On the rate of convergence for large coupling limits in quantum mechanics

M. Demuth, F. Jeske,
W. Kirsch

M. Demuth
Max-Planck-Arbeitsgruppe
Fachbereich Mathematik
Universität Potsdam
Am Neuen Palais
D-1571 Potsdam (Germany)

F. Jeske / W. Kirsch
Fakultät für Mathematik
Ruhr-Universität
D-4630 Bochum 1 (Germany)

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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by

M. Demuth*, F. Jeske, W. Kirsch****

* Max Planck–Arbeitsgruppe, Fachbereich Mathematik, Universität Potsdam, Am Neuen Palais, D–1571 Potsdam (Germany)
** Fakultät für Mathematik, Ruhr–Universität, D–4630 Bochum 1 (Germany)

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Abstract

Let $H_M = H + MU$ be a Schrödinger operator H additionally perturbed by a positive potential U, where M is a positive coupling parameter. The limit of H_M in the norm resolvent sense is a Dirichlet operator on the complement of the support of U. A quantitative estimate is given for the rate of this convergence as M is large.

1. Introduction

The main objective of this article is to estimate quantitatively the rate of convergence for Schrödinger operators if the positive part of the potential tends to infinity.
We consider Schrödinger operators of the form $H_M = H_0 + V + MU$ in $L^2(\mathbb{R}^d)$, H_0 is the selfadjoint realization of $-\frac{1}{2}\Delta$, V a potential in Kato's class, U a positive potential with support Γ. Γ is a closed subset of $\mathbb{R}^d$, called singularity region. H_M tends to an operator $(H_0 + V)_\Sigma$ in strong resolvent sense as M tends to infinity. $(H_0 + V)_\Sigma$ is the Dirichlet operator corresponding to $H_0 + V$ defined in $L^2(\Sigma)$, where $\Sigma = \mathbb{R}^d \setminus \Gamma$ is the complement of Γ.
For many estimates in spectral theory it is useful to have not merely the bare convergence but also the rate of convergence. Among others this is of interest for semiclassical limits where $\hbar^2 H_0 + V + U$ is compared with $(\hbar^2 H_0 + V)_\Sigma$ for small $\hbar^2$ (see e.g. [His/Sig] who studied also large coupling limits in relation to semiclassical considerations).
Therefore we estimate operator norms of resolvent differences, i.e.

$$(1.1) \qquad \left\|(H_M - z)^{-1} - ((H_0 + V)_\Sigma - z)^{-1}\right\| \leq r(M).$$

Our main technical tool is to estimate the corresponding operator norm of semigroup differences heavily using properties of Brownian motion.
Of course, the value for r depends strongly on the properties of the boundary $\partial\Gamma$. We tried to find very general conditions for $\partial\Gamma$. Therefore we allow Lipschitz continuous $\partial\Gamma$. In this general case we will prove

$$(1.2) \qquad r(M) \leq \text{const} \cdot \frac{1}{(\log M)^\gamma}, \qquad 0 < \gamma < \frac{1}{2}.$$

This rate can be improved if $\partial\Gamma$ becomes more regular. If for instance Σ is concave then

$$(1.3) \qquad r(M) \leq \text{const} \cdot M^{-\frac{1}{4}}.$$

In the special case of a halfspace one has

$$(1.4) \qquad r(M) \leq \text{const} \cdot M^{-\frac{1}{3}}.$$

In order to qualify the upper bounds we also estimate the resolvent and semigroup differences from below. One rough lower bound is

$$(1.5) \qquad \left\|(H_M - z)^{-1} - ((H_0 + V)_\Sigma - z)^{-1}\right\| \geq \text{const} \cdot M^{-\frac{d+2}{2}}.$$

The paper is organized as follows:
In Section 2, we collect some results concerning (1.1) which are preliminary to our discussion. Here, we point out that trace class properties and convergence of $e^{-tH_M} - e^{-tH_\Sigma}$ were treated in [Dei/Sim] for bounded Γ. In [Bau/Dem], H_Σ was identified as a suitable

Friedrichs extension. Furthermore, problems of the above kind are considered in [Dem/vCa 1] for a larger class of free operators H_0.
At the end of Section 2, we extend some results of the preceeding paper [Dem] and explain, where and why there are limitations for these methods with respect to dimension of the underlying Euclidean space and boundedness of the singularity regions. We recall that [Dem] treated the convergence of $(H_M - z)^{-1} - ((H_0 + V)_\Sigma - z)^{-1}$ with respect to trace class, Hilbert-Schmidt and uniform operator norm, but did not obtain convergence rates.

Our main result (1.2) is contained in Theorem 3.1. The corresponding Section 3, which is independent of Section 2, begins with the method basic to (1.2). In particular, we explain how the estimation of the semigroup difference leads to two expressions (see (3.2)).
The first of these terms,

$$\sup_{x\in\Sigma} \mathrm{P}_x\{\lambda - \varepsilon < A_\Gamma < \lambda\}, \tag{1.6}$$

where A_Γ is the time the Brownian path started at $x \in \Sigma$ hits Γ for the first time, is probabilistically of interest in its own right (see e.g. [LeG],[Kar/Shr]). We estimate (1.6) in Section 4, where we see in particular that it leads quite naturally to the Lipschitz regularity assumption imposed on $\partial\Gamma$ in Theorem 3.1.
The second term, the Laplace transform of (the distribution of) the time spent up to time ε in a cone by the Brownian trajectory started at the vertex of the cone, is estimated in Section 5 using strongly results and methods of T. Meyre [Mey].

Having thus finished the proof of our main result, we then shed some additional light on Theorem 3.1 in Section 6 by exhibiting the special case of the halfspace (Lemma 6.1), where the upper bound is improved considerably, and stating lower bounds on the operator norm of resolvent (Lemmas 6.2,3) and semigroup difference (Lemma 6.4).
The final section deals with applications of the above results to the semiclassical limit. After presenting a situation where semiclassical and large coupling limit are different (Theorem 7.1), we find asymptotic expressions as $\hbar \to 0$ (Theorem 7.2) and show lower bounds in $\hbar^2$ on the semiclassical approximations of resolvents in the ordinary (Theorem 7.3) as well as in the limit absorption case (Theorem 7.4). See [Nak],[Rob/Tam] for further examples of such results.

2. Assumptions and preliminary results

Throughout this text, we denote the following conditions on two potentials V, U and a singularity region Γ by

Assumption A: Let H_0 be the selfadjoint realization of $-\frac{1}{2}\Delta$ in $L^2(\mathbb{R}^d)$. Let V be a Kato class potential, i.e. $V = V_+ - V_-$, where

$$\lim_{\alpha \to 0} \sup_x \int_0^\alpha \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \; p(x, y|s) V_-(y) = 0$$

and

$$\lim_{\alpha \to 0} \sup_x \int_0^\alpha \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \; p(x, y|s) V_+(y) \chi_B(y) = 0$$

for any compact subset B of $\mathbb{R}^d$.
Moreover, we assume Γ to be a closed subset of $\mathbb{R}^d$ with positive Lebesgue measure and a piecewise C^1 boundary.
Finally, let U be nonnegative and such that $\operatorname{supp} U = \Gamma$, $U(x) = 0$ only for $x \in \partial\Gamma$.

Assumption B: H_0, V, Γ as in Assumption A, $U = \chi_\Gamma$.

Here and in the sequel $\chi_{\{\ldots\}}$ denotes the indicator function of the set $\{\ldots\}$, and p is the transition probability kernel for Brownian motion

$$p(x, y|t) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2t}}, \qquad x, y \in \mathbb{R}^d, \; t > 0.$$

Under Assumption A it is known that the limit of

$$H_M := H_0 + V + MU$$

exists in the strong resolvent sense as $M \to \infty$. Under mild conditions on $\partial\Gamma$ this limit coincides with the Friedrichs extension H_Σ of

$$H_0 + V \restriction L^2(\Sigma) \cap D(H_0 + V),$$

where $\Sigma = \mathbb{R}^d \setminus \Gamma$ is the complement of Γ (see [Bau/Dem]).

Since H_Σ is an operator on $L^2(\Sigma)$ while $H_0 + V$ acts on $L^2(\mathbb{R}^d)$, we can only compare functions of H_M and H_Σ via the restriction operator

$$Jf := f \restriction \Sigma, \quad f \in L^2(\mathbb{R}^d),$$

whose adjoint operator J^* is the natural embedding $L^2(\Sigma) \hookrightarrow L^2(\mathbb{R}^d)$.

2.1. Convergence in Hilbert-Schmidt sense

The possibility of using the Hilbert-Schmidt norm to measure the approximation of (functions of) H_Σ by H_M is restricted to very few situations. In particular, one should consider dimension $d \leq 3$.
We recall

Theorem 2.1: Let Assumption A be satisfied, Γ bounded. Then

$$\lim_{M\to\infty} \left\| J(H_M - z)^{-1} - (H_\Sigma - z)^{-1} J \right\|_p = 0, \quad z \in \rho(H_\Sigma), \tag{2.1}$$

where p indicates the usual operator norm for $d \geq 4$, the Hilbert-Schmidt norm for $d \leq 3$ and the trace class norm for $d = 1$.

This result can be extended to the limit absorption case $z = \lambda \pm i0$ for certain real λ (see [Dem]).
We will not repeat the proof of Theorem 2.1 (given in [Dem]) here but rather emphasize that it is strongly based upon the Hilbert-Schmidt estimate for differences of powers of resolvents

$$\left\| J(H_M - z)^{-q} - (H_\Sigma - z)^{-q} J \right\|_{\text{HS}}^2 \leq c \int_0^\infty d\lambda \ \lambda^\alpha \lambda^{2q-2} e^{-\operatorname{Re} z \cdot \lambda} \left\| J e^{-\lambda H_M} - e^{-\lambda H_\Sigma} J \right\|_{\text{HS}}^2 \tag{2.2}$$

(with $c > 0, 0 < \alpha < 1$) and the estimate

$$\left\| J e^{-\lambda H_M} - e^{-\lambda H_\Sigma} J \right\|_{\text{HS}}^2 = O(\lambda^{-\frac{d}{2}} + e^{A\lambda}) \tag{2.3}$$

for some constant $A > 0$.

The proof can be used to study a first very restrictive class of unbounded singularity regions Γ:

Theorem 2.2: Let Γ be the union of balls B_i of radius R_i around points $a_i \in \mathbb{R}^d$. Then we have the following estimate on the Hilbert-Schmidt norm of semigroup differences:

$$\left\| J e^{-\lambda H_M} - e^{-\lambda H_\Sigma} J \right\|_{\text{HS}}^2 \leq c(\lambda) \sum_i \left(R_i^{d-2-\alpha} + R_i^{d-1} \right), \tag{2.4}$$

where $\alpha > 0$, $c(\lambda)$ could be given explicitly.

Remark: We know that the l.h.s. of (2.4) tends to zero as $M \to \infty$ (see (2.5) below). Note that we can estimate the semigroup difference uniformly in M in dependence of R_i.

Proof (of Theorem 2.2): If (Ω_x, P_x) denotes the probability space corresponding to Brownian motion started at x, and $(\Omega_{x,0}^{y,\lambda}, \mathrm{P}_{x,0}^{y,\lambda})$ represents Brownian motion conditioned to start in x at time 0 and stop in y at time λ, we can evaluate the respective integral kernels and obtain

$$
(2.5)\qquad \begin{aligned} &\left\| Je^{-\lambda H_M} - e^{-\lambda H_\Sigma} J \right\|_{\mathrm{HS}}^2 = \\ &= \int_\Sigma \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \left| \int_{\Omega_{x,0}^{y,\lambda}} \mathrm{P}_{x,0}^{y,\lambda}(\mathrm{d}\omega) e^{-\int_0^\lambda V(\omega(s))\mathrm{d}s} e^{-M\int_0^\lambda U(\omega(s))\mathrm{d}s} \chi_{\{T_{\lambda,\Gamma}>0\}}(\omega) \right|^2 \end{aligned}
$$

since $\int_0^\lambda U(\omega(s))\mathrm{d}s$ vanishes iff the time $T_{\lambda,\Gamma}(\omega) = \text{meas}\,\{s \le \lambda \mid \omega(s) \in \Gamma\}$ of the path ω spent up to λ in Γ does.
Since V is Kato class,

$$
(2.6)\qquad \int_{\Omega_{x,0}^{y,\lambda}} \mathrm{P}_{x,0}^{y,\lambda}(\mathrm{d}\omega) e^{-2\int_0^\lambda V(\omega(s))\mathrm{d}s} \le c\lambda^{-\frac{d}{2}} e^{A\lambda}.
$$

Hence

$$
(2.7)\qquad \begin{aligned} \text{r.h.s. of } (2.5) &\le c\lambda^{-\frac{d}{2}} e^{A\lambda} \int_\Sigma \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y\ \mathrm{P}_{x,0}^{y,\lambda}\{T_{\lambda,\Gamma} > 0\} \\ &= c\lambda^{-\frac{d}{2}} e^{A\lambda} \int_\Sigma \mathrm{d}x\ \mathrm{P}_x\{T_{\lambda,\Gamma} > 0\} \\ &\le c\lambda^{-\frac{d}{2}} e^{A\lambda} \sum_i \int_{\Sigma_i} \mathrm{d}x\ \mathrm{P}_x\{T_{\lambda,B_i} > 0\}, \end{aligned}
$$

where we used $\Gamma = \bigcup_i B_i$ and $\Sigma \subset \Sigma_i := \mathbb{R}^d \setminus B_i$ in the last step. In order to estimate the summands above, let B denote a ball $\{x \in \mathbb{R}^d \mid |x-a| < R\}$. Then

$$
\begin{aligned} &\int_{\mathbb{R}^d\setminus B} \mathrm{d}x\ \mathrm{P}_x\{T_{\lambda,B} > 0\} \\ &\le \int_{|x-a|\ge R} \mathrm{d}x\ \mathrm{P}_x\{\omega \mid \omega(s) \in B \text{ for some } s \le \lambda\} \\ &= \int_{|u|\ge R} \mathrm{d}u\ \mathrm{P}_0\{\omega \mid |\omega(s)+u| < R \text{ for some } s \le \lambda\} \\ &= \int_{|u|\ge R} \mathrm{d}u \int_{\Omega_0} \mathrm{P}_0(\mathrm{d}\omega) \chi_{\{\omega \mid |\omega(s)+u|<R \text{ for some } s\le\lambda\}}(\omega) \times \\ &\qquad\qquad \times \chi_{\{\omega \mid |\omega(s)+u|<R \text{ for some } s\}}(\omega) \\ &\le \int_{|u|\ge R} \mathrm{d}u\ [\mathbf{P}_0\{\omega \mid |\omega(s)| > |u| - R \text{ for some } s \le \lambda\}]^{\frac{1}{p}} \times \\ &\qquad\qquad \times [\mathbf{P}_0\{\omega \mid |\omega(s)+u| < R \text{ for some } s\}]^{\frac{1}{q}} \end{aligned}
$$

with arbitrary $1 < p, q < \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Since

$$
(2.8)\qquad \mathrm{P}_0\{\omega \mid |\omega(s)+u| < R \text{ for some } s\} \le c\frac{R^{d-2}}{|u|^{d-2}}
$$

(see [Sim], p. 70) and

$$\text{(2.9)} \qquad P_0\{\omega \mid |\omega(s)| > r \text{ for some } s \le \lambda\} \le 2P_0\{\omega \mid |\omega(\lambda)| > r\}$$

(see e.g. [Por/Sto]) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus B} \mathrm{d}x\ P_x\{T_{\lambda,B} > 0\} &\le cR^{\frac{d-2}{q}} \int_{|u|\ge R} \mathrm{d}u\ |u|^{-\frac{d-2}{q}} e^{-\frac{(|u|-R)^2}{4p\lambda}} \\
&= cR^{\frac{d-2}{q}} \int_0^\infty \mathrm{d}r\ (R+r)^{\frac{d}{p}+\frac{2}{q}-1} e^{-\frac{r^2}{4p\lambda}} \\
&\le cR^{\frac{d-2}{q}} \left[\int_0^R \mathrm{d}r\ R^{\frac{d}{p}+\frac{2}{q}-1} e^{-\frac{r^2}{4p\lambda}} + \int_R^\infty \mathrm{d}r\ r^{\frac{d}{p}+\frac{2}{q}-1} e^{-\frac{r^2}{4p\lambda}} \right] \\
&\le cR^{\frac{d-2}{q}} \left[R^{\frac{d}{p}+\frac{2}{q}-1}\sqrt{\lambda} + \lambda^{\frac{1}{2}(\frac{d}{p}+\frac{2}{q})} \right] \\
&\le c(\lambda) \left[R^{d-1} + R^{\frac{d-2}{q}} \right].
\end{aligned}$$

Note that here and in the following we use the slight abuse of notation that the value of the constant c may (and in fact usually does) change from step to step.
Plugging this into (2.7) yields the desired bound with $\alpha = \frac{\varepsilon(d-2)}{1+\varepsilon}$, if we let $q = 1+\varepsilon$. □

2.2. Estimates on the operator norm

The proof employed in Section 2.1 using the Hilbert-Schmidt properties of semigroup differences fails if Γ has unbounded volume. But for applications in solid state physics or concerning N-body problems, one should also strive for results on potential barriers over unbounded Γ.
The approach we will use in the sequel is based once again on the Laplace transform

$$\text{(2.10)} \qquad \left\| J(H_M-z)^{-1} - (H_\Sigma - z)^{-1} J \right\| \le c \int_0^\infty \mathrm{d}\lambda\ e^{-\mathrm{Re}\, z\cdot\lambda} \left\| Je^{-\lambda H_M} - e^{-\lambda H_\Sigma} J \right\|.$$

Since the norm on the r.h.s. is not bounded by the Hilbert-Schmidt norm, we make use of the fact that $Je^{-\lambda H_M} - e^{-\lambda H_\Sigma} J$ is an integral operator. Using [Kat, eq. (III.2.8)] and noting that the kernel is symmetric, we have

$$\begin{aligned}
&\left\| Je^{-\lambda H_M} - e^{-\lambda H_\Sigma} J \right\| \le \\
\text{(2.11)} \qquad &\le \sup_{x\in\Sigma} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\Omega^{y,\lambda}_{x,0}} P^{y,\lambda}_{x,0}(\mathrm{d}\omega) e^{-\int_0^\lambda V(\omega(s))\mathrm{d}s} e^{-M\int_0^\lambda U(\omega(s))\mathrm{d}s} \chi_{\{T_{\lambda,\Gamma}>0\}}(\omega) \\
&\le \sup_{x\in\Sigma} \int_{\Omega_x} P_x(\mathrm{d}\omega) e^{-\int_0^\lambda V(\omega(s))\mathrm{d}s} e^{-M\int_0^\lambda U(\omega(s))\mathrm{d}s} \chi_{\{T_{\lambda,\Gamma}>0\}}(\omega).
\end{aligned}$$

The expression $\sup_x \int_{\mathbb{R}^d} \mathrm{d}y \; K(x,y)$ is an upper bound for the operator norm of the integral operator K.
By Dini's theorem, the estimate (2.11) provides a result dealing first with bounded Γ. But then it can be extended to unbounded Γ.

Theorem 2.3: Let H_0, V, U satisfy Assumption A, Γ compact.
Then

$$\left\|J(H_M-z)^{-1}-(H_\Sigma-z)^{-1}J\right\| \quad \text{and} \quad \left\|Je^{-\lambda H_M}-e^{-\lambda H_\Sigma}J\right\|$$

tend to zero as $M \to \infty$.

Proof: Since $\left\|Je^{-\lambda H_M}-e^{-\lambda H_\Sigma}J\right\| \leq Be^{A\lambda}$, it suffices to prove

$$\left\|Je^{-\lambda H_M}-e^{-\lambda H_\Sigma}J\right\| \xrightarrow[M\to\infty]{} 0$$

in view of (2.10).
If R is sufficiently large, then

$$\begin{aligned}
&\sup_{|x|\geq R} \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-\int_0^\lambda V(\omega(s))\mathrm{d}s} \chi_{\{T_{\lambda,\Gamma}>0\}} \\
&\quad \leq \sup_{|x|\geq R} \left[\int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-2\int_0^\lambda V(\omega(s))\mathrm{d}s}\right]^{\frac{1}{2}} \times \left[\mathrm{P}_x\left\{T_{\lambda,\Gamma}>0\right\}\right]^{\frac{1}{2}} \\
&\quad \leq c\lambda^{-\frac{d}{4}} e^{A\lambda} \left(\sup_{|x|\geq R} \mathrm{P}_x \left\{\omega \in \Omega_x \mid |\omega(\lambda)-x| \geq \operatorname{dist}(x,\Gamma)\right\}\right)^{\frac{1}{2}} \\
&\quad \leq c\lambda^{-\frac{d}{2}} e^{A\lambda} \sup_{|x|\geq R} e^{-\frac{\operatorname{dist}(x,\Gamma)^2}{8\lambda}}
\end{aligned}$$

becomes arbitrarily small independently of M.
Hence it remains to show that for R fixed $\sup_{|x|\leq R} f_M(x)$ tends to zero as $M \to \infty$, where

$$f_M(x) := \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-\int_0^\lambda V(\omega(s))\mathrm{d}s} e^{-M\int_0^\lambda U(\omega(s))\mathrm{d}s} \chi_{\{T_{\lambda,\Gamma}>0\}}(\omega).$$

Indeed, we have

$$\int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-\int_0^\lambda V(\omega(s))\mathrm{d}s} \leq Be^{A\lambda},$$

and $T_{\lambda,\Gamma}(\omega) > 0$ implies $e^{-M\int_0^\lambda U(\omega(s))\mathrm{d}s} \to 0$, so that $f_M(x)$ tends to zero nonincreasingly for all x and an application of Dini's theorem yields the result. □

In contrast to (2.5), there is no integration over x in (2.11). This enables us to deal with unbounded Γ as well. The simplest case is a sheet in $\mathbb{R}^d$:

Corollary 2.4: The conclusion of Theorem 2.3 remains valid, if

$$\Gamma = \{x \in \mathbb{R}^d \mid a \le x_1 \le b\},$$

where $-\infty \le a < b < \infty$.

Proof: For the sake of notational convenience, we assume $V \equiv 0$.
Due to the special form of Γ, we have

$$\begin{aligned}
&\sup_{x\in\Sigma} \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-M\int_0^\lambda \chi_\Gamma(\omega(s))\mathrm{d}s} \chi_{\{T_{\lambda,\Gamma}>0\}}(\omega) \\
&= \sup_{x\in\Sigma} \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-M\int_0^\lambda \chi_{[a,b]}(\omega_1(s))\mathrm{d}s} \chi_{\{\omega \mid T_{\lambda,[a,b]}(\omega_1)>0\}}(\omega) \qquad (2.12)\\
&= \sup_{x_1\notin[a,b]} \int_{\Omega_{x_1}} \mathrm{P}_{x_1}(\mathrm{d}\omega_1) e^{-M\int_0^\lambda \chi_{[a,b]}(\omega_1(s))\mathrm{d}s} \chi_{\{\omega_1 \mid T_{\lambda,[a,b]}(\omega_1)>0\}}(\omega_1),
\end{aligned}$$

where we used

$$\begin{aligned}
\int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) h(\omega_1) &= \int_{\Omega_{x_1}} \mathrm{P}_{x_1}(\mathrm{d}\omega_1) h(\omega_1) \int_{\Omega_{x_2}} \mathrm{P}_{x_2}(\mathrm{d}\omega_2) \dots \int_{\Omega_{x_d}} \mathrm{P}_{x_d}(\mathrm{d}\omega_d) \\
&= \int_{\Omega_{x_1}} \mathrm{P}_{x_1}(\mathrm{d}\omega_1) h(\omega_1).
\end{aligned}$$

The final expression in (2.3) tends to zero because of Theorem 2.1. $\square$

Having in mind many-body situations, the following property is of interest:

Lemma 2.5: Let $V \equiv 0$ and U denote the indicator function of Γ.
Then the class of sets Γ s.t.

$$\left\| J e^{-\lambda H_M} - e^{-\lambda H_\Sigma} J \right\| \xrightarrow[M\to\infty]{} 0$$

is closed with respect to finite unions.

Proof: If $\Gamma = \Gamma_1 \cup \Gamma_2$, we have

$$\begin{aligned}
&\sup_{x\in\Sigma} \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-M\int_0^\lambda \chi_\Gamma(\omega(s))\mathrm{d}s} \chi_{\{T_{\lambda,\Gamma}>0\}}(\omega) \\
&\le \sup_{x\in\Sigma} \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-\frac{M}{2}\int_0^\lambda \chi_{\Gamma_1}(\omega(s))\mathrm{d}s} e^{-\frac{M}{2}\int_0^\lambda \chi_{\Gamma_2}(\omega(s))\mathrm{d}s} \times \\
&\qquad\qquad \times \left[\chi_{\{T_{\lambda,\Gamma_1}>0\}}(\omega) + \chi_{\{T_{\lambda,\Gamma_2}>0\}}(\omega)\right] \\
&\le \sup_{x\in\mathbb{R}^d\setminus\Gamma_1} \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-\frac{M}{2}T_{\lambda,\Gamma_1}(\omega)} \chi_{\{T_{\lambda,\Gamma_1}>0\}}(\omega) + \\
&\qquad\qquad + \sup_{x\in\mathbb{R}^d\setminus\Gamma_2} \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-\frac{M}{2}T_{\lambda,\Gamma_2}(\omega)} \chi_{\{T_{\lambda,\Gamma_2}>0\}}(\omega). \quad \square
\end{aligned}$$

Using Lemma 2.5 and Jacobi coordinates, the following result on an N-body Hamiltonian

$$H = H_0 + \sum_{1 \le i < j \le N} V_{ij}(x_i - x_j) \tag{2.13}$$

with H_0 the free operator in $L^2(\mathbb{R}^{3N})$ can be reduced to Theorem 2.3.

Corollary 2.6: Consider an N-body Hamiltonian H from eq. (2.13) and

$$H_M = H + M \sum_{1 \le i < j \le N} \chi_{\Gamma_{ij}}(x_i - x_j),$$

where $\Gamma_{ij} = \{x \in \mathbb{R}^{3N} | x_i - x_j \in B_{ij}\}$ with bounded B_{ij}. Set $\Gamma = \bigcup_{i<j} \Gamma_{ij}$, $\Sigma = \mathbb{R}^{3N} \setminus \Gamma$, $Jf = f \upharpoonright \Sigma$, then

$$\lim_{M \to \infty} \left\| J(H_M - z)^{-1} - (H_\Sigma - z)^{-1} J \right\| = 0.$$

We refrain from giving the details of the proof, because Corollary 2.6 is actually included in Theorem 3.1.

3. General unbounded singularity regions

This section contains our main result Theorem 3.1, where the most general situation is treated s.t. resolvent or semigroup corresponding to H_M tends to the one corresponding to H_Σ. In particular, there is no restrictions on the dimension of the underlying Euclidean space nor do we assume boundedness of the singularity region.
We begin this section by explaining the overall strategy how to estimate the semigroup difference in the case of more general Γ. Here we specialize to operators of the form $H_M = H_0 + M\chi_\Gamma$.
As in section 2

$$(3.1)\qquad \left\|Je^{-\lambda H_M} - e^{-\lambda H_\Sigma}J\right\| \le \sup_{x\in\Sigma}\int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega)e^{-M\int_0^\lambda \chi_\Gamma(\omega(s))\mathrm{d}s}\chi_{\{T_{\lambda,\Gamma}>0\}}(\omega).$$

We recall that $T_{\lambda,\Gamma}$ denotes the occupation time of the path in Γ up to time λ,

$$T_{\lambda,\Gamma}(\omega) := \text{meas}\ \{s \le \lambda \mid \omega(s) \in \Gamma\}.$$

Furthermore, let A_Γ be the time the path hits Γ for the first time,

$$A_\Gamma(\omega) = \inf\{s > 0 \mid \omega(s) \in \Gamma\}.$$

We consider singularity regions Γ where the set of regular points for Γ coincides with the regular point set of the interior of Γ, that is $\Gamma^r = \left(\Gamma^{\text{int}}\right)^r$. In this case, $A_\Gamma(\omega)$ is equal to the penetration time (see [Dem/vCa 2]), i.e.

$$A_\Gamma(\omega) = \inf\{s > 0\,|T_{s,\Gamma}(\omega) > 0\}.$$

Clearly $T_{\lambda,\Gamma}(\omega) > 0$ implies $A_\Gamma(\omega) < \lambda$. Therefore

$$(3.2)\qquad \begin{aligned}&\left\|Je^{-\lambda H_M} - e^{-\lambda H_\Sigma}J\right\| \le \\ &\qquad \le \sup_{x\in\Sigma}\mathrm{P}_x\{\lambda - \varepsilon < A_\Gamma < \lambda\} + \\ &\qquad\qquad + \sup_{x\in\Sigma}\int_{\Omega_x}\mathrm{P}_x(\mathrm{d}\omega)e^{-M\int_{A_\Gamma(\omega)}^\lambda \chi_\Gamma(\omega(s))\mathrm{d}s}\chi_{\{A_\Gamma\le\lambda-\varepsilon\}}(\omega),\end{aligned}$$

where ε can be chosen at our convenience $(0 < \varepsilon = \varepsilon(M,\lambda) < \lambda)$.

Thus, the estimation of the semigroup difference is reduced to the consideration of two expressions.
The first term on the r.h.s. of (3.2), which is of interest in its own right, is the subject of Section 4. The analysis there allows Γ to have a boundary which is Lipschitz in a certain sense. We will call a set Γ with the properties (L1)-(L4) in Theorem 4.2 a uniform Lipschitz set (cf. the remarks on these conditions at the end of Section 4) and obtain

$$\sup_{x\in\Sigma}\mathrm{P}_x\{\lambda - \varepsilon < A_\Gamma < \lambda\} \le c\left(1 + \frac{1}{\sqrt{\lambda}}\right)\sqrt{\varepsilon}.$$

Using the Markov property, we estimate the second term as follows:

$$\begin{aligned}\int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-M\int_{A_\Gamma(\omega)}^{\lambda} \chi_\Gamma(\omega(s))\mathrm{d}s} \chi_{\{A_\Gamma \leq \lambda-\varepsilon\}}(\omega) \\ = \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) \int_{\Omega_{\omega(A_\Gamma)}} \mathrm{P}_{\omega(A_\Gamma)}(\mathrm{d}\tilde{\omega}) e^{-M\int_0^{\lambda-A_\Gamma(\omega)} \chi_\Gamma(\tilde{\omega}(s))\mathrm{d}s} \chi_{\{A_\Gamma \leq \lambda-\varepsilon\}}(\tilde{\omega}) \\ \leq \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) \int_{\Omega_{\omega(A_\Gamma)}} \mathrm{P}_{\omega(A_\Gamma)}(\mathrm{d}\tilde{\omega}) e^{-M\int_0^{\varepsilon} \chi_\Gamma(\tilde{\omega}(s))\mathrm{d}s}.\end{aligned}$$

We are able to get rid of the dependence on x of the latter expression, if we assume that Γ satisfies the following *uniform cone condition*:

$$(3.3) \qquad \begin{cases} \text{There is a cone } K \text{ of finite height in } \mathbb{R}^d \text{ with the origin as vertex} \\ \text{s.t. for any } y \in \partial\Gamma \text{ there is a motion } T \text{ of } \mathbb{R}^d \text{ with} \\ \text{(i) } y \text{ is the vertex of } T(K), \\ \text{(ii) } T(K)\setminus\{y\} \subset \Gamma. \end{cases}$$

Actually, each uniform Lipschitz set fulfils (3.3) (see e.g. [Wlo], Section 2.2). We emphasize that unbounded singularity regions are included in these considerations.

Under this assumption, the invariance properties of the Wiener measure imply for each $y = \omega(A_\Gamma) \in \partial\Gamma$

$$\begin{aligned}\int_{\Omega_y} \mathrm{P}_y(\mathrm{d}\tilde{\omega}) e^{-M\int_0^{\varepsilon} \chi_\Gamma(\tilde{\omega}(s))\mathrm{d}s} &\leq \int_{\Omega_y} \mathrm{P}_y(\mathrm{d}\tilde{\omega}) e^{-M\int_0^{\varepsilon} \chi_{T(K)}(\tilde{\omega}(s))\mathrm{d}s} \\ &= \int_{\Omega_0} \mathrm{P}_0(\mathrm{d}\tilde{\omega}) e^{-M\int_0^{\varepsilon} \chi_K(\tilde{\omega}(s))\mathrm{d}s},\end{aligned}$$

which is independent of $\omega(0)$.

Thus the second term on the r.h.s. of (3.2) is bounded in terms of the Laplace transform of (the distribution of) $T_{\varepsilon,K}$, which is dealt with in Section 5.
In fact, if C is a cone of infinite height, we can show (see Theorem 5.4)

$$\mathrm{E}_0\left(e^{-MT_{\varepsilon,C}}\right) \leq \frac{c}{(\log(M\varepsilon^{\frac{3}{2}+\gamma}))^\gamma}$$

for any $0 < \gamma < \frac{1}{2}$ and small ε.
Since we are interested in K rather than C, we will let $\varepsilon(M,\lambda) \to 0$ as $M \to \infty$ in order to obtain a useful bound on $\mathrm{E}_0\left(e^{-MT_{\varepsilon,K}}\right)$:

Theorem 3.1: Let Γ be a uniform Lipschitz set and suppose Assumption A.

Then we have for any $0 < \gamma < \frac{1}{2}$:

$$\|Je^{-\lambda H_M} - e^{-\lambda H_\Sigma} J\| \le c\left(1 + \frac{1}{\sqrt{\lambda}}\right)\frac{1}{(\log M)^\gamma} \quad \forall \lambda > 0, \tag{3.4}$$

$$\|J(a + H_M)^{-1} - (a + H_\Sigma)^{-1} J\| \le \frac{c}{(\log M)^\gamma} \quad \forall a > 0 \tag{3.5}$$

for large M.

Proof: Let K be a standard cone of finite height for Γ as in (3.3), $C = \{rx | r \ge 0, x \in K\}$ the cone extending K to infinity.
Using the above calculations and the main results of Sections 4,5 mentioned above, we have for arbitrary $0 < \gamma < \frac{1}{2}$

$$\begin{aligned}
\|Je^{-\lambda H_M} - e^{-\lambda H_\Sigma} J\| &\le \sup_{x\in\Sigma} \mathrm{P}_x\{\lambda - \varepsilon < A_\Gamma < \lambda\} + \mathrm{E}_0\left(e^{-MT_{\varepsilon,K}}\right) \\
&\le \sup_{x\in\Sigma} \mathrm{P}_x\{\lambda - \varepsilon < A_\Gamma < \lambda\} + \\
&\qquad + \mathrm{E}_0\left(e^{-MT_{\varepsilon,C}}\right) + \mathrm{P}_0\{\omega \mid |\omega(s)| > \operatorname{diam} K \text{ for some } s \le \varepsilon\} \\
&\le c\left[\left(m + \frac{\tilde{m}}{\sqrt{\lambda}}\right)\sqrt{\varepsilon} + \frac{1}{(\log(M\varepsilon^{\frac{3}{2}+\gamma}))^\gamma} + e^{-\frac{(\operatorname{diam} K)^2}{4\varepsilon}}\right]
\end{aligned}$$

with constants $m, \tilde{m}$ as in Theorem 4.2.
Now (3.4) follows by letting $\varepsilon = M^\alpha$ with $\alpha < 0$, hence the second assertion using (2.10). □

Remarks 3.2: All the results of Section 2 are contained in Theorem 3.1. In particular, N-body situations are treated to a satisfactory degree, because the singularity region may be unbounded and the smoothness condition on its boundary is relaxed.
Furthermore, we have a convergence rate for any such uniform Lipschitz singularity region. For a better convergence rate in a special case see Lemma 6.1 however.

4. Probability of late arrival

In this section, we provide the estimates needed in Section 3 on the probability of "late arrival" in Γ of Brownian motion starting in Σ, i.e. on

$$\sup_{x\in\Sigma} \mathrm{P}_x\{t-\varepsilon < A_\Gamma < t\}, \tag{4.1}$$

where

$$A_\Gamma(\omega) := \inf\{s>0 \mid \omega(s)\in\Gamma\}. \tag{4.2}$$

We start with the situation where the boundary of Γ is given *globally* by a Lipschitz continuous function.

Proposition 4.1: Let $d\geq 2$, $f: I\!R^{d-1}\to I\!R$ Lipschitz continuous, i.e. there is an $L>0$ s.t. $|f(a)-f(b)|\leq L|a-b|$ for every $a,b\in I\!R^{d-1}$. If Σ denotes the set below the graph of f in $I\!R^d$,

$$\Sigma = \{(x', f(x')+u)\in I\!R^d \mid x'\in I\!R^{d-1}, u<0\},$$

then there is a constant c s.t.

$$\sup_{x\in\Sigma} \mathrm{P}_x\{t-\varepsilon<A_\Gamma<t\}\leq c\frac{\sqrt{\varepsilon}}{\sqrt{t}} \tag{4.3}$$

for $0<\varepsilon<t$.

Proof: In the following, let $x_0=(x_0', f(x_0')+u_0)\in\Sigma$ be fixed, and $x=(x',f(x')+u)\in\Sigma$, $y=(y',f(y')+v)\in\Gamma$ arbitrary, i.e. $x_0',x',y'\in I\!R^{d-1}$, $u_0,u<0$, $v\geq 0$.
If a Brownian trajectory ω starting at x_0 hits Γ for the first time in the time interval $(t-\varepsilon,t)$, then we know that $\omega(t-\varepsilon)\in\Sigma$ and $\omega(s)\in\Gamma$ for some $s\in(t-\varepsilon,t)$. Thus

$$\begin{aligned}
&\mathrm{P}_{x_0}\{t-\varepsilon<A_\Gamma<t\}\leq\\
&\leq \int_\Sigma \mathrm{d}x\ p(x_0,x|t-\varepsilon)\mathrm{P}_x\{\omega \mid \omega(s)\in\Gamma \text{ for some } 0<s<\varepsilon\}\\
&\leq c\int_\Sigma \mathrm{d}x\ p(x_0,x|t-\varepsilon)\mathrm{P}_0\{\omega \mid |\omega(\varepsilon)|\geq \mathrm{dist}(x,\Gamma)\}\\
&\leq c\int_\Sigma \mathrm{d}x\ p(x_0,x|t-\varepsilon)e^{-\frac{1}{4\varepsilon}(\mathrm{dist}(x,\Gamma))^2}.
\end{aligned} \tag{4.4}$$

Concerning the distance between x and Γ, we now prove the existence of a constant $C_L>0$ s.t.

$$|y-x|^2 = |y'-x'|^2+|f(y')+v-f(x')-u|^2\geq C_L|v-u|^2 \quad \forall y\in\Gamma, x\in\Sigma. \tag{4.5}$$

In fact, if $|v-u| < L|y'-x'|$, then $|y-x|^2 \geq \frac{1}{L^2}|v-u|^2$. If, on the other hand, $|v-u| \geq L|y'-x'|$, then

$$\begin{aligned}
|y-x|^2 &\geq |y'-x'|^2 + [|v-u| - |f(y')-f(x')|]^2 \\
&\geq |y'-x'|^2 + [|v-u| - L|y'-x'|] \\
&= |v-u|^2 \left[\left(1 - L\frac{|y'-x'|}{|v-u|}\right)^2 + \left(\frac{|y'-x'|}{|v-u|}\right)^2 \right] \\
&\geq \kappa |v-u|^2
\end{aligned}$$

for some $\kappa > 0$, because $\alpha \mapsto (1-L\alpha)^2 + \alpha^2$ takes its global minimum.
Having completed the proof of (4.5), we now observe that $(\text{dist}(x,\Gamma))^2 = \inf\{|y-x|^2 | y' \in \mathbb{R}^{d-1}, v \geq 0\} \geq C_L|u|^2$. After a change of coordinates, we obtain

$$\begin{aligned}
\text{r.h.s. of (4.4)} &\leq c(t-\varepsilon)^{-\frac{d}{2}} \int_{-\infty}^{0} \mathrm{d}u \int_{\mathbb{R}^{d-1}} \mathrm{d}x' \, e^{-\frac{|x_0-x|^2}{2(t-\varepsilon)}} e^{-\frac{C_L}{4\varepsilon}|u|^2} \\
&\leq c(t-\varepsilon)^{-\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} \mathrm{d}x' \, e^{-\frac{|x_0'-x'|^2}{4(t-\varepsilon)}} \times \\
&\quad \times (t-\varepsilon)^{-\frac{1}{2}} \int_{-\infty}^{0} \mathrm{d}u \, e^{-\frac{C_L}{4\varepsilon}|u|^2} e^{-\frac{C_L}{4(t-\varepsilon)}|u_0-u|^2}.
\end{aligned} \tag{4.6}$$

Denoting the ν-dimensional transition probability kernel by $p_{(\nu)}(\cdot,\cdot \mid \cdot)$ and remarking that there is no norming factor corresponding to $e^{-\frac{C_L}{4\varepsilon}|u|^2}$, we end up with

$$\begin{aligned}
&\mathrm{P}_{x_0}\{t-\varepsilon < A_\Gamma < t\} \leq \\
&\leq c\sqrt{\varepsilon} \int_{\mathbb{R}^{d-1}} \mathrm{d}x' \, p_{(d-1)}(x_0', x' | 2(t-\varepsilon)) \int_{-\infty}^{0} \mathrm{d}r \, p_{(1)}\left(0, u \,\middle|\, \frac{2\varepsilon}{C_L}\right) p_{(1)}\left(u, u_0 \,\middle|\, \frac{2(t-\varepsilon)}{C_L}\right) \\
&\leq c\sqrt{\varepsilon} p_{(1)}\left(0, u_0 \,\middle|\, \frac{2t}{C_L}\right) \leq c\frac{\sqrt{\varepsilon}}{\sqrt{t}}
\end{aligned}$$

by Chapman-Kolmogorov's equation. □

Judging from the above proof, one should assume Γ to have a uniform locally Lipschitz boundary. To this end, we consider tubular neighborhoods of radius l around $\partial = \partial\Gamma = \partial\Sigma$

$$\partial_l = \{x \in \mathbb{R}^d \mid \text{dist}(x, \partial) \leq l\},$$

in which we assume locally a similar Lipschitz condition as in Proposition 4.1.

Definition L: We call Γ a uniform Lipschitz set if there is an $l > 0$ and subsets C_k, $k = 1, \ldots, N$ (N finite or infinite), of $\mathbb{R}^d$ whose union covers ∂_l s.t. the following conditions hold:

(L1) (uniform Lipschitz condition)
Up to congruence of $\mathbb{R}^d$, each C_k is of the form

$$C_k = \{(x', f_k(x') + u) \mid x' \in U_k, -l < u < l\}$$

with an open set $U_k \subset \mathbb{R}^{d-1}$ and a Lipschitz continuous function $f_k : U_k \to \mathbb{R}$ (with a Lipschitz constant L independent of k), and

$$\begin{aligned} C_k \cap \partial &= \{(x', f_k(x')) \mid x' \in U_k\}, \\ C_k \cap \Sigma &= \{(x', f_k(x') + u) \mid x' \in U_k, -l < u < 0\}, \\ C_k \cap \Gamma &= \{(x', f_k(x') + u) \mid x' \in U_k, 0 < u < l\}. \end{aligned}$$

(L2) (the C_k must neither be arbitrarily small nor large)
If $B_r(x)$ denotes the open ball of radius r around x, then there are constants $l_\pm > 0$ (independent of k) and "centers" $m_k \in \mathbb{R}^d$ s.t. $B_{l_-}(m_k) \subset C_k \subset B_{l_+}(m_k)$.

(L3) (the C_k must not have arbitrarily thin intersections)
For each $x \in \partial_l$ there is a k s.t. $B_{l_-}(x) \subset C_k$.

(L4) (uniform local finiteness)
There is a finite constant m s.t. each $x \in \partial_l$ lies in at most m of the sets C_k.

Theorem 4.2: Suppose Γ to be a uniform Lipschitz set in $\mathbb{R}^d$, and let m be as in Definition L.
Then there are constants $c, \tilde{m}$ s.t.

$$\sup_{x \in \Sigma} \mathrm{P}_x\{t - \varepsilon < A_\Gamma < t\} \le c\sqrt{\varepsilon}\left(m + \frac{\tilde{m}}{\sqrt{t}}\right) \tag{4.7}$$

for $0 < \varepsilon < t$.

Before giving the proof of Theorem 4.2, we want to discuss assumptions and result therein:

Example 4.3: Surely, neither of the estimates (4.3) and (4.7) is optimal. For example, if $d = 1$ and $\Gamma = [0, \infty)$, the distribution of A_Γ is known (see e.g. [Kar/Shr]). We obtain

$$\begin{aligned} &\sup_{x<0} \mathrm{P}_x\{t - \varepsilon < A_{[0,\infty)} < t\} \\ &= \sup_{x<0} \int_{t-\varepsilon}^{t} \frac{|x|}{\sqrt{2\pi s^3}} e^{-\frac{x^2}{2s}} \mathrm{d}s \\ &\le c\frac{\varepsilon}{t}. \end{aligned}$$

As usual, this result extends to the case of a halfspace in $\mathbb{R}^d$.

Since the conditions (L1)-(L4) in Definition L are somewhat lengthy and technical, it is worthwhile to note that they are not nearly so restrictive as they look at a first glance:

Remarks 4.4:

a) Let Γ be R-smooth in the sense of [vdB], i.e. for any $x_0 \in \partial$ there are open balls B_1, B_2 with radius R s.t. $B_1 \subset \Sigma$, $B_2 \subset \Gamma$, $\partial B_1 \cap \partial B_2 = \{x_0\}$.
Then Γ fulfils (L1)-(L4).

b) In view of (L1), ∂ need not be "smooth" in the usual sense but may very well have peaks as long as the corresponding angles don't become arbitrarily small. In particular, any parallelepiped or cone satisfies the assumptions of Theorem 4.2.

c) The class of sets Γ s.t. (4.7) holds for *some* constant c is closed with respect to finite unions.
In particular, the union Γ of two closed balls or cubes having exactly one point in common satisfies (4.7) although Γ is *not* an example for the conditions (L1)-(L4) in the first place.

Here, parts b) and c) are trivial, and we will supply the proof of Remark 4.4.a) after giving the proof of the main result of this section:

Proof (of Theorem 4.2): Let $x_0 \in \Sigma$ be fixed. Then

$$\begin{aligned} &\mathrm{P}_{x_0}\{t-\varepsilon < A_\Gamma < t\} \leq \\ (4.8) \qquad &\leq \int_\Sigma \mathrm{d}x\ p(x_0, x|t-\varepsilon)\mathrm{P}_x\{\omega \mid \omega(s) \in \Gamma \text{ for some } 0 < s < \varepsilon\} \\ &\leq \int_{\Sigma\cap\partial_l} \mathrm{d}x\ \ldots + \int_{\Sigma\setminus(\Sigma\cap\partial_l)} \mathrm{d}x\ \ldots \end{aligned}$$

Using (2.9) once again

$$\int_{\Sigma\setminus(\Sigma\cap\partial_l)} \mathrm{d}x\ \ldots \leq 2\int_{\Sigma\setminus(\Sigma\cap\partial_l)} \mathrm{d}x\ p(x_0, x|t-\varepsilon)e^{-\frac{l^2}{4\varepsilon}} \leq 2e^{-\frac{l^2}{4\varepsilon}},$$

and

$$\begin{aligned} \int_{\Sigma\cap\partial_l} \mathrm{d}x\ \ldots &\leq \int_{\Sigma\cap\partial_l} \mathrm{d}x\ p(x_0, x|t-\varepsilon)\mathrm{P}_x\{\omega \mid \omega(s) \in \Gamma \cap B_{l\_}(x) \text{ for some } s < \varepsilon\} + \\ &\quad + \int_{\Sigma\cap\partial_l} \mathrm{d}x\ p(x_0, x|t-\varepsilon)\mathrm{P}_x\{\omega \mid \omega(s) \in \Gamma \setminus B_{l\_}(x) \text{ for some } s < \varepsilon\} \\ &\leq \int_{\Sigma\cap\partial_l} \mathrm{d}x\ p(x_0, x|t-\varepsilon)\mathrm{P}_x\{\omega \mid \omega(s) \in \Gamma \cap B_{l\_}(x) \text{ for some } s < \varepsilon\} + 2e^{-\frac{l^2}{4\varepsilon}}. \end{aligned}$$

Thus, we only have to prove the desired estimate for

$$I := \int_{\Sigma\cap\partial_l} \mathrm{d}x\ p(x_0, x|t-\varepsilon)\mathrm{P}_x\{\omega \mid \omega(s) \in \Gamma \cap B_{l\_}(x) \text{ for some } s < \varepsilon\}.$$

To this end, for $x \in \Sigma \cap \partial_l$ let k be as in (L3). Then

$$I \leq \sum_k \int_{\Sigma\cap C_k} \mathrm{d}x\ p(x_0, x|t-\varepsilon)\mathrm{P}_x\{\omega \mid \omega(s) \in \Gamma \cap C_k \text{ for some } s \leq \varepsilon\}.$$

Using the transformations (L1), we see that

$$I \le c \sum_k \int_{U_k} \mathrm{d}x' \int_{-l}^{0} \mathrm{d}u \ (t-\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x_0-x|^2}{2(t-\varepsilon)}} e^{-\frac{C_L}{4\varepsilon}u^2} \tag{4.9}$$

just as in the global case (see (4.6)). Here, (L1) ensures uniformity in k.

If $x \in C_k$, we have $|x_0 - x| \ge |x_0 - m_k| - |m_k - x| \ge |x_0 - m_k| - l_+$ by (L2). On the other hand, if A_k is the congruence map assumed in (L1), and $x_0 = (x_0', u_0')$, $x = (x', u')$ in the sense of A_k-coordinates, then $|x_0 - x| \ge |x_0' - x'|$.
Thus

$$\begin{aligned} I \le c \sum_k \int_{U_k} \mathrm{d}x' \ (t-\varepsilon)^{-\frac{d-1}{2}} e^{-\frac{1}{4(t-\varepsilon)}\left[\max\left(|x_0-m_k|-l_+, |x_0'-x'|\right)\right]^2} \times \\ \times \int_{-\infty}^{\infty} \mathrm{d}u \ (t-\varepsilon)^{-\frac{1}{2}} e^{-\frac{C_L}{4(t-\varepsilon)}(u_0-u)^2} e^{-\frac{C_L}{4\varepsilon}u^2}. \end{aligned} \tag{4.10}$$

As in the global case, the one-dimensional integral is

$$\begin{aligned} &\le c\sqrt{\varepsilon} \int_{\mathbb{R}} \mathrm{d}u \ p_{(1)}\left(u_0, u \,\middle|\, \frac{2(t-\varepsilon)}{C_L}\right) p_{(1)}\left(u, 0 \,\middle|\, \frac{2\varepsilon}{C_L}\right) \\ &= c\sqrt{\varepsilon} p_{(1)}\left(u_0, 0 \,\middle|\, \frac{2t}{C_L}\right) = c\frac{\sqrt{\varepsilon}}{\sqrt{t}}. \end{aligned}$$

In the sum (4.10), we now consider two different cases with respect to k.
In the first place, if $|x_0 - m_k| \ge 6l_+$ (i.e. x_0 is far away from C_k), it is easily seen that $(|x_0 - m_k| - l_+)^2 \ge \frac{1}{2}|x_0 - x|^2$ for all $x \in C_k$, allowing us to return to the full-dimensional transition probability kernel in (4.10). By inserting unity, we obtain for such a k that the double integral in (4.10)

$$\begin{aligned} I(k) := &\int_{U_k} \mathrm{d}x' \ (t-\varepsilon)^{-\frac{d-1}{2}} e^{-\frac{1}{4(t-\varepsilon)}\left[\max\left(|x_0-m_k|-l_+, |x_0'-x'|\right)\right]^2} \times \\ &\times \int_{-\infty}^{\infty} \mathrm{d}u \ (t-\varepsilon)^{-\frac{1}{2}} e^{-\frac{C_L}{4(t-\varepsilon)}(u_0-u)^2} e^{-\frac{C_L}{4\varepsilon}u^2} \\ \le\ & c\sqrt{\varepsilon} \int_{U_k} \mathrm{d}x' \ (t-\varepsilon)^{-\frac{d}{2}} \left(\frac{1}{l}\int_{-l}^{0} \mathrm{d}u\right) e^{-\frac{|x_0-x|^2}{8(t-\varepsilon)}} \\ \le\ & c\sqrt{\varepsilon} \int_{\Sigma \cap C_k} \mathrm{d}x \ p(x_0, x | 4(t-\varepsilon)). \end{aligned}$$

If, on the other hand, $|x_0 - m_k| < 6l_+$ (i.e. x_0 is near C_k), we use the alternative for the maximum in (4.10). Then

$$\begin{aligned} I(k) &\le c\frac{\sqrt{\varepsilon}}{\sqrt{t}} \int_{\mathbb{R}^{d-1}} \mathrm{d}x' \ (t-\varepsilon)^{-\frac{d-1}{2}} e^{-\frac{|x_0'-x'|^2}{4(t-\varepsilon)}} \\ &= c\frac{\sqrt{\varepsilon}}{\sqrt{t}}. \end{aligned}$$

In total, we obtain

$$\begin{aligned} I &\le c\sqrt{\varepsilon}\left[\sum_{k;|x_0-m_k|\ge 6l_+}\int_{\Sigma\cap C_k} \mathrm{d}x\ p(x_0,x|4(t-\varepsilon)) + \frac{1}{\sqrt{t}}\#\{k \mid |x_0-m_k|<6l_+\}\right] \\ &\le c\sqrt{\varepsilon}\left[m\int_{\Sigma\cap\partial_{l_+}} \mathrm{d}x\ p(x_0,x|4(t-\varepsilon)) + \frac{\tilde{m}}{\sqrt{t}}\right] \\ &\le c\sqrt{\varepsilon}\left[m+\frac{\tilde{m}}{\sqrt{t}}\right] \end{aligned}$$

with m as in (L4) and a constant $\tilde{m}$ independent of x_0. Indeed, we have $\sup_{x_0}\#\{k \mid |x_0-m_k|<6l_+\} < +\infty$ because of (L2) and (L4). In order to prove this, we assume without loss of generality that balls are defined with respect to the supremum norm and that $Vl_- = l_+$ for some natural number V.
Consider a point $x\in\mathbb{R}^d$ and (pairwise distinct) integers $k_1,\ldots,k_n$ s.t. $|x-m_{k_i}|<6l_+$ for $i=1,\ldots,n$. Obviously, one can distribute at most $k(6V)^d$ "balls" of radius l_- onto $B_{6l_+}(x)$ in such a way that each point of $B_{6l_+}(x)$ is in at most k of the smaller "balls". Hence $\frac{n}{(6V)^d} \le \sup_{y\in B_{6l_+}(x)}\#\{k|y\in C_k\}\le m$. □

Proof (of Remark 4.4.a)): One can *construct* the necessary quantities appearing in (L1)-(L4), if Γ is R-smooth: let $l=\frac{R}{1000}$ and choose elements $y_k\in\partial$ with $|y_k-y_{k'}|\ge\frac{R}{10}$ for $k\ne k'$ and $\bigcup_k B_{\frac{R}{3}}(y_k)\supset\partial$.
For k fixed, assume w.l.o.g. that the defining balls B_1, B_2 meeting at y_k are of the form $B_R((\underline{0},\pm R))$, where $\underline{0}$ in the origin in $\mathbb{R}^{d-1}$. Obviously,

$$\partial\cap\left\{(y,y')\in\mathbb{R}^{d-1}\times\mathbb{R}\,\middle|\, |y|<\frac{R}{3}, |y'|<\frac{R}{3}\right\}$$

is given as the graph of a function $f_k:\{y\in\mathbb{R}^{d-1}|\ |y|<R/3\}\to\mathbb{R}$. If we let

$$C_k := \left\{(y, f_k(y)+u)\,\middle|\, y\in\mathbb{R}^{d-1}, |y|<\frac{R}{3}, |u|<l\right\},$$

then (L1) is an easy consequence of the mean value theorem and (L2)-(L4) are trivial. □

5. Occupation time in a cone

The aim of this section is to demonstrate how the method of [Mey] to study the asymptotic behavior can also be used to estimate the Laplace transform of (the distribution of) the occupation time

$$T_{t,C}(\omega) := \text{meas } \{s \le t \mid \omega(s) \in C\}$$

of d-dimensional Brownian motion starting at 0 in a cone C given by

$$C = \{rx | r \ge 0, x \in \mathcal{F}\},$$

where $\mathcal{F}$ is a closed subset of the unit sphere S^{d-1} in $\mathbb{R}^d$ having a nonempty interior. In view of our applications in Section 3, it is sufficient to think $\mathcal{F}$ to be of the form $\{x \in S^{d-1} | \, |x - x_0| \le r\}$.
It is hard to determine the distribution of $T_{t,C}$ precisely; for example, to our best knowledge this problem is still open in the easy-looking case where $t = 1$ and $C = \{(x,y)|x,y \ge 0\}$ a quadrant in $\mathbb{R}^2$ (see e.g. p. 108 in [Mey]).

At the end of this section, we will prove the following quantitative version of Proposition 4.3 in [Mey]:

Proposition 5.1: Let $t_n = 2^{-n}$ for $n \in \mathbb{N}$. If q is large enough, then there is a constant c s.t.

$$\text{P}_0 \left\{ \omega \in \Omega_0 \left| |\log t_k|^q \frac{T_{t_k,C}(\omega)}{t_k} < 1 \text{ for some } k > n \right. \right\} \le c \frac{\log n}{n} \tag{5.1}$$

for large n.

By interpolation, we obtain

Proposition 5.2: If q is sufficiently large, there are positive constants c, η s.t.

$$\text{P}_0 \left\{ T_{\varepsilon,C} < \eta \frac{\varepsilon}{|\log \varepsilon|^q} \right\} \le c \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \tag{5.2}$$

for small $\varepsilon > 0$.

Proof: If $\varepsilon \in (0,1)$, let n be the natural number with $t_n \le \varepsilon < t_{n-1}$. From $T_{\varepsilon,C}(\omega) < \eta \frac{\varepsilon}{|\log \varepsilon|^q}$, we infer

$$T_{t_n,C}(\omega) \le T_{\varepsilon,C}(\omega) < \eta \frac{t_{n-1}}{|\log t_{n-1}|^q} \le 2^{q+1} \eta \frac{t_n}{|\log t_n|^q} < \frac{t_n}{|\log t_n|^q}$$

for η sufficiently small. Hence, (5.1) implies the result. □

Since any polynomial is increasing faster than the logarithm at infinity, we immediately obtain

Corollary 5.3: Let $\alpha > 0$ be arbitrary. Then there are $c, \eta > 0$ s.t.

$$\text{(5.3)} \qquad \mathrm{P}_0 \left\{ T_{\varepsilon,C} < \eta \varepsilon^{1+\alpha} \right\} \leq \frac{c}{|\log \varepsilon|^{1-\alpha}}$$

for small $\varepsilon > 0$.

The above assertion on how large $T_{\varepsilon,C}$ usually is for small times enables us to show how small the Laplace transform of $T_{\varepsilon,C}$ is in certain parameter regions.

Theorem 5.4: Let $0 < \gamma < \frac{1}{2}$. Then there are positive constants c, ε_0, K_0 s.t.

$$\text{(5.4)} \qquad \mathrm{E}_0 \left(e^{-MT_{\varepsilon,C}} \right) \leq \frac{c}{(\log(M\varepsilon^{\frac{3}{2}+\gamma}))^{\gamma}}$$

as soon as $\varepsilon < \varepsilon_0$ and $M\varepsilon^{\frac{3}{2}+\gamma} > K_0$.

Proof: Let $\varepsilon > 0$ be sufficiently small in the sense of Corollary 5.3. Setting $\varepsilon_k := \varepsilon e^{-k^2}$ for $k \in I\!N$, we will use the fact that by (5.3) P_0-a.e. $\omega \in \Omega_0$ satisfies $T_k(\omega) \geq \eta \varepsilon_k^{1+\alpha}$ for large k, where $\alpha > 0$ may be chosen arbitrarily small and T_k is a shorthand notation for $T_{\varepsilon_k,C}$:

$$\begin{aligned} \mathrm{E}_0 \left(e^{-MT_{\varepsilon,C}} \right) = & \int_{T_0 \geq \eta \varepsilon^{1+\alpha}} \mathrm{P}_0(\mathrm{d}\omega) e^{-MT_{\varepsilon,C}(\omega)} + \\ & + \sum_{k=1}^{\infty} \int_{T_k \geq \eta \varepsilon_k^{1+\alpha}; T_{k-1} < \eta \varepsilon_{k-1}^{1+\alpha}} \mathrm{P}_0(\mathrm{d}\omega) e^{-MT_{\varepsilon,C}(\omega)} \\ \leq & \; e^{-M\eta\varepsilon^{1+\alpha}} + \sum_{k=1}^{\infty} e^{-M\eta\varepsilon_k^{1+\alpha}} \mathrm{P}_0 \left\{ T_{k-1} < \eta \varepsilon_{k-1}^{1+\alpha} \right\}. \end{aligned}$$

On the r.h.s., every single summand is fine with respect to (5.4). Thus we still have to prove that

$$R_K(\varepsilon, M) := \sum_{k=K+1}^{\infty} \frac{1}{|\log \varepsilon_{k-1}|^{1-\alpha}} \, e^{-M\eta\varepsilon_k^{1+\alpha}}$$

satisfies the desired bound for some K. Letting $N = M\eta\varepsilon^{1+\alpha}$, we have

$$\begin{aligned} R_K(\varepsilon, M) \leq & \sum_{k=K+1}^{\infty} \frac{e^{-Ne^{-(1+\alpha)k^2}}}{(k-1)^{2-2\alpha}} \\ \text{(5.5)} \qquad \leq & \; c \int_K^{\infty} \frac{e^{-Ne^{-(1+\alpha)x^2}}}{x^{2-2\alpha}} \mathrm{d}x \\ = & \; c \int_0^{e^{-(1+\alpha)K^2}} e^{-Ny} \frac{1}{y} \frac{1}{(-\log y)^{\frac{3}{2}-\alpha}} \mathrm{d}y \end{aligned}$$

via the substition $y = e^{-\lambda x^2}$.
Thus we have for arbitrary $\kappa < 1$

$$\begin{aligned} R_K(\varepsilon, M) &\le c\int_0^{N^{-\kappa}} \frac{1}{y}\frac{1}{(-\log y)^{\frac{3}{2}-\alpha}}\mathrm{d}y + ce^{-N^{1-\kappa}}\int_{N^{-\kappa}}^{e^{-(1+\alpha)K^2}} \frac{1}{y}\frac{1}{(-\log y)^{\frac{3}{2}-\alpha}}\mathrm{d}y \\ &\le c(\kappa \log N)^{\alpha-\frac{1}{2}} + cN^{\kappa}e^{-N^{1-\kappa}}. \end{aligned}$$

Hence the result by resubstituting $N = \eta M \varepsilon^{1+\alpha}$. □

Problem: For the halfspace $I\!H = \{x \in I\!R^d | x_d \ge 0\}$, we know by the arcsine law (see e.g. Section 4.4 of [Kar/Shr]) that

$$\mathrm{E}_0\left(e^{-MT_{\varepsilon,H}}\right) \le \frac{c}{\sqrt{\varepsilon M}}. \tag{5.6}$$

We believe that for the case of Theorem 5.4 the Laplace transform decays much faster than logarithmically and might be similar to (5.6).

Proof (of Proposition 5.1): In the rest of this section, we will work our way through [Mey] in order to prove Proposition 5.1. For ease of reference, we employ Meyre's notation as introduced in Sections (3.1),(3.2) of [Mey]. We recall that the underlying idea is to construct for a given t_n well-chosen random variables $\tau_n, \sigma_n \ge t_n$ s.t.

i) $\omega(s) \in C$ for all $s \in [\tau_n(\omega), \sigma_n(\omega)]$,
ii) $\sigma_n(\omega) - \tau_n(\omega)$ is large.

To this end, choose $\delta > 0$ small enough s.t.

$$\mathcal{F}_\delta = \{x \in S^{d-1} | \mathrm{dist}(x, \mathcal{F}^c) \ge \delta\}$$

has a nonempty interior and consider the following chain of real numbers, where q_1, q are arbitrary:

$$E_1(\mathcal{F}^c) < E_1(\mathcal{F}_\delta^c) < q_2 := 2E_1(\mathcal{F}_\delta^c) < q_1 < q,$$

where $E_1(\mathcal{F}^c)$ denotes the smallest eigenvalue of $-\frac{1}{2}\Delta$ (Δ being the Laplacian on S^{d-1}) on $S^{d-1} \setminus \mathcal{F}$ with Dirichlet boundary conditions.
Letting C_δ be the cone in $I\!R^d$ determined by $\mathcal{F}_\delta$, the line of the proof is to show that the following sequence of stopping times

$$\begin{aligned} T_n^0(\omega) &:= t_n = 2^{-n}, \\ U_n^p(\omega) &:= \inf\{t \ge T_n^{p-1}(\omega) \mid |\omega(t)| \ge \sqrt{t_n}\}, \quad p = 1, 2, \dots, \\ T_n^p(\omega) &:= \inf\{t \ge U_n^p(\omega) \mid \omega(t) \in C_\delta\}, \qquad p = 1, 2, \dots, \end{aligned}$$

"soon" becomes stationary, i.e. soon after time t_n the path ω will be found "far away" from the origin and "safely within" the cone C. Therefore we consider

$$N_n(\omega) = \inf\{p \in I\!N \mid \omega(T_n^p) \ge \sqrt{t_n}\}.$$

The following estimate immediately follows from $P_0\{N_n > k\} \le \rho^k, \quad k \in \mathbb{N}$, for some $\rho \le \frac{1}{2}$, which is shown on p. 120 of [Mey]:

$$(5.7) \qquad \begin{cases} \text{For any } K < 1 \text{ there is a constant } c > 0 \quad \text{s.t.} \\ P_0\{N_k \ge c \log k \text{ for some } k > n\} \le cn^{1-K}. \end{cases}$$

Thus, the stopping time

$$\tau_n := T_n^{[c \log n]}$$

(where $[\lambda]$ denotes the largest integer smaller than λ) usually has the following properties for large n:

$$(5.8) \qquad \begin{cases} \text{i)} & \tau_n \ge t_n, \\ \text{ii)} & \omega(\tau_n) \in C_\delta, \\ \text{iii)} & |\omega(\tau_n)| \ge \sqrt{t_n}. \end{cases}$$

Furthermore it is bounded from above in the following way:

Lemma 5.5: For large n

$$(5.9) \qquad P_0\{\tau_k > t_k |\log t_k|^{q_1} \text{ for some } k > n\} \le c\frac{\log n}{n}.$$

Proof: In the proof of Lemme 3.3 in [Mey] it is shown that $P_0(A_{n,p,i}) \le \frac{C}{|\log t_n|^2}$ for $i = 1, 2,\ p \in \mathbb{N}$ and large n, where

$$A_{n,p,1} = \{U_n^p - T_n^{p-1} \ge t_n (\log |\log t_n|)^2\},$$
$$A_{n,p,2} = \{T_n^p - U_n^p \ge t_n |\log t_n|^{q_2}\}$$

(note that we – in contrast to Meyre – have *chosen* q_2).
For $\omega \in B_n = \bigcap_{p=1}^{[c \log n]} (A_{n,p,1}^c \cap A_{n,p,2}^c)$ we obtain as usual $\tau_n(\omega) \le t_n |\log t_n|^{q_1}$.
Since $P_0(B_n^c) \le \frac{2C[c \log n]}{|\log t_n|^2}$, (5.9) follows. □

Due to (5.8), the stopping time

$$\sigma_n(\omega) := \inf\{t > \tau_n(\omega) \mid \omega(t) \notin C\}$$

usually should be much larger than τ_n. In fact $P_0\left\{\sigma_n - \tau_n \le \frac{t_n}{(\log |\log t_n|)^2}\right\} \le \frac{c}{n^2}$ (see the proof of Lemme 3.4 in [Mey]) implies

$$(5.10) \qquad P_0\left\{\sigma_k - \tau_k \le \frac{t_k}{(\log |\log t_k|)^2} \text{ for some } k > n\right\} \le \frac{c}{n}$$

for large n.

In order to utilize (5.10) for a lower bound on the occupation time up to time t_n (recall $\tau_n \geq t_n$), one introduces for $n \in \mathbb{N}$ the unique number $m(n)$ s.t.

$$t_{m(n)} \leq \frac{t_n}{2|\log t_n|^{q_1}} < 2t_{m(n)}.$$

Now $\tau_n(\omega) \leq t_n |\log t_n|^{q_1}, n \geq n_0(\omega)$, implies $\tau_{m(n)}(\omega) \leq \frac{2}{3} t_n$ for large n. From Lemma 5.5 we deduce

$$\text{(5.11)} \qquad \mathrm{P}_0 \left\{ \tau_{m(n)} > \frac{2}{3} t_n \right\} \leq c \frac{\log n}{n}$$

for large n.
Obviously

$$T_{t_n, C}(\omega) \geq \min(\sigma_{m(n)}, t_n) - \tau_{m(n)}.$$

Thus, if $\sigma_{m(n)}(\omega) > t_n$, one usually has $T_{t_n,C}(\omega) \geq \frac{1}{3} t_n$ for large n due to (5.11).
On the other hand, if $\sigma_{m(n)} \leq t_n$, (5.10) usually implies $T_{t_n,C}(\omega) \geq \frac{t_{m(n)}}{(\log |\log t_{m(n)}|)^2}$, if n is sufficiently large.
In each of these cases, we conclude $|\log t_n|^q \frac{T_{t_n,C}(\omega)}{t_n} \geq 1$. Hence, the estimates (5.7), (5.10) and (5.11) now prove Proposition 5.1. □

6. Estimates from below

Before stating our results concerning lower bounds on semigroup and resolvent differences, we present a version of Theorem 3.1 in the special case of the halfspace, which can easily be treated probabilistically:

Lemma 6.1: Let $\Gamma = \mathbb{H} = \{x \in \mathbb{R}^d | x_d \geq 0\}$. Then

$$\left\| Je^{-\lambda H_M} - e^{-\lambda H_\Sigma} J \right\| \leq \frac{c}{(M\lambda)^{\frac{1}{3}}} \quad \text{if } M\lambda \geq 1,$$

$$\left\| J(H_M - z)^{-1} - (H_\Sigma - z)^{-1} J \right\| \leq \frac{c}{M^{\frac{1}{3}}} \quad \text{if } M \text{ is large.}$$

Proof: Proceeding as in the general case in Section 3 and using Example 4.3 and (5.6), we have

$$\begin{aligned}\left\| Je^{-\lambda H_M} - e^{-\lambda H_\Sigma} J \right\| &\leq \sup_{x \in \Sigma} \mathrm{P}_x \{\lambda - \varepsilon < A_{\mathbb{H}} < \lambda\} + \mathrm{E}_0 \left(e^{-MT_{\varepsilon,\mathbb{H}}}\right) \\ &\leq c \left(\frac{\varepsilon}{\lambda} + \frac{1}{\sqrt{\varepsilon M}}\right).\end{aligned}$$

Letting $\varepsilon = M^\alpha \lambda^\beta$, we obtain the best result for $\alpha = -\frac{1}{3}$, $\beta = +\frac{2}{3}$. This proves the first assertion. The resolvent estimate follows by integration. □

This example clearly measures the quality of the lower bounds in the rest of this section. For the lower bounds, we return to the situation of Assumption B, i.e. V Kato class and $U = \chi_\Gamma$.

We recall that $\|\cdot\|_{\mathrm{HS}} \geq \|\cdot\|$, i.e. it is sufficient to bound the operator norm from below. Moreover, let P denote multiplication by χ_Γ in $L^2(\mathbb{R}^d)$. Employing the obvious notations for the resolvent of H_M and H_Σ respectively and suppressing the dependence on the point in the resolvent set for a moment, we have

$$R_M - J^* R_\Sigma J = PR_M + J^*(JR_M - R_\Sigma J). \tag{6.1}$$

Hence

$$\begin{aligned}\|R_M - J^* R_\Sigma J\|^2 &= \|PR_M\|^2 + \|J^*(JR_M - R_\Sigma J)\|^2_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\ &= \|PR_M\|^2 + \|JR_M - R_\Sigma J\|^2_{L^2(\mathbb{R}^d) \to L^2(\Sigma)}.\end{aligned}$$

In particular

$$\|R_M - J^* R_\Sigma J\| \geq \max\left(\|PR_M\|, \|JR_M - R_\Sigma J\|_{L^2(\mathbb{R}^d) \to L^2(\Sigma)}\right) \tag{6.2}$$

and

$$\|JR_M - R_\Sigma J\|^2 = \|R_M - J^* R_\Sigma J\|^2 - \|PR_M\|^2. \tag{6.3}$$

Lemma 6.2: In addition to Assumption B assume $\sup_x V(x) \leq b$.
If $-a \in \rho(H_0 + V)$ is sufficiently small, we have as $M \to \infty$

$$\|R_M(-a) - J^* R_\Sigma(-a) J\| \geq \|P R_M(-a)\| \geq \frac{c}{a+b+M} = O\left(\frac{1}{M}\right). \tag{6.4}$$

Proof: For any $B \subset I\!R^d$ with finite Lebesgue measure and any ball $\Gamma_0 \subset \Gamma$ we may estimate

$$\begin{aligned}
\left\|P(H_M + a)^{-1}\right\|^2 &\geq c\left\|P(H_M + a)^{-1}\chi_B\right\|^2 \\
&\geq c\int_\Gamma \mathrm{d}x \left|\int_0^\infty \mathrm{d}\lambda\ e^{-a\lambda} e^{-M\lambda} e^{-b\lambda} \int_B \mathrm{d}y\ p(x,y|\lambda)\right|^2 \\
&\geq c\int_\Gamma \mathrm{d}x \left|\int_0^\infty \mathrm{d}\lambda\ e^{-(a+b+M)\lambda} \int_{\{y\in B|\ |y-x|\leq\sqrt{\lambda}\}} \mathrm{d}y\ \frac{1}{\lambda^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2\lambda}}\right|^2 \\
&\geq c\int_{\Gamma_0} \mathrm{d}x \left|\int_0^1 \mathrm{d}\lambda\ e^{-(a+b+M)\lambda} \frac{1}{\lambda^{\frac{d}{2}}} \left|\{y\in B|\ |y-x|\leq\sqrt{\lambda}\}\right|\right|^2.
\end{aligned}$$

Choosing $B := \{y \in I\!R^d | \mathrm{dist}(y, \Gamma_0) \leq 1\}$ we note that $\frac{1}{\lambda^{\frac{d}{2}}}\left|\{y\in B|\ |y-x|\leq\sqrt{\lambda}\}\right| = \frac{1}{\lambda^{\frac{d}{2}}}\left|\{y\in I\!R^d|\ |y-x|\leq\sqrt{\lambda}\}\right| = \text{const}$ and obtain for sufficiently large M

$$\left\|P(H_M + a)^{-1}\right\|^2 \geq c\int_{\Gamma_0} \mathrm{d}x \left|\int_0^1 \mathrm{d}\lambda\ e^{-(a+b+M)\lambda}\right|^2 \geq \frac{c|\Gamma_0|}{2}\frac{1}{(a+b+M)^2}.$$

Recalling (6.2), this completes the proof of (6.4). □

Lemma 6.3: In addition to Assumption B assume $\sup_x V(x) \leq b$ and there is a $x_0 \in \partial\Gamma$ s.t. for some cones K_1, K_2 of finite height and with vertex x_0 one has $K_1 \subset \Gamma$, $K_2 \setminus \{x_0\} \subset \Sigma$.
If $-a \in \rho(H_0 + V)$ is sufficiently small, then

$$\|J R_M(-a) - R_\Sigma(-a) J\| \geq \frac{c}{(a+b+M)^{\frac{d}{2}+1}}.$$

Proof: In the course of the calculations, we will choose subsets $B \subset I\!R^d$, $\Sigma_0 \subset \Sigma$ and $\Gamma_0 \subset \Gamma$ of finite Lebesgue measure.

For any such candidate we have similar to the proof of Lemma 6.2

$$
\begin{aligned}
&\|JR_M - R_\Sigma J\|^2 \geq \\
&\geq c\int_\Sigma \mathrm{d}x \left| \int_0^\infty \mathrm{d}\lambda\ e^{-a\lambda} \int_B \mathrm{d}y \int_{\Omega^{y,\lambda}_{x,0}} \mathrm{P}^{y,\lambda}_{x,0}(\mathrm{d}\omega) \left(e^{-MT_{\lambda,\Gamma}(\omega)} - \chi_{\{T_{\lambda,\Gamma}=0\}}(\omega) \right) e^{-\int_0^\lambda V(\omega(s))\mathrm{d}s} \right|^2 \\
&= \int_\Sigma \mathrm{d}x \left| \int_0^\infty \mathrm{d}\lambda\ e^{-a\lambda} \int_B \mathrm{d}y \int_{T_{\lambda,\Gamma}>0} \mathrm{P}^{y,\lambda}_{x,0}(\mathrm{d}\omega) e^{-MT_{\lambda,\Gamma}(\omega)} e^{-\int_0^\lambda V(\omega(s))\mathrm{d}s} \right|^2 \\
&\geq c\int_\Sigma \mathrm{d}x \left| \int_0^\infty \mathrm{d}\lambda\ e^{-(a+b+M)\lambda} \int_B \mathrm{d}y\ \mathrm{P}^{y,\lambda}_{x,0} \left\{ \omega \in \Omega^{y,\lambda}_{x,0} \middle| \omega\left(\frac{\lambda}{2}\right) \in \Gamma \right\} \right|^2 \\
&\geq c\int_{\Sigma_0} \mathrm{d}x \left| \int_0^1 \mathrm{d}\lambda\ e^{-(a+b+M)\lambda} \int_B \mathrm{d}y \int_{\Gamma_0} \mathrm{d}u\ p\left(x,u \middle| \frac{\lambda}{2}\right) p\left(u,y \middle| \frac{\lambda}{2}\right) \right|^2 .
\end{aligned}
$$

Now we choose $B = \{y \in I\!R^d | \mathrm{dist}(y,\Gamma_0) \leq 1\}$ and obtain as in the proof of Lemma 6.2

$$
\begin{aligned}
\|JR_M - R_\Sigma J\|^2 &\geq c\int_{\Sigma_0} \mathrm{d}x \left| \int_0^1 \mathrm{d}\lambda\ e^{-(a+b+M)\lambda} \int_{\Gamma_0} \mathrm{d}u\ p\left(x,u \middle| \frac{\lambda}{2}\right) \right|^2 \\
&\geq c\left(\int_{\Sigma_0} \mathrm{d}x \int_0^1 \mathrm{d}\lambda\ e^{-(a+b+M)\lambda} \int_{\Gamma_0} \mathrm{d}u\ p\left(x,u \middle| \frac{\lambda}{2}\right) \right)^2 \\
&= c\left(\int_0^1 \mathrm{d}\lambda\ e^{-(a+b+M)\lambda} \int_{\Sigma_0} \mathrm{d}x \int_{\Gamma_0} \mathrm{d}u\ p\left(x,u \middle| \frac{\lambda}{2}\right) \right)^2 .
\end{aligned}
$$

In principle, we let Σ_0, Γ_0 be the cones in the assumption on $\partial\Gamma$. In order to estimate $p(x,u|\frac{\lambda}{2})$ appropriately from below, we rather integrate over the λ-dependent sets

$$
\begin{aligned}
\Sigma_0(\lambda) &= \left\{ x \in \Sigma_0 \middle| \ |x - x_0| \leq \frac{\sqrt{\lambda}}{2} \right\}, \\
\Gamma_0(\lambda) &= \left\{ u \in \Gamma_0 \middle| \ |u - x_0| \leq \frac{\sqrt{\lambda}}{2} \right\}.
\end{aligned}
$$

Then for any such u, x, we have $p(x,u|\frac{\lambda}{2}) \geq c\lambda^{-\frac{d}{2}} e^{-1}$. Hence for large M

$$
\begin{aligned}
\|JR_M - R_\Sigma J\| &\geq c\int_0^1 \mathrm{d}\lambda\ e^{-(a+b+M)\lambda} \cdot \lambda^{\frac{d}{2}} \\
&= \frac{c}{(a+b+M)^{\frac{d}{2}+1}} \int_0^{a+b+M} \mathrm{d}s\ e^{-s} s^{\frac{d}{2}} \\
&\geq \frac{c}{(a+b+M)^{\frac{d}{2}+1}} \\
&\geq \frac{c}{M^{\frac{d}{2}+1}}. \ \Box
\end{aligned}
$$

Lemma 6.4: In addition to Assumption B assume $\sup_x V(x) \leq b$ and there is a cone K of finite height h with vertex $x_0 \in \partial\Gamma$ s.t. $K \subset \Gamma$. Then

$$\left.\begin{array}{r} \left\|e^{-\lambda H_M} - J^* e^{-\lambda H_\Sigma} J\right\| \\ \left\|P e^{-\lambda(H_0+V+MP)}\right\| \\ \left\|J e^{-\lambda H_M} - e^{-\lambda H_\Sigma} J\right\| \end{array}\right\} \geq \left\{ \begin{array}{ll} c e^{-2M\lambda}, & \lambda \leq h^2, \\ \frac{c}{\lambda^{\frac{d}{2}}} e^{-2M\lambda}, & \lambda \geq h^2. \end{array}\right.$$

Proof: Since the proofs of these estimates are similar to one another and to the preceding Lemmas, we only give some details concerning the first semigroup difference:

$$\begin{aligned} \left\|e^{-\lambda H_M} - J^* e^{-\lambda H_\Sigma} J\right\|^2 &\geq c\left\|\left(e^{-\lambda H_M} - J^* e^{-\lambda H_\Sigma} J\right)\chi_K\right\|^2 \\ &= c \int_{\mathbb{R}^d} \mathrm{d}x \left| \int_{T_{\lambda,\Gamma}>0} \mathrm{P}_x(\mathrm{d}\omega) e^{-\int_0^\lambda V(\omega(s))\mathrm{d}s} e^{-MT_{\lambda,\Gamma}} \chi_K(\omega(\lambda))\right|^2 \\ &\geq c e^{-\lambda b} e^{-M\lambda} \int_{\mathbb{R}^d} \mathrm{d}x \left| \int_{\omega(\lambda)\in\Gamma} \mathrm{P}_x(\mathrm{d}\omega)\chi_K(\omega(\lambda))\right|^2 \\ &\geq c e^{-(M+b)\lambda} \int_{\mathbb{R}^d} \mathrm{d}x \left| \int_K \mathrm{d}y\, p(x,y|\lambda)\right|^2 \\ &= c e^{-(M+b)\lambda} \int_K \mathrm{d}y \int_K \mathrm{d}u\, p(u,y|2\lambda). \end{aligned}$$

Thus

$$\left\|e^{-\lambda H_M} - J^* e^{-\lambda H_\Sigma} J\right\|^2 \geq \left\{ \begin{array}{ll} \frac{c}{\lambda^{\frac{d}{2}}} e^{-(M+b)\lambda} \int_K \mathrm{d}y \int_K \mathrm{d}u e^{-\frac{|y-u|^2}{2h}}, & \lambda \geq h^2, \\ c e^{-(M+b)\lambda} \int_K \mathrm{d}y \int_{u\in K; |u-y|\leq\sqrt{\lambda}} \mathrm{d}u\, p(u,y|2\lambda), & \lambda \leq h^2, \end{array}\right.$$

completing the proof of the first assertion (cf. proof of Lemma 6.2). □

7. Applications to the semiclassical limit

The large coupling limit is strongly related to the semiclassical limit. However, these two limits are not equivalent.
We set $M \to \infty$ and $\hbar \to 0$ for

$$H_{\hbar,M} = \hbar^2 H_0 + MU, \tag{7.1}$$

where H_0, U are given as in Assumption A.
We will frequently use the following projection P in $L^2(\mathbb{R}^d)$:

$$(Pf)(x) = \chi_\Gamma(x) f(x), \quad f \in L^2(\mathbb{R}^d). \tag{7.2}$$

Theorem 7.1: In addition to Assumption A, let $V \equiv 0$, $\sup_x U(x) = C$, $\operatorname{supp} U = \Gamma$, $|\Gamma| < \infty$. Then

$$\lim_{M\to\infty} \|Pe^{-tH_{\hbar,M}}|f|\| = 0, \tag{7.3}$$

but

$$\lim_{\hbar\to 0} \|Pe^{-tH_{\hbar,M}}|f|\| \geq e^{-MtC}\|P|f|\|. \tag{7.4}$$

Proof: In order to prove (7.3), it is convenient to use the Hilbert Schmidt norm:

$$\begin{aligned}
\|Pe^{-tH_{\hbar,M}}\|_{\mathrm{HS}}^2 &= \int_\Gamma \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \left| e^{-t\hbar^2(H_0 + \frac{M}{\hbar^2}U)}(x,y)\right|^2 \\
&= \int_\Gamma \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \left| \int_{\Omega_{x,0}^{y,t\hbar^2}} \mathrm{P}_{x,0}^{y,t\hbar^2}(\mathrm{d}\omega) e^{-\frac{M}{\hbar^2}\int_0^{t\hbar^2} U(\omega(s))\mathrm{d}s}\right|^2 \\
&\leq \frac{1}{(2\pi t\hbar^2)^{\frac{d}{2}}} \int_\Gamma \mathrm{d}x \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-\frac{2M}{\hbar^2}\int_0^{t\hbar^2} U(\omega(s))\mathrm{d}s}.
\end{aligned}$$

Now we choose sets $\Gamma_n \subset \Gamma_{n+1} \subset \Gamma$ with a boundary as regular as $\partial\Gamma$, $\operatorname{dist}(x,\Sigma) > 0$ for any $x \in \partial\Gamma_n$ and $|\Gamma_n \setminus \Gamma| \to 0$ as $n \to \infty$. Then $x \in \Gamma_n$ implies $\int_0^{t\hbar^2} U(\omega(s))\mathrm{d}s > 0$ for any $t > 0$, $\omega \in \Omega_x$.
By dominated convergence we infer

$$\int_{\Gamma_n} \mathrm{d}x \int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-\frac{2M}{\hbar^2}\int_0^{t\hbar^2} U(\omega(s))\mathrm{d}s} \xrightarrow[M\to\infty]{} 0.$$

Hence (7.3) by letting $n \to \infty$ and using that the integrand is bounded.

On the other hand, (7.4) is an immediate consequence of

$$\left\|Pe^{-tH_{\hbar,M}}|f|\right\|^2 = \int_\Gamma \mathrm{d}x \left|\int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega) e^{-\frac{M}{\hbar^2}\int_0^{t\hbar^2} U(\omega(s))\mathrm{d}s} |f(\omega(t\hbar^2))|\right|^2$$
$$\geq e^{-2MtC}\left\|Pe^{-t\hbar^2 H_0}|f|\right\|^2 . \text{Kasten}$$

On the other hand, the large coupling asymptotics may improve estimates on the semiclassical limit, as the following result shows:

Theorem 7.2: Let $0 \leq U(x) \leq C$, $\operatorname{supp} U = \Gamma$.
Assume that there is a $\gamma > 0$ s.t. as $M \to \infty$

(7.5) $$\left\|J(H_0 + MU - z)^{-1} - ((H_0)_\Sigma - z)^{-1}J\right\| = O\left(M^{-\gamma}\right)$$

and

(7.6) $$\left\|P(H_0 + MU - z)^{-1}\right\| = O\left(M^{-\gamma}\right).$$

Then we obtain for the semiclassical limit

(7.7) $$\left\|J\left(\hbar^2 H_0 + U - z\right)^{-1} - \left(\hbar^2(H_0)_\Sigma - z\right)^{-1}J\right\| = O\left(\hbar^{-2+2\gamma}\right)$$

for any $z \in \rho((H_0)_\Sigma) = \mathbb{C} \setminus [0,\infty)$.

Proof: Writing

(7.8) $$J\left(\hbar^2 H_0 + U - z\right)^{-1} - \left(\hbar^2(H_0)_\Sigma - z\right)^{-1}J = \frac{1}{\hbar^2}\left[J\left(H_0 + \frac{1}{\hbar^2}U - \frac{z}{\hbar^2}\right)^{-1} - \left((H_0)_\Sigma - \frac{z}{\hbar^2}\right)^{-1}J\right],$$

we can separate the high energy limit:

(7.9) $$\begin{aligned}\left[J\left(H_0 + \frac{1}{\hbar^2}U - \frac{z}{\hbar^2}\right)^{-1} - \left((H_0)_\Sigma - \frac{z}{\hbar^2}\right)^{-1}J\right] &= J\left[1 + \left(\frac{z}{\hbar^2} - z\right)\left(H_0 + U - \frac{z}{\hbar^2}\right)^{-1}\right] \times \\ \times \left[(H_0 + U - z)^{-1} - J^*((H_0)_\Sigma - z)^{-1}J\right] &\times \left[1 + \left(\frac{z}{\hbar^2} - z\right)J^*\left((H_0)_\Sigma - \frac{z}{\hbar^2}\right)^{-1}J\right].\end{aligned}$$

Since $\left|\frac{z}{\hbar^2} - z\right|\left\|\left(H_0 + U - \frac{z}{\hbar^2}\right)^{-1}\right\| \leq \text{const} \cdot |z|$, the asymptotics for (7.9) are given by the asymptotics of

$$\begin{aligned}&\left\|\left(H_0 + \frac{1}{\hbar^2}U - z\right)^{-1} - J^*((H_0)_\Sigma - z)^{-1}J\right\| \leq \\ &\leq \left\|P\left(H_0 + \frac{1}{\hbar^2}U - z\right)^{-1}\right\| + \left\|J\left(H_0 + \frac{1}{\hbar^2}U - z\right)^{-1} - ((H_0)_\Sigma - z)^{-1}J\right\| \\ &= O\left(\hbar^{2\gamma}\right),\end{aligned}$$

where we used $1-P=J^*J$. Now (7.8) completes the proof. □

In order to apply Theorem 7.2 it is indispensable to determine not only the convergence rate of the resolvent difference but also of $\left\|P(H_0+MU-z)^{-1}\right\|$ in M.
We will not go into detail here, because this term is not as interesting in its own right as the resolvent or semigroup difference. Furthermore, it leads to very similar considerations. For example,

$$\begin{aligned}\left\|P(H_0+MU-z)^{-1}f\right\|^2 &= \int_\Gamma \mathrm{d}x \left|\int_0^\infty \mathrm{d}\lambda\ e^{-z\lambda}\left(e^{-\lambda(H_0+MU)}f\right)(x)\right|^2 \\ &\le c\int_\Gamma \mathrm{d}x \int_0^\infty \mathrm{d}\lambda\ e^{-\mathrm{Re}\,z\cdot\lambda}\left|\left(e^{-\lambda(H_0+MU)}f\right)(x)\right|^2 \\ &= c\int_0^\infty \mathrm{d}\lambda\ e^{-\mathrm{Re}\,z\cdot\lambda}\left\|Pe^{-\lambda(H_0+MU)}\right\|^2\|f\|^2.\end{aligned}$$

Hence the rate of $\left\|P(H_0+MU-z)^{-1}\right\|$ with respect to M is determined by the one of $\left\|Pe^{-\lambda(H_0+MU)}\right\|$.
Due to

$$\left\|Pe^{-\lambda(H_0+MU)}\right\| \le \sup_{x\in\Gamma}\int_{\Omega_x} \mathrm{P}_x(\mathrm{d}\omega)e^{-M\int_0^\lambda U(\omega(s))\mathrm{d}s},$$

we end up in the situation of Section 3 (see e.g. (3.1), where the restriction $T_{\lambda,\Gamma}>0$ can be neglected).

Theorem 7.3: In addition to Assumption B, assume $V\equiv 0$. Then

$$\left\|\left(\hbar^2H_0+U+a\right)^{-1} - J^*\left(\hbar^2(H_0)_\Sigma+a\right)^{-1}J\right\| \ge c\hbar^4 \tag{7.10}$$

for any $a<0$.

Proof: Writing H_Σ instead of $(H_0)_\Sigma$ for convenience, Hilbert's identity yields

$$\begin{aligned}\left(H_0+\frac{1}{\hbar^2}U+a\right)^{-1} &- J^*(H_\Sigma+a)^{-1}J \\ &= \hbar^2\left[\left(\hbar^2H_0+U+\hbar^2a\right)^{-1} - J^*\left(\hbar^2H_\Sigma+\hbar^2a\right)^{-1}J\right] \\ &= \hbar^2\left[1+(a-\hbar^2a)\left(\hbar^2H_0+U+\hbar^2a\right)^{-1}\right]\times \\ &\qquad\times\left[\left(\hbar^2H_0+U+a\right)^{-1} - J^*\left(\hbar^2H_\Sigma+a\right)^{-1}J\right]\times \\ &\qquad\times\left[1+(a-\hbar^2a)J^*\left(\hbar^2H_\Sigma+\hbar^2a\right)^{-1}J\right].\end{aligned}$$

Since for small $\hbar$ e.g.

$$\left\|1+(a-\hbar^2a)\left(\hbar^2H_0+U+\hbar^2a\right)^{-1}\right\| \le 1+|a|\cdot\left|1-\hbar^2\right|\cdot\frac{1}{\hbar^2}\cdot\frac{1}{|a|} \le \frac{c}{\hbar^2},$$

this implies

$$\left\| \left(H_0 + \frac{1}{\hbar^2} U + a \right)^{-1} - J^*(H_\Sigma + a)^{-1} J \right\| \leq$$

$$\leq c\hbar^2 \cdot \frac{1}{\hbar^2} \cdot \frac{1}{\hbar^2} \left\| \left(\hbar^2 H_0 + U + a\right)^{-1} - J^* \left(\hbar^2 H_\Sigma + a\right)^{-1} J \right\|.$$

Now the l.h.s. is bounded below via (6.2). The first alternative for the maximum in (6.2), i.e. an application of Lemma 6.2 rather than Lemma 6.3, yields the better result, namely (7.10). □

Theorem 7.4: In addition to Assumption B, assume $V \equiv 0$. If $E \in \mathbb{R}$, $\gamma > 0$, then

$$\left\| \langle x \rangle^{-\gamma} \left[\left(\hbar^2 H_0 + U - E \pm i0\right)^{-1} - J^* \left(\hbar^2 H_\Sigma - E \pm i0\right)^{-1} J \right] \langle x \rangle^{-\gamma} \right\| \geq c\hbar^4$$

for small $\hbar$, whenever the expression in $\|\cdot\|$ makes sense, and where $\langle x \rangle = \sqrt{1 + |x|^2}$, $H_\Sigma = (H_0)_\Sigma$.

Proof: Since

$$\left(H_0 + \frac{1}{\hbar^2} U + a \right)^{-1} - J^*(H_\Sigma + a)^{-1} J =$$

$$= \hbar^2 \left[\left(\hbar^2 H_0 + U + \hbar^2 a\right)^{-1} - J^* \left(\hbar^2 H_\Sigma + \hbar^2 a\right)^{-1} J \right]$$

$$= \hbar^2 \left[1 + (-\hbar^2 a - E \pm i\varepsilon) \left(\hbar^2 H_0 + U + \hbar^2 a\right)^{-1} \right] \times$$

$$\times \left[\left(\hbar^2 H_0 + U - E \pm i\varepsilon\right)^{-1} - J^* \left(\hbar^2 H_\Sigma - E \pm i\varepsilon\right)^{-1} J \right] \times$$

$$\times \left[1 + (-\hbar^2 a - E \pm i\varepsilon) \left(\hbar^2 H_\Sigma + \hbar^2 a\right)^{-1} \right],$$

we have

$$\left\| \langle x \rangle^{-\gamma} \left[\left(H_0 + \frac{1}{\hbar^2} U + a \right)^{-1} - J^*(H_\Sigma + a)^{-1} J \right] \langle x \rangle^{-\gamma} \right\| \leq$$

$$\leq \hbar^2 \left[1 + \left|\hbar^2 a + E\right| \cdot \left\| \langle x \rangle^{-\gamma} \left(\hbar^2 H_0 + U + \hbar^2 a\right)^{-1} \langle x \rangle^{\gamma} \right\| \right] \times$$

$$\times \lim_{\varepsilon \to 0} \left\| \langle x \rangle^{-\gamma} \left\{ \left(\hbar^2 H_0 + U - E \pm i\varepsilon\right)^{-1} - J^* \left(\hbar^2 H_\Sigma - E \pm i\varepsilon\right)^{-1} J \right\} \langle x \rangle^{-\gamma} \right\| \times$$

$$\times \left[1 + \left|\hbar^2 a + E\right| \cdot \left\| \langle x \rangle^{\gamma} \left(\hbar^2 H_\Sigma + \hbar^2 a\right)^{-1} \langle x \rangle^{-\gamma} \right\| \right]$$

$$\leq \hbar^2 \left[1 + \frac{\left|\hbar^2 a + E\right|}{\hbar^2} \left\| \langle x \rangle^{-\gamma} \left(H_0 + \frac{1}{\hbar^2} U + a \right)^{-1} \langle x \rangle^{\gamma} \right\| \right] \times$$

$$\times \left\| \langle x \rangle^{-\gamma} \left\{ \left(\hbar^2 H_0 + U - E \pm i0\right)^{-1} - J^* \left(\hbar^2 H_\Sigma - E \pm i0\right)^{-1} J \right\} \langle x \rangle^{-\gamma} \right\| \times$$

$$\times \left[1 + \frac{\left|\hbar^2 a + E\right|}{\hbar^2} \left\| \langle x \rangle^{\gamma} (H_\Sigma + a)^{-1} \langle x \rangle^{-\gamma} \right\| \right].$$

Now

$$\left\| \langle x\rangle^{-\gamma}\left(H_0+\frac{1}{\hbar^2}U+a\right)^{-1}\langle x\rangle^{\gamma}\right\| \le$$

$$\le \sup_x \frac{1}{(1+|x|^2)^{\frac{\gamma}{2}}}\int_{\mathbb{R}^d} dy \int_0^\infty d\lambda\, e^{-\lambda a}\int_{\Omega^{y,\lambda}_{x,0}} \mathrm{P}^{y,\lambda}_{x,0}(d\omega) e^{-\frac{1}{\hbar^2}\int_0^\lambda U(\omega(s))ds}(1+|y|^2)^{\frac{\gamma}{2}}$$

$$\le \sup_x \frac{1}{(1+|x|^2)^{\frac{\gamma}{2}}}\int_{\mathbb{R}^d} dy \int_0^\infty d\lambda\, e^{-\lambda a}\frac{1}{\lambda^{\frac{d}{2}}}e^{-\frac{|x-y|^2}{2\lambda}}(1+|y|^2)^{\frac{\gamma}{2}}$$

$$\le \sup_x \frac{1}{(1+|x|^2)^{\frac{\gamma}{2}}}\int_0^\infty d\lambda\, \frac{e^{-\lambda a}}{\lambda^{\frac{d}{2}}}\int_{\mathbb{R}^d} du\, e^{-\frac{|u|^2}{2\lambda}}(1+|x-u|^2)^{\frac{\gamma}{2}}$$

$$\le c\int_0^\infty d\lambda\, \frac{e^{-\lambda a}}{\lambda^{\frac{d}{2}}}\int_{\mathbb{R}^d} du\, e^{-\frac{|u|^2}{2\lambda}}(1+|u|^2)^{\frac{\gamma}{2}}$$

$$= c\int_0^\infty d\lambda\, e^{-\lambda a}\lambda^{\frac{d}{2}-1}\int_0^\infty ds\, e^{-\frac{s^2}{2}}(1+\lambda^2 s^2)^{\frac{\gamma}{2}}s^{d-1}$$

$$< \infty$$

and the corresponding calculation for $\left\|\langle x\rangle^{\gamma}(H_\Sigma+a)^{-1}\langle x\rangle^{-\gamma}\right\|$ (replace "$e^{-\frac{1}{\hbar^2}\int_0^\lambda U(\omega(s))ds}$" by "$\chi_{\{T_{\lambda,\Gamma}=0\}}(\omega)$") show that

$$\left\| \langle x\rangle^{-\gamma}\left(H_0+\frac{1}{\hbar^2}U+a\right)^{-1}\langle x\rangle^{\gamma}\right\|,$$

$$\left\|\langle x\rangle^{\gamma}(H_\Sigma+a)^{-1}\langle x\rangle^{-\gamma}\right\|$$

are bounded uniformly in $\hbar$. Thus we have

$$\left\| \langle x\rangle^{-\gamma}\left[\left(H_0+\frac{1}{\hbar^2}U+a\right)^{-1} - J^*(H_\Sigma+a)^{-1}J\right]\langle x\rangle^{-\gamma}\right\| \le$$

$$\le \frac{c(E)}{\hbar^2}\left\| \langle x\rangle^{-\gamma}\left\{\left(\hbar^2 H_0+U-E\pm i0\right)^{-1} - J^*\left(\hbar^2 H_\Sigma - E\pm i0\right)^{-1}J\right\}\langle x\rangle^{-\gamma}\right\|.$$

An imitation of the proof of Lemma 6.2 finally shows that

$$\left\| \langle x\rangle^{-\gamma}\left[\left(H_0+\frac{1}{\hbar^2}U+a\right)^{-1} - J^*(H_\Sigma+a)^{-1}J\right]\langle x\rangle^{-\gamma}\right\| \ge c\hbar^2$$

(note that Γ_0, B in the proof of Lemma 6.2 are *bounded*). □

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