible for elements of the dense lineal V_+.

THEOREM 3.5. *There exists a generalized measure on* ∂*Spec A* *(= a bounded functional on the subalgebra* $Im \wedge \subset C(\partial \text{Spec } A)$ *) such that for every* $x \in V_+$ *and every* $x' \in V^+$

$$\langle x, x' \rangle = \int_{\partial \mathrm{Spec} A} \langle x, e^\lambda \rangle \langle e_\lambda, x' \rangle \, d\mu(\lambda)$$

$$\left(\text{or } x = \int_{\partial \mathrm{Spec} A} \langle x, e^\lambda \rangle e_\lambda \, d\mu(\lambda) \right)$$

PROOF. As $x \in V_+$, $x' \in V^+$ then there exist $B, C \in J(0)$ such that $x = B\Delta$, $x' = C'\nabla$. Consider the following functional on $J(0)$: $D \mapsto \langle D\Delta, \nabla \rangle$. This is certainly a bounded linear functional, therefore there exists a generalized measure $d\mu$ on $\partial\mathrm{Spec}\, A$ such that

$$\langle D\nabla, \Delta \rangle = \int_{\partial \mathrm{Spec} A} \hat{D}(\lambda)\, d\mu(\lambda) = \lim_{\alpha} \int_{\partial \mathrm{Spec} A} \hat{D}(\lambda)\, d\mu_\alpha(\lambda).$$

Then, recalling that $\langle e_\lambda, \nabla \rangle = \langle \Delta, e^\lambda \rangle = 1$ for all $\lambda \in \mathrm{Spec}\, A$, we obtain

$$\langle x, x' \rangle = \langle B\Delta, C'\nabla \rangle = \langle CB\Delta, \nabla \rangle = \int_{\partial \mathrm{Spec} A} {}^\wedge(CB)(\lambda)\, d\mu(\lambda) =$$

$$= \int_{\partial \mathrm{Spec} A} \hat{C}(\lambda)\, \hat{B}(\lambda)\, d\mu(\lambda) = \int_{\partial \mathrm{Spec} A} \hat{C}(\lambda) \langle e_\lambda, \nabla \rangle \hat{B}(\lambda) \langle \Delta, e^\lambda \rangle\, d\mu(\lambda) =$$

$$= \int_{\partial \mathrm{Spec} A} \langle Ce_\lambda, \nabla \rangle \langle \Delta, B'e^\lambda \rangle\, d\mu(\lambda) = \int_{\partial \mathrm{Spec} A} \langle B\Delta, e^\lambda \rangle \langle e_\lambda, C'\nabla \rangle d\mu(\lambda) =$$

$$= \int_{\partial \mathrm{Spec} A} \langle x, e_\lambda \rangle \langle e_\lambda, x' \rangle\, d\mu(\lambda).$$ ∎

§ 4. Jordan decomposition - main constructions .

Root vectors . What can we do in the case of a nonsemisimple algebra $J(0)$? Here it is definitely nonsufficient to use only eigenvectors.

The finite dimensional situation shows that it is very useful to consider also root vectors - nontrivial solutions of equations $(A - \lambda\mathbb{I})^k x = 0$. But they may not exist. The most important idea of the preceding sections was to continuously extend the action of all related operators to a broader space and try to find the missing vectors there. We show how to construct the needed broader spaces. First we describe the situation in geometric terms.

We started from the space V and the operator A yielding the Bicyclicity Condition. We managed to construct a broader space V_- and extend the action of A to the space V_- by continuity. Then for every $\lambda \in \text{Spec } A$ we found an eigenvector $e_\lambda \in V_-$, $Ae_\lambda = \lambda e_\lambda$. Let E_λ denote the related eigensubspace - it is 1-dimensional due to the BC. Consider the quotient space V_- / E_λ. The operator A may be naturally lifted to V_- / E_λ. Let $A(1,\lambda)$ denote the related operator :

$$A(1,\lambda): V_- / E_\lambda \longrightarrow V_- / E_\lambda .$$

There may be two possibilities :

(i) $\lambda \in \text{Spec } A(1,\lambda)$,

(ii) $\lambda \notin \text{Spec } A(1,\lambda)$.

As for the case (ii), we cannot even expect that there exists an eigenvector of the operator $A(1,\lambda)$ with the eigenvalue λ - so the related associate vector of the initial operator A cannot exist. And as for the case (i) we do can expect the existence of the related eigenvector, but it is not difficult to give examples when there is no such eigenvectors in V_- / E_λ, so there is no associate vectors in V_-. What to do? As it was already shown in the Theorem 2.3, it is possible to extend the operator $A(1,\lambda)$ by continuity to a broader space $(V_- / E_\lambda)_-$ and to find the needed eigenvector there, we must only find a cyclic vector in $E_\lambda^\perp$ - a predual space of V_- / E_λ (one can easily see that $(E_\lambda^\perp)' = V_- / E_\lambda$). Note that if we have only one cyclic vector

$\nabla \in V'$ ($\{ B'\nabla : B \in R(A) \}$ is $\sigma(V',V)$-dense in V') then we already have a possibility to construct the inclusions $V^+ \subset V'$ and $V \subset V_-$ such that V is $\sigma(V_-,V^+)$-dense in V_-. So, in order to be able to construct the space $(V_- / E_\lambda)_-$ we need only a cyclic vector in the space $E_\lambda^\perp \subset V^+$.

Let us first do everything in terms of the algebra J(0) and then use once again the rigging constructed above. So, V_- is isometric to $J(0)'$, E_λ corresponds to the one-dimensional subspace $\{ \nu\varphi_\lambda : \nu \in \mathbb{C} \}$, V_- / E_λ is therefore isometric to

$$J(0)'/\{\nu\varphi_\lambda : \nu \in \mathbb{C} \} = (\mathrm{Ker}\ \varphi_\lambda)'$$

The operator $A(1,\lambda)$ corresponds to the operator conjugate to the multiplication by A in the subspace $\{ \nu\varphi_\lambda : \nu \in \mathbb{C} \}^\perp = \mathrm{Ker}\ \varphi_\lambda \subset J(0)$. Really , for any $\varphi \in J(0)'$ and any $B \in \mathrm{Ker}\ \varphi_\lambda$

$$\langle A(1,\lambda)\ [(\tau^\nabla)']^{-1}(\varphi)\ ,\ (\tau^\nabla)(B) \rangle = \langle [(\tau^\nabla)']^{-1}(\varphi)\ ,\ A'\tau^\nabla(B) \rangle =$$

$$= \langle [(\tau^\nabla)']^{-1}(\varphi)\ ,\ \tau^\nabla(AB) \rangle = \varphi(AB).$$

Let $A(1,\lambda)$ denote also the operator of multiplication by A in the subspace $\mathrm{Ker}\ \varphi_\lambda$.

So we have a pair of Banach spaces $(\mathrm{Ker}\ \varphi_\lambda)'$, $\mathrm{Ker}\ \varphi_\lambda$ and a bounded linear operator $A(1,\lambda) : \mathrm{Ker}\ \varphi_\lambda \to \mathrm{Ker}\ \varphi_\lambda$.

LEMMA 4.1. *Spec* $A \supseteq$ *Spec* $A(1,\lambda) \supseteq$ *Spec* $A \setminus \{\lambda\}$.

PROOF. If $\mu \notin \mathrm{Spec}\ A$ then $(A - \mu\mathbb{I})^{-1} \in J(0)$ and the subspace $\mathrm{Ker}\ \varphi_\lambda$ is invariant under the multiplication by $(A - \mu\mathbb{I})^{-1}$, so

$\mu \notin \mathrm{Spec}\, A(1,\lambda)$. If $\mu \neq \lambda$ and $\mu \notin \mathrm{Spec}\, A(1,\lambda)$ then we construct bounded operators in J(0) , which are left and right inverses to the multiplication by $(A - \mu \mathbb{1})$.

Take any $B \in J(0)$ and decompose it

$$B = \varphi_\lambda(B)\cdot \mathbb{1} + (B - \varphi_\lambda(B)\cdot \mathbb{1})$$

Obviously , the second term belongs to Ker φ_λ . We obtain a decomposition into a direct sum

$$J(0) = \{\ \nu \mathbb{1} : \nu \in \mathbb{C}\ \} \oplus \mathrm{Ker}\ \varphi_\lambda$$

We want to solve the equation - to find B :

$$(A - \mu\ \mathbb{1})B = C$$

Decomposing the entries as above we obtain

$$(A - \mu \mathbb{1})(\ \varphi_\lambda(B)\mathbb{1} + (B - \varphi_\lambda(B)\mathbb{1})\ ) = \varphi_\lambda(C)\mathbb{1} + (C - \varphi_\lambda(C)\mathbb{1})$$

$$\varphi_\lambda(B)\ (A - \lambda \mathbb{1}) + (\lambda - \mu)\varphi_\lambda(B)\mathbb{1} + (A - \mu \mathbb{1})(\ B - \varphi_\lambda(B)\mathbb{1}\ ) =$$

$$= \varphi_\lambda(C)\mathbb{1} + (\ C - \varphi_\lambda(C)\mathbb{1}\ ).$$

The first and the third terms in the left hand side belong to Ker φ_λ , therefore

$$(\lambda - \mu)\ \varphi_\lambda(B) = \varphi_\lambda(C)$$

and

$$\varphi_\lambda(B)(A - \lambda \mathbb{1}) + (\ A(1,\lambda) - \mu \mathbb{1}\ )(\ B - \varphi_\lambda(B)\mathbb{1}\ ) = C - \varphi_\lambda(C)\mathbb{1}$$

So ,

$$\varphi_\lambda(B) = \frac{1}{\lambda - \mu}\,\varphi_\lambda(C)$$

$$B - \varphi_\lambda(B)\mathbb{1} = (A(1,\lambda) - \mu \mathbb{1})^{-1}[\ C - \varphi_\lambda(C)\mathbb{1} - \varphi_\lambda(B)(A - \lambda \mathbb{1})\ ]$$

The operator $(A(1,\lambda) - \mu \mathbb{1})^{-1}$: Ker $\varphi_\lambda \to$ Ker φ_λ exists and is bounded because $\mu \notin \operatorname{Spec} A(1,\lambda)$. So

$$B = \varphi_\lambda(B)\mathbb{I} + (B - \varphi_\lambda(B)\mathbb{I}) =$$

$$= \frac{1}{\lambda - \mu}\ \varphi_\lambda(C) + (A(1,\lambda) - \mu\mathbb{I})^{-1}\ [\ C - \varphi_\lambda(C)\mathbb{I} - \frac{\varphi_\lambda(C)}{\lambda - \mu}\ (A - \lambda\mathbb{I})\ ].$$

So we have constructed a left inverse operator for the multiplication by $(A - \mu\mathbb{I})$ in $J(0)$, this operator is obviously bounded. The right inverse operator may be constructed in the same way. So, $\mu \notin \operatorname{Spec} A$ and the Lemma is proved. ∎

REMARK 4.2. Now we may assume that $\operatorname{Rat}(A) = \operatorname{Rat}(A(1,\lambda))$ because the connected components of $\mathbb{C} \setminus \operatorname{Spec} A$ and of $\mathbb{C} \setminus \operatorname{Spec} A(1,\lambda)$ are in an obvious one-to-one correspondence.

COROLLARY 4.3. *If λ is a non-isolated point of Spec A then $\lambda \in \operatorname{Spec} A(1,\lambda)$.*

There is a natural cyclic vector in $\operatorname{Ker} \varphi_\lambda$ for the action of the algebra $R(\ A(1,\lambda)\ )$ – it is the element $(A - \lambda\mathbb{I}) \in \operatorname{Ker} \varphi_\lambda$. Really one can easily show that $R(\ A(1,\lambda)\ )\cdot(A - \lambda\mathbb{I})$ $(= R(A)\cdot(A - \lambda\mathbb{I}))$ is dense in $\operatorname{Ker} \varphi_\lambda$. Let $\bar{R}(\ A(1,\lambda)\ )$ denote the closure of $R(\ A(1,\lambda)\ )$ in the operator norm topology on $\operatorname{Ker} \varphi_\lambda$

$$\|\ \varphi\ (A(1,\lambda))\ \|_{\bar{R}(A(1,\lambda))} = \sup_{B \in R(A)} \frac{\|\ \varphi(A)\cdot(A - \lambda\mathbb{I})\cdot B\ \|_{J(0)}}{\|\ (A - \lambda\mathbb{I})B\ \|_{J(0)}}$$

We imbed $\overline{R(A(1,\lambda))}$ into $\operatorname{Ker} \varphi_\lambda$ with the help of the cyclic vector $(A-\lambda\mathbb{I})$: $\tau(1,\lambda)B = B\ (A - \lambda\mathbb{I})$, this is certainly a continuous imbedding with a dense range (it contains $R(A)(A - \lambda \mathbb{1})$). Equip $\operatorname{Im} \tau(1,\lambda)$ with the norm induced by this imbedding : for $C \in \operatorname{Im} \tau(1,\lambda)$

$$\| C \|_{1,\lambda} = \sup_{B \in R(A)} \frac{\| C B \|}{\| (A - \lambda \mathbb{1})B \|} .$$

Let $J(1,\lambda)$ denote the range of $\tau(1,\lambda)$ (= the completion of $(A-\lambda\mathbb{1})R(A)$ with respect to the norm $\|\cdot\|_{1,\lambda}$). Then $\lambda \in \operatorname{Spec} A(1,\lambda)$ if and only if there exists $\varphi_{1,\lambda} \in J(1,\lambda)'$, such that

$$\varphi_{1,\lambda} \neq 0 \text{ and } A(1,\lambda)\varphi_{1,\lambda} = \lambda \cdot \varphi_{1,\lambda} .$$

Unfortunately, this element $\varphi_{1,\lambda}$ belongs to an extension of the space $(J(0)/\langle \nu\varphi_\lambda : \nu \in \mathbf{C} \rangle)'$, we would prefer to have it in an extension of the space $J(0)'$. In order to improve the situation note that $\operatorname{Ker} \varphi_\lambda$ is naturally complemented in $J(0)$:

$$J(0) = \{ \nu\mathbb{1} : \nu \in \mathbf{C} \} \oplus \operatorname{Ker} \varphi_\lambda \quad , \quad (\lambda \in \operatorname{Spec} A)$$

$$B = \varphi_\lambda(B)\mathbb{1} + (B - \varphi_\lambda(B)\mathbb{1})$$

therefore $J(0)'$ is naturally split into a direct sum

$$J(0)' = \mathbb{1}^\perp \oplus \{ \nu\varphi_\lambda : \nu \in \mathbf{C} \}$$

$$\varphi = (\varphi - \varphi(\mathbb{1})\varphi_\lambda) + \varphi(\mathbb{1})\varphi_\lambda$$

(φ_λ is normalized by the condition $\varphi_\lambda(\mathbb{1}) = 1$) . So, $(\operatorname{Ker} \varphi_\lambda)'$ is naturally isometric to $\mathbb{1}^\perp$. Extend the norm $\|\cdot\|_{1,\lambda}$ to a seminorm on the whole $R(A)$ in the following way

$$\| \varphi(A) \|_{1,\lambda} = \sup_{B \in R(A)} \frac{\| (\varphi(A) - \varphi(\lambda)\mathbb{1})B \|}{\| (A - \lambda\mathbb{1})B \|}$$

Note that $\| \varphi(A) \|_{1,\lambda} = 0$ for $\varphi(A) \in \{ \nu\mathbb{I} \ : \ \nu \in \mathbb{C} \}$.

$J(1,\lambda)$ is continuously imbedded into Ker φ_λ and therefore into $J(0)$. So, (Ker $\varphi_\lambda)'$ is continuously and weak-densely imbedded into $J(1,\lambda)'$, therefore

$$J(0)' = (\text{Ker } \varphi_\lambda)' \oplus \{ \nu\varphi_\lambda : \nu \in \mathbb{C} \}$$

is continuously and weak densely imbedded into

$$J(1,\lambda)' \oplus \{ \nu\varphi_\lambda : \nu \in \mathbb{C} \},$$

so the vector $\varphi_{1,\lambda} \in J(1,\lambda)'$ may be considered to belong to an extension of the space $J(0)'$, and moreover, we may consider

$$\varphi_{1,\lambda}(\mathbb{I}) = 0 \ , \ \varphi_{1,\lambda}(A - \lambda\mathbb{I}) = 1.$$

We may naturally iterate this construction and consequently extend our operators to broader and broader spaces and find all possible root vectors in such spaces.

So, the construction is the following :

we start with the space V and the operator $A : V \to V$. We consider the algebra R(A) and using the cyclic vector $\Delta \in V$ we construct the dense injection of the B(V)-closure of R(A) (B(V) is the algebra of all bounded operators in V)into the space V

$$\tau_\Delta : \overline{R(A)}^{\,B(V)} \to V \quad , \quad \tau_\Delta(B) = B\Delta$$

J(0) denotes the space $\overline{R(A)}^{B(V)}$ and let A(0) denote the operator of multiplication by A in the space J(0)

$$A(0) : J(0) \to J(0)$$

It is obvious that Spec A = Spec A(0) and we may take

$$\text{Rat}(A) = \text{Rat } A(0)$$

Take $\lambda \in \text{Spec } A = \text{Spec } A(0)$ and find $\varphi_\lambda \in J(0)'$, $\varphi_\lambda(\mathbb{I}) = 1$, $\varphi_\lambda((A - \lambda\mathbb{I})\cdot B) = 0$ for any $B \in R(A(0))$. Consider the subspace $\text{Ker } \varphi_\lambda \subset J(0)$. It is invariant under the operators from $R(A(0))$. Let $A(1,\lambda)$ denote the restriction of the operator $A(0)$ to $\text{Ker } \varphi_\lambda$. We have shown (Lemma 4.1) that

$$\text{Spec } A = \text{Spec } A(0) \supset \text{Spec } A(1,\lambda) \supset \text{Spec } A(0)\setminus\{\lambda\},$$

and therefore, in particular, we may take Rat $A(1,\lambda)$ = Rat $A(0)$ = Rat A

So we have

$$A(1,\lambda) : \text{Ker } \varphi_\lambda \to \text{Ker } \varphi_\lambda$$

and we may construct a dense injection $\tau(1,\lambda)$ with the help of the $R(A(1,\lambda))$-cyclic vector $(A - \lambda\mathbb{I}) \in \text{Ker } \varphi_\lambda$.

$$\tau(1,\lambda) : \overline{R(A(1,\lambda))}^{B(\text{Ker}\varphi_\lambda)} \to \text{Ker } \varphi_\lambda \ , \ \tau(1,\lambda)(B) = B(A - \lambda\mathbb{I})$$

We can also restrict the operator $A(1,\lambda)$ to $\text{Im } \tau(1,\lambda)$.

Let $J(1,\lambda)$ denote the lineal $\text{Im } \tau(1,\lambda)$ equipped with the norm transferred from $\overline{R(A(1,\lambda))}^{B(\text{Ker}\varphi_\lambda)}$ by the mapping $\tau(1,\lambda)$. We get the operator

$$A(1,\lambda) : J(1,\lambda) \to J(1,\lambda)$$

Obviously , Spec $\{ A(1,\lambda) : J(1,\lambda) \to J(1,\lambda) \}$ = Spec $A(1,\lambda)$ and if $\lambda \in \text{Spec } A(1,\lambda)$ we find a vector $\varphi_{1,\lambda} \in J(1,\lambda)'$ such that

$$\varphi_{1,\lambda}(A - \lambda\mathbb{I}) = 1 \ , \ \ \varphi_{1,\lambda}((A - \lambda\mathbb{I})^2 B) = 0,$$

for any $B \in R(A(1,\lambda)) = R(A)$.

Consider the subspace $\text{Ker } \varphi_{1,\lambda} \subset J(1,\lambda)$. It is invariant under the operators from $R(A(1,\lambda))$. Let $A(2,\lambda)$ denote the restricton of $A(1,\lambda)$ to $\text{Ker } \varphi_{1,\lambda}$, and so on . In general, suppose that we have already obtained

$$\varphi_\lambda , \varphi_{1,\lambda} , \varphi_{2,\lambda} , \dots ,\varphi_{k,\lambda} , \quad (\varphi_{p,\lambda} \in J(p,\lambda)' , \forall p)$$

$$\varphi_{k,\lambda}((A - \lambda\mathbb{I})^k) = 1 , \quad \varphi_{k,\lambda}((A - \lambda\mathbb{I})^{k+1}B) = 0 , \forall B \in R(A(k,\lambda))$$

We have the operator

$$A(k,\lambda) : J(k,\lambda) \to J(k,\lambda)$$

and let $A(k+1,\lambda)$ denote its restriction to Ker $\varphi_{k,\lambda}$

$$A(k+1,\lambda) : \text{Ker } \varphi_{k,\lambda} \to \text{Ker } \varphi_{k,\lambda}$$

We construct a dense injection $\tau(k,\lambda)$ with the help of the $R(A(k,\lambda))$-cyclic vector $(A - \lambda\mathbb{I})^k \in$ Ker $\varphi_{k,\lambda}$.

$$\tau(k,\lambda) : \overline{R(A(k,\lambda))}^{B(\text{Ker } \varphi_{k,\lambda})} \to \text{Ker } \varphi_{k,\lambda} ,$$

$$\tau(k,\lambda)(B) = B(A - \lambda\mathbb{I})^k.$$

We also may restrict the operator $A(k,\lambda)$ to $\text{Im } \tau(k,\lambda)$, equipped with the norm transferred from the space $\overline{R(A(k,\lambda))}^{B(\text{Ker}\varphi_{k,\lambda})}$ by the operator $\tau(k,\lambda)$, we let $J(k+1,\lambda)$ denote this Banach space. So, we get the operator

$$A(k+1,\lambda) : J(k+1,\lambda) \to J(k+1,\lambda)$$

Obviously , $\text{Spec } \{ A(k+1,\lambda) : J(k+1,\lambda) \to J(k+1,\lambda) \} = \text{Spec } A(k+1,\lambda)$, and if $\lambda \in \text{Spec } A(k+1,\lambda)$ we find a vector $\varphi_{k+1,\lambda} \in J(k+1,\lambda)'$ such that

$$\varphi_{k+1,\lambda}((A - \lambda\mathbb{I})^{k+1}) = 1 , \quad \varphi_{k+1,\lambda}((A - \lambda\mathbb{I})^{k+2}\cdot B) = 0$$

for any $B \in R(A)$, and so on .

Let us note that the ideals

$$\{ (A - \lambda \mathbb{I})^k \cdot B : B \in R(A) \} := I(k,\lambda) \subset R(A)$$

are dense in $J(k,\lambda)$ (with respect to the norm transferred from $\overline{R(A(k-1,\lambda))}^{B(Ker\varphi_{k-1,\lambda})}$). The ideals $I(k,\lambda)$ are naturally complemented in $R(A)$

$$f(A) = \left(\sum_{l=0}^{k-1} \frac{1}{l!} f^{(l)}(\lambda)(A - \lambda \mathbb{I})^l \right) + \left(f(A) - \sum_{l=0}^{k-1} \frac{1}{l!} f^{(l)}(\lambda)(A - \lambda \mathbb{I})^l \right)$$

So,

$$R(A) = T(k,\lambda) \oplus I(k,\lambda),$$

where $T(k,\lambda) = \{ \sum_{l=0}^{k-1} c_l (A - \lambda \mathbb{I})^l : c_l \in \mathbf{C} \}$.

The first summand is k-dimensional , so we consider the following spaces

$$R(k,\lambda) = T(k,\lambda) \oplus J(k,\lambda)$$

($J(k,\lambda)$ is equipped with the natural norm),

$$J(0) = R(0,\lambda) \supset R(1,\lambda) \supset R(2,\lambda) \supset \ldots$$

all inclusions are dense and continuous.

So, the intersection $\bigcap_{k,\lambda} R(k,\lambda)$ - we shall denote it $\mathfrak{A}_\infty$ - contains $R(A)$ and it carries a natural locally convex topology of the projective limit. We can find all possible root vectors in the dual space of $\mathfrak{A}_\infty$ - we denote it $\mathfrak{A}^\infty$. In the next paragraph we describe the whole construction in analytic terms, permitting further study of the Jordan decomposition.

§ 5. Algebra $\mathfrak{A}_\infty$.

Consider the algebra R(A) and introduce the following seminorms on R(A)

$$\| f(A) \|_{0,\lambda} = \| f(A) \| \qquad (\lambda \in \mathbb{C})$$

$$\| f(A) \|_{1,\lambda} = \sup_{B \in R(A)} \frac{\| (f(A) - f(\lambda) I) B \|_{0,\lambda}}{\| (A - \lambda I) B \|_{0,\lambda}}, \qquad (\lambda \in \mathbb{C} \setminus \Lambda)$$

$$\| f(A) \|_{n,\lambda} = \sup_{B \in R(A)} \frac{\| [f(A) - \sum_{i=0}^{n-1} \frac{f^{(i)}(\lambda)}{i!} (A - \lambda I)^i] B \|_{n-1,\lambda}}{\| (A - \lambda I)^n B \|_{n-1,\lambda}}, \qquad (\lambda \in \mathbb{C} \setminus \Lambda).$$

The origin of these seminorms was explained previously.

Let $\mathfrak{A}_\infty$ denote the completion of R(A) with respect to this family of seminorms - this definition obviously coincides with the one given above.

We again consider the following ideals in the algebra R(A):

$$I(n,\lambda) = \{ (A - \lambda I)^n B : B \in R(A) \}$$

The seminorm $\| \cdot \|_{n,\lambda}$ is a norm if restricted to $I(n,\lambda)$, moreover for $f(A) \in I(n,\lambda)$

$$\| f(A) \|_{n,\lambda} = \sup_B \frac{\| f(A) B \|_{n-1,\lambda}}{\| (A - \lambda I)^n B \|_{n-1,\lambda}}$$

LEMMA 5.1. *If* $B \in I(n,\lambda)$, $C \in R(A)$ *then*

$$\| BC \|_{n,\lambda} \le \| B \|_{n,\lambda} \, \| (A - \lambda I)^n C \|_{n,\lambda}.$$

PROOF.

$$\| BC \|_{n,\lambda} = \sup_D \frac{\| BCD \|_{n-1,\lambda}}{\| (A-\lambda I)^n D \|_{n-1,\lambda}} \le$$

$$\le \sup_D \frac{\| BCD \|_{n-1,\lambda}}{\| (A-\lambda I)^n CD \|_{n-1,\lambda}} \sup_D \frac{\| (A-\lambda I)^n CD \|_{n-1,\lambda}}{\| (A-\lambda I)^n D \|_{n-1,\lambda}} \le$$

$$\le \| B \|_{n,\lambda} \, \| (A-\lambda I)^n C \|_{n,\lambda}$$ ∎

Really many of these seminorms are equivalent . In order to see this we must introduce some notions.

DEFINITION 5.2. We say that $\lambda \in \mathrm{Spec}^k A$ if $\lambda \in \mathrm{Spec}\, A(k,\lambda)$.

PROPOSITION 5.3. $\lambda \in \mathrm{Spec}^k A$ *if and only if for every* $f \in Rat(A)$

$$\| f(A) \|_{k,\lambda} \ge \frac{| f^{(k)}(\lambda) |}{k!}$$

PROOF.

$$\lambda \in \mathrm{Spec}^k A \Leftrightarrow \lambda \in \mathrm{Spec}\, A(k,\lambda) \Leftrightarrow$$

$$\Leftrightarrow \forall\, \varphi \in \mathrm{Rat}(A) = \mathrm{Rat}\, A(k,\lambda)\ \| \varphi(A(k,\lambda)) : J(k,\lambda) \longrightarrow J(k,\lambda) \| \ge |\varphi(\lambda)| \Leftrightarrow$$

$$\Leftrightarrow \forall\, \varphi \in \mathrm{Rat}(A) \quad \sup_B \frac{\| \varphi(A)(A - \lambda I)^k B \|_{k-1,\lambda}}{\| (A - \lambda I)^k B \|_{k-1,\lambda}} \ge | \varphi(\lambda) | \Leftrightarrow$$

$$\Leftrightarrow \forall f \in I(k,\lambda) \quad \| f(A) \|_{k,\lambda} \geq \frac{| f^{(k)}(\lambda) |}{k!} \Leftrightarrow$$

$$\Leftrightarrow \forall f \in R(A) \quad \| f(A) \|_{k,\lambda} \geq \frac{| f^{(k)}(\lambda) |}{k!} .$$

■

PROPOSITION 5.4. $\mu \in \operatorname{Spec} A(k,\lambda)$ if and only if for any $f \in \operatorname{Rat}(A)$

$$\| f(A) \|_{k,\lambda} \geq \begin{cases} \left| \dfrac{f(\mu) - \sum_{i=0}^{k-1} f^{(i)}(\lambda)\dfrac{(\mu-\lambda)^i}{i!}}{(\mu-\lambda)^k} \right| & , \mu \neq \lambda \\ \left| \dfrac{f^{(k)}(\lambda)}{k!} \right| & , \mu = \lambda \end{cases}$$

PROOF. Let $\mu \neq \lambda$ (or else see the previous Proposition 5.3).

$$\mu \in \operatorname{Spec} A(k,\lambda) \Leftrightarrow$$

$$\Leftrightarrow \forall \varphi \in \operatorname{Rat}(A) = \operatorname{Rat} A(k,\lambda) \ \| \varphi (A(k,\lambda)) : J(k,\lambda) \to J(k,\lambda) \| \geq | \varphi(\mu) | \Leftrightarrow$$

$$\Leftrightarrow \forall \varphi \in \operatorname{Rat}(A) \quad \sup_B \frac{\| \varphi(A) (A - \lambda I)^k B \|_{k-1,\lambda}}{\| (A - \lambda I)^k B \|_{k-1,\lambda}} \geq | \varphi(\mu) | \Leftrightarrow$$

$$\Leftrightarrow \forall f \in I(k,\lambda) \quad \sup_B \frac{\| f(A) B \|_{k-1,\lambda}}{\| (A-\lambda I)^k B \|_{k-1,\lambda}} = \| f(A) \|_{k,\lambda} \geq \left| \frac{f(\mu)}{(\mu-\lambda)^k} \right| \Leftrightarrow$$

$$\Leftrightarrow \forall f \in \operatorname{Rat}(A) \quad \| f(A) \|_{k,\lambda} \geq \left| \frac{f(\mu) - \sum_{i=0}^{k-1} f^{(i)}(\lambda) \dfrac{(\mu-\lambda)^i}{i!}}{(\mu-\lambda)^k} \right|$$

■

PROPOSITION 5.5. *Let $\lambda \notin \operatorname{Spec}^k A$. Then the norms $\|\cdot\|_{k,\lambda}$ and $\|\cdot\|_{k-1,\lambda}$ are equivalent on $I(k,\lambda)$.*

PROOF. Let $f \in I(k,\lambda)$. Then

$$\| f(A) \|_{k,\lambda} = \sup_B \frac{\| f(A)\, B \|_{k-1,\lambda}}{\| (A-\lambda 1)^k\, B \|_{k-1,\lambda}} \geq$$

$$\geq \frac{1}{\| (A-\lambda 1)^k \|_{k-1,\lambda}} \| f(A) \|_{k-1,\lambda}$$

(without any assumptions on λ).

Now let $\lambda \notin \mathrm{Spec}^k A$. Then $\lambda \notin \mathrm{Spec}\, A(k,\lambda)$, i.e.

$$\| (A(k,\lambda) - \lambda 1)\, C \|_{k-1,\lambda} \geq c\, \| C \|_{k-1,\lambda} \quad \text{for } C \in I(k-1,\lambda)$$

or

$$\| (A-\lambda 1)(A-\lambda 1)^{k-1} D \|_{k-1,\lambda} \geq c\, \| (A-\lambda 1)^{k-1} D \|_{k-1,\lambda}$$

for $D \in R(A)$. So

$$\| f(A) \|_{k,\lambda} = \sup_D \frac{\| f(A)\, D \|_{k-1,\lambda}}{\| (A-\lambda 1)^k\, D \|_{k-1,\lambda}} \leq \frac{1}{c} \sup_D \frac{\| f(A)\, D \|_{k-1,\lambda}}{\| (A-\lambda 1)^{k-1}\, D \|_{k-1,\lambda}} \leq$$

$$\leq \frac{1}{c} \sup_D \frac{\| f(A) \|_{k-1,\lambda}\, \| (A-\lambda 1)^{k-1} D \|_{k-1,\lambda}}{\| (A-\lambda 1)^{k-1}\, D \|_{k-1,\lambda}} = \frac{1}{c} \| f(A) \|_{k-1,\lambda} \qquad \blacksquare$$

So , really we need only the seminorms $\| \cdot \|_{k,\lambda}$, $\lambda \in \mathrm{Spec}^k A$, because other seminorms do not contribute anything new to the topology .

Space $J_\infty(M_k)$. Consider $(M_k)_{k=0}^{\infty}$ – a sequence of compact subsets of $\mathbb{C}$,

$$M_o \supseteq M_1 \supseteq M_2 \supseteq \ldots$$

such that $M_k \setminus (\bigcap_{l=0}^{\infty} M_l) \subset \mathrm{Isol}\, M_k$ – the set of isolated points of M_k .

DEFINITION 5.6. Let $J_\infty\langle M_k\rangle$ denote the following locally convex space of sequences of functions :

$$J_\infty\langle M_k\rangle = \{ (f_k)_{k=0}^\infty \mid f_k : M_k \to \mathbb{C} ,\forall \lambda \in M_k ,$$

$$\sup_{\mu \in M_0 \setminus (\lambda)} \frac{\left| f_0(\mu) - \sum_{l=0}^{k-1} f_l(\lambda)(\mu-\lambda)/l! \right|}{|\mu-\lambda|^k} < \infty \ \}$$

The above supremums define a fundamental family of seminorms on $J_\infty\langle M_k\rangle$.

Let us list several simple properties of this space:

(i) *f_0 is continuous on M_0 .*

Really, one must only check the continuity at nonisolated points of M_0 , but all such points belong to $\bigcap_{k=0}^{\infty} M_k$, so for every $\lambda \in M_0 \setminus (\mathrm{Isol}\ M_0) \subset M_1$ and for every $\mu \in M_0$

$$| f_0(\mu) - f_0(\lambda) | \le C(\lambda) \ | \mu - \lambda |$$

and the assertion is proved.

(ii) *for every $\lambda \in \bigcap_{p=0}^{\infty} M_p$ the function*

$$\frac{f_0(\mu) - \sum_{l=0}^{k-1} f_l(\lambda)(\mu-\lambda)^l / l!}{(\mu - \lambda)^k}$$

is continuous on M_0 , and its limit at λ equals $\dfrac{f_k(\lambda)}{k!}$.

The continuity at all points besides λ is obvious (see (i)).

As for $\mu \to \lambda$

$$\infty > \sup_{\substack{\mu \in M_0 \\ \mu \neq \lambda}} \left| \frac{f_0(\mu) - \sum_{i=0}^{k} f_i(\lambda) \frac{(\mu - \lambda)^i}{i!}}{(\mu - \lambda)^{k+1}} \right| =$$

$$= \sup_{\substack{\mu \in M_0 \\ \mu \neq \lambda}} \left| \frac{\dfrac{f_0(\mu) - \sum_{i=0}^{k-1} f_i(\lambda) \frac{(\mu-\lambda)^i}{i!}}{(\mu-\lambda)^k} - \dfrac{f_k(\lambda)}{k!}}{\mu - \lambda} \right|$$

So the limit of the numerator is zero .

(iii) *f_0 is analytic in $\overset{\circ}{M}$ - the interior of M - and for every $\lambda \in \overset{\circ}{M}$ $f_i(\lambda) = f_0^{(i)}(\lambda)$ (for all $i = 0,1,...$).*

A proof may be easily obtained from (i,ii).

(iv) *The functions $f_k|_{\text{Isol } M_k}$ are arbitrary ($k = 1,2,...$).*

(v) *The space $J_\infty\{ M_k \}$ is naturally isomorphic to the space $\mathcal{A}_0 \times (\prod_{k=1}^{\infty} \mathbb{C}^{|\text{Isol}M_k|})$, were $\mathcal{A}_0$ is a locally convex algebra, consisting of all functions f_0, ($\{ f_i \}_{i=0}^{\infty} \in J_\infty\{ M_k \}$) and $|$ Isol M_k $|$ = the number of points in Isol M_k. $\mathcal{A}_0 \subset A(M_0)$ the space of functions , continuous on M_0 and analytic inside M_0, the imbedding is continuous and the range is dense in $A(M_0)$.*

(vi) *The dual space $(J_\infty\{ M_k \})'$ is naturally isomorphic to $(\mathcal{A}_0)' \oplus (\sum_{m=1}^{\infty} \mathbb{C}^{|\text{Isol}M_k|})$, i.e. for every continuous linear* *functional* $\varphi : J_\infty\{ M_k \} \to \mathbf{C}$ *there exist a "generalized measure"* $d\mu \in (\mathcal{A}_0)'$ *and complex numbers* $c(k,\lambda)$, $k = 1,2, \dots,$ $\lambda \in Isol\ M_k$, *such that*

$$\varphi (\{f_k\}) = \int_{M_0} f_0(\lambda)\, d\mu(\lambda) + \sum_{\substack{k=1,2,\dots \\ \lambda \in IsolM_k}} f_k(\lambda)\, c(k,\lambda)$$

(only finitely many of the numbers $c(k.\lambda)$ *are nonzero).*

§ 6 Generalized Gelfand Transform.

Let $B \in \mathfrak{B}_\infty$. Consider the space $J_\infty\{ \mathrm{Spec}^k A \}$ and define the mapping

$$R(A) \ni f \mapsto \{ f^{(k)}|_{\mathrm{Spec}^k A} \} \in J_\infty\{ \mathrm{Spec}^{\,k} A \}$$

The Proposition 5.3 shows that this mapping is continuous and therefore it may be extended by continuity to the following mapping - we shall call it the generalized Gelfand Transform:

$$\wedge : \ \mathfrak{B}_\infty \to J_\infty\{ \mathrm{Spec}^k A \}$$

It may be also described as follows :

$$(\hat{B})_k(\lambda) = \varphi_{k,\lambda}(B) \text{ for } k = 0,1, \dots , \lambda \in \mathrm{Spec}^{\,k} A.$$

DEFINITION 6.1. We call Ker $\wedge$ the **small radical** of $\mathfrak{B}_\infty$ and we call the operator A **semicomplicated** if Ker $\wedge = \{ 0 \}$.

Ker $\wedge$ may be described as follows:

Ker $\wedge = \{ B: V \to V \mid$ *there exists a net* $\{ g_\beta \}_{\beta \in \mathfrak{B}} \subset Rat\ (A)$ *such that* $\| g_\beta(A) - B \|_{\lambda,k} \to 0$, $\forall\ k = 0,1, \dots , \forall\ \lambda \in \mathbf{C}$ *and* $g_\beta^{(k)}(\lambda) \to 0$, $\forall\ k = 0,1, \dots , \forall\ \lambda \in Spec^k\ A \}$.

THEOREM 6.2. *The system of generalized root vectors*

$$\{ \varphi_{\lambda,k} : k = 0,1, \dots , \lambda \in Spec^k A \}$$

is $\sigma (\mathfrak{A}^{\infty},\mathfrak{A}_{\infty})$ *- complete if and only if the operator A is semicomplicated , i.e. for any net* $\{ g_{\beta} \}_{\beta\in\mathfrak{B}} \subset Rat(A)$ *, such that the net* $\{ g_{\beta} \}_{\beta\in\mathfrak{B}}$ *is fundamental in every seminorm* $\|\cdot\|_{k,\lambda}$ $(k = 0,1, \dots, \lambda \in \mathbb{C})$ *and* $g_{\beta}^{(k)}(\lambda) \to 0$, $\forall\ k = 0,1, \dots , \forall\ \lambda \in Spec^k A$ *, the following equality holds :*

$$\lim_{\beta} g_{\beta}(A) = 0 .$$

The proof is obvious.

THEOREM 6.3. *The system of generalized root vectors*

$$\{ \varphi_{\lambda,k} : k = 0,1, \dots , \lambda \in Spec^k A \}$$

is $\sigma (\mathfrak{A}^{\infty},\mathfrak{A}_{\infty})$ *- complete if and only if the following subsystem is* $\sigma(\mathfrak{A}^{\infty},\mathfrak{A}_{\infty})$ *- complete :*

$$\{ \varphi_{o,\lambda} : \lambda \in \partial Spec\ A \ ; \varphi_{k,\lambda} : k = 1,2, \dots , \lambda \in Isol\ Spec^k A \}.$$

PROOF. Suppose the first system is complete , but the second is not . Then there exists $B \in \mathfrak{A}_{\infty}$, orthogonal to all vectors of the second system, i.e. $\varphi_{k,\lambda}(B) = 0$ for all $\lambda \in \partial$Spec A , k = 0 , and for all k = 1,2, ... , $\lambda \in$ Isol $Spec^k$ A . $B \neq 0$, therefore there exist k_0 and $\lambda_0 \in Spec^{k_0} A$ such that $\varphi_{k_0,\lambda_0}(B) \neq 0$.

First, let $k_0 = 0$. Then λ_0 belongs to the interior of Spec A. The function $\mu \mapsto \varphi_{0,\mu}(B)$ is analytic in the interior of Spec A and it vanishes on ∂Spec A, therefore it vanishes identically , and $\varphi_{0,\lambda_0}(B) = 0$. Then take the minimal possible $k_0 \geq 1$ and

$$\lambda_0 \in \text{Spec}^{k_0} A \setminus \text{Isol}(\text{Spec}^{k_0} A) \subset \text{Spec } A \setminus \text{Isol}(\text{Spec } A),$$

$\lambda \in \partial$Spec A (or else the previous argument is valid) . Take $z_l \in \partial$Spec A , $z_l \to \lambda_0$. Then

$$\infty > \sup_l \left| \frac{\varphi_{0,z_l}(B) - \sum_{k=0}^{k_0} \varphi_{k,\lambda_0}(B) \frac{(z_l-\lambda_0)^k}{k!}}{(z_l - \lambda_0)^{k_0+1}} \right| =$$

$$= \sup_l \left| \frac{0 - \frac{\varphi_{k_0,\lambda_0}(B)}{k_0!} (z_l-\lambda_0)^{k_0}}{(z_l - \lambda_0)^{k_0+1}} \right| = \sup_l \left| \frac{\varphi_{k_0,\lambda_0}(B)}{(z_l-\lambda_0)\ k_0!} \right|$$

so $\varphi_{k_0,\lambda_0}(B) = 0$ what contradicts the choice of k_0,λ_0 . ∎

§ 7. Jordan decomposition

Jordan decomposition in $\mathfrak{A}^\infty$. Suppose the system

$$\{ \varphi_{\lambda,0} : \lambda \in \partial(\text{Spec } A),\ \varphi_{\lambda,k} : k = 1,2, \dots , \lambda \in \text{Isol Spec}^k A \}$$

is $\sigma(\mathfrak{A}^\infty,\mathfrak{A}_\infty)$-complete , i.e. the generalized Gelfand Transform is injective. Then $\wedge(\mathfrak{A}_\infty)$ may be viewed as a subalgebra of $J_\infty(\text{Spec}^k A)$, therefore the conjugate mapping $\wedge'$ maps $J^\infty(\text{Spec}^k A)$ onto a $\sigma(\mathfrak{A}^\infty,\mathfrak{A}_\infty)$-dense subspace of $\mathfrak{A}^\infty$.

So for any $\varphi \in \mathfrak{A}^{\infty}$ there exists a net $\{ \psi_\beta \}_{\beta \in \mathfrak{B}} \subset J^{\infty}(\operatorname{Spec}^k A)$ such that $\Lambda' \psi_\beta \xrightarrow{\sigma(\mathfrak{A}^{\infty}, \mathfrak{A}_{\infty})} \varphi$.

Every ψ_β is defined by a generalized measure $d\mu_\beta \in (\mathcal{A}_0)'$ and a set of numbers $c_\beta(k,\lambda)$, $k = 1,2, \ldots$, $\lambda \in \operatorname{Isol} \operatorname{Spec}^k A$ (only a finite number of them are nonzero).

$$\psi_\beta \{f_k\} = \int_{\operatorname{Spec} A} f_0(\lambda)\, d\mu_\beta(\lambda) + \sum_{\substack{k=1,2,\ldots \\ \lambda \in \operatorname{Isol} \operatorname{Spec}^k A}} f_k(\lambda)\, c_\beta(k,\lambda).$$

So,

$$\varphi(B) = \lim_\beta (\Lambda' \psi_\beta)(B) = \lim_\beta \psi_\beta(\hat{B}) =$$

$$\lim_\beta \Big[\int_{\operatorname{Spec} A \setminus \operatorname{Isol} \operatorname{Spec} A} \varphi_{0,\lambda}(B)\, d\mu_\beta(\lambda) + \sum_{\lambda \in \operatorname{Isol} \operatorname{Spec} A} \varphi_{0,\lambda}(B)\, c_\beta(0,\lambda) +$$

$$+ \sum_{\substack{k=1,2,\ldots \\ \lambda \in \operatorname{Isol} \operatorname{Spec}^k A}} \varphi_{k,\lambda}(B)\, c_\beta(k,\lambda) \Big]$$

One can prove that there exists $\lim_\beta c_\beta(k,\lambda)$ for every possible k,λ. Really, for any $k = 1,2,\ldots$, $\lambda \in \operatorname{Isol} \operatorname{Spec}^k A$, choose a function f analytic in a neighborhood of Spec A such that f vanishes outside of a small disk centered at λ (and not containing other points of Spec A) and $f(\mu) = \dfrac{(\mu - \lambda)^k}{k!}$ in a smaller disk centered at λ. Then $f(A)$ is correctly defined and one can easily show that $f(A) \in \mathfrak{A}_{\infty}$. Then

$$\varphi_{p,\mu}(f(A)) = 0 \quad \text{for all} \quad \mu \neq \lambda \quad \text{and all } p$$

$$\varphi_{p,\lambda}(f(A)) = 0 \quad \text{for all} \quad p \neq k, \ \varphi_{k,\lambda}(f(A)) = 1.$$

So,

$$\varphi\ (\ f(A)\ ) = \lim_{\beta}\ c_{\beta}(k,\lambda)$$

and so the limit in the right hand part exists for any k,λ .

Jordan decomposition in the initial spaces. Now we return to the initial spaces V ,V' and to the inclusions τ , τ^{∇} .

$$R(A) \xrightarrow{\tau_{\Delta}} V$$

$$V' \xleftarrow{\tau^{\nabla}} R(A)$$

One can easily see that the mappings τ_{Δ} , τ^{∇} are continuous if we equip $R(A)$ with the system of seminorms $\{ \|\ .\ \|_{n,\lambda} \}$ and V ,V' are supposed to be equipped with the weak topologies . So we again obtain a rigging

$$\mathfrak{A}_{\infty} \xrightarrow{\tau_{\Delta}} V \xrightarrow{(\tau^{\nabla})'} \mathfrak{A}^{\infty}$$

$$\mathfrak{A}^{\infty} \xleftarrow{(\tau_{\Delta})'} V' \xleftarrow{\tau^{\nabla}} \mathfrak{A}_{\infty}$$

Let $\mathcal{V}_{+}$ denote the range of τ_{Δ} , equipped with the topology transferred from $\mathfrak{A}_{\infty}$ by τ_{Δ} , let $\mathcal{V}_{-}$ denote the completion of V with respect to the topology transferred from $\mathfrak{A}^{\infty}$ by $[(\tau^{\nabla})']^{-1}$, let $\mathcal{V}^{+}$ denote the range of τ^{∇} (with the topology transferred by τ^{∇} from $\mathfrak{A}_{\infty}$), let $\mathcal{V}^{-}$ denote the completion of V' with respect of the topology transferred from $\mathfrak{A}^{\infty}$ by $[(\tau_{\Delta})']^{-1}$.

We obtain the following scheme

$$\mathcal{V}_{+} \subset V \subset \mathcal{V}_{-}$$

$$\mathcal{V}^{-} \supset V' \supset \mathcal{V}^{+}$$

$\mathcal{V}_{-}$ (resp., $\mathcal{V}^{-}$) is the dual space for $\mathcal{V}^{+}$ (resp., $\mathcal{V}_{+}$), the duality being extended from the initial duality between V and V'. τ_{Δ} (resp., τ^{∇}) is an isomorphism between $\mathfrak{A}_{\infty}$ and $\mathcal{V}_{+}$ (resp., $\mathcal{V}^{+}$). $(\tau^{\nabla})'$ (resp., $(\tau_{\Delta})'$) is an isomorphism between $\mathfrak{A}^{\infty}$ and $\mathcal{V}_{-}$ (resp., $\mathcal{V}^{-}$).$^{\square}$

Put $e_{\lambda,k} = [\ (\tau^{\nabla})'\ ]^{-1} (\varphi_{\lambda,k})$, $e^{\lambda,k} = [\ (\tau_{\Delta})'\ ]^{-1} (\varphi_{\lambda,k})$. Then

$$\langle e_{\lambda,k}, \nabla \rangle = \langle [\ (\tau^{\nabla})'\ ]^{-1} (\varphi_{\lambda,k}) , \nabla \rangle =$$

$$= \varphi_{\lambda,k}(\ (\tau^{\nabla})^{-1}(\nabla)\ ) = \varphi_{\lambda,k}(\mathbb{I}) = \begin{cases} 0 , & k \geq 1 \\ 1 , & k = 0 \end{cases}$$

Similarly , $\langle \Delta , e^{\lambda,k} \rangle = \begin{cases} 0 , & k \geq 1 \\ 1 , & k = 0 \end{cases}$

Let us compute

$$\langle (\ A - \lambda\mathbb{I}\ )^{p} e_{\lambda,k} , \tau^{\nabla}(B) \rangle =$$

$$= \langle (\ A - \lambda\mathbb{I}\ )^{p} [(\tau^{\nabla})']^{-1} (\varphi_{\lambda,k}) , \tau^{\nabla}(B) \rangle =$$

$$\langle [(\tau^{\nabla})']^{-1} (\varphi_{\lambda,k}) , \tau^{\nabla}(\ (A - \lambda\mathbb{I})^{p} B\ ) \rangle = \varphi_{\lambda,k}(\ (A - \lambda\mathbb{I})^{p} B\ ) =$$

$$= \varphi_{\lambda,k-p} (B) = \langle [(\tau^{\nabla})']^{-1} (\varphi_{\lambda,k-p}) , \tau^{\nabla}(B) =$$

$$= \langle e_{\lambda,k-p} , \tau^{\nabla}(B) \rangle \qquad (\text{ if } k < p \text{ , then put } e_{\lambda,k-p} = 0\ ).$$

So , $(\ A - \lambda\mathbb{I}\ )^{p} e_{\lambda,k} = e_{\lambda,k-p}$. Similarly

$$(\ A' - \lambda\mathbb{I}\ )^{p} e^{\lambda,k} = e^{\lambda,k-p} .$$

Therefore

$$\langle C \Delta , e^{\lambda,k} \rangle = \langle \nabla , C' e^{\lambda,k} \rangle =$$

$$\langle \Delta, [\ \sum_{s<k} \hat{C}_{s}(\lambda) \frac{(A - \lambda\mathbb{I})^{s}}{s!} + \hat{C}_{k}(\lambda) \frac{(A - \lambda\mathbb{I})^{k}}{k!} + (A - \lambda\mathbb{I})^{k+1} D\ ]\ e^{\lambda,k} \rangle =$$

$$= \frac{\hat{C}_k(\lambda)}{k!}$$

Now we want to obtain a version of the Jordan decomposition for vectors from V . This is not possible for all vectors from V , but this turns to be possible for vectors from $\mathcal{V}_+$ - a dense lineal in V.

DEFINITION 7.1. Let Lim Spec A denote the set

$$\partial(\text{Spec } A \setminus \text{Isol Spec } A).$$

Consider the linear functional on $\mathfrak{A}_\infty$

$$\varphi_{\Delta,\nabla} \ : B \longmapsto \langle B\Delta \ , \ \nabla \rangle.$$

It is obvious that $\varphi_{\Delta,\nabla} \in \mathfrak{A}^\infty$ and therefore $\varphi_{\Delta,\nabla}$ may be $\sigma(\mathfrak{A}^\infty,\mathfrak{A}_\infty)$-approximated by $\wedge'$-images of a net of functionals on $J_\infty\{ \text{Spec}^k A \}$

$$\wedge'(d\mu_\beta \ ,c_\beta(k,\lambda)) \xrightarrow{\sigma(\mathcal{V}_-,\mathcal{V}^+)} \varphi_{\Delta,\nabla} \ .$$

Take any $x \in \mathcal{V}_+$, $y \in \mathcal{V}^+$. Then there exist B , C $\in \mathfrak{A}_\infty$, such that $x = B\,\Delta$, $y = C'\nabla$. Then

$$\langle x,y \rangle = \langle B\ \Delta \ , \ C'\nabla \rangle = \langle (CB)\Delta \ , \ \nabla \rangle = \varphi_{\Delta,\nabla} (CB) =$$

$$= \lim_\beta [\ \wedge'(d\mu_\beta \ , \ c_\beta(k,\lambda) \ ] \ (CB) =$$

$$= \lim_\beta (\int_{\text{LimSpec} A} \varphi_\lambda \ (CB) \ d\mu_\beta(\lambda) + \sum_{\substack{k=0,1,2,\dots \\ \lambda\in \text{Isol Spec}^k A}} \varphi_{\lambda,k} \ (CB) \ c_\beta(k,\lambda) \ ] =$$

$$= \lim_\beta (\int_{\text{limSpec} A} \varphi_\lambda(C) \ \varphi_\lambda(B) \ \langle \Delta,e^{\lambda,0} \rangle \ \langle e_{\lambda,0},\nabla \rangle \ d\mu_\beta(\lambda) +$$

$$+ \sum_{\substack{k=0,1,2,\dots \\ \lambda\in \text{Isol Spec}^k A \\ l_1+l_2=k}} \frac{(\hat{C})_{l_1}(\lambda)}{l_1!} \ \frac{(\hat{B})_{l_2}(\lambda)}{l_2!} \ c_\beta(k,\lambda) \ k! \ \ ) =$$

$$= \lim_{\beta} \Big(\int_{\mathrm{LimSpecA}} \langle \Delta , B' e^{\lambda,0} \rangle \langle Ce_{\lambda,0} , \nabla \rangle \, d\mu_{\beta}(\lambda) +$$

$$+ \sum_{\substack{k=0,1,\dots \\ \lambda \in \mathrm{IsolSpec}^k A \\ l_1 + l_2 = k}} \frac{\hat{(C)}_{l_1}(\lambda)}{l_1!} \frac{\hat{(B)}_{l_2}(\lambda)}{l_2!} c_{\beta}(k,\lambda) \, k! \Big) =$$

$$= \lim_{\beta} \Big(\int_{\mathrm{LimSpecA}} \langle x , e^{\lambda,0} \rangle \langle e_{\lambda,0} , y \rangle \, d\mu_{\beta}(\lambda) +$$

$$+ \sum_{\substack{k=0,1,\dots \\ l_1 + l_2 = k \\ \lambda \in \mathrm{IsolSpec}^k A}} \langle e_{\lambda,l_1} , C'\nabla \rangle \langle B\Delta , e^{\lambda,l_2} \rangle c_{\beta}(k,\lambda) \, k! \Big) =$$

$$= \lim_{\beta} \Big(\int_{\mathrm{LimSpecA}} \langle x , e^{\lambda,0} \rangle \langle e_{\lambda,0}, y \rangle \, d\mu_{\beta}(\lambda) +$$

$$+ \sum_{\substack{l_1, l_2 = 0,1,\dots \\ \lambda \in \mathrm{Isol\ Spec}^{l_1+l_2} A}} \langle x , e^{\lambda,l_2} \rangle \langle e_{\lambda,l_1} , y \rangle c_{\beta}(l_1 + l_2, \lambda) \, (l_1 + l_2)! \Big).$$

So we have obtained the following

THEOREM 7.2. *Suppose the operator A is semicomplicated . There exists a net of measures $\{\mu_\beta\}_{\beta\in\mathfrak{B}}$ on Lim Spec A and of complex numbers $\{ c_\beta(k,\lambda) : k = 0,1, \dots , \lambda \in \mathrm{Isol\ Spec}^k A \}_{\beta\in\mathfrak{B}}$ such that for every $x \in V_+$, $y \in V^+$ the following Jordan decomposition holds:*

$$\langle x,y \rangle = \lim_{\beta} \Big(\int_{\mathrm{LimSpecA}} \langle x , e^{\lambda,0} \rangle \langle e_{\lambda,0}, y \rangle \, d\mu_{\beta}(\lambda) +$$

$$+ \sum_{\substack{l_1, l_2 = 0,1,\dots \\ \lambda \in \mathrm{Isol\ Spec}^{l_1+l_2} A}} \langle x , e^{\lambda,l_2} \rangle \langle e_{\lambda,l_1} , y \rangle c_{\beta}(l_1 + l_2, \lambda) \, (l_1 + l_2)! \Big).$$

and for every $f \in Rat(A)$

$$\langle f(A)x,y\rangle = \lim_{\beta} \Big(\int_{\mathrm{LimSpec}A} f(\lambda)\langle x , e^{\lambda,0} \rangle \langle e_{\lambda,0}, y \rangle \, d\mu_{\beta}(\lambda) +$$

$$+ \sum_{\substack{l_1, l_2 = 0,1,\dots \\ \lambda \in \mathrm{Isol\,Spec}^{l_1+l_2} A \\ s=0,1,\dots,l_1}} \frac{f^{(s)}(\lambda)}{s!} \langle x , e^{\lambda, l_1 - s} \rangle \langle e_{\lambda, l_2}, y \rangle c_{\beta}(l_1 + l_2, \lambda) \, (l_1 + l_2)! \Big)$$

These formulae are exact analogues of the usual formulae appearing in the finite dimensional Jordan decompositions and they give an infinite dimensional analogue of such a decomposition.

REFERENCES

[1] YU.M. BEREZANSKII, " Eigenfunction Expansions for Selfadjoint Operators", Naukova Dumka, Kiev, 1965. (Russian)

[2] N. DUNFORD, J.T. SCHWARTZ, " Linear Operators. Part 3." Wiley Interscience, 1971.

[3] I.M. GELFAND, A.G. KOSTYUCHENKO, On eigenfunction expansions for differential and other operators// *Doklady AN SSSR*, 103:3(1955), 349 - 352. (Russian).

[4] I.M. GELFAND, G.E. ŠILOV, D.A. RAIKOV, "Commutative Normed Rings", Fizmatgiz, Moscow, 1960. (Russian).

[5] I.Z. GOHBERG, M.G. KREIN, "Introduction to the Theory of Linear Nonselfadjoint Operators", Transl. Math. Mono., vol. 18, Amer. Math. Soc., Providence, R.I., 1969.

[6] L. Hörmander, "An Introduction to Complex Analysis in Several Variables", Van Nostrand, Princeton, 1966.

[7] B. Sz. - Nagy, C. Foiaş, "Harmonic Analysis of Operators on Hilbert Space", North-Holland, Amsterdam, 1970.

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