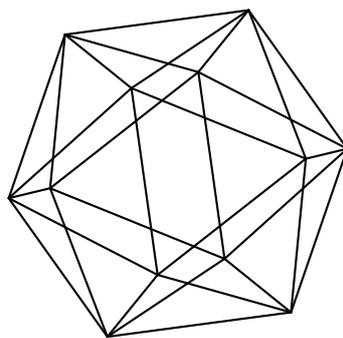


# Max-Planck-Institut für Mathematik Bonn

*Gauss Skizze* - Operad and monodromy on semisimple  
Frobenius manifolds

by

Noémie C. Combe





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# GAUSS SKIZZE - OPERAD AND MONODROMY ON SEMISIMPLE FROBENIUS MANIFOLDS

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I wish to thank the Max Planck Institute for Mathematics for supporting my research. All my gratitude goes towards Yuri Ivanovitch Manin, with whom I had amazing discussions. Yuri Ivanovitch suggested to name the *Gauss-Skizze* - a certain type of graphs, introduced by Gauss - by *dessins du vieillard*, in contrast to the Grothendieck's *dessins d'enfant*: similarities between those two objects being striking. I wish to thank Pascal Lambrechts for having introduced me to this cosimplicial model, to Hanna Nencka for valuable remarks, and also to Felix Wierstra and José Manuel Moreno for discussions around tea time at the Max Planck Institute, about the braid operad. I wish also to thank Hans Henrik Rugh for many important discussions concerning the monodromy and the intertwining of roots; to Leila Schneps and Pierre Lochak for discussions on the *dessins d'enfant*.

ABSTRACT. In this paper we consider the configuration space of  $n$ -marked points in the complex plane and introduce the *Gauss Skizze* -operad, which is homotopically equivalent to the little disk operad. This construction leads us to define a *Gauss Skizze*-braid operad, which is an operad equivalent to the classical Artin braid operad. This is done by merging together completely different approaches : a cosimplicial model for configuration spaces and one of the constructions of Frobenius manifolds using classical complex space of polynomials, and the monodromy on the Frobenius manifold, around the discriminant locus. Our construction is done using a special category of graphs. We introduce a new way of understanding the little disk operad and related algebraic structures. Through this work, we point out hidden geometrical similarities between the approaches given for  $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$  and  $\mathcal{M}_{0,n}(\mathbb{C})$ .

## 1. INTRODUCTION

In this paper we introduce the *Gauss Skizze* operad (Gs-op, in short) for the space of configurations of  $n$  marked points in the complex plane. The Gs-op is a topological operad, which is homotopically equivalent to little disc operad [5] and which gives the possibility of having information concerning the critical values and critical points of polynomials of high degree. There exist important similarities between this *Gauss-Skizze* and the Grothendieck *dessins d'enfant* [28]. A detailed study concerning the *Gauss-Skizze* can be found in [9, 10, 11] and are used more implicitly in [12, 13, 14].

Not only this approach establishes a more explicit bridge between the geometric results found for the real locus of the moduli space  $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$  [17, 18, 20, 29] and in the complex version  $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$  (see for example [30, 31, 16, 22]), but also gives a great deal of information about the monodromy occurring for small loops around the discriminant locus of the configuration space. The study of the monodromy on the Frobenius manifold, using the Gs-op is a construction which leads us to define the *Gauss Skizze*-braid operad being equivalent to the classical Artin braid operad.

A stepping stone towards the construction of this new Gs-op is to use the categorical language introduced in [26, 27, 34, 35, 43, 44] for configuration spaces of marked points in the Euclidean space. This approach fits perfectly the language used in one of the constructions of Frobenius manifolds [39] and which was extensively used by Dubrovin [19] and Manin [39], as well as Arnold and his school in the seventies [1, 2, 3, 8, 47, 48, 49], to study configuration spaces and marked points as a space of complex polynomials.

Merging those two approaches, and using the technique relying on Gauss drawings (see [23, 24, 25, 4]), we show the existence of a new point of view on the little disc operad and related algebraic structures.

First, we recall the cosimplicial model from [43, 44, 34, 35], constructed from the configuration spaces. Historically, the utility of this model was to consider the topology of

spaces of knots (note that for that approach the real dimension of the affine space is of dimension greater or equal to 4). We adopt this approach, however in a completely different perspective: the real dimension of the affine space being 2 there, are no relations to spaces of knots.

Second, we introduce our tool: the category of special decorated graphs and operations on those graphs. Third, we explain how, by using this tool, we can develop a new interpretation of the little 2-disc operad and use it as a bridge connecting results for  $M_{0,n}(\mathbb{R})$  and the complex version  $M_{0,n}(\mathbb{C})$ . We wish to point out some hidden relations between the real and complex configuration spaces (resp. moduli spaces) of  $n$  marked points on the genus 0 curve; as well as to establish deeper bridges between the known results for  $\overline{M}_{0,n}(\mathbb{R})$  and  $\overline{M}_{0,n}(\mathbb{C})$ .

**1.1. Cosimplicial model for configuration spaces.** The consideration of so-called configuration spaces of points on the projective complex (or real) line, emerged from the interest of physicists in Conformal Field theory. A configuration space  $\text{Conf}_d(\mathbb{C})$  of  $d$  marked points on the complex plane is :

$$\text{Conf}(\mathbb{C}) = \{(x_1, \dots, x_d) \in \mathbb{C}^d \mid x_i \neq x_j\}.$$

One can investigate those configuration spaces using cosimplicial models. We borrow the cosimplicial model which has been constructed for the space of long knots  $\mathbb{R} \hookrightarrow \mathbb{R}^d$ , for  $d \geq 4$  (presented in [43, 44, 35]) and apply it to the more specific case of configurations spaces with marked points on the complex plane. We briefly recall some general facts about cosimplicial models. For more details we recommend [6] and [7].

A cosimplicial object over a category  $C$  consists of a diagram in  $C$  indexed by the category  $\Delta$  of finite ordinal numbers. It is formed from objects  $X^n \in C$  for  $n \geq 0$  with *coface* maps  $d^i : X^n \rightarrow X^{n+1}$  for  $1 \leq i \leq n$  and *codegeneracy* maps  $s_j : X^{n+1} \rightarrow X^n$ ,  $1 \leq j \leq n$  satisfying the usual cosimplicial identities ([7] p. 267). To a cosimplicial object over  $C^{op}$ , corresponds a cosimplicial object over  $C$ .

By a simplicial model category, we mean a model category  $C$  which is also a simplicial category satisfying axioms:

- (1) the object  $X \otimes K$  and  $\text{hom}(K, X)$  exist for each  $X \in C$  and each finite  $K \in S$ .
- (2) if  $i : A \rightarrow B \in C$  is a cofibration and  $p : X \rightarrow Y \in C$  is a fibration, then:

$$\text{map}(B, X) \rightarrow \text{map}(A, X) \times_{\text{map}(A, Y)} \text{map}(B, Y),$$

is a fibration in  $S$  which is trivial if either  $i$  or  $p$  are trivial.

As is mentioned in [6], when  $C$  is a model category, there is an induced model category structure on the category of cosimplicial objects over  $C$ . By simplicial category, is meant a category  $C$  enriched over  $S$ , and we write  $\text{map}(X, Y) \in S$  for the mapping space of  $X, Y \in C$ .

Let  $M$  be a smooth manifold with non - empty boundary (2-disc in our case). The  $n$ -th entry of the considered cosimplicial model is given by the Cartesian product  $M^n$ . Fix two unit tangent vectors  $\alpha \in UTM$  (resp.  $\beta \in UTM$ ), located on the boundary of  $M$  pointing inward (resp. outward).

We consider configurations spaces  $\text{Conf}(M)$ . Elements of  $UTM$ , the unit tangent bundle to  $M$ , are denoted by  $\xi = (x, v)$ , with  $x \in M$  and  $v \in UT_x M$  with  $\|v\| = 1$ .

We consider the configuration space  $\text{Conf}_q(M)$ , consisting of  $(q + 2)$ -tuples,

$$\xi_0 = (x_0, v_0), \xi_1 = (x_1, v_1), \dots, \xi_{q+1} = (x_{q+1}, v_{q+1}) \in (UTM)^{q+2}$$

such that  $\xi_0 = \alpha, \xi_{q+1} = \beta$  and  $x_i \neq x_j$ .

For our purposes, it is necessary to take a Fulton-MacPherson compactification  $\overline{\text{Conf}}(M)$ . The idea of the construction is that elements of  $\overline{\text{Conf}}(M)$  consist of some ‘‘virtual’’ configurations, where the marked points  $x_i$  and  $x_j$  may be equal - in which case, some extra data serves to distinguish two points infinitesimally close, as explained in the definitions 4.1 and 4.12 of [43]).

Very important to us, in the construction for the doubling maps for this cosimplicial model. Let  $0 \leq i \leq q$ :

$$(1) \quad d^i : \overline{\text{Conf}}_{q-1}(M) \rightarrow \overline{\text{Conf}}_q(M)$$

$$(\xi_0, \dots, \xi_i, \dots, \xi_q) \mapsto (\xi_0, \dots, \xi_i, \xi'_i, \dots, \xi_q),$$

where  $\xi'_i = (x'_i, v_i)$  with  $x_i = x'_i$  but infinitesimally  $x'_i - x_i = v_i$ .

Whereas, the forgetting maps are defined as follows:

$$(2) \quad s_j : \overline{\text{Conf}}_q(M) \rightarrow \overline{\text{Conf}}_{q-1}(M),$$

$$(\xi_0, \dots, \xi_i, \dots, \xi_q) \mapsto (\xi_0, \dots, \hat{\xi}_i, \dots, \xi_q),$$

where  $\hat{\xi}_i$  means that this point is ‘‘forgotten’’.

**Remark 1.** *As was pointed out in [43] definition 6, by defining forgetting maps  $s_j$ , a natural guess would be that this gives a cosimplicial space with  $d_i$  as cofaces and  $s_j$  as codegeneracies. However, this is not true: certain cosimplicial identities are not satisfied. To remedy, it is possible to replace  $\overline{\text{Conf}}_q(M)$  by a homotopy equivalent quotient, for which the induced map satisfy the cosimplicial identities. This latter space is denoted by  $C' \langle [M, \partial] \rangle$ . The cosimplicial space is given by*

$$X_{\bullet} \{C' \langle [M, \partial] \rangle, d_i, s_j\}_{q \geq 0}.$$

Two important tools are used in the cosimplicial model: the maps  $\theta$  and  $\delta$ . We denote by  $S$  a finite set of points. Let  $x$  be a given configuration.

(1) Take two distinct elements  $a$  and  $b$  in  $S$ :

$$\theta_{a,b} : \text{Conf}_S(M) \rightarrow \mathbb{S}^1$$

$$\theta_{a,b} : x \mapsto \frac{x(b) - x(a)}{\|x(b) - x(a)\|}.$$

This map gives the direction between vectors given two points of the configuration  $x$ .

(2) For three distinct elements  $a, b, c$  in  $S$ , we define their relative distance map:

$$\delta_{a,b,c} : \text{Conf}_S(M) \rightarrow [0, +\infty)$$

$$x \mapsto \frac{\|x(a) - x(b)\|}{\|x(a) - x(c)\|}.$$

For three different points  $a, b, c$  in  $S$  and  $x \in \overline{\text{Conf}}_S$  we adopt the notation:

$$x(a) \cong x(b) \text{ rel } x(c),$$

if their relative distance is zero.

**Remark 2.** *Note that up to translations and dilatation, any configuration can be recovered from direction  $\theta$  and the relative distances  $\delta_{a,b,c}$ .*

**Proposition 1.** *Let  $x$  be a given point in the Fulton-MacPherson compactified configuration space. We have the following equivalent conditions:*

- (1)  $x$  lies in the boundary of the configurations space,
- (2) (if and only if) there exist three points  $a, b, c$  in  $S$ , such that  $a$  and  $b$  are relatively close, with respect to  $c$ . This is denoted by  $x(a) \cong x(b) \text{ rel } x(c)$ .

**1.2. Configuration spaces and moduli spaces.** Fadell and Neuwirth [21] were first to investigate these spaces, in a formal manner.

There are many different ways of looking at those configuration spaces. One way of looking at configuration spaces is by using their relation to moduli spaces. This relation is first given by noticing that :

$$\text{Conf}_n(\mathbb{C}) \cong \text{Conf}_{n+1}(\mathbb{P}).$$

Note that the points are non-coincident. So, both spaces are non compact and taking the quotient of those spaces by the action of  $PGL_2(\mathbb{C})$  (which is also non-compact) is problematic: it leads to a (natural) compactification of the spaces:

$$\overline{\mathcal{M}}_{0,n} \cong \overline{\text{Conf}}_{n+1}(\mathbb{P})/PGL_2(\mathbb{C})$$

Different approaches to the compactification have been given (Deligne-Mumford, Keel, Knudsen, Fulton-MacPherson, Kapranov [16, 22, 29, 31]) but for our purposes we will focus only on the compactification of Fulton-MacPherson. For the genus 0 case, the different methods of compactifications were shown to be equivalent.

We wish to point out some hidden relations between the real and complex configuration spaces (resp. moduli spaces) of  $n$  marked points on the genus 0 curve; as well as to establish deeper bridges between the known results for  $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$  and  $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ .

Let us start by recalling a very basic step, given by the following commutative diagram.

$$\begin{array}{ccc} \text{Conf}(\mathbb{R}) & \xrightarrow{\phi} & \text{Conf}(\mathbb{C}) \\ \downarrow & & \downarrow \psi \\ \overline{\mathcal{M}}_{0,*+1}(\mathbb{R}) & \xrightarrow{\eta} & \mathcal{M}_{0,*+1}(\mathbb{C}) \end{array}$$

The upper right corner is the homotopy equivalence of the little discs operad; whereas the space in the upper left corner is the classical  $A_\infty$  operad  $\{\Sigma_* \times K_{*-1}\}$ . The moduli space  $\overline{\mathcal{M}}_{0,k}(\mathbb{R})$  is described by the surjective map:

$$\Sigma_n \times_{D_n} K_{n-1} \rightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{R}),$$

where the symbol  $D_n$  stands for the dihedral group of order  $2n$  and  $K_{n-1}$  is the associahedron ( $D_n$  acts as a group of isometries on  $K_{n-1}$ ).

More precisely, this map is a bijection on the interior of the cells and, from Theorem 3.1.3 [15] it is known that these real moduli spaces are tessellated by the Stasheff associahedra (there are  $\frac{(n-1)!}{2}$  copies of  $K_{n-1}$  tessellating  $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ ).

For the next step, it is necessary to mention the definition of the operadic structure on this configuration space,  $\overline{\text{Conf}}_\bullet = \{\overline{\text{Conf}}_n\}_{n \geq 0}$ . We will principally be interested in the Fulton-MacPherson operad. In particular, we heavily rely on the construction explicited in Lambrechts [35]. It is mostly convenient to our approach, since it establishes a relation between the real approach  $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$  and  $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ .

This operad is homotopy equivalent to many other operads such that the Little  $N$ -balls operad (or the little  $N$ -cube operad) [5]. The space  $\overline{\text{Conf}}_n$  is a compact manifold with corners, obtained by adding a boundary to the regular (and open) manifold  $\text{Conf}_n$  (i.e. where points of  $\mathbb{C}$  do not collide). The operad of structure of  $\overline{\text{Conf}}_n$  corresponds to the inclusions of various faces of the boundary.

**1.3. Operads, weak partitions and totally ordered sets.** Consider a symmetric monoidal category  $C$ . Let  $\text{Fin}$  be the category of finite sets with the bijections. Given any subset  $S \subset S'$ , we write  $S/S' := S \setminus S' \sqcup \{*\}$ .

An operad in  $C$  is a presheaf  $\mathcal{F} : \text{Fin}^{op} \rightarrow C$  endowed with partial composition,

$$\mathcal{F}(S/S') \otimes \mathcal{F}(S) \rightarrow \mathcal{F}(S'),$$

for any  $S \subset S'$  and unit  $\eta : 1_C \rightarrow \mathcal{F} : (\{*\})$  such that equivariance, associativity axioms are satisfied [40].

1.3.1. *The little disc operad.* Boardman and Vogt have constructed a topological operad called little disc operad [5].

Consider a unit disc  $D$  in  $\mathbb{C}$  and let  $O(n)$  be a topological space.

$$O(n) = \left\{ \begin{pmatrix} z_1 & \cdots & z_n \\ r_1 & \cdots & r_n \end{pmatrix} \in \begin{pmatrix} D^n \\ \mathbb{R}_+^n \end{pmatrix} \mid \text{the discs } r_i D + z_i \text{ are disjoint subsets of } D \right\}.$$

The symmetric group  $\mathbb{S}_n$  acts on  $\mathbb{P}^n$  by permuting the discs. The product operation in this operad is defined by glueing disks.

If

$$a = \begin{pmatrix} z_1 & \cdots & z_k \\ r_1 & \cdots & r_k \end{pmatrix} \quad \text{and} \quad b_i = \begin{pmatrix} w_1^i & \cdots & w_{n_i}^i \\ s_1^i & \cdots & s_{n_i}^i \end{pmatrix},$$

then, the final output of the product operation is:

$$\begin{pmatrix} r_1 w_1^1 + z_1 & \cdots & r_1 w_{n_1}^1 + z_1 & \cdots & r_k w_1^k + z_1 & r_k w_{n_k}^k + z_k \\ r_1 s_1^1 & \cdots & r_1 s_{n_1}^1 & \cdots & r_k s_1^k & r_k s_{n_k}^k \end{pmatrix}.$$

The map from the topological spaces  $O(n)$  to the configuration space  $\text{Conf}_n(\mathbb{C})$  defined by

$$\begin{pmatrix} z_1 & \cdots & z_k \\ r_1 & \cdots & r_k \end{pmatrix} \rightarrow (z_1, \dots, z_k)$$

is a homotopy equivalence. One can easily move from those topological operads onto the braid operad (where the collection of objects are braid groups  $B_n$  given by  $\pi_1(\text{Conf}_n(\mathbb{C})) = B_n$ ). The product operation is given by cabling  $B_d \times B_{i_1} \times \dots \times B_{i_d} \rightarrow B_{i_1 + \dots + i_d}$  as composition. We will show in the next section that we have a braid operad, defined in terms of the *Gauss Skizze*-operad, in short *Gs-op*.

1.3.2. *Weak partition and totally ordered set.* We define some notions such as the weak partition, following [35]. To do this, we fix two objects: a finite set  $S$  and a weak partition  $v : S \rightarrow P$ , where  $P$  is a linearly ordered finite set.

A *weak partition of a finite set*  $S$  is a map  $v : S \rightarrow P$  between two finite sets (here  $P$  is not required yet to be a linearly ordered finite set), where for a given  $p \in P$ , the preimages  $v^{-1}(p)$  are elements of the partition.

It is not required that  $v$  is surjective, so some of the elements  $v^{-1}(p)$  are allowed to be empty. The weak partition is degenerate if  $v$  is not surjective, and non-degenerate otherwise. From now on, we will simply say “partition” instead of a non-degenerate weak partition.

The (weak) partition  $v$  is ordered if its codomain  $P$  is equipped with a linear order. The undiscrte partition is the partition  $v : S \rightarrow \{1\}$  whose only element is  $S$ . We adopt the following notations:

$$(3) \quad P^* = \{0\} \star P, A_P = v^{-1}(p), \text{ and } A_0 = P.$$

A *linearly ordered* (or a totally ordered) set is a pair  $(L, \leq)$  where  $L$  is a set and  $\leq$  is a reflexive, transitive, and antisymmetric relation on  $L$  such for any  $x, y \in L$  we have  $x \leq y$  or  $y \leq x$ . We write  $x < y$  when  $x \leq y$  and  $x \neq y$ .

Given two disjoint linearly ordered sets  $(L_1, \leq_1)$  and  $(L_2, \leq_2)$  their ordered sum is the linearly ordered set  $L_1 \star L_2 := (L_1 \cup L_2, \leq)$  such that the restriction of  $\leq$  to  $L_i$  is the given order  $\leq_i$  and such that  $x_1 \leq x_2$  when  $x_1 \in L_1$  and  $x_2 \in L_2$ .

More generally, if  $\{L_p\}_{p \in P}$  is a family of linearly ordered sets, indexed by a linearly ordered set  $P$ , its ordered sum  $\star_{p \in P} L_p$  is the disjoint union  $\sqcup_{p \in P} L_p$  equipped with linear order  $\leq$  whose restriction to each  $L_p$  is the given order on that set such that  $x < y$  when  $x \in L_p$  and  $y \in L_q$  with  $p < q$  in  $P$ .

We have:

$$(4) \quad \prod_{p \in P^*} \overline{\text{Conf}}_{A_p} = \overline{\text{Conf}}_P \times \prod_{p \in P} \overline{\text{Conf}}_{v^{-1}(p)},$$

which defines the operad structure:

$$\Phi_v : \prod_{p \in P^*} \overline{\text{Conf}}_{A_p} \rightarrow \overline{\text{Conf}}_A,$$

As a more intuitive explanation, the configuration  $x = \Phi_v((x_p)_{p \in P^*})$  is obtained by proceeding by a replacement, the  $p$ -th component  $x_0(p)$  of the configuration  $x_0 \in \overline{\text{Conf}}_P$ , for any point  $p \in P$ , by the configuration  $x_p \in \overline{\text{Conf}}_{A_p}$  made infinitesimal, see section 5.2 of [35].

The case of inserting two points - which are infinitesimally close - enter the perspectives described in [34, 35] where is defined a cosimplicial space. The  $n$ -th component  $\overline{\text{Conf}}_n$  of the cosimplicial space is some compactification of the configuration space of points in  $I \times \mathbb{R}^{d-1} = [0, 1] \times \mathbb{R}^{d-1}$ .

## 2. MONODROMY AND FROBENIUS MANIFOLDS

**2.1. Frobenius manifolds.** We start with reminding some basic notation and facts from [39].

Let us recall that an affine flat structure on the (super)manifold  $M$  is a subsheaf  $T_M^f \subset T_M$  of linear spaces of pairwise (super)commuting vector fields, such that  $T_M = O_M \otimes T_M^f$ . Sections of  $T_M^f$  are called flat vector fields.

A *Frobenius manifold* is a quadruple  $(M, T_M^f, g, A)$ , where  $M$  is a supermanifold in one of the (classical) categories;  $g$  is a flat Riemannian metric compatible with the structure  $T_M^f$ ;  $A$  is an even symmetric tensor  $A : S^3(T_M) \rightarrow O_M$ .

This data must satisfy the following conditions :

- Potentiality of  $A$ . There exists a function  $\Phi$  verifying, for any flat vector fields  $X, Y, Z$ :

$$(5) \quad A(X, Y, Z) = (XYZ)\Phi.$$

- Associativity. The symmetric tensor  $A$  together with the flat Riemannian metric  $g$ , define a unique symmetric multiplication  $\circ : T_M \otimes T_M \rightarrow T_M$  such that :

$$(6) \quad A(X, Y, Z) = g(X \circ Y, Z) = g(X, Y \circ Z).$$

This multiplication must be associative.

In the analytic case, an affine flat structure can be described by a complete atlas whose transition functions are affine linear, one can find local coordinates  $x_a$  such that  $e_a = \frac{\partial}{\partial x^a} = \partial_a$  are commuting sets of linear independent vector fields. In flat local coordinates, the equation 5, becomes  $A_{abc} = \partial_a \partial_b \partial_c \Phi$  and equation 6, is rewritten as:

$$\partial_a \circ \partial_b = \sum_c A_{ab}^c \partial_c,$$

where  $A_{ab}^c := \sum_e A_{abe} g^{ec}$ ,  $(g^{ab}) := (g_{ab})^{-1}$ .

$M$  is called *semisimple* if an isomorphism of the sheaves of  $O_M$ -algebras:

$$(T_M, \circ) \rightarrow (O_M^n, \text{componentwise multiplication})$$

exists everywhere, locally.

The semisimplicity of the Frobenius manifold implies that, in the local basis  $(e_1, \dots, e_n)$  of  $T_M$ , the multiplication takes the form:

$$\left( \sum f_i e_i \right) \circ \left( \sum g_j e_j \right) = \sum f_i g_i e_i,$$

and in particular

$$e_i \circ e_j = \delta_{ij} e_j.$$

In the dual basis, the metric  $(g(e_i, e_k) = \delta_{ik} g_{ii})$  is diagonal, and  $A$  also.

Moreover, we can introduce a local function (metric potential)  $\eta$ , defined up to addition of a constant, such that:  $g_{ii} = e_i \eta$ .

**2.2. The space of polynomials.** The space of polynomials is an example of semisimple Frobenius manifolds of arbitrary dimension (the construction is due to Dubrovin [19] and Saito [41]). Consider the  $n$ -dimensional affine space  $\mathbb{A}^n$  with coordinate functions  $a_1, \dots, a_n$ . We identify the space  $\mathbb{A}^n$  with the space of degree  $n + 1$  polynomials  $P(z) = z^{n+1} + a_1 z^{n-1} + \dots + a_n$ . Let

$$\pi : \tilde{\mathbb{A}}^n \rightarrow \mathbb{A}^n$$

be a covering space of degree  $n!$  whose fiber over a point  $P(z)$  consists of the total orderings of the roots of  $P'(z)$ . In other words,  $\tilde{\mathbb{A}}^n$  supports functions  $\rho_1, \dots, \rho_n$  such that:

$$\pi^*(P'(z)) = (n+1) \prod_{i=1}^n (z - \rho_i);$$

$$\pi^*(a_i) = (-1)^{i+1} \frac{n+1}{n-i} \sigma_{i+1}(\rho_1, \dots, \rho_n), i = 1, \dots, n-1$$

and  $\sigma_1(\rho_1, \dots, \rho_n) = \rho_1 + \dots + \rho_n = 0$ . We will omit  $\pi^*$  in the notation of lifted functions.

Let  $M \subset \tilde{\mathbb{A}}^n$  be the open dense subspace in which :

- (1)  $\forall i, P''(z_i) \neq 0$  that is  $\rho_i \neq \rho_j$  for  $i \neq j$ .
- (2)  $u^i := P(\rho_i)$  form local coordinates at any point.

$M$  is a *semisimple Frobenius manifold* with the following structure data:

- (1) The  $n$ -tuple  $(u^i)$ , generate the canonical coordinates, the basis is given by  $e_i = \frac{\partial}{\partial u^i}$ , and the dual basis is generated by  $(du^i)$ .
- (2) Flat metric

$$g = \sum \frac{(du^i)^2}{P''(\rho_i)}$$

with metric potential

$$\eta = \frac{a_1}{n+1} = \frac{1}{n-1} \sum_{i < j} \rho_i \rho_j = \frac{1}{2(n-1)} \sum \rho_i^2.$$

**2.3. Monodromy.** The main instrument to calculate the monodromy of a Frobenius manifold coming from loops winding around the discriminant is a differential equation, with rational coefficients. Let  $\Sigma = \{t | \det(g^{ij}(t)) = 0\}$  be a proper analytic subset of  $M$ , which is the discriminant locus of the Frobenius manifold. For the space of polynomials, the discriminant locus is the space of polynomials with multiple roots.

**Definition 1.** Let  $(P_t)_{t \in [0,1]}$  be a continuous path in the space of polynomials i.e. the configuration space. By intertwining of a pair of roots  $z_i(t)$  and  $z_j(t)$  of  $P_t$  we mean the following:

$$\begin{cases} z_i(t) &= r \exp(2i\pi(\frac{t}{2} + t_1)) + z_0, \\ z_j(t) &= r \exp(2i\pi(\frac{t}{2} + t_2)) + z_0, \end{cases}$$

for  $t \in [0, 1]$  where  $t_1$  and  $t_2$  are the values of  $t$  such that  $z_1(0) = z_1$  and  $z_2(0) = z_2$  (up to exchanging the numbering of the roots, we may assume that  $t_2 - t_1 = \frac{1}{2}$ ) and  $z_0$  is a complex point in the complex plane.

Following [19], for flat coordinates, the equation of the linear pencil of the metrics is:

$$(7) \quad (g^{ij} - \lambda \eta^{ij}),$$

as the functions of the parameter  $\lambda$ . The coordinates  $x_1 = x_1(t), \dots, x_n = x_n(t)$  are the orthogonal flat coordinates for the metric  $g^{ij}(t)$ .

More precisely, the functions

$$(8) \quad \tilde{x}_i(t, \lambda) = x_i(t - \lambda, t^2, \dots, t^n), i = 1, \dots, n.$$

form flat coordinates for the metric in equation 7. Their gradient  $\xi_k = \partial_k x(t, P)$ , satisfies the following system of linear differential equations in  $P$ :

$$(9) \quad (t\eta^{ik} - g^{ik}(t)) \frac{d}{d\lambda} \xi_k = \eta^{ik} \left( \frac{-1}{2} + \mu_k \right) \xi_k,$$

where  $\mu : \pi_1(M \setminus \Sigma) \rightarrow Isom(\mathbb{A}^n)$  is the representation, obtained from the action of the fundamental group on the universal covering, which is lifted by the isometries on the affine space.

This equation is a system of linear ordinary differential equations with rational coefficients depending on parameters  $t^1, \dots, t^n$ . The coefficients have poles on the shifted discriminant locus

$$\Sigma_\lambda = \{t|\Delta(t^1 - \lambda, t^2, \dots, t^n) = 0\}.$$

By [19], we know that:

**Lemma 1.** *The monodromy system in equation (9) of differential equations with rational coefficients around the  $\Sigma_\lambda$  coincides with the monodromy of the Frobenius manifold around the discriminant  $\Sigma$ . The monodromy does not depend on the parameter  $t$ .*

### 3. Gauss Skizze-OP, GRAPH MODIFICATIONS AND MONODROMY

We explain the monodromy in the semisimple Frobenius manifold formed from the space of polynomials, using graphs and their Whitehead modifications. It is known that (Appendix G [19]) the monodromy along a small loop around the discriminant on an analytic Frobenius manifold satisfying semisimplicity condition is a reflection. We can prove this property, for the space of polynomials using the graphs that we introduce in this section. This property is strengthened because we can show that the small loop around the discriminant is invariant under the Klein group.

**3.1. Graphs.** In the language of Frobenius manifolds, the configuration space is interpreted as the space of complex polynomials with  $n$  roots (which are distinct- if we omit the discriminant locus). We use this remark in the following way. Taking the inverse image of  $\mathbb{R} \cup i\mathbb{R}$  under a given polynomial gives so-called *Gauss Skizze* which are objects reminiscent of the Grothendieck *dessins d'enfant* [28], but which have been introduced by Gauss. Let  $P$  be a degree  $n$  polynomial. Then the properties of any Gauss-skizze are that:

- $P^{-1}(\mathbb{R})$  (resp.  $P^{-1}(i\mathbb{R})$ ) gives a system of  $n$  red (blue) curves (properly embedded in  $\mathbb{C}$ ).

- The drawing is geometric realization of a graph which is a forest.
- The curves are oriented, and each red curve intersects a unique blue curve, once.
- The asymptotic directions are  $\frac{2k\pi}{4n}$ , where  $k \in \{0, \dots, 4n - 1\}$ .

We start by recalling some classical definition and terminology for graphs.

Let  $V$  and  $E$  be two finite sets. A finite graph  $G$  is defined by  $V$  (the set of vertices) and  $E$  (the set of edges) and a mapping  $d$  that interprets edges as pairs of vertices. A forest, is a graph, such that there exists no subgraph forming a  $k$ -gon, where  $1 \leq k \leq m$ , for a given  $m \in \mathbb{N}^*$ .

$Card(V)$  stands for the cardinality of the set  $V$ . The notation  $val(v)$ , where  $v \in V$ , indicates the valency of the vertex  $v$  i.e. how many edges are incident to the vertex  $v$ . By *flag* we mean the union of a vertex and the set of half-edges incident to it.

We now introduce the objects of our study: decorated  $n$ -forests and their morphisms.

**Definition 2.** *The decorated  $n$ -forest  $\sigma^n$  is a graph, defined by a sextuple of sets and maps:*

$$(V_\sigma = V_{roots} \cup V_{crit} \cup \{*\}, E_\sigma, F_\sigma, \partial_1, \partial_2, n),$$

where:

- $V_\sigma$  are vertices;  $0 \leq Card(V_{roots}) \leq n$  and  $0 \leq Card(V_{crit}) \leq n - 1$ .
- $E_\sigma$  are (colored) edges,
- $F_\sigma$  are (colored) 2-cells,
- the map  $\partial_1 : F_\sigma \rightarrow V_\sigma$ ,  $\partial_2 : F_\sigma \rightarrow E_\sigma$  are the boundary maps.

We call this graph *decorated* since, two additional (coloring) maps exist:

$$E_\sigma \rightarrow \{R, B\}$$

$$e \mapsto e_R \quad (\text{resp. } e_B)$$

$$F_\sigma \rightarrow \{A, B, C, D\}$$

$$f \mapsto A \quad (\text{resp. } B, C, D.)$$

Precisely, the rules of the coloring are given below:

- For any  $e_R \in E$ ,  $\partial_1^{-1}(e_R)$  is  $\{A, D\}$  or  $\{B, C\}$ ,
- For any  $e_B \in E$ ,  $\partial_1^{-1}(e_B)$  is  $\{A, B\}$  or  $\{C, D\}$ .
- For any  $v_1 \in V_{roots}$ ,  $\partial_1^{-1}(v_1) = \{A, B, C, D\}$ ,  $val(v_1) = 4k$ ,  $k \in \mathbb{N}^*$ ;
- $\partial_1^{-1}(*) = \{A, B, C, D\}$ ,  $val(*) = 4n$ ;
- for any  $v_2 \in V_{crit}$ , such that  $v_2 \notin V_{roots}$ , we have:  $\partial_1^{-1}(v_2) = \{B, C\}$  or  $\{A, D\}$ , if edges incident to  $v_2$  are colored  $R$
- $\partial_1^{-1}(v_2) = \{A, B\}$  or  $\{C, D\}$ , if edges incident to  $v_2$  are colored  $B$
- for any  $v_2 \in V_{crit}$ , such that  $v_2 \notin V_{roots}$ :  $4 \leq val(v_2) \leq 2n$ .

**Remark 3.** We call those graphs generic if  $V_{crit}$  is empty. Flags, for vertices in  $V_{roots}$  have  $4k$  half-edges of alternating colors. Flags for vertices of  $V_{crit}$  have  $2m$  edges of a given fixed color, where  $m \geq 2$ .

**Definition 3.** We call a red (or blue) half-graph the above graph, obtained by omitting all the edges of blue (or red) color.

**3.2. Geometric realisations.** The *geometric realisation*  $|\sigma^n|$  of a graph  $\sigma^n$  refers to a Gauss drawing (or Gauss skizze), i.e. a pre-image of  $\mathbb{C}$  under a polynomial. We can label the regions in the complementary part of the drawing by  $A, B, C, D$  (each region being the inverse image of a quadrant of the complex plane).

- The roots of the polynomial are intersections of a red curve with a blue curve; a root is a vertex in  $V_{roots}$ .
- The critical points  $z_0$  of  $P$  verifying  $Re(P)(z_0) = 0$  (resp.  $Im(P)(z_0) = 0$ ) i.e. such that the critical values  $P(z_0)$  are imaginary numbers (resp. real)- intersection of blue curves (resp. red); it is a vertex in  $V_{crit}$ .
- the point at  $\infty$  corresponds to  $\{*\}$ .
- The roots of the polynomial are given by the intersection of a red and a blue curve. The adjacent 2-faces are respectively colored  $A, B, C, D$  in the trigonometric orientation.
- The critical points of real (imaginary) critical values are represented by the intersection of red (blue) curves, the adjacent 2-faces are colored  $A, D, A, D, A, D$ , or  $B, C, B, C, B, C$  ( $A, B, A, B, A, B$ , or  $C, D, C, D, C, D$ ).

**Remark 4.** The *geometric realisation* of a red (or blue) half-graph is obtained by taking  $P^{-1}(\mathbb{R})$  (or  $P^{-1}(i\mathbb{R})$ ).

### 3.3. Gs-op.

**Definition 4.**  $Gs - op$  is a collection of objects  $\{\mathcal{F}(S) | S \in Set\}$ , which belong to the monoidal category of the topological spaces of genus 0 Riemann surfaces, on which are drawn topological realisations of the  $|S|$ -graphs verifying the definition 2.

**Proposition 2.**  $Gs-op$  is a topologico-combinatorial operad.

*Proof.* The collection of objects  $\{\mathcal{F}(S) | S \in Set\}$ , belong to the monoidal category of the topological spaces of genus 0 Riemann surfaces, on which are drawn topological realisations of the  $|S|$ -graphs verifying the definition 2. This collection of objects is endowed with some extra structures:

- (1)  $\mathcal{F}(S)$ , where  $|S| = n$  carries an action of the symmetric group  $\mathbb{S}_n$  on the vertices of  $V_{roots}$ .
- (2) The product  $\mathcal{F}(S/S') \otimes \mathcal{F}(S) \rightarrow \mathcal{F}(S')$  is inherited from section 1.3.2 and defined as follows:

Let  $P$  be the linear set from section 1.3.2. Consider a point in  $\text{Conf}_P$ . It corresponds to a Riemann surface on which is uniquely drawn a topological realisation of a graph  $\sigma_0^n$ .

Consider a weak partition  $v : S \rightarrow P$  and determine  $v^{-1}(p)$ , for all  $p \in P$ . The cardinality of  $v^{-1}(p)$  gives the multiplicity of the  $p$ -th point in the Fulton MacPherson compactified space  $\overline{\text{Conf}}_P$ . Cut out a small disc centred at the  $p$ -th point such that except for this special point, the combinatorial data of the graph remains unchanged, and insert instead of this small disc a starlike tree, such that the inner node is of valency  $2|v^{-1}(p)|$  and edges alternate colors. The resulting graph (after replacement of the 4-edged starlike graph of inner node the  $p$ -th point and insertion of the new starlike tree) is required to be an object of the considered category. From the combinatorial point of view, this operation may seem not well defined because there exist many different possibilities in the manner of inserting the starlike tree. However, from the configuration space point of view, this operation is well-defined because the way of inserting the starlike tree defines uniquely a polynomial, and therefore corresponds to a unique configuration of points.

The product operation satisfies the well known axioms.  $\square$

**3.4. Modifications of graphs.** We now turn our attention to morphisms between those graphs. Take two graphs  $\sigma_1^n$  and  $\sigma_2^n$ .

Modifications are done on the graphs (and therefore their geometric realisations), using so-called *Whitehead moves* (WH moves). We have two types of such moves, which we call respectively contracting  $WH_-$  and expanding  $WH_+$  moves. The composition of a contracting and an expanding Whitehead move on edges of a color  $R$  (or  $B$ ) will be denoted  $WH_R$  (resp.  $WH_B$ ).

**Definition 5.** A **contracting half - Whitehead** move  $WH_-$ , is a modification done in 2 steps.

*STEP 1.* Add a  $k$ -gon within a 2-cell  $f$  in  $F_\sigma$ , with vertices lying on edges of  $\sigma$ . Those edges of color  $R$  or  $B$ , are incident to  $f$ .

*STEP 2.* The  $k$ -gon is contracted to one vertex.

However, we ought to stress a degenerate case: if there exist two vertices in  $V_{crit}$  in the boundary of a blue (red) edge, then we contract the edge to one vertex.

**Definition 6.** A **expanding half - Whitehead** move  $WH_+$ , is a modification done in 2 steps.

*STEP 1.* Consider a vertex in  $V_{crit}$  of valency  $2k$ .

*STEP 2.* Replace it by a (small enough)  $2k$ -gon, vertices agreeing with the incoming red (or blue) edges. Draw diagonals, in the  $2k$ -gon, such that all vertices are connected and such that there are no cycles (i.e no  $m$ -gons).

We define the category  $F$  of decorated  $n$ -forests, where the objects  $Ob(F)$  are decorated  $n$ -forests defined from definition 2, and morphisms are described as follows. Let

$\sigma_1^n, \sigma_2^n \in Ob(F)$ . The morphisms  $Hom(\sigma_1^n, \sigma_2^n)$  are given by a finite sequence of contracting and expanding WH moves starting on the graph  $\sigma_1^n$  and ending on  $\sigma_2^n$ . If  $\sigma_1^n$  and  $\sigma_2^n$  are isomorphic (or  $\sigma_1^n = \sigma_2^n$ ) then  $Hom(\sigma_1^n, \sigma_2^n)$  includes also the isomorphism (resp. the identity).

Note that due to the 1-connectivity of the configuration space,  $Hom(\sigma_1^n, \sigma_2^n)$  is never empty.

**3.5. Metamorphosis of graphs.** We introduce some terminology. A mixed WH move is such that there is a WH move operating alternatively on red and on blue edges.

**Definition 7.** *Let  $\sigma^n \in Ob(F)$  be a generic  $n$ -decorated forest. We call metamorphosis of  $\sigma^n$ , an element of  $Hom(\sigma^n, \sigma^n)$ , different from the identity, and given by the smallest number of mixed WH moves.*

**Proposition 3.** *An element  $Hom(\sigma^n, \sigma^n)$ , different from the identity, is formed from a number of mixed WH moves which is a multiple of 4.*

*Proof.* An element from  $Hom(\sigma^n, \sigma^n)$  is obtained by using a certain number of pairs of WH moves - one WH move which operates on  $R$ -edges and one on  $B$ -edges - and then, by taking the inverse operations. This is a multiple of 4.  $\square$

**Corollary 1.** *Let  $Hom(\sigma^n, \sigma^n)$  be different from the identity. Then, the minimal number of mixed WH moves is 4.*

**Theorem 1 (Metamorphosis theorem).** *Let  $\sigma^n \in Ob(F)$  be a generic decorated  $n$ -forest. Consider the set of all complex polynomials having Gauss drawings  $|\sigma^n|$ . Let  $(v_1, v_2, \dots, v_{n-1})$  be their set of critical values in  $\mathbb{C}^{n-1}$ . Then,  $\phi \in Hom(\sigma^n, \sigma^n)$  is a metamorphosis if there exists a critical value verifying  $v_1(t) = r_1 \exp(t)$ , for  $t \in [0, 2\pi]$ . Reciprocally, there exists a metamorphosis in  $Hom(\sigma^n, \sigma^n)$  for which  $v_1(t) = r_1 \exp(t)$ , for  $t \in [0, 2\pi]$ .*

*Proof.* The proof will be done in two parts. Suppose that  $\phi \in Hom(\sigma^n, \sigma^n)$  is a metamorphosis such that the superimposition of  $WH_-^R \cup WH_-^B$  forms, up to isotopy - with respect to the asymptotic direction - a star-like subgraph in  $\sigma^n$ , with eight branches of alternating colors (red, blue). Consider a pair of roots of the polynomial in a small neighbourhood. The edges of some color, say blue are incident to the same face  $f_A$ . The geometric realisation of the contracted graph  $\sigma^n$  (by  $WH_-$  move) indicates that, in terms of complex polynomials, there exists a critical point on the real part of the polynomial. This node  $v_2 \in V_{crit}$ , verifies:  $\partial_1^{-1}(v_2) = \{A, B\}$ . The expansion step modifies the graph in such a way that the new pair of blue edges are both incident to a face of color  $B$ . The same reasoning is applied for the three other expansion and contraction WH moves. Therefore, by continuity of the loop in the space of polynomials and continuity of the map  $P : \mathbb{C} \rightarrow \mathbb{C}$ , there exists a critical value, being the image of the critical point  $c_1(t)$ , and given by  $P_t(c_1(t)) = v_1(t)$  forming a loop around the origin in  $\mathbb{C}$ .

Secondly, consider a simple smooth path  $\mathbf{v}(t) = (v_1(t), v_2, \dots, v_{d-1})$ . Suppose that all the critical values are distinct and fixed, for all  $t \in [0, 1]$ . Suppose without loss of generality that  $v_1(t)$  moves from one quadrant of the complex plane colored  $A$  to its adjacent cell colored  $D$ . Thus, there exists a unique  $t_0 \in (0, 1)$  such that  $v_1(t_0)$  lies on the real (resp. imaginary) axis. The system of  $d$  red curves is obtained by taking  $P^{-1}(\mathbb{R})$ . Since  $P^{-1}(v_1(t_0)) \subset P^{-1}(\mathbb{R})$ , the point  $P^{-1}(v_1(t_0))$  lies on the red curves. This point is a critical point, thus a singular point for the real polynomial  $ReP(x, y)$ . For  $t \neq t_0$ , the critical value  $v_1(t)$  is strictly contained in the interior of:

$$\begin{cases} \text{the cell } A & \text{if } t < t_0 \\ \text{the cell } D & \text{if } t > t_0. \end{cases}$$

The pullback of  $A$  (resp.  $D$ ) under  $P_t(z)$  is again a cell colored  $A$  (resp.  $D$ ). Therefore this defines a metamorphosis (one contracting Whitehead move followed by one expanding Whitehead move of a couple of red curves).  $\square$

**Proposition 4.** *A pair of roots intertwine if and only if a critical value forms a loop around the origin.*

*Proof.* (1) if  $z_1(t)$  and  $z_2(t)$  intertwine then for  $t \in [0, 1]$  the polynomial  $P_t(z)$  forms a loop in  $P_{c,v}$  around the polynomial  $P = (z - z_0)^2 Q(z)$  where  $z_1 = z_2 = z_0$ . We have:

$$\begin{cases} z_1(t) &= r \exp(2i\pi(\frac{t}{2} + t_1)) + z_0 \\ z_2(t) &= r \exp(2i\pi(\frac{t}{2} + t_2)) + z_0, \end{cases}$$

Now, if  $z_0 = z_1 = z_2$ , then  $z_0$  is a critical point of critical value  $v = 0$ . Therefore, if  $P_t(z)$  moves along a non contractible loop around  $P$ , then  $0 \notin v(t)$ . From the non-autonomous ODE [37, 45] it follows that the critical value  $v(t)$  forms a loop around 0.

(2) Suppose that  $v(t)$  forms a loop around 0. Consider the Taylor expansion around the critical point  $z = c$ :

$$\begin{aligned} P(z) &= P(c) + P'(c)(z - c) + P''(c)(z - c)^2 Q_{a(z),c(z)}(z) \\ &= v + (z - c)^2 Q. \end{aligned}$$

If  $P \in P_c$  then  $v \mapsto (c_v, a_v)$  is analytic in  $v$  (smooth). For  $v = 0$ ,  $P_a(z) = 0 + (z - c)^2 Q$ . So, we have double roots  $z = c_0$ . In a small neighborhood of  $v = 0$ , the roots are solution of:  $v + (z - c_0)^2 Q = 0$ , where  $Q_{a_0, c_0}(z) = K_v$ . Thus, we solve :

$$\begin{aligned} v + (z - c_0)^2 K_v &= 0 \\ z &= \pm \frac{-v}{K_v} = \pm \frac{-v}{K_0 + \dots}. \end{aligned}$$

Therefore, one loop  $v(t) = r \exp 2i\pi t$  implies that  $z_t = \pm \frac{-r}{K_0} \exp(\pi i t)$ . So, the roots intertwine. □

**Corollary 2.** *The intertwining of a pair of roots of a polynomial, lying in a small  $\epsilon$ -ball forms a metamorphosis.*

**Proposition 5.** *The metamorphosis is invariant under a Klein group.*

*Proof.* It is known that - the monodromy along a small loop around the discriminant on an analytic Frobenius manifold, satisfying the semisimplicity condition - is a reflection [?]. We have two types of reflections, for the case of the space of polynomials. This property can be easily showed by studying the geometric realisations of the pair of subgraphs occurring in the neighbourhood of the multiplicity 2 point. Indeed, let us consider a disc strictly containing only the pair of intertwining roots; let us label from 1 to 8 (in some given order) the (end)-vertices lying on the boundary of the disc, and obtained by intersecting the colored curves with this boundary. Then, the edges of the subgraphs connecting the vertices labeled 1 and 7 (respectively 3 and 5) are mapped by the modification induced by the metamorphosis, to a pair of edges connecting 1 and 3 (respectively 5 and 7). As for the edges of the opposite color, we proceed in the same way: the edges labeled 2 and 4 (resp. 6 and 8) are mapped to 4 and 6 (resp. 2 and 8). Geometrically, this procedure is equivalent to having a pair of reflections and, therefore, the metamorphosis is invariant under the Klein group. □

Let  $\tilde{\mathbb{A}}^n$  be the affine space corresponding to the space of critical values.

We have the following triangular diagram relating the space of polynomials, the space of their critical points and values as follows. Lifting a point in  $\tilde{\mathbb{A}}^n$  gives a finite family of polynomials:

$$\begin{array}{ccc}
 & (c_1, \dots, c_n) \in \tilde{\mathbb{A}}^n & \\
 \swarrow \pi & & \searrow P \\
 P \in \mathbb{A}^n & \xrightarrow{\tilde{\pi}} & (u^1, \dots, u^n) \in \tilde{\mathbb{A}}^n.
 \end{array}$$

#### 4. Gauss Skizze BRAID-OPERAD, ASSOCIAHEDRA AND THE MOSAIC OPERADS

**4.1. From Gs-op to Braid Gauss Skizze-operad.** We reinvestigate the operad structures (homotopy equivalent to little disc operad or cube operad) under the light of the graphs which were introduced in the section 3 and the model 1.1. This operad is called Gs-operad.

Our construction, using the graphs, relies on the “infinitesimally close” argument, concerning given points. In particular, to avoid any source of confusion, we choose an initial point  $x$  in  $\text{Conf}_n(\mathbb{C})$  such that no pair of points lie in the interior of an  $\epsilon$ -ball. Then, for

a chosen  $i$ -th point, which we duplicate, via the coface map, we apply the “infinitesimally close” argument for this new pair of points. In fact, the hidden idea is to pass onto the compactified version i.e. to the case where the duplicated point is now a multiplicity two point, and to interpret this situation using the decorated graphs. The multiplicity two point, turns out to be the inner node of a starlike graph, with eight branches of alternating colours. Winding around this point gives some monodromy and, interpreting this using the metamorphosis procedure, we have eight different possible new graphs. For points of higher multiplicities, we proceed in a similar way.

If we start with a configuration of points such that there exists a subset of points lying in an  $\epsilon$ -neighbourhood, then we proceed the same way as above. However, there are more possibilities of new graphs, since it may occur that a Whitehead move (or a sequence of Whitehead moves) modifies the new subgraph with a given other tree of the initial graph.

**Proposition 6.** *Consider a configuration of  $n$  marked points on the complex plane and the coface map  $\delta^i(m+1) : \overline{\text{Conf}}_n(\mathbb{C}) \rightarrow \overline{\text{Conf}}_{n+m+1}(\mathbb{C})$ . Then, omitting the cases where graphs can be obtained one from another by using the degenerate Whitehead contracting move, there are at  $8^{m+1}$  possible graphs which can be created.*

Suppose that we are given  $n$  marked points on the complex plane. Then using the construction depicted before, we chose one point and duplicate it using the  $i$ -th coface map. In the language of our graphs, duplicating one point, is the same as replacing the flag of four edges of alternating colors by a flag of eight edges of alternating colors. The other components of the graph remain unchanged. As in the construction given by [35], we distinguish those two points “infinitesimally”. Applying our construction to this infinitesimal modification, we have any of the eight possible graphs given by the metamorphosis which are possibly inserted in the  $\epsilon$ -neighbourhood of the duplicated point.

For points of higher multiplicities, the  $i$ -th coface map can also be used. Indeed, take a triple  $i$ -th point. It can be considered as a composition of two  $i$ -th coface maps: one which splits the multiplicity 3 point into a pair of points - one of which is a double point- and then, apply a second time the coface map in order to split the double point into two infinitesimally close points. Now again, for the first splitting, we have eight possible graphs coming from the metamorphosis (note that the flag around the multiplicity two point forms an eight-branched star like graph, with edges of alternating color). The second splitting concerns this double point, i.e. the 8-edged flag, which splits into two infinitesimally close points, and which we can apply a metamorphosis and have eight new possibilities.

We continue by induction, on the multiplicity of the  $i$ -th point. Consider an  $i$ -th point of multiplicity  $m$ . By hypothesis, we apply successively  $m - 1$  infinitesimal splittings at the multiple point (i.e. we split each multiple point into a pair of points - one of which is of multiplicity 1 - and we continue this procedure until all points are of multiplicity 1). From this operation and via metamorphosis we have  $8^{m-1}$  possible graphs, which can be constructed. Suppose that we have we have an  $i$ -th point of multiplicity  $m + 1$ . Then, we

split it into a pair of infinitesimally close points: one of multiplicity  $m$  and the other of multiplicity 1. For this case, the metamorphosis gives 8 possible graphs. We can apply the induction hypothesis to the multiplicity  $m$  point. In conclusion, we have  $8^{m+1}$  possible graphs that can be obtained.

Conclusion: going back to the operad, whenever we have a duplication of points by the map  $d$ , we insert in a small neighbourhood of this point any of the eight possible modifications of the star-like tree with 8 branches of alternating colors (given by WH moves). We therefore can construct the operad, in this way.

The Gs-operad furnishes an operadic model for the braid operad.

**Definition 8.** We call a Gauss Skizze-braid-op (in short Gsb-op) the collection  $\{Gsb_S | S \in Set\}$  of objects which are generated by  $|S| - 1$  pairwise different metamorphosis operations.

**Definition 9.** The Artin braid operad is a collection of  $\{B_S | S \in Set\}$  such that  $B_S$  is the Artin braid group on  $|S|$  strings with cabling as composition

**Proposition 7.** Gauss Skizze-braid-op (or Gsb-op) is an operad, coinciding with the classical Artin braid operad.

*Proof.* We have defined Gs-op previously, in section 3.3. Now, we can add to this operad a supplementary property: that the roots, which have been duplicated by the coface map, intertwine. If those points intertwine then, we have a metamorphosis occurring. This metamorphosis is a non-trivial loop in the configuration space  $Conf_S$  and therefore a generator for the braid group  $B_S$ . This forms a braid operad, since:

1. The action of the symmetric group on the strands of the braid corresponds at most to a rotation of the discs which are formed from the infinitesimally close points of the cosimplicial model (and at least to a permutation of the points inside those discs).
2. The product operation is obtained, in the classical way, by cabling. It is easy to verify that this product operation satisfies the axioms.  $\square$

**4.2. Associahedra and mosaic operads in Gauss-skizze operad.** We wish to point out the rich similarities between the mosaic operad introduced in [17] and our Gs-operad. The richness of their interaction, comes from the fact that, as in  $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ , the structure of the decomposition in Gauss Skizze of the configuration space of marked points on  $\mathbb{C}$ , contains associahedral structures.

An associahedron  $K_n$  is a CW-ball with codimension  $k$  faces corresponding to using  $k$  parentheses meaningfully on  $n$  letters.

For example,  $K_2$  is a point;  $K_3$  is a line;  $K_4$  is a pentagon. Let us show that in the decomposition provided by Gauss drawings, we have the structure of an associahedron.

We start with a “canonical” half-graph i.e. a diagram of the following type:

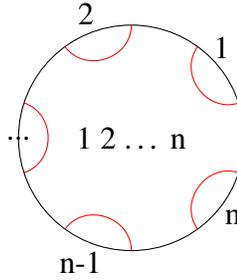


FIGURE 1. Canonical (red) half-graph  
 ]

where the curves in the diagram are labeled from 1 to  $n$ , clockwise. This forms a word in  $n$  letters:  $1, 2, \dots, n$ . To each pair of curves (being deformed until they intersect) corresponds a pair of parentheses in the given word.

**Example 1.** *In the word  $1, 2, \dots, n$ , if the curves 1 and 2 are deformed until they intersect, then this word becomes  $(1, 2)\dots, n$ .*

This procedure gives exactly the construction of an associahedron. For example, see the case of the  $K_4$  associahedron in connection with the half-graph decomposition, in figure 2.

Our decomposition is richer. Using the analogy of associahedron, there exist different drawings which are attributed to the same word (for the case of  $K_4$  in the figure 2, the dashed half-graph corresponds to the second drawing associated to a parenthesised word). Therefore, the decomposition of the semi-simple Frobenius space of complex polynomials and which is given by the Gauss drawings of  $P^{-1}(\mathbb{R})$  (resp.  $P^{-1}(i\mathbb{R})$ ), contains the associahedron structure  $K_n$ . We prove this statement.

Let  $\{C_i\}_{i \in I}$  be a cell decomposition (and even a good cover in the sense of Čech [9]) of the space of degree  $n$  polynomials indexed by the graphs corresponding to *Gauss Skizze* (i.e. verifying definition 2). Let  $\{V_j\}_{j \in J}$  be the cover of this space indexed by half-graphs of degree  $n$  of a given colour.

Let us consider the associahedron  $K_n$ , where an interior point of  $K_n$  corresponds to the word  $1 2 \dots n + 1$  ( it can also be identified with an  $n + 1$ -gon) and an interior point of a codim  $k$  face corresponds to  $k$  pairs of parenthesis on the  $1 2 \dots n + 1$  ( word (this can also be identified with an  $n + 1$ -gon containing  $k$  non-crossing diagonals, within the polygon).

Let us construct the following mapping. Let us map the half-graphs with empty  $V_{crit}$  and which verify the property of figure 1 to the interior of  $K_n$ ; and let us map the set of half-graphs corresponding to a given word, with a certain combination of  $k$  pairs of parenthesis to a codim  $k$  face of  $K_n$ . This is a surjective map. Indeed, it is enough to notice the case of the subword  $((ij)k)$  corresponds already two different drawings. Since the sets of  $n$ -half-graphs and of faces in  $K_n$  are finite, we have a surjective map from the

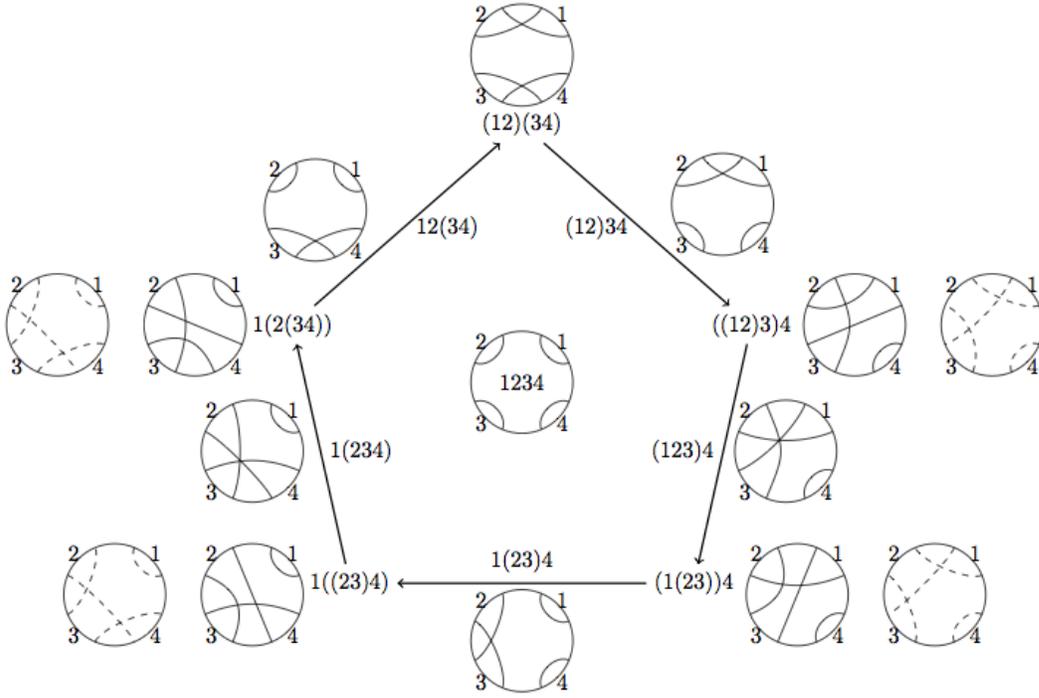


FIGURE 2. Associahedra  $K_4$  and the half-graph decomposition

set of half-graphs to the set of faces of  $K_n$ . We call the associahedron red (or blue) if the half-edges of the graphs are red (or blue).

So, we have proved that:

**Proposition 8.** *The cell decomposition  $\{C_i\}_{i \in I}$  of the space of degree  $n$  polynomials indexed by the decorated  $n$ -forests, contains an underlining pair of red and blue associahedra  $K_n$ .*

**Corollary 3.** *The Čech nerve of decomposition of the space of degree  $n$  polynomials indexed by the decorated  $n$ -forests contains a substructure, isomorphic to the dual of an associahedron  $K_n$ .*

We show now the relation between the mosaic operad and the Gs-operad. Let us recall the definition of the mosaic operad.

Let  $\mathcal{G}^L(m, l)$  be the space of  $m$ -gons with  $l$  non-crossing diagonals (in the case of the moduli space of  $m$  points on  $\mathbb{R}$ , this corresponds to  $l$  points having coalesced).

**Definition 10.** [17] Given  $G \in \mathcal{G}^L(m, l)$  and  $G_i \in \mathcal{G}^L(n_i, k_i)$  (where  $1 \leq i \leq m$ ), there are composition maps

$$G \circ_{a_1, b_1} G_1 \circ_{a_2, b_2} G_2 \circ_{a_3, b_3} \dots \circ_{a_m, b_m} G_m \mapsto G_t,$$

where  $G_t \in \mathcal{G}^L(-m + \sum n_i, m + l + \sum k_i)$ . The object  $G_t$  is obtained by gluing the side  $a_i$  of  $G$  along side  $b_i$  of  $G_i$ . The symmetric group acts on  $G_n$  by permuting the labelling of the sides. These operations define the mosaic operad  $\{\mathcal{G}^L(n, k)\}$ .

The relation between the  $n$ -gon and the associahedron  $K_{n-1}$  follows from [36] and has been mentioned in the above paragraphs.

We recall the following proposition:

**Proposition 9** ([36], §5). *The dihedral group  $D_n$  acts as a group of isometries on  $K_{n-1}$ .*

This relation to polygons, is particularly important because it enables the use of the mosaic operad structure on  $K_{n-1}$  and, therefore we can state that:

**Proposition 10.** *The structure of the  $G$ s-operad contains a pair of red and blue mosaic operads.*

**Remark 5.** *A red (blue) mosaic operad means that it is an operadic structure for the red (blue) curves of  $G$ s-op.*

*Proof.* Let us consider the decomposition for  $P^{-1}(\mathbb{R})$  and apply proposition 8. Each face of  $K_{n-1}$  is a product of lower dimensional associahedra ([17], prop 2.4.1). In general, a codimension  $k - 1$  face of the associahedron  $K_{m-1}$  will decompose as

$$K_{n_1-1} \times \dots \times K_{n_k-1} \hookrightarrow K_{m-1},$$

where  $\sum n_i = m + 2(k - 1)$  and  $n_i \geq 3$ . This parallels the mosaic operad structure:

$$G(n_1) \circ \dots \circ G(n_k) \mapsto G(m),$$

where  $G(n_i)$  is an  $n_i$ -gon and  $G(m)$  is a  $m$ -gon, containing  $k - 1$  non-intersecting diagonals in its interior. The glueing of sides is arbitrary. We proceed similarly for  $P^{-1}(\mathbb{R})$  and  $P^{-1}(i\mathbb{R})$ , so we have a pair of red and blue mosaic operads.  $\square$

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