Existence and uniqueness of a regular solution of Cauchy-Dirichlet problem for doubly nonlinear parabolic equations

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Introduction

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 1$, $Q_T = \Omega \times (0,T]$, $S_T = \partial\Omega \times (0,T]$, $\Gamma_T = S_T \cup (\overline{\Omega} \times \{t = 0\})\})$ (Γ_T is the parabolic boundary of the cylinder Q_T). Consider in Q_T the equation

$$F[u] \doteqdot \frac{\partial u}{\partial t} - \operatorname{div} a(u, \nabla u) = f \tag{1.1}$$

where $\nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})$, $f(x,t)$ is a given function, $a = (a^1, \ldots, a^n)$, $a^i = a^i(u,p)$ are continuous on $\mathbb{R} \times \mathbb{R}^n$ and satisfy for all $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ the inequalities

$$\begin{gathered} a(u,p) \cdot p \geq \nu_0 |u|^l |p|^m - \phi_0(u), \qquad \nu_0 > 0, \\ |a(u,p)| \leq \mu_1 |u|^l |p|^{m-1} + \phi_1(u), \qquad m > 1,\ l \geq 0,\ \phi_i(u) \geq 0,\ i = 0,1. \end{gathered} \tag{1.3}$$

Equations (1.1), (1.2) are known as doubly nonlinear parabolic equations (DNPE). The prototype of DNPE is

$$F_0[u] \doteqdot \frac{\partial u}{\partial t} - \operatorname{div} [|u|^l |\nabla u|^{m-2} \nabla u] = 0. \tag{1.3}$$

In this paper we consider a special case of DNPE. In particular we limit ourselves by consideration equations (1.1), (1.2) only for $m > 1$, $l \geq 0$ (instead of more general conditions $m > 1$, $1 > 1 - m$).

Equations (1.1), (1.2) and in particular (1.3) arise in the study of turbulent filtration of a gas or of a fluid through porous media and non-Newtonian flows (see [1]).

Existence of generalized solutions of Cauchy-Dirichlet problem for DNPE were established first by Raviart [2] and J.-L. Lions [3] and then by many authors. In particular Bamberger stated in [4] his results on existence and uniqueness of some nonnegative generalized solution of Cauchy-Dirichlet problem for a nonhomogeneous equation $F_0[u] = f$ (see (1.3)).

Up to recent time there were no regularity results for DNPE. The simple modification of the Barenblatt explicit solutions lets to show that at least in the case $l > 1$ hölderness is the best possible smoothness of generalized solutions of equation (1.3). Hence the key question of the regularity theory for DNPE is establishing Hölder

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estimates for their generalized solutions. At first such estimates were established in [5] for the case of, so-called, doubly degenerate parabolic equations, i.e. for (1.1), (1.2) in the case $m > 2$, $l > 0$.

This paper is devoted to the proof of existence and uniqueness of some Hölder continuous generalized solution of Cauchy-Dirichlet problem for equations of the type (1.1), (1.2). The crucial role is played by the Hölder estimates established by the author in [6]-[8].

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2. The statement of the main result

Assume that for any $u, v \in \mathbb{R}$ and any $p, q \in \mathbb{R}^n$ we have

$$\text{(G)} \qquad |a(u,p)| \le \mu(|u|^l |p|^{m-1} + \overline{\mu}(|u|)),\ \mu = \text{const} \ge 0,\ m > 1,\ l \ge 0,$$

$\overline{\mu}(s) \ge 0$ is nondecreasing.

Definition 2.1. Any nonnegative bounded in Q_T function u is a weak solution of equation (1.1), (G) with $f \in L_1(Q_T)$ if

(a) $u \in C([0,T]; L_2(\Omega)), \nabla u^{\sigma+1} \in L_m(Q_T), \sigma = \frac{l}{m-1}$;

(b) for any $\phi \in C^1(\overline{Q}_T), \phi = 0$ on S_T, and any $t_1, t_2 \in [0,T]$

$$\int_\Omega u\phi dx|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega [-u\phi_t + a(u,u_x)\cdot \nabla\phi - f\phi] dxdt = 0 \tag{2.1}$$

where $u_x = (u_{x_1}, \dots, u_{x_n})$ and u_{x_i} are defined by

$$u_{x_i} = \begin{cases} (1+\sigma)^{-1} u^{-\sigma} \frac{\partial u^{\sigma+1}}{\partial x_i} & \text{in } [Q_T : u > 0], \\ 0 & \text{in } [Q_T : u = 0], i = 1, \dots, n. \end{cases} \tag{2.2}$$

Consider Cauchy-Dirichlet problem

$$F[u] = \frac{\partial u}{\partial t} - \operatorname{div} a(u, \nabla u) = f \text{ in } Q_T, u = \Psi \quad \text{on} \quad \Gamma_T \tag{2.3}$$

where

$$f \in L_1(Q_T), \Psi \in W_1^1(Q_T), \Psi \ge 0 \quad \text{in} \quad Q_T. \tag{2.4}$$

Definition 2.2. Function u is a weak solution of Cauchy-Dirichlet problem (2.3), (2.4) if u is a weak solution of equation (1.1), (G) and $u = \Psi$ on Γ_T.

Remark 2.1. Every weak solution of (1.1), (G) and every $\Psi \in W_1^1(Q_T)$ have traces on Γ_T.

Definition 2.3. Let $\inf(\Psi, \Gamma_T) > 0$. We say that function u is a strong solution of Cauchy-Dirichlet problem (2.3) if u is a weak solution of (2.3) and moreover

$$\inf(u, Q_T) > 0 \,(\text{and hence } u \in W_m^{1,0}(Q_T)).$$

Definition 2.4. Let $\Psi \in \overset{0}{W}{}_1^1(Q_T)$. We say that function u is a quasistrong solution of Cauchy-Dirichlet problem (2.3) if u is a weak solution of (2.3) and moreover there exists a sequence of strong solutions of problems

$$F[u_n] = f_n \text{ in } Q_T, u_n = \Psi_n \text{ on } \Gamma_T$$

such that

$$u_n \to u \text{ in } C([0,T]; L_1(\Omega)); f_n \in L_1(Q_T), f_n \to f \text{ in } L_1(Q_T);$$

(2.5)

$$\Psi_n = \Psi + \varepsilon_n(x,t), \varepsilon_n \in W_1^1(Q_T) \cap C(\overline{Q}_T),\ \inf(\varepsilon_n, \Gamma_T) > 0, \sup(\varepsilon_n, \Gamma_T) \to 0.$$

Definition 2.5. Let $\Psi \in \overset{0}{W}{}_1^1(Q_T)$. We say that function u is a regular solution of Cauchy-Dirichlet problem (2.3) if u is Hölder continuous in $\overline{Q}_T$ and u is a quasistrong solution of (2.3).

Introduce the following assumptions:

(Ω) $\exists \rho_0 > 0 \, \exists \alpha_0 \in (0,1) \forall x_0 \in \partial\Omega \, \forall \rho \in (0, \rho_0) : |B_\rho(x_0) \cap \Omega| \leq (1-\alpha_0)|B_\rho(x_0)|;$

(BI) $$\Psi \geq 0, \Psi \in \overset{0}{W}{}_2^1(Q_T) \cap C_{\beta,\beta/m}(\Gamma_T), \beta \in (0,1);$$

(RHS) $$f \geq 0, f \in L_\infty(Q_T).$$

Moreover assume that the following conditions are fulfilled for equation (1.1):

0) functions $u^{-\alpha} a^i(u, u^{-\alpha} p), \alpha = \frac{l}{m}$, are continuous on $\bar{\mathbb{R}}_+ \times \mathbb{R}^n$;

1) **(the growth condition)** for any $u \in \bar{\mathbb{R}}_+, p \in \mathbb{R}^n$

$$a(u,p) \cdot p \geq \nu_0 |u|^l |p|^m - \mu_0(|u|^\delta + 1), \nu_0 > 0,$$
$$2 < \delta < m + l \text{ if } m + l > 2, \delta = 2 \text{ if } m + l \leq 2;$$
$$|a(u,p)| \leq \mu_1 |u|^l |p|^{m-1} + \mu(|u|)|u|^\alpha,$$
$$\alpha = \frac{l}{m}, \mu(s) \geq 1 \text{ is a nondecreasing on } \mathbb{R}_+;$$

2) **(the strict monotonicity condition)** there exists $\nu_1 > 0$ and continuous vector-function $b(u) \in \mathbb{R}^n$ such that for any $u \in \mathbb{R}$ and any $p, q \in \mathbb{R}$

$$[a(u,p) - a(u,q)] \cdot (p - q) \geq \nu_1 |u|^l |p - q|^\kappa (|p - b|^m + |q - b|^m)^{1 - \frac{\kappa}{m}}$$

where $\kappa = m$ if $m \geq 2$ and $\kappa = 2$ if $m \in (1,2)$;

3) **(the local Lipschitz condition)** for any $u, v \in [\varepsilon, M], \varepsilon > 0, M > \varepsilon$, and any $p \in \mathbb{R}^n$

$$|a(u,p) - a(v,p)| \leq \Lambda |u - v|(1 + |p|^{m-1}), \Lambda = \Lambda(\varepsilon, M) \geq 0.$$

4)
$$(m,l) \in D \backslash \omega, D \doteqdot \{m > 1, l \geq 0\},$$
$$\omega \doteqdot \left\{(m,l) \in D : \frac{\sigma+1}{\sigma+2} \leq \frac{1}{m} - \frac{1}{n}, \sigma = \frac{l}{m-1}\right\}.$$

Theorem 2.1. **(existence and uniqueness of regular solution).** *Let conditions $(\Omega), (BI), (RHS)$ and 0)-4) hold. Then Cauchy-Dirichlet problem (2.3) has exactly one regular solution.*

Remark 2.2. Conditions 0) - 3) are fulfilled for equation (1.3).

Remark 2.3. It is easy to see that $\omega \subset F \doteqdot \{(m,l) \in D : m + l < 2\}$. We constructed a counter-example (see [9]) showing that for every $(m,l) \in \omega$ the local boundedness of generalized solutions of equation (1.3) fails to be true.

Remark 2.4. Existence of Hölder continuous weak solution of Cauchy-Dirichlet problem for some class of equations of the type (1.1), (1.2) in the case $m \geq 2, l \geq 0$ was proved in [10]. Existence and uniqueness of regular solution of Cauchy-Dirichlet problem (2.3) under conditions $(\Omega), (BI), (RHS)$, 0) -3) and for $l \geq 0$, $\max(1, \frac{2n}{n+2}) < m < 2$, $m + l > 2$ can be derived from results of [11]. The proofs of the results of [10] and [11] are based on using Hölder estimates established in [5] and [6]-[8] respectively.

3. Uniqueness of quasistrong solution

In this section we state the uniqueness results of paper [11]. Assume at first that for any $u, v \in \mathbb{R}$ and any $p, q \in \mathbb{R}^n$ functions $a^i(u,p)$ satisfy conditions

($\tilde{G}$) $|a(u,p)| \leq \mu(|p|^{m-1} + 1), \mu \geq 0$;

(M) $[a(u,p) - a(u,q)] \cdot (p - q) \geq 0$;

(L) $|a(u,p) - a(v,p)| \leq \Lambda |u - v|(|p|^{m-1} + 1), \Lambda = \text{const} \geq 0, m > 1$.

Definition 3.1. Function u is a generalized solution (subsolution, supersolution) of equation (1.1), $(\tilde{G})$ if $u \in W_m^{1,0}(Q_T) \cap C([0,T]; L_1(\Omega))$ and for all $\phi \in \overset{0}{W}{}_m^1(Q_T) \cap L_\infty(Q_T)(\phi \geq 0)$ and any $t_1, t_2 \in [0,T]$

$$\int_\Omega u\phi dx|_{t_1}^{t_2} + \int_\Omega [-u\phi_t + a(u, \nabla u) \cdot \nabla\phi - f\phi] dxdt = 0 (\leq 0, \geq 0) \quad (3.1)$$

Proposition 3.1. *(Comparison Principle, [11]). Assume that conditions $(\tilde{G})$, (M), and (L) hold. Let u_1 and u_2 are generalized subsolution and supersolution such that*

$$F[u_1] \leq f_1, F[u_2] \geq f_2$$

where $f_1, f_2 \in L_1(Q_T)$. *If*

$$u_1 \le u_2 \, on \; S_T = \partial\Omega \times (0,T]$$

then for any $\tau \in (0,T]$ *we have*

(3.2)

$$\int_\Omega (u_1 - u_2)^+ dx|^{t=\tau} \le \int_\Omega (u_1 - u_2)^+ dx|^{t=0} + \int_0^\tau \int_\Omega (f_1 - f_2)\mathrm{sign}\,(u_1 - u_2)^+ dxdt.$$

Proof. Let $\eta \in \overset{0}{W}{}^{1,0}_m(Q_T) \cap L_\infty(Q_T), \eta \ge 0, 0 < h < t_1 < t_2 < T - h, Q_{t_1,t_2} \doteqdot \Omega \times [t_1, t_2]$. Then from conditions of Proposition 3.1 it follows (see also [12]) that

$$(3.3) \quad \iint_{Q_{t_1,t_2}} \{(u_1 - u_2)_{\overline{h}t}\eta + [(a(u_1, \nabla u_1))_{\overline{h}} - (a(u_2, \nabla u_2))_{\overline{h}}] \cdot \nabla\eta\} \, dxdt \le$$

$$\iint_{Q_{t_1,t_2}} (f_1 - f_2)_{\overline{h}} \eta dxdt$$

where $g_{\overline{h}} \doteqdot (1/h)\int_{t-h}^t g(x,\tau)d\tau$. Denote

$$H_\delta(s) = \begin{cases} 1, \; s \ge \delta \\ s/\delta, \; 0 < s < \delta \\ 0, \; s \le 0 \end{cases}, \qquad G_\delta(s) = \begin{cases} s - \delta/2, \; s \ge \delta \\ s^2/2\delta, \; 0 < s < \delta \\ 0, \; s \ge 0 \end{cases}$$

so that $G'_\delta(s) = H_\delta(s)$ on $\mathbb{R}$. Set in (3.3)

$$(3.4) \qquad \eta = H_\delta(u_1 - u_2).$$

Obviously that test function (3.4) is admissible. In view of concavity of function $G_\delta(w)$ we have

$$(u_1 - u_2)_{\overline{h}t} H_\delta(u_1 - u_2) \ge (G_\delta(u_1 - u_2))_{\overline{h}t}.$$

Then from (3.3) it follows that

$$(3.5) \quad \iint_{Q_{t_1,t_2}} (G_\delta(u_1 - u_2))_{\overline{h}t} dxdt + \iint_{Q_{t_1,t_2}} [(a(u_1, \nabla u_1))_{\overline{h}} - (a(u_2, \nabla u_2))_{\overline{h}}] \cdot$$

$$\cdot \nabla(u_1 - u_2) H'_\delta(u_1 - u_2) dxdt \le \iint_{Q_{t_1,t_2}} (f_1 - f_2)_{\overline{h}} H_\delta(u_1 - u_2) dxdt.$$

Integrating in t the first term in (3.5) and then letting $h \to 0$ we obtain for any $\tau \in (0,T]$

$$(3.6) \quad \int_\Omega G_\delta(u_1 - u_2) dx|_0^\tau +$$

$$+ (1/\delta) \iint_{\{Q_{0,\tau}: 0 < u_1 - u_2 < \delta\}} [a(u_1, \nabla u_1) - a(u_2, \nabla u_2)] \cdot \nabla(u_1 - u_2) dxdt \le$$

$$\le \iint_{Q_{0,\tau}} (f_1 - f_2) H_\delta(u_1 - u_2) dxdt.$$

Taking into account that $G_\delta(u_1 - u_2) \to (u_1 - u_2)^+, H_\delta(u_1 - u_2) \to \text{sign}(u_1 - u_2)^+$ as $\delta \to 0$ we derive from (3.6) and conditions (M) and (L) that inequality (3.2) holds. Proposition 3.1 is proved.

Consider now Cauchy-Dirichlet problem (2.3) assuming that condition $(\widetilde{G})$ holds and $f \in L_1, \Psi \in W_1^1(Q_T)$.

Definition 3.2. Function u is a generalized solution of Cauchy-Dirichlet problem (2.3) if u is a generalized solution of equation (1.1) and $u = \Psi$ on Γ_T.

From Proposition 3.1 we can derive directly the following

Proposition 3.2. *Let conditions $(\widetilde{G}), (M)$, and (L) are fulfilled. Then there is at most one generalized solution of Cauchy-Dirichlet problem (2.3).*

Replace now condition $(\widetilde{G})$ by condition (G) (see sect. 2) and consider instead of assumption (L) the local Lipschitz condition

for any $u, v \in [\varepsilon, M], \varepsilon > 0, M > \varepsilon,$ and any $p \in \mathbb{R}^n$

$$(\widetilde{L}) \qquad |a(u,p) - a(v,p)| \le \Lambda(|u - v|(1 + |p|^{m-1}), \Lambda = \Lambda(\varepsilon, M) \ge 0.$$

From Proposition 3.2 we can derive the following

Proposition 3.3. *Let $\inf(\Psi, \Gamma_T) > 0$ and let conditions $(G), (M)$, and $(\widetilde{L})$ hold. Then there is at most one strong (in sense of Definition 2.3) solution of Cauchy-Dirichlet problem (2.3).*

The main uniqueness result for DNPE is

Theorem 3.1. *(uniqueness of quasistrong solution, [11]). Let $\Psi \in \overset{0}{W}{}^1_m(Q_T)$ and let conditions $(G), (M)$, and $(\widetilde{L})$ are fulfilled. Then there is at most one quasistrong (in sense of Definition 2.4) solution of Cauchy-Dirichlet problem (2.3).*

Proof. Let u and $\widetilde{u}$ are two quasistrong solutions of (2.3). Let $(u_n, f_n, \Psi_n) \to (u, f, \Psi)$ and $(\widetilde{u}_n, \widetilde{f}_n, \widetilde{\Psi}_n) \to (\widetilde{u}, f, \Psi)$ in sense of (2.5) Obviously we can choose subsequences $\{\Psi_n\}$ and $\{\widetilde{\Psi}_n\}$ such that $\sup(\Psi_n, S_T) \le \inf(\widetilde{\Psi}_n, S_T), n = 1, 2, \ldots$. Then we can apply Proposition 3.1, i.e., for any $\tau \in (0, T]$

$$\int_\Omega (u_n - \widetilde{u}_n)^+ dx|^{t=\tau} \le \int_\Omega (\Psi_n - \widetilde{\Psi}_n) dx + \int_0^\tau \int_\Omega |f_n - \widetilde{f}_n| dx dt.$$

Letting $n \to \infty$ and using (2.5) we obtain that $(u - \widetilde{u})^+ = 0$ *a.e.*in Q_T. Theorem 3.1 is proved.

Remark 3.1. In some sense Definition 2.4 of quasistrong solution and Theorem 3.1 are similar to definition of "the limit of strong solutions" and the corresponding uniqueness theorem given by Bamberger in[4] for equation (1.3). However instead of our condition $\inf(u, Q_T) > 0$ in the definition of strong solution Bamberger used condition "u has $\partial u/\partial t \in L_1(Q_T)$".

4. Hölder estimates for DNPE.

Establishing Hölder estimates is the key question of the regularity problem for DNPE not only in view of the fact that hölderness is the best possible smoothness for a large class of such equations. In fact Hölder estimates for bounded generalized solutions are crucial and the best difficult step in proving of existence of regular solution of Cauchy-Dirichlet problem for DNPE.

Directly from our results [6]-[8] for DNPE of the full type

$$\frac{\partial u}{\partial t} - \operatorname{div}\, a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u) = 0 \tag{4.1}$$

with the limit growth conditions we can derive the following estimates for equations of the type (1.1), (1.2). Introduce condition

$$(H)\begin{cases} a^i(u,p) \text{ are continuous on } \mathbb{R}\times\mathbb{R}^n, i=1,\dots,n;\\ a(u,p)\cdot p \geq \nu_0|u|^l|p|^m - \varphi_0, \nu_0>0;\\ |a(u,p)| \leq \mu_1|u|^l|p|^{m-1} + |u|^\alpha\varphi_1, \alpha = \dfrac{l}{m};\\ |f(x,t)| \leq \varphi_2, \varphi_i = \text{const } \geq 0, l=0,1,2;\\ m>1, l\geq 0. \end{cases}$$

For the sake of brevity we stated here only global Hölder estimates (i.e. Hölder estimates up to the boundary) for equations (1.1), (1.2).

Theorem 4.1. *([6], [7]). Assume that $m+l\geq 2$ and let conditions (H) and (Ω) hold. Let u be a weak solution of equation (1.1) (in sense of Definition 2.1) such that its trace on the parabolic boundary Γ_T is Hölder continuous. Then function u belongs to the class $C^{\lambda,\lambda/m}(\overline{Q}_T)$ for some $\lambda\in(0,1)$. Moreover*

$$\langle u\rangle_{\lambda,\overline{Q}_T} \doteq \sup_{(x,t),(x',t')\in Q_T} \frac{|u(x,t)-u(x',t')|}{(|x-x'|^m+|t-t'|)^{\lambda/m}} \leq K \tag{4.2}$$

where $\lambda\in(0,1)$ and $K>0$ depend only on $\sup(u,Q_T)$, *n, m, l, ν_0, μ_0, φ_0, φ_1, φ_2, $|\Omega|$, T, α_0, ρ_0 and the Hölder constant and exponent of the trace of function u on Γ_T.*

Theorem 4.2. *([8]). Assume that $m+l<2$ and let conditions $(H),(M),(L)$, and (Ω) hold. Let $u\in W^{1,0}_m(Q_T)$ be a weak solution of equation (1.1) (in sense of Definition 2.1) such that its trace on the parabolic boundary Γ_T is Höldercontinuous. Then function u belongs to $C^{\lambda,\lambda/m}(Q_T)$ for some $\lambda\in(0,1)$. Moreover estimate (4.2) holds with some constants $\lambda\in(0,1)$ and $K>0$ depending on the same data as in the case of Theorem 4.1 (in particular λ and K are independent of $\|\nabla\|_{L_m(Q_T)}$ and constant Λ from condition (L).*

Remark 4.1. Theorems 4.1 and 4.2 remain valid if the inequalities in condition (H) are fulfilled only for values u from the range of weak solution under consideration.

Remark 4.2. The proofs of Theorems 4.1 and 4.2 (as well as Hölder estimates for general equations (4.1) in [6] - [8]) are concerned with some development of the methods of papers by De Giorgi, Ladyzhenskaya-Ural'tseva (see [12]), DiBenedetto [13], Chen-DiBenedetto [14], and [5].

Remark 4.3. Other results on Hölder estimates for some classes of DNPE are obtained in [15], [16].

5. The auxiliary Cauchy-Dirichlet problem

This section has an auxiliary character. At first we prove some generalization of well-known Friedrieks inequality (cf. [12]], p. 529 -530) which will be used not only in this section.

Lemma 5.1. *Let $\{\Psi_\kappa(x)\}$ is an orthonormal basis in $L_2(\Omega)$. Let $\beta \geq 0$ is fixed. Then for any $\varepsilon > 0$ there exists number $\mathcal{N}_\varepsilon$ such that for function $u(x)$ satisfying condition*

$$|u|^\beta u \in \overset{0}{W}{}^1_m(\Omega), m > 1, 1/m < 1/n + \frac{1+\beta}{2}, \tag{5.1}$$

we have

$$\|u\|_{L_2(\Omega)} \leq \left(\sum_{k=1}^{\mathcal{N}_\varepsilon}(u,\Psi_\kappa)^2\right)^{1/2} + \varepsilon\|\nabla(|u|^\beta u)\|^{\frac{1}{\beta+1}}_{L_m(\Omega)} \tag{5.2}$$

where $(u,\Psi_\kappa) \doteqdot \int_\Omega u\Psi_\kappa dx$ and $\mathcal{N}_\varepsilon$ does not depend on u.

Proof. It is sufficient to prove that for any $\delta > 0$ and $\varepsilon > 0$

$$\|u\|_{L_2(\Omega)} \leq (1+\delta)\left(\sum_{k=1}^{\mathcal{N}_{\varepsilon,\delta}}(u,\Psi_\kappa)^2\right)^{1/2} + \varepsilon\|\nabla(|u|^\beta u)\|^{\frac{1}{\beta+1}}_{L_m(\Omega)}. \tag{5.3}$$

Really for function $v \doteqdot |u|^\beta u$ we have well-known Sobolev inequality

$$\|v\|_{L_2(\Omega)} \leq c\|\nabla v\|_{L_m(\Omega)}, r \doteqdot \frac{2}{1+\beta} > 0, \tag{5.4}$$

because from condition $1/m < 1/n + \frac{1+\beta}{2}$ it follows that $1/r > 1/m - 1/n$. Rewrite (5.4) as

$$\|u\|_{L_2(\Omega)} \leq c_1\|\nabla(|u|^\beta u)\|^{\frac{1}{\beta+1}}_{L_m(\Omega)}. \tag{5.5}$$

Then from (5.3) and (5.5) it follows that

$$\|u\|_{L_2(\Omega)} \leq \left(\sum_{k=1}^{\mathcal{N}_\varepsilon}(u,\Psi_\kappa)^2\right)^{1/2} + (c_1\delta+\varepsilon)\|\nabla(|u|^\beta u)\|^{\frac{1}{\beta+1}}_{L_m(\Omega)}, \tag{5.6}$$

i.e., result of Lemma 5.1 is true. So prove that (5.3) holds. If (5.3) is violated then there exist $\varepsilon_0 > 0$ and sequence of functions $\{u_\nu(\kappa)\}$ satisfying condition (5.1) such that for some fixed $\delta > 0$ and any $\nu = 1, 2, \ldots$

$$(5.7) \qquad \|u_\nu\|_{L_2(\Omega)} > (1+\delta)\left(\sum_{k=1}^{\nu}(u_\nu, \Psi_k)^2\right)^{1/2} + \varepsilon_0 \|\nabla(|u_\nu|^\beta u_\nu\|_{L_m(\Omega)}^{\frac{1}{\beta+1}}.$$

Then for functions $\hat{u}_\nu$, where $\hat{u}_\nu = u_\nu / \|u_\nu\|_{L_2(\Omega)}$, we have

$$(5.8) \qquad 1 = \|\hat{u}_\nu\|_{L_2(\Omega)} > (1+\delta)\left(\sum_{k=1}^{\nu}(\hat{u}_\nu, \Psi_k)^2\right)^{1/2} + \varepsilon_0 \|\nabla(|\hat{u}_\nu|^\beta \hat{u}_\nu)\|_{L_m(\Omega)}^{\frac{1}{\beta+1}}.$$

Denote $v_\nu = |\hat{u}_\nu|^\beta \hat{u}_\nu$. In view of (5.8) the norms $\|\nabla v_\nu\|_{L_m(\Omega)}$ are uniformly bounded and hence (taking into account that $1/r > 1/m - 1/n$ for $r = 2/(1+\beta)$) there exists some subsequence $\{v_{\nu_s}\}$ converging strongly in $L_r(\Omega)$. It is easy to see that then subsequence $\{u_{\nu_s}\}$ converges strongly in $L_2(\Omega)$ to some function $\hat{u} \in L_2(\Omega)$. Really in view of a strict monotonicity of function $x \to |x|^\beta x, \beta > 0$, we have

$$c^{-1}|\hat{u}_\nu - \hat{u}_\mu|^{2+\beta} \le [|\hat{u}_\nu|^\beta \hat{u}_\nu - |\hat{u}^\mu|^\beta \hat{u}_\mu](\hat{u}_\nu - \hat{u}_\mu) \le |v_\nu - v_\mu||\hat{u}_\nu - \hat{u}_\mu|$$

with some constant $c > 0$ and hence

$$|\hat{u}_\nu - \hat{u}_\mu|^2 \le c|v_\nu - v_\mu|^r, r = \frac{2}{1+\beta}.$$

Moreover it is obvious that $|\hat{u}\|_{L_2(\Omega)} = 1$. Functions $P_{\nu_s}\hat{u}_{\nu_s} \doteq \sum_{k=1}^{\nu_s}(\hat{u}_{\nu_s}, \Psi_k)\Psi_k$ also converge strongly in $L_2(\Omega)$ to $\hat{u}$ because

$$\|\hat{u} - P_{\nu_s}\hat{u}_{\nu_s}\|_{L_2(\Omega)} = \|P_{\nu_s}(\hat{u} - \hat{u}_{\nu_s}) + (E - P_{\nu_s})\hat{u}\|_{L_2(\Omega)} \le$$
$$\le \|\hat{u} - \hat{u}_{\nu_s}\|_{L_2(\Omega)} + \|(E - P_{\nu_s})\hat{u}\|_{L_2(\Omega)} \to 0 \text{ as } s \to \infty.$$

Then

$$(5.9) \qquad \left(\sum_{k=1}^{\nu_s}(\hat{u}_{\nu_s}, \Psi_k)^2\right)^{1/2} = \|P_{\nu_s}\hat{u}_{\nu_s}\|_{L_2(\Omega)} \to \|\hat{u}\|_{L_2(\Omega)} = 1 \text{ as } s \to \infty.$$

In view of (5.8), (5.9) we obtain then impossible inequality $1 \ge 1 + \delta$. Lemma 5.1 is proved.

Now we consider the Cauchy-Dirichlet problem

$$F[u] = f \text{ in } Q_T, u = \Psi \text{ on } \Gamma_T,$$

assuming

0') functions $a^i(u,p)$ are continuous on $\mathbb{R} \times \mathbb{R}^n$;

1') for any $u \in \mathbb{R}, p \in \mathbb{R}^n$

$$a(u,p) \cdot p \ge \nu_0 |p|^m - \mu_0, \nu_0 > 0; |a(u,p)| \le \mu_1(|p|^{m-1} + 1);$$

2') for any $u \in \mathbb{R}$ and any $p, q \in \mathbb{R}^n$

$$[a(u,p) - a(u,q)] \cdot (p - q) \ge \nu_1 |p - q|^m, \nu_1 > 0;$$

3') for any $u, v \in \mathbb{R}$ and any $p, q \in \mathbb{R}^n$

$$|a(u,p) - a(v,p)| \le \Lambda |u - v|(|p|^{m-1} + 1), \Lambda \ge 0;$$

4') $m > max(1, 2n/(n+2))$.

Proposition 5.1. *Let f is measurable and bounded in Q_T and let $\Psi \in \overset{0}{W}{}_2^1(Q_T)$. Assume that conditions 0') -4') hold. Then Cauchy-Dirichlet problem (5.10) has exactly one generalized (in sense of Definition 3.1) generalized solution u such that $u \in C([O,T]; L_2(\Omega))$.*

Proof. Uniqueness of generalized solution of (5.10) follows from Proposition 2.2. So we have to prove only existence of solution cited. The forthcoming proof is a suitable adaptation of the proof of theorem 6.7 of Chapter 5 in [12].

Let $\{\Psi_k(x)\}$ is a basis in $\overset{0}{W}{}_m^1(\Omega)$ such that $\int_\Omega \Psi_k \Psi_l dx = \delta_k^l$, where δ_k^l is the Kronecker delta, and

$$\sup(|\Psi_k|, \Omega) + \sup(|\nabla \Psi_k|, \Omega) \leq c_k = \text{ const }, k = 1, 2, \dots .$$

Set

$$u^{\mathcal{N}} = \sum_{k=1}^{\mathcal{N}} c_k^{\mathcal{N}}(t)\Psi_k(x) \tag{5.11}$$

where $\{c_k^N(t)\}_{k=1,\dots,N}$ is solution of the system of ordinary differential equations

$$(u_t^N, \Psi_k) + \left(a^i(u^N, \nabla u^N), \frac{\partial \Psi_k}{\partial x_i}\right) = (f, \Psi_k), k = 1, \dots, N \tag{5.12}$$

with initial conditions

$$c_k^N(0) = (\Psi(x,0), \Psi_k), k = 1, \dots, N. \tag{5.13}$$

From conditions of Proposition 5.1 it follows that the second and third terms in (5.12) are bounded and measurable functions of variables t, c_k^N on any set $[0,T] \times \{|c_k^N| \leq \text{ const }, k = 1, \dots, N\}$; moreover these functions are continuous in $c_k^N, k = 1, \dots, N$. Therefore existence at least of one solution of (5.12), (5.13) will be established if we could show that all possible solutions of this problem are uniformly bounded on $[0,T]$. Exactly in the same way as in [12], p. 533 - 535, we can prove that a priori estimate

$$\sup_{t\in[0,T]} \|u^N\|_{L_2(\Omega)}^2 + \|\nabla u^N\|_{L_m(Q_T)}^m \leq c \tag{5.14}$$

holds with some constant c independent of N. Then from (5.14) it follows that

$$\sup_{t\in[0,T]} \sum_{k=1}^{N} |c_k^N(t)|^2 = \sup_{t\in[0,T]} \|u^N\|_{L_2(\Omega)}^2 \leq c \tag{5.15}$$

and hence existence at least of one solution (5.12), (5.13) is established. From (5.14) it follows (see [12],p.534) that

$$\|u^N\|_{L_{m(n+2)/n}(Q_T)} \leq c \tag{5.16}$$

where constant c is independent of N. Moreover for any fixed k functions

$$l_{N,k}(t) = (u^N(x,t), \Psi_k(x)), N, k = 1, 2, \dots . \tag{5.17}$$

are equicontinuous (with respect to N) in t on $[0,T]$. Together with (5.14) it gives possibility (see [12], p. 535) to choose some subsequence $\{u^N\}$ that converges weakly in $L_2(\Omega)$ uniformly with respect to t on $[0,T]$ to some function u such that

$$\sup(\|u\|_{L_2(\Omega)}, [0,T]) \leq c. \tag{5.18}$$

Moreover using again (5.14) we can count that

$$\frac{\partial u^N}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ weakly in } L_m(Q_T) \text{ as } N \to \infty \tag{5.19}$$

and hence $u \in \overset{0}{W}{}^{1,0}_m(Q_T)$ and

$$\|\nabla u\|_{L_m(Q_T)} \leq c \tag{5.20}$$

with some constant c depending only on the data (see [12], p. 535).

Obviously from (5.12) it follows that the integral identity

$$\int_\Omega u^N \varphi dx|_0^\tau + \iint_{Q_\tau} [-u^N \varphi_t + a(u^N, \nabla u^N) \cdot \nabla \varphi] dxdt = \iint_{Q_\tau} f \varphi dxdt \tag{5.21}$$

holds for any $\tau \in (0,T]$ and $\varphi = \sum_{k=1}^N d_k(t) \Psi_k(x)$ where $d_k(t)$ are arbitrary continuous in t on $[0,T]$ functions having bounded on $[0,T]$ generalized derivatives $d_k'(t)$. Denote the class of such functions φ as $\mathcal{P}_N$. Obviously u^N belong to $\mathcal{P}_N$. Denote $A_N^i \doteq a^i(u^N, \nabla u^N), i = 1, \dots, N$. In view of the second inequality in condition 1′) and estimate (5.14) we have uniform (with respect to N) estimate

$$\|A_i^N\|_{L_{m'}(Q_T)} \leq c, i = 1, \dots, N, N = 1, 2, \dots . \tag{5.22}$$

Therefore we can count that there exist functions $A_i \in L_{m'}(Q_T)$ such that

$$A_i^N \to A_i \text{ weakly in } L_{m'}(Q_T). \tag{5.23}$$

Using estimate (5.14) and taking into account that $u^N \to u$ weakly in $L_2(\Omega)$ (uniformly with respect to t on $[0,T]$) we derive from inequality (5.2) in the case $\beta = 0$ for difference $u^N - u^{N_1}$ that

$$u^N \to u \text{ strongly in } L_{2,m}(Q_T) \tag{5.24}$$

and hence we can count that

$$u^N \to u \text{ strongly in } L_2(\Omega) \text{ for } a.e.t \in [0,T] \tag{5.25}$$

and

$$u^N \to u \text{ a.e. in } Q_T; \tag{5.26}$$

moreover in view of (5.16) and condition 4')

$$u^N \to u \text{ weakly in } L_2(Q_T). \tag{5.27}$$

Then from (5.21) and (5.23)-(5.27) we can conclude that for a.e. $\tau \in (0,T]$ and $\varphi \in \cup_{k=1}^{\infty} \mathcal{P}_k$

$$\int_\Omega u\varphi dx|_0^\tau + \iint_{Q_\tau} (-u\varphi_t + A_i\varphi_{x_i})dxdt = \iint_{Q_\tau} f\varphi dxdt. \tag{5.28}$$

In the same way as in [12], p. 538 we can derive from (5.28), (5.18) that

$$u \in C([0,T]; L_2(\Omega)) \tag{5.29}$$

and to prove that identity (5.28) holds for any $\tau \in (0,T]$; moreover we establish that for every $\tau \in (0,T]$

$$\frac{1}{2}\int_\Omega u^2 dx|_0^\tau + \iint_{Q_\tau} A_i u_{x_i} dxdt = \iint_{Q_\tau} fudxdt. \tag{5.30}$$

To prove that u is a generalized solution of (5.10) it is sufficient to establish that

$$\iint_{Q_\tau} A_i\varphi_{x_i} dxdt = \iint_{Q_\tau} a^i(u,\nabla u)\varphi_{x_i} dxdt \tag{5.31}$$

for any $\varphi \in \bigcup_{k=1}^{\infty} \mathcal{P}_k$ because $\bigcup_{k=1}^{\infty} \mathcal{P}_k$ is dense in $\overset{0}{W}{}^1_m(Q_T)$. To prove (5.31) it is sufficient to establish that

$$\partial u^N/\partial x_i \to \partial u/\partial x_i \text{ a.e. in } Q_\tau, i = 1, \dots, N \tag{5.32}$$

because in view of (5.32), (5.26), the continuity functions $a^i(u,p)$, condition 1'), estimate (5.14) and the Vitali theorem we obtain that for any $\varphi \in \bigcup_{k=1}^{\infty} \mathcal{P}_k$

$$\lim_{N\to\infty} \iint_{Q_\tau} a^i(u^N, \nabla u^N)\varphi_{x_i} dxdt = \iint_{Q_\tau} a^i(u,\nabla u)\varphi_{x_i} dxdt.$$

On the other hand in view of (5.23)

$$\lim_{N\to\infty} \iint_{Q_\tau} a^i(u^N, \nabla u^N)\varphi_{x_i} dxdt = \iint_{Q_\tau} A_i\varphi_{x_i} dxdt.$$

Hence (5.32) implies (5.31). The remainder of this section is devoted to proving of (5.32).

Choosing $\varphi = u^N$ in (5.21) we obtain

$$\frac{1}{2}\int_\Omega (u^N)^2 dx|_0^\tau + \iint_{Q_\tau} a(u^N, \nabla u^N)\cdot \nabla u^N dxdt = \iint_{Q_\tau} f u^N dxdt. \tag{5.33}$$

Using (5.25), (5.27) we derive from (5.33) and (5.30) that for any $\tau \in (0, T]$

$$\lim_{N\to\infty}\iint_{Q_\tau} a(u^N, \nabla u^N)\cdot \nabla u^N dxdt = \iint_{Q_\tau} A_i u_{x_i} dxdt. \tag{5.34}$$

Using now condition 2') we have

(5.35)

$$\nu_1 \iint_{Q_\tau} |\nabla u^N - \nabla u|^m dxdt \le \iint_{Q_\tau} [a(u^N, \nabla u^N) - a(u^N, \nabla u)]\cdot(\nabla u^N - \nabla u)dxdt.$$

Using (5.19), (5.23), (5.34) and taking into account (in view of 1'), (5.14) and (5.26)) that

$$a^i(u^N, \nabla u) \to a^i(u, \nabla u) \text{ strongly in } L_{m'}(Q_T) \text{ as } N \to \infty \tag{5.36}$$

we derive from (5.35)

$$\lim_{N\to\infty}\iint_{Q_\tau} |\nabla u^N - \nabla u|^m dxdt = 0. \tag{5.37}$$

But from (5.37) it follows that (5.32) holds for some subsequence $\{u^N\}$. Proposition 5.1 is proved.

6. A priori estimates for solutions of regularized Cauchy-Dirichlet problems

In view of Theorem 3.1 to prove theorem 1.1 it is sufficient to establish the following

Theorem 6.1. *Let conditions $(\Omega), (BI), (RHS)$ and 0) - 4) hold. Then Cauchy-Dirichlet problem*

$$F[u] = f \text{ in } Q_T, u = \Psi \text{ on } \Gamma_T \qquad (CD)$$

has at least one regular (in sense of Definition 2.5) solution.

The result of Theorem 6.1 correspondent to the case

$$m \ge 2, l \ge 0 \tag{6.1}$$

can be derived from the proof of the main theorem of paper [10] if to use Theorem 4.1 of given paper. Therefore we shall prove Theorem 6.1 only in the case when

$$m \in (1, 2), l \ge 0. \tag{6.2}$$

It is easy to see that

$$\omega \subset (1 < m < 2) \times (l \geq 0).$$

The proof of theorem 6.1 correspondent to the case (6.2) can be easily transformed in one applicable in the case (6.1).

In the remainder of this paper we assume that all conditions $(\Omega), (BI), (RHS)$, 0) -4) of Theorem 6.1 and also condition (6.2) are fulfilled. Consider the following regularized Cauchy-Dirichlet problems

$$F_{\delta,\epsilon,N}[u] \doteqdot \frac{\partial u}{\partial t} - \delta \nabla u - \text{ div } a(\chi(u), \nabla u) = f \text{ in } Q_T, u = \Psi + \epsilon \text{ on } \Gamma_T, (RCD)_{\delta,\epsilon,N}$$

where

$$\delta > 0, \chi(u) = \min\{\max(u,\epsilon), N\}, \epsilon > 0, N > \epsilon. \tag{6.3}$$

Without loss of generality we can and shall count that $\delta \leq 1, \epsilon \leq 1$. It is easy to see that in view of conditions 0) - 4) and (6.2) and structure of the left-hand side of equation in $(RCD)_{\delta,\epsilon,N}$ assumptions 0') - 3') of Proposition 5.1 are fulfilled in the case $m = 2$ because $\epsilon \leq \chi(u) \leq N$ and $|p|^{m-1} + 1 \leq |p| + 1$ for any $m \in (1, 2)$. Denote $v = u - \epsilon$ and consider Cauchy-Dirichlet problem

$$\partial v / \partial t - \delta \nabla v - \text{ div } a(\chi(v + \epsilon), \nabla v) = f \text{ in } Q_T, v = \Psi \text{ on } \Gamma_T, \tag{6.4}$$

where $\Psi \in \overset{0}{W}{}_2^1(Q_T)$. In view of previous conclusions it follows obviously that for the problem (6.4) all conditions of Proposition 5.1 are fulfilled in the case $m = 2$. Hence there exists exactly one generalized solution v of this problem (such that $v \in C([0,T]; L_2(\Omega)) \cap \overset{0}{W}{}_2^1(Q_T)$. But then Cauchy-Dirichlet problem $(RCD)_{\delta,\epsilon,N}$ has exactly one generalized solution u such that $u \in C([0,T]; L_2(\Omega)) \cap W_2^{1,0}(Q_T)$, i.e., we prove the following

Lemma 6.1. *For any $\delta > 0, \epsilon > 0, N > \epsilon$ Cauchy-Dirichlet problem $(RCD)_{\delta,\epsilon,N}$ has exactly one generalized solution $u \in C([0,T]; L_2(\Omega)) \cap W_2^{1,0}(Q_T)$.*

In the remainder of this section we consider problem $(RCD)_{\delta,\epsilon,N}$ for $\delta \geq 0, \epsilon > 0, N > \epsilon$. Now the term "generalized solution u" means in particular that $u \in C([0,T]; L_2(\Omega)) \cap W_2^{1,0}(Q_T)$ in the case $\delta > 0$ and $u \in C([0,T]; L_2(\Omega)) \cap W_m^{1,0}(Q_T)$ in the case $\delta = 0$.

Lemma 6.2. *Let u be a generalized solution of $(RCD)_{\delta,\epsilon,N}$ for any fixed $\delta \geq 0, \epsilon > 0, N > \epsilon$. Then*

$$\inf(u, Q_T) \geq \epsilon. \tag{6.5}$$

Proof. Obviously that conditions of Theorem 6.1 imply validity of assumptions $(G), (M)$, and (L) of Proposition 3.1 for the operator $F_{\delta,\epsilon,N}[u]$ (with $m = 2$ if $\delta > 0$). Then taking into account that $F_{\delta,\epsilon,N}[u] = f, F_{\delta,\epsilon,N}[\epsilon] = 0$, and $u = \epsilon$ on S_T, we can apply Proposition 3.1 for $u_1 = \epsilon, u_2 = u$ and $f_1 = 0, f_2 = f$. Using that $u_1 = \epsilon \leq \Psi + \epsilon = u_2$ on $\Omega \times \{t = 0\}$ (because $\Psi \geq 0$) we derive from (3.2) that $(\epsilon - u)^+ \leq 0$ a.e. in Q_T, i.e., $u \geq \epsilon$ a.e. in Q_T. Lemma 6.1 is proved.

Lemma 6.3. *There exist constants c_1 and c_2 depending on n, m, l, the parameters from conditions 1) - 3), and* $\sup(\Psi, \overline{Q}_T)$ *such that for any generalized solution u of $(RCD)_{\delta,\epsilon,N}$ with any fixed $\delta \geq 0, \epsilon > 0, N \geq c_1$ we have*

$$\sup(u, Q_T) \leq c_1 \tag{6.6}$$

and

$$\epsilon^l \iint_{Q_T} |\nabla u|^m dxdt \leq \iint_{Q_T} |\nabla u^{\alpha+1}|^m dxdt \leq c_2, \alpha = \frac{l}{m}. \tag{6.7}$$

Proof. The proof of validity of estimates (6.6) and (6.7) in the case $m + l \geq 2$ is given in [10]. The case $m + l < 2$ required to find new (a more difficult) version of the Moser method of establishing L_∞- estimates. It was made in paper [9]. Lemma 6.3 in the case $m + l < 2$ follows from theorems 1.1 and 1.2 of [9].

Remark 6.1. In the remainder of this paper we consider $(RCD)_{\delta,\epsilon,N}$ with $N = c_1$ where constant c_1 is defined by Lemma 6.2. Then in view of estimates (6.5) and (6.6) we can rewrite $(RCD)_{\delta,\epsilon,N}$ as

$$F_{\delta,\epsilon}[u] \doteqdot \partial u/\partial t - \delta \nabla u - \text{div } a(u, \nabla u) = f \text{ in } Q_T, u = \Psi + \epsilon \text{ on } \Gamma_T, (RCD)_{\delta,\epsilon}$$

where $\delta > 0, \epsilon > 0$.

Lemma 6.4. *Let u be a generalized solution of Cauchy-Dirichlet problem $(RCD)_{\delta,\epsilon}$ for $\delta = 0, \epsilon > 0$. There exist constant $\lambda \in (0,1)$ and $K > 0$ independent of ϵ such that (see (4.21)*

$$\langle u \rangle_{\lambda, \overline{Q}_T} \leq K. \tag{6.8}$$

Proof. In view of conditions 1)-3), Remark 6.1, estimates (6.5)-(6.7) and Remark 4.1 we can apply either Theorem 4.1 of Theorem 4.2 and hence establish (6.8) with some $\lambda \in (0,1)$ and $K > 0$ independent of ε. Lemma 6.3 is proved.

7. The passing to the limit as $\delta \to 0$

In this section we show that generalized solutions u_δ of Cauchy-Dirichlet problems $(RCD)_{\delta,\epsilon}$ (for any fixed $\epsilon > 0$) tend to generalized solution of Cauchy-Dirichlet problem

$$F_\epsilon[u] \doteqdot \partial u/\partial t - \text{div } a(u, \nabla u) = f \text{ in } Q_T, u = \Psi + \epsilon \text{ on } \Gamma_T (RCD)_\epsilon$$

as $\delta \to 0$. For proving this we use estimates (6.5) - (6.7) and Lemma 5.1 with appropriate $\beta > 0$.

Obviously functions u_δ satisfy for any $\tau \in (0, T]$ and every $\phi \in \overset{0}{W}{}_2^1(Q_T)$ the integral identity

$$\int_\Omega u_\delta \phi dx|_0^\tau + \iint_{Q_T} [-u_\delta \psi + \delta \nabla u_\delta \cdot \nabla \phi + a(u_\delta, \nabla u_\delta) \cdot \nabla \phi - f\phi] dxdt = 0. \tag{7.1}$$

Set in (7.1) $\phi = \Psi(x) \in C_0^1(\Omega)$. Then from (7.1), condition 1) and estimates (6.6), (6.7) it follows that for any $t_1, t_2 \in [0,T]$ we have

$$
\begin{aligned}
|\int_\Omega u_\delta \Psi dx|_{t_1}^{t_2}| \le c\int_{t_1}^{t_2}\int_\Omega [|\nabla u_\delta| + |\nabla u_\delta^{\alpha+1}|^{m-1} + 1]dxdt \le \\
\le c[(|t_2 - t_1||\Omega|)^{1/2} + (|t_2 - t_1||\Omega|)^{1/m} + |t_2 - t_1||\Omega|]. \qquad (7.2)
\end{aligned}
$$

From (7.2) it follows that integrals $\int_\Omega u_\delta \Psi dx, \delta \in (0,1)$, are equicontinuous (with respect to δ) in t on $[0,T]$ for any fixed $\Psi(x) \in C_0^1(\overline{\Omega})$. Using density of $C_0^1(\bar{\Omega})$ in $L_2(\Omega)$ and uniform boundedness of $\{u_\delta\}$ in Q_T (see (6.6)) we can derive from here that there exists a sequence $\{u_\delta\}$ which converges weakly in $L_2(\Omega)$ uniformly with respect to t on $[0,T]$ to some function u satisfying inequality (5.18) with a constant c independent of δ. Moreover in view of (6.5) - (6.7) we can count that

$$\nabla u_\delta^{\alpha+1} \to \nabla u^{\alpha+1} \text{ weakly in } L_m(Q_T) \text{ as } \delta \to 0; \qquad (7.3)$$

$$\nabla u_\delta \to \nabla u \text{ weakly in } L_m(Q_T) \text{ as } \delta \to 0; \qquad (7.4)$$

$$\sqrt{\delta}\nabla u_\delta \to 0 \text{ weakly in } L_2(Q_T) \text{ as } \delta \to 0; \qquad (7.5)$$

$$\sup(u_\delta, Q_T) + \sup(u, Q_T) \le c_1; \qquad (7.6)$$

$$
\begin{aligned}
\epsilon^l \iint_{Q_T} |\nabla u_\delta|^m dxdt + \epsilon^l \iint_{Q_T} |\nabla u|^m dxdt + \iint_{Q_T} |\nabla u_\delta^{\alpha+1}|^m dxdt + \\
+ \iint_{Q_T} |\nabla u^{\alpha+1}|^m dxdt \le c_2 \qquad (7.7)
\end{aligned}
$$

where $\alpha = l/m$. Denote $A_\delta^i \doteqdot a^i(u_\delta, \nabla u_\delta), i = 1, \dots, n$. In view of condition 1) and (7.6), (7.7) we have the uniform with respect to δ estimate

$$\|A_\delta^i\|_{L_{m'}(Q_T)} \le c, i = 1, \dots, n, \delta > 0. \qquad (7.8)$$

Then we can count that there exist functions $A^i \in L_{m'}(Q_T), i = 1, \dots, n$, such that

$$A_\delta^i \to A^i \text{ weakly in } L_{m'}(Q_T) \text{ as } \delta \to 0, i = 1, \dots, n. \qquad (7.9)$$

On the other hand from (7.6), (7.7) it follows that for any $\delta, \delta' > 0$

$$\iint_{Q_T} |\nabla(|u_\delta - u_{\delta'}|^\beta (u_\delta - u_{\delta'}))|^m dxdt \le c, \beta \doteqdot \frac{\sigma}{\sigma+2}, \qquad (7.10)$$

with some constant c independent of δ. Really in view of definition β we have $(\beta+1)/2 = \frac{\sigma+1}{\sigma+2}$ and hence conditions $m > 1, 1/m < \frac{1}{n} + \frac{1+\beta}{2}$ of Lemma 5.3 are fulfilled for $\beta = \frac{\sigma}{\sigma+2}$ in view of condition 4). It is easy to see that from (7.7) it follows that constant c in (7.10) is independent of δ. Using (7.10) and taking into account that $u_\delta \to u$ weakly in $L_2(\Omega)$ uniformly with respect to t on $[0,T]$ we derive from inequality (5.2) in the case $\beta = \frac{\sigma}{\sigma+2}$ for difference $u_\delta - u_{\delta'}$ that

$$u_\delta \to u \text{ strongly in } L_{2,m}(Q_T), \tag{7.11}$$

$$u_\delta \to u \text{ a.e. in } Q_T, \tag{7.12}$$

$$u_\delta \to u \text{ weakly in } L_2(Q_T), \tag{7.13}$$

$$u_\delta \to u \text{ strongly in } L_2(\Omega) \text{ for a.e. } t \in [0,T]. \tag{7.14}$$

Then from (7.2), (7.5), and (7.12) - (7.14) we can derive that for a.e. $\tau \in (0,T]$ and any $\phi \in \overset{0}{W}{}_2^1(Q_T)$

$$\int_\Omega u\phi dx|_0^\tau + \iint_{Q_\tau} (-u\phi_t + A^i\phi_{x_i} - f\phi)dxdt = 0. \tag{7.15}$$

The following proposition is well-known (see, for example, [10]).

Proposition 7.1. *Let function $g(u)$ satisfies a Lipschitz condition uniformly on $\mathbb{R}$ and its derivative $g'(u)$ be continuous everywhere on $\mathbb{R}$ with possible exception of finitely many points at which $g'(u)$ has a discontinuity of the first order. Let $u \in C([0,T]; L_2(\Omega)) \cap W_m^{1,0}(Q_T), \varphi \in W_m^1(Q_T), f_i \in L_{m'}(Q_T), i = 0,1,\dots,n, 1/m + 1/m' = 1, m > 1$. Assume that for any $t_1, t_2 \in [0,T]$ and any $\phi \in \overset{0}{W}{}_m^1(Q_T)$*

$$\int_\Omega u\phi dx|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega (-u\phi_t + f_i\phi_{x_i} + f_0\phi)dxdt = 0$$

and let $u = \varphi$ on S_T. Then for any $t_1, t_2 \in [0,T]$ we have

$$\int_\Omega [G(u) - ug(\varphi)]dx|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega [ug'(\varphi)\varphi_t + f_i(g'(u)u_{x_i} - g'(\varphi)\varphi_{x_i}) + f_0(g(u) - g(\varphi))]dxdt = 0 \tag{7.16}$$

where $G(u) = \int_0^u g(\xi)d\xi$.

Using Proposition 7.1 we can conclude (in the same way as in [12], p. 538) that in view of (7.15) and (5.18) or (7.6) condition (5.29) holds for function u. Moreover using Proposition 7.1 we can derive from (7.15) that for any $\tau \in (0,T]$ we have

$$\int_\Omega \left(\frac{1}{2}u^2 - u\epsilon\right) dx|_0^\tau + \iint_{Q_\tau} [A^i u_{x_i} - f(u - \epsilon)]dxdt = 0. \tag{7.17}$$

In view of (5.29) integral identity (7.15) holds for any $\tau \in (0,T]$.

To prove that u is a generalized solution of $(RCD)_\epsilon$ it is sufficient to establish that

$$\iint_{Q_\tau} A^i \phi_{x_i} dxdt = \iint_{Q_\tau} a^i(u, \nabla u)\phi_{x_i} dxdt \tag{7.18}$$

for any $\phi \in C_0^1(\Omega)$ (because $C_0^1(\Omega)$ is dense in $\overset{0}{W}{}^1_m(Q_T)$). To prove (7.18) it is sufficient to establish that

$$\partial u_\delta/\partial x_i \to \partial u/\partial x_i \text{ a.e. in } Q_T, i = 1, \dots, N, \tag{7.19}$$

because in view of (7.19), (7.12), the continuity functions $a^i(u,p)$, condition 1'), estimates (7.6) and (7.7) and the Vitali theorem we obtain that for any $\phi \in \overset{0}{W}{}^1_m(Q_T)$ and every $\tau \in (0,T]$

$$\lim_{\delta\to 0} \iint_{Q_\tau} a^i(u_\delta, \nabla u_\delta)\phi_{x_i} dxdt = \iint_{Q_\tau} a^i(u, \nabla u)\phi_{x_i} dxdt. \tag{7.20}$$

On the other hand in view of (7.9) the left-hand side in (7.20) is equal to the left-hand side in (7.18). Hence (7.19) implies (7.18).

Choosing $\phi = u_\delta - \epsilon$ in (7.1) we obtain with the aid of Proposition 7.1 that

$$\int_\Omega \left(\frac{1}{2}u_\delta^2 - u_\delta \epsilon\right) dx|_0^\tau + \iint_{Q_\tau} \left[a^i(u_\delta, \nabla u_\delta)\frac{\partial u_\delta}{\partial x_i} - f(u_\delta - \epsilon)\right] dxdt = 0. \tag{7.21}$$

Using (7.13), (7.14) we derive from (7.21) and (7.17) that for any $\tau \in (0,T]$

$$\lim_{\delta\to 0} \iint_{Q_\tau} a^i(u_\delta, \nabla u_\delta)\frac{\partial u_\delta}{\partial x_i} dxdt = \iint_{Q_\tau} A^i \frac{\partial u}{\partial x_i} dxdt. \tag{7.22}$$

Using now condition 2) we have

$$\begin{aligned} \nu_1 J_\delta \doteqdot \nu_1 \iint_{Q_\tau} \frac{|\nabla u_\delta - \nabla u|^2}{[|\nabla u_\delta|^m + |\nabla u|^m]^{\frac{2}{m}-1}} dxdt \le \\ \le \iint_{Q_\tau} [a(u_\delta, \nabla u_\delta) - a(u, \nabla u)] \cdot (\nabla u_\delta - \nabla u) dxdt \doteqdot \mathfrak{H}_\delta. \end{aligned} \tag{7.23}$$

Using (7.3) - (7.5),(7.9),(7.22) and taking into account that in view of (7.12), (7.6), (7.7), condition 1) and the Vitali theorem

$$a^i(u_\delta, \nabla u) \to a^i(u, \nabla u) \text{ strongly in } L_{m'}(Q_T) \text{ as } \delta \to 0, \tag{7.24}$$

we derive from (7.23) that

$$\lim_{\delta\to 0} \mathfrak{H}_\delta = 0. \tag{7.25}$$

Using (7.25) and inequalities $0 \leq J_\delta \leq \nu_1^{-1} \mathfrak{H}_\delta$ we obtain

$$\lim_{\delta \to 0} J_\delta = 0. \tag{7.26}$$

Show that from (7.26) it follows that (7.19) is true. Denote

$$\frac{|\nabla u_\delta - \nabla u|^2}{(|\nabla u_\delta|^m + |\nabla u|^m)^{\frac{2}{m}-1}} \doteqdot h_\delta(x,t). \tag{7.27}$$

From (7.26) it follows that there exist some subsequence $\{\delta\}$ and subset $\tilde{Q} \subset Q_\tau$, $|\tilde{Q}| = |Q_\tau|$, such that

$$\lim_{\delta \to 0} h_\delta(x,t) = 0 \qquad \text{on } \tilde{Q}. \tag{7.28}$$

Without loss of generality we can count that $\frac{\partial u}{\partial x_i}$ are finite on $\tilde{Q}$, i.e., $|\nabla u|$ is bounded (nonuniformly) at any point $(x,t) \in \tilde{Q}$. In view of (7.27) we have for any $(x,t) \in \tilde{Q}$

$$h_\delta(x,t) \geq \frac{(|\nabla u_\delta| - c)^2}{(|\nabla u_\delta| + c)^{2-m}} \tag{7.29}$$

with constant c depending on $(x,t) \in \tilde{Q}$. Suppose now that $|\nabla u_\delta|$ is unbounded in some point $(x,t) \in \tilde{Q}$. Then $|\nabla u_\delta| \to \infty$ for some subsequence $\{\delta\}$ and hence in view of (7.29) we obtain that for this subsequence $\lim_{\delta \to \infty} h_\delta(x,t) = \infty$ i.e., we obtain contradiction with (7.28). Hence

(7.30) $|\nabla u_\delta|$ are bounded (nonuniformly) at any point of $\tilde{Q}$.

Then from (7.27), (7.28) and (7.30) it follows that numerators of h_δ tend to zero on $\tilde{Q}$ as $\delta \to 0$, i.e. (7.19) is true. Therefore function $u \in C([0,T]; L_2(\Omega)) \cap W_m^{1,0}(Q_T)$ is a generalized solution of $(RCD)_\epsilon$. From lemmas 6.2 and 6.3 it follows that this function satisfies estimates (6.5) - (6.8). In view of (6.5) and Proposition 3.3 function u is an unique strong solution of $(RCD)_\epsilon$. So we proved the following

Lemma 7.1. *For any fixed $\epsilon > 0$ there exist exactly one strong solution (in sense of Definition 2.3) of $(RCD)_\delta$ satisfying estimates (6.5) - (6.8) with constants $c_1, c_2, \lambda \in (0,1)$ and K independent of ϵ.*

8. The passing to the limit as $\epsilon \to 0$.

Now we are ready to prove Theorem 6.1 and hence Theorem 1.1. In the remainder of this section we denote solution of $(RCD)_\epsilon$ as u_ϵ. We are going to realize the passing to the limit as $\epsilon \to 0$ using a priori estimates (6.5) - (6.8). This passing can be done in the same way as one in [11] where existence of regular solution of (CD) was proved in the case $l \geq 0$, $\max\left(1, \frac{2n}{n+2}\right) < m < 2$, $m + l \geq 2$.

In view of estimates (6.5) - (6.8) we can conclude that there exists function u such that

$$u_\epsilon \to u \text{ uniformly in } Q_T; \tag{8.1}$$

$$\frac{\partial}{\partial x_i}(u_\epsilon^{\alpha+1}) \to u^\alpha u_{x_i} \text{ weakly in } L_m(Q_T), i = 1, \dots, n, \alpha = \frac{l}{m}; \tag{8.2}$$

$$0 \le \inf(u, Q_T) \le \sup(u, Q_T) \le c_1; \tag{8.3}$$

$$\iint_{Q_T} u^l |u_x|^m dx dt \le c_2; \tag{8.4}$$

and (see (4.2))

$$\langle u \rangle_{\lambda, \overline{Q}_T} \le K. \tag{8.5}$$

In (8.2) and (8.4) we used the following notation similar to one from Definition 2.3:

$$u_x = (u_{x_1}, \dots, u_{x_n}),\ u_{x_i} = \begin{cases} (\alpha+1)^{-1} u^{-\alpha} & \text{on } [Q_T : u > 0] \\ 0 & \text{on } [Q_T : u = 0] \end{cases}, \alpha = \frac{l}{m}. \tag{8.6}$$

Obviously $u^\alpha u_{x_i} \in L_m(Q_T), i = 1, \dots, n$ (in view of (8.4)). In view of boundedness of u and inequality $\sigma = \frac{l}{m-1} > \alpha$ expressions for u_{x_i} in (8.6) and (2.2) coincide; moreover from condition $u^\alpha u_{x_i} \in L_m(Q_T)$ it follows that $u^\sigma u_{x_i} \in L_m(Q_T), i = 1, \dots, n$. We use below the following auxiliary propositions (see [10] or [11]).

Proposition 8.1. *Let function u be bounded and nonnegative in Q_T and such that $\nabla u^{\alpha+1} \in L_m(Q_T)$ for some $\alpha \ge 0$. Let function $\overline{u}$ is defined by*

$$\overline{u} = \sup(u - \epsilon_1, 0),\ \epsilon_1 = const > 0. \tag{8.7}$$

Then $\overline{u}$ has generalized derivatives $\partial\overline{u}/\partial x_i \cdot \in L_m(Q_T), i = 1, \dots, n$, such that

$$\frac{\partial \overline{u}}{\partial x_i} = \begin{cases} u_{x_i} & in\ [Q_T : u > \epsilon_1] \\ 0 & in\ [Q_T : 0 \le u \le \epsilon_1] \end{cases} \tag{8.8}$$

where u_{x_i} are defined by (8.6); moreover

$$\lim_{\epsilon_1 \to 0} \| u^\alpha \frac{\partial \overline{u}}{\partial x_i} - u^\alpha u_{x_i} \|_{L_m(Q_T)} = 0. \tag{8.9}$$

Proposition 8.2. *Let $A^i \in L_{m'}(Q_T)$, $i = 1, \ldots, n$, $B \in L_{m'}(Q_T)$, $1/m + 1/m' = 1$, $m > 1$, and let function u is bounded and nonnegative in Q_T and such that $\nabla u^{\alpha+1} \in L_m(Q_T)$ for some $\alpha \geq 0$. Assume that for any $t_1, t_2 \in [0,T]$ and any $\phi \in \overset{0}{W}{}^1_m(Q_T)$*

$$\int_\Omega u\phi dx|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega (-u\phi_t + u^\alpha A^i \phi_{x_i} + B\phi) dx dt = 0. \tag{8.10}$$

Let $\varphi \in W^1_m(Q_T)$ and $u = \varphi$ on S_T. Then for any $t_1, t_2 \in [0,T]$

$$\int_\Omega \left(\frac{1}{2}u^2 - u\varphi\right) dx|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega [u\varphi_t + u^\alpha A^i(u_{x_i} - \varphi_{x_i}) + B(u-\varphi)] dx dt = 0 \tag{8.11}$$

where u_{x_i} are defined by (8.8).

Returning to (8.1)-(8.5) we see that function u is nonnegative and bounded in $Q_T, u \in C^{\lambda,\lambda/m}(\overline{Q}_T), \nabla u^{\alpha+1} \in L_m(Q_T), \alpha = \frac{l}{m}$ (so that $\nabla u^{\sigma+1} \in L_m(Q_T), \sigma = \frac{l}{m-1}$) and $u = \psi$ on Γ_T. Hence to prove Theorem 6.1 it is sufficient to show that for any $t_1, t_2 \in [0,T]$ and $\phi \in \overset{0}{W}{}^1_m(Q_T)$

$$\int_\Omega u\phi\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega [-u\phi_t + a(u, u_x)\cdot\nabla\phi - f\phi]\, dx\, dt = 0, \tag{8.12}$$

where u_x is defined by (8.8) in the case $\alpha = \frac{l}{m}$. Really in this case from the kind of $(\mathrm{RCD})_\epsilon$ it will follow that u is a quasistrong and hence regular solution of (CD).

To prove that (8.12) holds denote

$$A^i_\epsilon = u_\epsilon^{-\alpha} a^i(u_\epsilon, \nabla u_\epsilon), \quad \alpha = \frac{l}{m}, \quad i = 1, \ldots, n. \tag{8.13}$$

In view of the second inequality in condition 1) and estimate (8.4) we have uniform estimate

$$\|A^i_\epsilon\|_{L_{m'}(Q_T)} \leq c, \quad i = 1, \ldots, n. \tag{8.14}$$

Then we can count that there exist functions $a^i \in L_{m'}(Q_T)$ such that

$$A^i_\epsilon \to A^i \quad \text{weakly in } L_{m'}(Q_T) \text{ as } \epsilon \to 0,\ i = 1, \ldots, n. \tag{8.15}$$

Letting $\epsilon \to 0$ in the integral identity

$$\int_\Omega u_\epsilon\phi\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega [-u_\epsilon\phi_t + a(u_\epsilon, \nabla u_\epsilon)\cdot\nabla\phi - f\phi]\, dx\, dt = 0,\ \phi \in \overset{\circ}{W}{}^1_m(Q_T), \tag{8.16}$$

we obtain in view of (8.1), (8.15) that for any $t_1, t_2 \in [0,T]$ and $\phi \in \overset{\circ}{W}{}^1_m(Q_T)$

$$\int_\Omega u\phi\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega [-u\phi_t + u^\alpha A^i\phi_{x_i} - f\phi]\, dx\, dt = 0. \tag{8.17}$$

To prove Theorem 6.1 it is sufficient to show that
(8.18)

$$\int_{t_1}^{t_2}\int_\Omega u^\alpha A^i \phi_{x_i} dx\,dt = \int_{t_1}^{t_2}\int_\Omega a^i(u,\nabla u)\phi_{x_i} dx\,dt = 0 \quad \text{for any } \phi \in \overset{\circ}{C}{}^1(\overline{Q_T}),$$

because A^i, $a^i(u,\nabla u) \in L_{m'}(Q_T)$ and $\overset{\circ}{C}{}^1(\bar{Q_T})$ is dense in $\overset{\circ}{W}{}^1_m(Q_T)$. To prove (8.18) it is sufficient to establish that for some subsequence $\{\epsilon\}$

$$u_\epsilon^\alpha \frac{\partial u_\epsilon}{\partial x_i} \to u^\alpha u_{x_i} \quad \text{a.e. in } Q_T, \quad i = 1,\dots,n, \tag{8.19}$$

because in view of (8.19), (8.1), the continuity functions $u^{-\alpha}a^i(u, u^{-\alpha}p)$, $\alpha = \frac{l}{m}$, on $\bar{\mathbb{R}}_+ \times \mathbb{R}^n$, condition 1), uniform estimate (6.7) for $u = u_\epsilon$ and the Vitali theorem we obtain that for any $\phi \in \overset{\circ}{C}{}^1(\bar{Q_T})$ the integral

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega a(u_\epsilon,\nabla u_\epsilon)\cdot\nabla\phi\, dx\,dt &= \int_{t_1}^{t_2}\int_\Omega u_\epsilon^\alpha A_\epsilon^i \phi_{x_i} dx\,dt = \\ &= \int_{t_1}^{t_2}\int_\Omega u_\epsilon^\alpha(u_\epsilon^{-\alpha}a^i(u_\epsilon, {u_\epsilon}^{-\alpha}({u_\epsilon}^\alpha\nabla u_\epsilon)))\phi_{x_i} dx\,dt\end{aligned} \tag{8.20}$$

tends to the integral $\int_{t_1}^{t_2}\int_\Omega a^i(u,u_x)\phi_{x_i} dx\,dt$. On the other hand in view of (8.1), (8.15) integral (8.20) tends to the integral $\int_{t_1}^{t_2}\int_\Omega u^\alpha A^i\phi_{x_i} dx\,dt$. Hence (8.19) implies (8.18). The remainder of this section is devoted to the proof of (8.19). Applying Proposition 7.1 with $g(\xi) = \xi - \epsilon$ we derive from (7.16)

$$\int_\Omega (\frac{1}{2}u_\epsilon^2 - \epsilon u_\epsilon)\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega [a^i(u_\epsilon,\nabla u_\epsilon)\frac{\partial u}{\partial x_i} - f(u-\epsilon)]\,dx\,dt = 0. \tag{8.21}$$

Applying Proposition 8.2 with $g(\xi) = \xi$ and using that $u = 0$ of S_T (in view of (8.1) because $u_\epsilon = \epsilon$ on S_T) we derive from (8.11)

$$\int_\Omega \frac{1}{2}u^2 dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega [u^\alpha A^i u_{x_i} - fu]\,dx\,dt = 0. \tag{8.22}$$

Using (8.1) we derive from (8.21), (8.22) that

$$\lim_{\epsilon\to 0}\int_{t_1}^{t_2}\int_\Omega a^i(u_\epsilon,\nabla u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i} dx\,dt = \int_{t_1}^{t_2}\int_\Omega u^\alpha A^i u_{x_i} dx\,dt. \tag{8.23}$$

Let $\bar{u}$ is defined by (8.7). Obviously that the following proposition holds (see also [10]).

Proposition 8.3. *We have*

$$u_\epsilon{}^{-\alpha}a^i(u_\epsilon,\nabla\bar{u}) \to u^{-\alpha}a^i(u,\nabla\bar{u}) \quad \textit{strongly in } L_{m'}(Q_T) \textit{ as } \epsilon\to 0, \tag{8.24}$$

$$u_\epsilon{}^{-\alpha}a^i(u,\nabla\bar{u}) \to u^{-\alpha}a^i(u,u_x) \quad \textit{strongly in } L_{m'}(Q_T) \textit{ as } \epsilon_1\to 0. \tag{8.25}$$

Using now condition 2) we have

$$\nu_1 \mathcal{H}_{\epsilon,\epsilon_1} \dot{=} \nu_1 \int_{t_1}^{t_2} \int_\Omega u_\epsilon{}^l |\nabla u_\epsilon - \nabla \bar{u}|^2 [|\nabla u_\epsilon - b(u_\epsilon)|^m + |\nabla \bar{u} - b(u_\epsilon)|^m]^{1-2/m} dx\, dt \leq$$

$$\leq \int_{t_1}^{t_2} \int_\Omega [a^i(u_\epsilon, \nabla u_\epsilon) - a^i(u_\epsilon, \nabla \bar{u})](\frac{\partial u}{\partial x_i} - \frac{\partial \bar{u}}{\partial x_i})\, dx\, dt =$$

$$= \int_{t_1}^{t_2} \int_\Omega [a^i(u_\epsilon, \nabla u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i} - A^i_\epsilon u_\epsilon{}^\alpha \frac{\partial \bar{u}}{\partial x_i} -$$

$$-u_\epsilon{}^{-\alpha} a^i(u_\epsilon, \nabla \bar{u})(u_\epsilon^\alpha \frac{\partial u_\epsilon}{\partial x_i} - u_\epsilon^\alpha \frac{\partial \bar{u}}{\partial x_i})]\, dx\, dt \dot{=} J_{\epsilon,\epsilon_1}. \tag{8.26}$$

Using (8.1), (8.2), (8.15), (8.23), (8.24) and letting $\epsilon \to 0$ we obtain

$$\lim_{\epsilon \to 0} J_{\epsilon,\epsilon_1} =$$

(8.27)

$$= \int_{t_1}^{t_2} \int_\Omega [A^i(u^\alpha u_{x_i} - u^\alpha \frac{\partial \bar{u}}{\partial x_i}) - u^{-\alpha} a^i(u, \nabla \bar{u})(u^\alpha u_{x_i} - u^\alpha \frac{\partial \bar{u}}{\partial x_i})]\, dx\, dt \dot{=} \tilde{J}_{\epsilon_1}.$$

Using (8.25), (8.9) we derive from (8.27)

$$\lim_{\epsilon_1 \to 0} \tilde{J}_{\epsilon_1} = 0. \tag{8.28}$$

From (8.27), (8.28) it follows that there exist subsequences $\{\epsilon_k\}$ and $\{\epsilon_{1k}\}$ tending to zero such that $\lim_{k\to\infty} J_{\epsilon_k,\epsilon_{1k}} = 0$. Because $0 \leq \mathcal{H}_{\epsilon_k,\epsilon_{1k}} \leq J_{\epsilon_k,\epsilon_{1k}}$ we derive from here that

$$\lim_{k\to\infty} \mathcal{H}_{\epsilon_k,\epsilon_{1k}} = 0. \tag{8.29}$$

Rewrite $\mathcal{H}_{\epsilon_k,\epsilon_{1k}}$ as

$$\mathcal{H}_{\epsilon_k,\epsilon_{1k}} = \int_{t_1}^{t_2} \int_\Omega \frac{|u^\alpha_{\epsilon_k} \nabla u_{\epsilon_k} - u^\alpha_{\epsilon_k} \nabla \bar{u}|^2}{[|u^\alpha_{\epsilon_k} \nabla u_{\epsilon_k} - u^\alpha_{\epsilon_k} b(u_{\epsilon_k})|^m + |u^\alpha_{\epsilon_k} \nabla \bar{u} - u^\alpha_{\epsilon_k} b(u_{\epsilon_k})|^m]^{\frac{2-m}{m}}} dx\, dt \dot{=}$$

$$\dot{=} \int_{t_1}^{t_2} \int_\Omega h_k(x,t)\, dx\, dt. \tag{8.30}$$

Is should be recalled that $\bar{u} = \sup(u - \epsilon_{1k}, 0)$ in (8.30). From (8.28), (8.29) it follows that there exist subsequence $\{k\}$ and subset $\tilde{Q} \subset Q_{t_1,t_2}$, $Q_{t_1,t_2} \dot{=} \Omega \times [t_1, t_2]$, $|\tilde{Q}| = |Q_{t_1,t_2}|$, such that

$$\lim_{k\to\infty} h_k(x,t) = 0 \quad \text{on } \tilde{Q}. \tag{8.31}$$

Without loss of generality we can count that $\frac{\partial u^{\alpha+1}}{\partial x_i}$ are finite on $\tilde{Q}$, $i = 1, \ldots, n$. Then using (8.1), (8.3), the definition of $\bar{u}$ (see (8.7)), (8.8), (8.6) and continuity of vector-function $b(u)$ we can conclude that

$$|u^\alpha_{\epsilon_k} \nabla \bar{u}|, \quad |u^\alpha_{\epsilon_k} b(u_{\epsilon_k})| \text{ are bounded (nonuniformly) on } \tilde{Q}. \tag{8.32}$$

On the other hand in view of definition of functions $h_k(x,t)$ we can estimate

$$h_k(x,t) \geq \frac{[|u^\alpha_{\epsilon_k}\nabla u_{\epsilon_k}| - c]^2}{c[|u^\alpha_{\epsilon_k}\nabla u_{\epsilon_k}| + c]^{2-m}} \quad \text{at } (x,t) \in \tilde{Q} \tag{8.33}$$

with some constant c depending on (x,t). Assume that $|u^\alpha_{\epsilon_k}\nabla u_{\epsilon_k}|$ are unbounded at some point $(x,t) \in \tilde{Q}$. Then for some subsequence $\{k\}$ $|u^\alpha_{\epsilon_k}\nabla u_{\epsilon_k}| \to \infty$ as $k \to \infty$ and hence (using that $m \in (1,2)$) we derive from (8.33) that

$$\lim_{k\to\infty} h_k(x,t) = \infty \quad \text{on } \tilde{Q}. \tag{8.34}$$

But (8.34) gives a contradiction with (8.31). Hence

$$|u^\alpha_{\epsilon_k}\nabla u_{\epsilon_k}| \text{ are bounded (nonuniformly) on } \tilde{Q}. \tag{8.35}$$

Then from (8.31), (8.30) and (8.35) it follows that the numerators of h_k tend to zero on $\tilde{Q}$ as $k \to \infty$, i.e.,

$$\lim_{k\to\infty} |u^\alpha_{\epsilon_k}\frac{\partial u_{\epsilon_k}}{\partial x_i} - u^\alpha_{\epsilon_k}\frac{\partial \bar{u}}{\partial x_i}| = 0 \text{ on } \tilde{Q},\ i = 1, \ldots, n. \tag{8.36}$$

Remark now that

$$\lim_{k\to\infty} |u^\alpha_{\epsilon_k}\frac{\partial \bar{u}}{\partial x_i} - u^\alpha\frac{\partial u}{\partial x_i}| = 0 \text{ on } \tilde{Q},\ i = 1, \ldots, n. \tag{8.37}$$

Really if $(x,t) \in \tilde{Q}$ and $u(x,t) > 0$ than $\frac{\partial \bar{u}(x,t)}{\partial x_i} = u_{x_i}(x,t)$ for all sufficiently large k and hence (8.37) follows from (8.1). On the other hand if $(x,t) \in \tilde{Q}$ and $u(x,t) = 0$ then $\frac{\partial \bar{u}(x,t)}{\partial x_i} = 0$ for any k and hence (8.37) follows from (8.1) and the definition of $\bar{u}_{x_i}$.

Finally from (8.36) and (8.37) it follows obviously that (8.19) holds. Theorem 6.1 (and hence Theorem 1.1) is proved.

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