ON THE RANK OVER $\mathbb{Q}$ OF SOME REAL MATRICES

PART 1.

LEONID A. GUTNIK

To my daughters Tanya and Natasha

"While there is life there is hope," said Sir Henry.
H.Rider Haggard. King Solomon's mines.

TABLE OF CONTENTS

§0. INTRODUCTION.

The arithmetic nature of the special values of the polylogarithm

$$F(z,k)=L_k(z)=\sum_{n=1}^{\infty}n^{-k}z^n$$

in the rational points $z=\frac{b}{a}$, where $b\in\mathbb{N}, a\in\mathbb{Z}, b<|a|$, and $(a,b)=1$, has interested many mathematicians. An informative survey of the subject can be found in an interesting paper by Steffen Eckmann [11]. We cite:

"Neben vielen anderen Irrationalitätsaussagen konnte Maier [9] durch explicite Konstruktion von rationalen Approximationen zeigen, daß Dilogarithmusfunktion $F(x,2)$ an rationalen Stellen $x_0=\frac{b}{a}$ mit $|a|>b^3 2^8 e^2$ irrationale verte annimt. Chudnovski [1] entwickelte Maiers Methode weiter und bewies u.a.: Gilt $a>>k^k(k+1)^{-k-1}\exp(k^2)$ für $a\in\mathbb{N}$, so ist $F(a^{-1},k)$ irrational. Gutnik [8] verallgemeinerte diese Ergebnisse und zeigte, daß die Werte der Polylogarithmusfunktionen an bestimmten Stellen linear unabhängig über dem Körper rationalen Zahlen (kurz: $\mathbb{Q}$ - l.u.) sind:

Satz 1a

Ist $x_0 = \frac{b}{a}$ *mit* $(a, b) \in \mathbb{Z} \times \mathbb{N}$, *so sind für* $s \in \mathbb{N}$ *die* $s+1$ *Zahlen*

$$1, F(x_0, 1), F(x_0, 2), \ldots, F(x_0, s)$$

$\mathbb{Q}$ - *l.u., falls* $|a| > b^{s+1}s^{-1}\exp(s^2)$.

In [1] wurde ein ähnliches Resultat über obige Funktionen an gewissen Stellen a^{-1} mit $a \in \mathbb{N}$ behauptet. Durch eine andere Beweismethode, welche Padé-Approximation benutzt, konnte Nikishin bereits in [10] eine etwas schwächere Version des Satzes 1a zeigen:

Satz 1b

Ist $x_0 = \frac{b}{a}$ *mit* $(a, b) \in \mathbb{Z} \times \mathbb{N}$,*so sind für* $s \in \mathbb{N}$ *die* $s+1$ *Zahlen*

$$1, F(x_0, 1), F(x_0, 2), \ldots, F(x_0, s)$$

$\mathbb{Q}$-*l.u.,falls* $|a| > b^{s+1}\exp\{s(s-1)(lns + 1 + \log 4)\}$.

Man beachte hierbei, daß in [10] gegebene Vorraussetzung eine Korrektur erfahren musste, da dort auf seite 388 bei einer Abschätzung zwei Faktoren unberücksichtig geblieben waren."

The proof of Theorem 1a mentioned above, which was given in [8], made use of Mejer's function; later I have found an elementary proof of this result [15]. That elementary proof, with some generalisations, is presented here.

Acknowledgments. I am obliged to Prof. P. Bundshuh for a reprint of the paper [11], to Prof. B.Z. Moroz for translating this paper from the Russian, and to Prof. A.G. Aleksandrov for his constant support.

§1. Quasi-irrational matrices.

Let $m \in \mathbb{N}$, $n \in \mathbb{N}$. Let E_m denote the unit $m \times m-$ matrix.

Definition 1. *A real* $m \times n-$*matrix* $A = (a_{i,k})$ *is said to be quasi-irrational, if the columns of the matrix*

$$(A, E_m) \tag{1}$$

are linearly independent over $\mathbb{Q}$.

Thus the real numbers

$$1, a_1, \ldots, a_n \tag{2}$$

are linearly independent over $\mathbb{Q}$ if and only if the row $(a_1, \ldots, a_n)$ is quasi-irrational.

Lemma 1. *The following assertions are equivalent:*

(i) for no column X in $\mathbb{Q}^n\backslash\{0\}$ all the entries of the column AX are rational numbers;

(ii) there are $m+n$ sequences of integers

$$\alpha_l(\nu),\ \nu\in\mathbb{N},$$

where $l=1,\ldots,m+n$, satisfying the condition

$$\lim_{\nu\to\infty}\left|-\alpha_{m+k}(\nu)+\sum_{j=1}^{m}a_{j,k}\alpha_j(\nu)\right|=0, \tag{3}$$

where $k=1,\ldots,n$, and such that for each $\kappa\in\mathbb{N}$ the $m+n$ sequences

$$\alpha_l(\kappa+\mu),\ \mu\in\mathbb{N}, \tag{4}$$

where $l=1,\ldots,m+n$, are linearly independent over $\mathbb{R}$;

(iii) the matrix A is quasi-irrational.

Proof. Condition (i) is clearly equivalent to (iii).

Suppose (ii) hold true, then there is a column X in $\mathbb{Q}^n\backslash\{0\}$ such that all the entries of the column AX are rational numbers. On choosing a suitable $d\in\mathbb{N}$ and replacing X by dX, if necessary, we may assume that

$$X\in\mathbb{Z}^n\backslash\{0\}\ ,AX\in\mathbb{Z}^m.$$

Let x_k, for $k=1,\ldots,n$, be the $k-$th entry of the column X. It follows from relation (3) that

$$\lim_{\nu\to\infty}\left(\left(-\sum_{k=1}^{n}x_k\alpha_{m+k}(\nu)\right)+\sum_{j=1}^{m}\alpha_j(\nu)\sum_{k=1}^{n}a_{j,k}x_k\right)= \tag{5}$$

$$\sum_{k=1}^{n}x_k\lim_{\nu\to\infty}\left(-\alpha_{m+k}(\nu)+\sum_{j=1}^{m}a_{j,k}\alpha_j(\nu)\right)=0.$$

Therefore there is such an integer $\kappa\in\mathbb{N}$ that for any $\mu\in\mathbb{N}$ the absolute value of the integer

$$\left(-\sum_{k=1}^{n}x_k\alpha_{m+k}(\kappa+\mu)\right)+\sum_{j=1}^{m}\alpha_j(\kappa+\mu)\sum_{k=1}^{n}a_{j,k}x_k \tag{6}$$

is smaller than 1; consequently, the integer (6) is equal to 0. However the number (6) can not be equal to 0 for every μ in $\mathbb{N}$ because sequences (4) are linearly independent over $\mathbb{R}$ and $x_k \neq 0$ for some $k = 1, \dots, n$. Suppose (iii) hold true. Let us consider, for each $i \in \{1, \dots, m\}$, a system of n linear forms

$$L_{i,k} = y_0 a_{i,k} - y_k, \ k = 1, \dots, n, \tag{7}$$

of $n+1$ independent variables $y_0, \dots, y_n$. Since the number $(n+1)$ of the variables exceeds the number n of the linear forms, it follows from the well-known lemma (cf., for instance, [12, p.131]; the cited lemma is a consequence of the "pigeon-hole" Dirichlet principle) that, given $\mu \in \mathbb{N}$, for each $k = 1, \dots, n$ there are real numbers

$$\theta_k(i + m(\mu - 1)) \in [-1, 1], \tag{8}$$

and integers $y_0, \dots, y_n$ such that

$$L_{i,k} = \frac{\theta_k(i + m(\mu - 1)}{i + m\mu},$$

and $y_i \neq 0$ for some i; therefore $y_0 \neq 0$. For these values of the variables $y_0, \dots, y_n$, we let

$$\alpha_i(i + m(\mu - 1)) = y_0, \alpha_{m+k}(i + m(\mu - 1)) = y_k,$$

with $k = 1, \dots, n$, and

$$\alpha_l(i + m(\mu - 1)) = 0$$

for $l \in \{1, \dots, m\} \backslash \{i\}$. We have thus constructed $m + n$ sequences of integers

$$\alpha_l(\nu), \ \nu \in \mathbb{N},$$

with $l = 1, \dots, m + n$, satisfying the following conditions:

$$\alpha_i(\nu) \neq 0$$

for $\nu \equiv i \ (mod \ m)$ and $i \in \{1, \dots, m\}$,

$$\alpha_l(\nu) = 0$$

for $\nu \equiv i \ (mod \ m)$ and $l \in \{1, \dots, m\} \backslash \{i\}$, and

$$-\alpha_{m+k}(\nu) + \sum_{j=1}^{m} a_{j,k} \alpha_j(\nu) = \tag{9}$$

$$-\alpha_{m+k}(\nu) + a_{i,k}\alpha_i(\nu) = \frac{\theta_k(\nu)}{\nu+1},$$

for $\nu \equiv i\ (mod\ m)$, $k = 1, \ldots, n$.

Suppose that for some $\kappa \in \mathbb{N}$ the $m+n$ sequences (4), arising from the sequences $\alpha_l(\nu)$ we have just constructed when the first κ terms are removed, turn out to be linearly dependent over $\mathbb{R}$. Since $\alpha_l(\nu) \in \mathbb{Z}$ for $l = 1, \ldots, m+n$, $\nu \in \mathbb{N}$, the sequences (4) are actually linearly dependent over $\mathbb{Q}$; therefore one can find $m+n$ integers t_l, $l = 1, \ldots, m+n$, such that

$$\sum_{i=1}^{m} t_i\alpha_i(\kappa+\mu) + \sum_{k=1}^{n} t_{k+m}\alpha_{m+k}(\kappa+\mu)) = 0 \tag{10}$$

and $t_j \neq 0$ for some j.

It follows from (9) and (10) that

$$\sum_{i=1}^{m} \alpha_i(\kappa+\mu)(t_i + \sum_{k=1}^{n} t_{k+m}a_{i,k}) = -(\kappa+\mu+1)^{-1}\sum_{k=1}^{n} t_{k+m}\theta_k(\kappa+\mu). \tag{11}$$

For $i \in [1, m] \cap \mathbb{N}$, let

$$\mu \equiv -\kappa + i\ (mod\ m). \tag{12}$$

It follows from (9) that

$$|t_i + \sum_{k=1}^{n} t_{k+m}a_{i,k}\Bigg| \leq (\kappa+\mu+1)^{-1}\sum_{k=1}^{n} |t_{k+m}\Bigg| . \tag{13}$$

On passing to the limit $\mu \to \infty$, one obtains from (12) and (13) the following equation

$$t_i + \sum_{k=1}^{n} t_{k+m}a_{i,k} = 0,$$

with $i = 1, \ldots, m$, in contradiction to (iii). This proves the Lemma.

Remark. *If the condition (iii) holds true, then it follows from [12, p. 131, Lemma 1] and the above construction that, for $l = 1, \ldots, m+n$, there is a sequence $\alpha_l(\nu)$ satisfying (ii) and the extra condition*

$$\alpha_l(\nu) = O(\nu^{n+1}).$$

§2. On the zeros of some functions.

Suppose that $0 < \alpha, 0 < \delta, 0 \le \eta, 0 < D$, and $n \in \mathbb{N}$. Let

$$f_0 = f_0(\eta) = f_0(\alpha, \delta, D, \eta) = (\eta + \delta)^{\alpha+1} - (1 + \delta)^{\alpha+1} D\eta,$$

(14)

$$f^{\wedge}_{0,n} = f^{\wedge}_{0,n}(\eta) = f^{\wedge}_{0,n}(\alpha, \delta, D, \eta) = \sum_{k=0}^{n-1} (\eta + \delta)^{(\alpha+1)k}((1 + \delta)^{\alpha+1} D\eta)^{n-k}.$$

Since $(\frac{\partial}{\partial\eta})^2 f_0 > 0$, the function f_0 is convex and has no more than two zeros, counted according to their multiplicity, in the interval $[0, +\infty)$. Consider the function

$$f_1 = f_1(\eta) = f_1(\alpha, \delta, \eta) = (\eta + \delta)^{\alpha+1}\eta^{-1}(1 + \delta)^{-\alpha-1} \tag{15}$$

defined on the interval $(0, +\infty)$; since

$$\left(\frac{\partial}{\partial\eta} f_1\right)(\alpha, \delta, \eta) = (\eta + \delta)^{\alpha}\eta^{-2}(1 + \delta)^{-\alpha-1}(\alpha\eta - \delta),$$

the function f_1 attains the global minimum

$$A_1(\alpha, \delta) = \left(\frac{\alpha + 1}{1 + \delta}\right)^{\alpha+1} \left(\frac{\delta}{\alpha}\right)^{\alpha}. \tag{16}$$

in the point $\eta = \frac{\delta}{\alpha}$. Thus

$$f_1(\alpha, \delta, \eta) \ge A_1(\alpha, \delta) \tag{17}$$

for $\eta \in (0, +\infty)$; moreover, relation (17) is a strict inequality if and only if $\eta \ne \frac{\delta}{\alpha}$. Since

$$f_0(\alpha, \delta, D, \eta) = \eta(1 + \delta)^{\alpha+1}(f_1(\alpha, \delta, \eta) - D), \tag{18}$$

$$\left(\frac{\partial}{\partial\eta} f_0\right)(\alpha, \delta, D, \eta) = \frac{f_0(\alpha, \delta, D, \eta)}{\eta} + \eta(1 + \delta)^{\alpha+1} \left(\frac{\partial}{\partial\eta} f_1\right)(\alpha, \delta, \eta)$$

for $\eta \in (0, +\infty)$, the function $f_0(\eta) = f_0(\alpha, \delta, \eta)$ has no zeros in the interval $[0, +\infty)$ if $D < A_1(\alpha, \delta)$ and has precisely two zeros in this interval, counted in accordance with their multiplicity, if $D \ge A_1(\alpha, \delta)$; let $\eta_1(\alpha, \delta, D)$ and $\eta_2(\alpha, \delta, D)$ stand for the smallest and the biggest of these zeros accordingly. Relations (17) and (18) show that if $D > A_1(\alpha,\delta)$, then each of the zeros $\eta_k(\alpha,\delta,D)$, $k=1,2$, is a simple zero and

$$\eta_1(\alpha,\delta,D) < \frac{\delta}{\alpha} < \eta_2(\alpha,\delta,D). \tag{19}$$

On the other hand, if $D = A_1(\alpha,\delta)$, then

$$\eta_1(\alpha,\delta,D) = \frac{\delta}{\alpha} = \eta_2(\alpha,\delta,D), \tag{20}$$

and therefore it follows from (18) that $\eta = \frac{\delta}{\alpha}$ is a double zero of the function f_0. Let

$$A_0 = A_0(\eta) = \eta^{\alpha}(1+\eta)^{-\alpha-1} \tag{21}$$

for $\eta \in [0,+\infty)$. Since

$$\left(\frac{\partial}{\partial\eta}A_0\right)(\eta) = \eta^{\alpha-1}(1+\eta)^{-\alpha-2}(\alpha-\eta)$$

for $\eta \in (0,+\infty)$, the function $A_0(\eta)$ has a global minimum in that interval at the point $\eta=\alpha$. Thus $A_0(\alpha) \geq A_0(\eta)$ for $\eta \in [0,+\infty)$, where the equality is attained in the only point $\eta = \alpha$. Since $A_1(\alpha,\delta) = (A_0(\delta))/A_0(\alpha)$, it follows that

$$A_1(\alpha,\delta) \leq 1, \tag{22}$$

the equality in (22) being attained in the only point $\delta = \alpha$.

Lemma 2. *Suppose that $0 < \delta \leq \alpha, 1 \leq D$. Then $\eta_1(\alpha,\delta,D)$ is a simple root of the function f_0 and $\eta_1(\alpha,\delta,D) < \delta/\alpha \leq 1$ as soon as $\alpha - \delta + D > 1$, but $\eta_1(\alpha,\alpha,1)$ is a double root of f_0 and $\eta_1(\alpha,\alpha,1) = 1$.*

Proof. If $\delta < \alpha$, then $A_1(\alpha,\delta) < 1 \leq D$ and it follows from (19) that $\eta_1(\alpha,\delta,D)$ is a simple root of the function $f_0(\alpha,\delta,D,\eta)$, satisfying the inequality

$$\eta_1(\alpha,\delta,D) < \frac{\delta}{\alpha} < 1.$$

If $\delta = \alpha$ and $D > 1$, then $A_1(\alpha,\delta) = 1 < D$ and it follows from (19) that $\eta_1(\alpha,\delta,D)$ is a simple root of the function $f_0(\alpha,\delta,D,\eta)$ satisfying the relation

$$\eta_1(\alpha,\delta,D) < \frac{\delta}{\alpha} = 1.$$

Finally, if $\alpha=\delta$ and $D=1$, then $A_1(\alpha,\delta)=1=D$ and it follows from (20) that $\eta_1(\alpha,\delta,D)=\frac{\delta}{\alpha}=1$ is a double root of the function $f_0(\alpha,\delta,D,\eta)$. The lemma is proved.

Let $\eta=u\delta^{\alpha+1}/D$ and let

$$\tilde f_0(u)=f_0(u\delta^{\alpha+1}/D)=\delta^{\alpha+1}((u\delta^{\alpha}/D+1)^{\alpha+1}-(1+\delta)^{\alpha+1}u). \tag{23}$$

If

$$D\ge\delta^{\alpha-1}, \tag{24}$$

then $\tilde f_0(1)\le 0$ and

$$\eta_1(\alpha,\delta,D)\le\delta^{\alpha+1}/D. \tag{25}$$

§3. Statement of the main results.

For $0<\eta\le 1$, $0<\delta\le\alpha$, let

$$\tilde\rho(\alpha,\delta,\eta)=\left(\frac{\eta}{\delta}\right)^{\delta}\left(\frac{1-\eta}{1+\delta}\right)^{\alpha-\delta}$$

if $\delta<\alpha$, and let

$$\tilde\rho(\alpha,\alpha,\eta)=\left(\frac{\eta}{\alpha}\right)^{\alpha}.$$

For $k\in\mathbb{N}$, $|x|<1$, let

$$L_k(x)=\sum_{n=1}^{\infty}n^{-k}x^n,$$

and let $\Lambda_{m,s}(x)=(a_{i,j}(x))$ denote the $m\times s-$ matrix, whose element at the intersection of its i-th row and its j-th column is equal to

$$a_{i,j}(x)=\binom{i+j-2}{i-1}L_{i+j-1}(x),$$

for each $i=1,\dots,m$ and each $j=1,\dots,s$.

The following two theorems are the main results of this paper.

Theorem 1. *Suppose that* $m\in\mathbb{N}, s\in\mathbb{N}, n\in s-1+\mathbb{N}, \alpha=m/n, 0<\delta\le\alpha, a\in\mathbb{Z}, b\in\mathbb{N}, z=\frac{a}{b}$, *and* $D=|z|^{1/n}>1$. *If*

$$\tilde\rho(\alpha,\delta,\eta_1(\alpha,\delta,D))\exp\left(\frac{s}{n}+\left(\alpha-\frac{1}{n}\right)\max(\delta,1)\right)b^{1/n}<1, \tag{26}$$

then the matrix $\Lambda_{m,s}\left(\frac{b}{a}\right)$ *is quasi-irrational.*

Theorem 2. *Let* $m\in\mathbb{N}, n\in\mathbb{N}, a\in\mathbb{Z}, b\in\mathbb{N}$. *If*

$$|a|>\left(\frac{m}{n}\right)^{m}\exp\left(\frac{n}{m}\left(n+(m-1)\max\left(\frac{m}{n},1\right)\right)\right)b^{\frac{m+n}{m}}, \tag{27}$$

then the matrix $\Lambda_{m,n}\left(\frac{b}{a}\right)$ *is quasi-irrational.*

On letting $m=1$, one observes that Theorem 1a in the Eckmann's survey [11], cited in § 1, is a corollary of our Theorem 2.

§4. Deduction of theorem 2 from theorem 1.

Let us start by deducing theorem 2 from theorem 1 when $\delta = \alpha$ and $s = n$. Since the inequality (25) follows from (24), the following inequality

$$\eta_1(\alpha, \alpha, D) \leq \alpha^{\alpha+1}/D \tag{28}$$

is a consequence of the inequality $D \geq \alpha^{\alpha-1}$, and therefore it follows from the inequality

$$|z| > \left(\frac{m}{n}\right)^{m-n}. \tag{29}$$

Thus it follows from (29) that

$$\tilde{\rho}(\alpha, \alpha, \eta_1(\alpha, \alpha, D)) \exp\left(\frac{s}{n} + (\alpha - \frac{1}{n}) max(\delta, 1)\right) b^{1/n} \leq$$

$$(\alpha^\alpha/D)^\alpha \exp(n/n + (\alpha - 1/n) \max(\alpha, 1)) b^{1/n}.$$

Since the following inequality

$$(\alpha^\alpha/D)^\alpha \exp(n/n + (\alpha - 1/n)) \max(\alpha, 1)) b^{1/n} < 1 \tag{30}$$

is equivalent to (27), it suffices to prove that the inequality (27) implies (29). But it follows from (27) that

$$|z| > \left(\frac{m}{n}\right)^m e^{\frac{n}{m}(n+(m-1))} \geq \left(\frac{m}{n}\right)^m \left(\frac{n}{m}\right)^n = \left(\frac{m}{n}\right)^{m-n}.$$

Thus Theorem 2 is indeed a consequence of Theorem 1.

§5. Theorem 1 (beginning of the proof).

It follows from the relation (19) and Lemma 2 that

$$\left(\frac{\partial}{\partial \eta} f_0\right)(\alpha, D, \eta_1(\alpha, \delta, D)) = \frac{\alpha}{\eta_1(\alpha, \delta, D)} (\eta + \delta)^\alpha (\eta_1(\alpha, \delta, D) - \delta/\alpha)$$

and, consequently, that the function $\tilde{\rho}(\alpha, \delta, \eta_1(\alpha, \delta, D))$ is continuous at the point (α, δ, D) as soon as the following condition

$$0 < \delta \leq \alpha, 1 \leq D, \alpha - \delta + D > 1 \tag{31}$$

is satisfied. Therefore, in view of the supposition of Theorem 1, it may be assumed that

$$\delta \in \mathbb{Q}, \delta = d^\vee/d^\wedge, \tag{32}$$

where $d^\vee \in \mathbb{N}, d^\wedge \in \mathbb{N}$.

§6. Upper bounds for the sums of certain series.

For $\nu\in\mathbb{N}$, let

$$R(t,\nu)=\frac{\prod\limits_{k=1}^{\nu d^{\vee}}(t-k)^{n}}{\prod\limits_{k=0}^{\nu d^{\wedge}+1}(t+k)^{m}}\frac{((\nu d^{\wedge}+1)!)^{m}}{((\nu d^{\vee})!)^{n}}.$$

Given $\mu\in\mathbb{N}-1$, let us consider the series

$$\sum_{t=1}^{\infty}\frac{(-1)^{\mu}}{\mu!}\left(\left(\frac{\partial}{\partial t}\right)^{\mu}R\right)(t,\nu)z^{-t}. \tag{33}$$

Since $d^{\vee}n\le d^{\wedge}m$, the series (33) converges for $|z|\ge 1$; let $\psi_{\mu}(z;\nu)$ stand for the sum of this series. Clearly,

$$|\psi_0(z;\nu)|\le\psi_0(|z|;\nu). \tag{34}$$

Let

$$R_1(t,\nu)=\frac{\prod\limits_{k=1}^{\nu d^{\vee}}(t-k)^{n}}{\prod\limits_{k=0}^{\nu d^{\wedge}-1}(t+k)^{m}}\frac{((\nu d^{\wedge})!)^{m}}{((\nu d^{\vee})!)^{n}},$$

(35)

$$R_2(t,\nu)=\frac{(\nu d^{\wedge}+1)^{m}}{(t+\nu d^{\wedge})^{m}(t+\nu d^{\wedge}+1)^{m}}.$$

Since $R_1(t,\nu)=0$ for $t=1,\ ,\dots,d^{\vee}\nu$, it follows that

$$\psi_0(z;\nu)=\sum_{t=1+\nu d^{\vee}}^{\infty}R_1(t,\nu)z^{-t}R_2(t,\nu), \tag{36}$$

and

$$\sum_{t=1+\nu d^{\vee}}^{\infty}R_2(t,\nu)\le(\nu d^{\wedge}+1)^{m}\sum_{t=2}^{\infty}\frac{1}{(t+\nu d^{\wedge})^{2m}}\le$$

(37)

$$(\nu d^{\wedge}+1)^m\int\limits_1^{\infty}\frac{1}{(t+\nu d^{\wedge})^{2m}}=\frac{1}{(2m-1)(1+\nu d^{\wedge})^{m-1}}\le 1,$$

$$\psi_0(|z|;\nu)\le\sup_{t\ge 1+\nu d^{\vee}}R_1(t,\nu)|z|^{-t}, \tag{38}$$

$$R_1(t+1,\nu)|z|^{-t-1}=R_1(t,\nu)|z|^{-t}t^{m+n}(t-\nu d^{\vee})^{-n}(t+\nu d^{\wedge})^{-m}z^{-1}. \tag{39}$$

Let $t=\nu\tau d^{\wedge}$. Then it follows from (39) that

$$R_1(t+1,\nu)|z|^{-t-1}=R_1(t,\nu)|z|^{-t}\tau^{m+n}(\tau-\delta)^{-n}(\tau+1)^{-m}|z|^{-1}. \tag{40}$$

Let further

$$g(\tau)=\tau^{m+n}(\tau-\delta)^{-n}(\tau+1)^{-m}|z|^{-1}-1 \tag{41}$$

for $\delta<\tau<+\infty$ and let

$$\tau=\frac{\delta+\eta}{1-\eta} \tag{42}$$

with $0\le\eta<1$. Let $D=|z|^{1/n}$. On substituting relation (42) in (41), one obtains

$$g(\tau)=h(\eta)=\frac{(\eta+\delta)^{m+n}-(1+\delta)^{m+n}(D\eta)^n}{(1+\delta)^{m+n}(D\eta)^n}=$$

$$\frac{f_0(\alpha,\delta,D,\eta)f^{\wedge}_{0,n}(\alpha,\delta,D,\eta)}{(1+\delta)^{m+n}(D\eta)^n},$$

where $D=|z|^{1/n}$, and the functions $f_0(\alpha,\delta,D,\eta)$, $f^{\wedge}_{0,n}(\alpha,\delta,D,\eta)$ are given by the equalities (14). Let

$$\tau_1(\alpha,\delta,D)=\frac{\delta+\eta_1(\alpha,\delta,D)}{1-\eta_1(\alpha,\delta,D)}.$$

Then

$$\delta<\tau_1(\alpha,\delta,D),$$

$$\eta_1(\alpha,\delta,D)=\frac{\tau_1(\alpha,\delta,D)-\delta}{\tau_1(\alpha,\delta,D)+1},$$

$g(\tau)<0$ for $\tau\in(\delta,\tau_1(\alpha,\delta,D))$, and $g(\tau)>0$ for $\tau>\tau_1(\alpha,\delta,D)$. Therefore

$$R_1(t+1,\nu)|z|^{-t-1}\le R_1(t,\nu)|z|^{-t}$$

for $t \geq \nu d^\wedge \tau_1 > \nu d^\vee$ and

$$R_1(t+1,\nu)|z|^{-t-1} \geq R_1(t,\nu)|z|^{-t}$$

for $\nu d^\vee + 1 \leq t \leq \nu d^\wedge \tau_1$. Consequently, one obtains

$$\sup_{t \geq 1+\nu d^\vee} R_1(t,\nu)|z|^{-t} = R_1(\nu d^\wedge \tau_1 + \theta_\nu; \nu)|z|^{-t-1} \geq R_1(t,\nu)|z|^{-\nu d^\wedge \tau_1 - \theta_\nu} \tag{43}$$

for $\nu d^\wedge \tau_1 + \theta_\nu \in \mathbb{N}$ and $0 \leq \theta_\nu < 1$. It follows from (35) that

$$R_1(t,\nu)|z|^{-t} = \frac{(\Gamma(t))^{m+n}(\Gamma(1+\nu d^\wedge))^m}{(\Gamma(t-\nu d^\vee))^n(\Gamma(t+\nu d^\wedge))^m \Gamma(1+\nu d^\vee)^n}|z|^{-t}.$$

To estimate that quantity for $t = \nu d^\wedge \tau, \tau = \tau_1 + \theta_\nu/(\nu d^\wedge), |z| > 1, \nu \geq 2$ we use the Stirling formula:

$$log\Gamma(x) = (x - \frac{1}{2})logx - x + O(1).$$

In view of the inequality the inequality $\tau - \delta \geq \tau_1 - \delta > 0$, it follows that

$$\log(R_1(t,\nu)|z|^{-t}) =$$

$$(m+n)((\nu d^\wedge \tau - 1/2)log(\nu d^\wedge \tau) - \nu d^\wedge \tau) -$$

$$n((\nu d^\wedge(\tau - \delta) - 1/2)\log(\nu d^\wedge(\tau-\delta)) - \nu d^\wedge(\tau - \delta)) -$$

$$m((\nu d^\wedge(\tau+1) - 1/2)\log(\nu d^\wedge(\tau+1)) - \nu d^\wedge(\tau+1)) +$$

$$m((\nu d^\wedge + 1/2)\log(\nu d^\wedge(\tau+1)) - \nu d^\wedge) -$$

$$n((\nu d^\vee + \frac{1}{2})(\log(\nu d^\vee) - \nu d^\vee) -$$

$$\nu d^\wedge \tau \log(|z|) + O(1) =$$

$$A(\tau)\nu\log(\nu) + B(\tau)\nu + C(\tau)\log(\nu) + O(1),$$

where

$$A(\tau) = (m+n)d^\wedge \tau - nd^\wedge(\tau - \delta) - md^\wedge(\tau+1) - nd^\vee + md^\wedge =$$

$$(m+n)d^\wedge \tau - nd^\wedge \tau - md^\wedge \tau + nd^\wedge \delta - nd^\vee - md^\wedge + md^\wedge = 0,$$

$$B(\tau) = (\log(d^\wedge) - 1)\times$$

$$((m+n)d^\wedge \tau - nd^\wedge(\tau-\delta) - md^\wedge(\tau+1) + md^\wedge - nd^\vee) -$$

$$-nd^\wedge\delta log(\delta)+$$
$$(m+n)d^\wedge\tau\log(\tau)-nd^\wedge(\tau-\delta)\log(\tau-\delta)-$$
$$-md^\wedge(\tau+1)\log(\tau+1)-d^\wedge\tau\log(|z|)=$$
$$-nd^\wedge\delta\log(\delta)+$$
$$(m+n)\left(d^\wedge\tau_1+\frac{\theta_\nu}{\nu}\right)\log\left(\tau_1\left(1+\frac{\theta_\nu}{d^\wedge\tau_1\nu}\right)\right)-$$
$$n\left(d^\wedge(\tau_1-\delta)+\frac{\theta_\nu}{\nu}\right)\log\left((\tau_1-\delta)\left(1+\frac{\theta_\nu}{d^\wedge(\tau_1-\delta)\nu}\right)\right)-$$
$$m\left(d^\wedge(\tau_1+1)+\frac{\theta_\nu}{\nu}\right)\log\left((\tau_1+1)\left(1+\frac{\theta_\nu}{d^\wedge(\tau_1+1)\nu}\right)\right)-$$
$$\left(d^\wedge\tau_1+\frac{\theta_\nu}{\nu}\right)\log(|z|)=$$
$$(m+n)d^\wedge\tau_1\log(\tau_1)-nd^\wedge(\tau_1-\delta)\log(\tau_1-\delta)-md^\wedge(\tau_1+1)\log(\tau_1+1)-$$
$$nd^\wedge\delta\log(\delta)-d^\wedge\tau_1\log(|z|)+O(1/nu),$$

and

$$C(\tau)=-(m+n)/2+n/2+m/2+m/2-n/2=(m-n)/2.$$

Thus

$$t=\nu d^\wedge\tau,\tau=\tau_1+\theta_\nu/(\nu d^\wedge),|z|>1,\nu\geq 2,$$
$$\log(R_1(t,\nu)|z|^{-t})=$$
$$\nu(m+n)d^\wedge\tau_1\log(\tau_1)-\nu nd^\wedge(\tau_1-\delta)\log(\tau_1-\delta)-\nu md^\wedge(\tau_1+1)\log(\tau_1+1)-$$
$$\nu nd^\wedge\delta\log(\delta)-\nu d^\wedge\tau_1\log(|z|)+O(1)$$

for

$$t=\nu d^\wedge\tau,\tau=\tau_1+\theta_\nu/(\nu d^\wedge),|z|>1,\nu\geq 2.$$

It follows from (41) that

$$(m+n)\log(\tau_1)-n\log(\tau_1-\delta)-m\log(\tau_1+1)-\log(|z|)=0. \tag{44}$$

On multiplying (44) by

$$\nu d^\wedge\tau_1$$

and then substracting the resulting equality from the previous one, we obtain the following equality:

$$\psi_0(|z|;\nu)=\nu^{(m-n)/2}\left(\frac{(\tau_1-\delta)^\delta}{(\tau_1+1)^\alpha(\delta)^\delta}\right)^{n\nu d^\wedge}\exp(O(1))=$$

$$\nu^{(m-n)/2}\left(\left(\frac{\eta_1}{\delta}\right)^{\delta}\left(\frac{1-\eta_1}{1+\delta}\right)^{\alpha-\delta}\right)^{n\nu d^{\wedge}}\exp(O(1)).$$

Since, for $t\geq \nu d^{\vee}+1$, the product $R(t,\nu)$ consists of $O(1)\nu$ factors and since the quotient of each of the factors to its partial derivative with respect to t can be bounded above as $O(1)$, it follows that

$$\left(\left(\frac{\partial}{\partial t}\right)^{\mu}R\right)(t,\nu)=\nu^{\mu}R(t,\nu)O(1)$$

for $\mu\in\mathbb{N}-1$ and $t\geq \nu d+1$. Therefore

$$\psi_{\mu}(|z|;\nu)=\nu^{\mu}\psi_0(|z|;\nu)O(1)=$$

(45)

$$\nu^{s-1+(m-n)/2}\left(\left(\frac{\eta_1}{\delta}\right)^{\delta}\left(\frac{1-\eta_1}{1+\delta}\right)^{\alpha-\delta}\right)^{n\nu d^{\wedge}}O(1)$$

for $\mu=0\,,\dots,s-1$, where $s\leq n$.

§7. On the arithmetic properties of the coefficients of some simple fractions.

Let

$$c_1=c_1(\nu)=\nu d^{\vee},c_2=c_2(\nu)=\nu d^{\wedge}+1,u(\nu)=c_1(\nu)+c_2(\nu). \tag{46}$$

It follows that

$$R(t,\nu)=\sum_{k=0}^{c(\nu)}\sum_{\lambda=1}^{m}\frac{\alpha_{k,\lambda}^{\sim}(\nu)}{(t+k)^{\lambda}}, \tag{47}$$

with

$$\alpha_{k,\lambda}^{\sim}=\frac{1}{(m-\lambda)!}\lim_{t\to -k}\left(\frac{\partial}{\partial t}\right)^{m-\lambda}((t+k)^mR)(t,\nu)$$

for $k=0\,,\dots,c_2(\nu)$ and $\lambda=1\,,\dots,m$. It follows from the well-known Leibniz differentiation rule that

$$\alpha_{k,\lambda}^{\sim}=(-1)^{nc_1(\nu)+mk}\binom{c_1(\nu)+k}{k}^n\binom{c_2(\nu)}{k}^m\times \tag{48}$$

$$\sum_{\substack{\lambda_1+\ldots+\lambda_{u(\nu)}=m-\lambda \\ \lambda_1\in\mathbb{N}-1,\ldots,\lambda_{u(\nu)}\in\mathbb{N}-1 \\ \lambda_{c_2(\nu)-k+\kappa}\leq n, \\ k=0,\ldots,c_2(\nu),\kappa=k+1,\ldots,c_1(\nu)+k}} \Pi_{k,\lambda}(\nu;\lambda_1,\ldots,\lambda_{u(\nu)})$$

with

$$\Pi_{k,\lambda}(\nu;\lambda_1,\ldots,\lambda_{c_1(\nu)+c_2(\nu)}) = \tag{49}$$

$$\left(\prod_{\kappa=1}^{k}\binom{m+\lambda_\kappa-1}{\lambda_\kappa}\kappa^{-\lambda_\kappa}\right)\times$$

$$\left(\prod_{\kappa=1}^{c_2(\nu)-k}\binom{m+\lambda_{k+\kappa}-1}{\lambda_{k+\kappa}}(-\kappa)^{-\lambda_{k+\kappa}}\right)\times$$

$$\left(\prod_{\kappa=k+1}^{c_1(\nu)+k}\binom{n}{\lambda_{c_2(\nu)-k+\kappa}}(-\kappa)^{-\lambda_{c_2(\nu)-k+\kappa}}\right)$$

for $k=0,\ldots,c_2(\nu)$ and $\lambda=1,\ldots,m$. Let

$$c_3(\nu)=min(c_1(\nu),c_2(\nu),c_4(\nu)=max(c_1(\nu),c_2(\nu)),$$

(50)

$$c_5(\nu)=u(\nu)=c_1(\nu)+c_2(\nu))=c_3(\nu)+c_4(\nu).$$

For a rational prime p, let v_p stand for the p-adic valuation on $\mathbb{Q}$.

Lemma 3. *If $p>c_4(\nu)$, then*

$$v_p(\alpha_{k,\lambda}^{\sim})\geq 0 \tag{51}$$

for $k=0,\ldots,c_2(\nu)$ and $\lambda=1,\ldots,m$.

Proof. If $p>max(c_4(\nu),c_1(\nu)+k)$, then the asserted inequality (51) is an immediate consequence of the equations (48) and (49). Suppose that

$$c_4(\nu)<p\leq c_1(\nu)+k. \tag{52}$$

Then

$$c_1(\nu)+c_2(\nu))\leq 2c_4(\nu)<2p<p^2, \tag{53}$$

and

$$v_p\left(\binom{c_1(\nu)+k}{k}^n\right) = n. \tag{54}$$

Since the variable κ in the equation (49) satisfies the inequality

$$\kappa \leq max(k, c_2(\nu) - k, c_1(\nu) + k) \tag{55}$$

and

$$\binom{n}{\lambda_{c_2(\nu)-k+\kappa}} = 0$$

for $\lambda_{c_2(\nu)-k+\kappa} > n$,then the assertion of the lemma now follows immediately from (48) and (49). The lemma is proved.

Let, as usual, $\pi(x)$ stand for the number of rational primes p with $p \leq x$, $x \in \mathbb{R}$, so that $\pi(x) = 0$ for $x < 2$. Let $\{\alpha, \beta, \gamma\} \subset \mathbb{R}$. If $\pi(\alpha) < \pi(\beta)$ and $1 \leq \gamma$, let $d_{\alpha,\beta,\gamma}$ denote the smallest positive integer d satisfying the condition

$$v_p\left(\frac{d}{\kappa}\right) \geq 0 \tag{56}$$

for all primes p in the interval $\alpha < p \leq \beta$ and all positive integers κ with $\kappa \leq \gamma$. We let, by definition, $d_{\alpha,\beta,\gamma} = 1$ if ether $\gamma < 1$, or $\pi(\beta) \leq \pi(\alpha)$. Clearly,

$$d_{\alpha,\beta,\gamma} = \prod_{\alpha<p\leq\beta} p^{[ln(\gamma)/ln(p)]} \leq \gamma^{max(\pi(\beta)-\pi(\alpha),0)}. \tag{57}$$

Lemma 4. *If*

$$1 \leq \alpha, 1 \leq \beta, 1 \leq \gamma \leq \gamma_0\alpha \tag{58}$$

for some $\gamma_0 > 0$, then for every $\varepsilon > 0$ there exists $C_0(\gamma_0, \varepsilon) > 0$ such that

$$d_{\alpha,\beta,\gamma} \leq C_0(\gamma_0, \varepsilon) \exp(\alpha((\max(\beta/\alpha - 1, 0) + \varepsilon)),$$

(59)

$$d_{1,\alpha,\gamma} \leq C_0(\gamma_0, \varepsilon) \exp \alpha(1 + \varepsilon)).$$

Proof. We can assume that $\beta \leq \gamma$. It follows from the relations (57) and (58) that

$$d_{\alpha,\beta,\gamma} \leq$$

$$\exp((\log(\gamma_0/e)+\log(\alpha e))(\max(\beta-\alpha,0)/\log(e\alpha)+O(\max(\beta-\alpha,0)/\log^2(e\alpha)))) \leq$$
$$C_0(\gamma_0,\varepsilon)\exp(\alpha(\max(\beta/\alpha-1,0)+\varepsilon)),$$

(60)

$$d_{1,\alpha,\gamma} \leq \exp((\log(\gamma_0/e)+\log(\alpha e))(\alpha/\log(e\alpha)+O(\alpha/\log^2(e\alpha)))) \leq$$
$$C_0(\gamma_0,\varepsilon)\exp(\alpha(1+\varepsilon)).$$

The lemma is proved.

Lemma 5. *Each of the numbers*

$$\alpha^{\sim}_{k,\lambda}(d_{1,c_2(\nu),c_5(\nu)})^{m-\lambda}(d_{c_2(\nu),c_1(\nu),c_5(\nu)})^{min(m-\lambda,nc_1(\nu))},$$

for $k=0,\ldots,c_2(\nu)$, $\lambda=1,\ldots,m$, *is contained in* $\mathbb{Z}$.

Proof. Let us consider the equality

$$\Pi_{k,\lambda}(\nu;\lambda_1,\ldots,\lambda_{u(\nu)})(d_{1,c_2(\nu),c_5(\nu)})^{m-\lambda}(d_{c_2(\nu),c_1(\nu),c_5(\nu)})^{min(m-\lambda,nc_1)} = \tag{61}$$
$$\left(\prod_{\kappa=1}^{k}\binom{m+\lambda_\kappa-1}{\lambda_\kappa}\left(\frac{d_{1,c_2(\nu),c_5(\nu)}}{\kappa}\right)^{\lambda_\kappa}\right)\times$$
$$\left(\prod_{\kappa=1}^{c_2(\nu)-k}\binom{m+\lambda_{k+\kappa}-1}{\lambda_{k+\kappa}}\left(\frac{-d_{1,c_2(\nu),c_5(\nu)}}{\kappa}\right)^{\lambda_{k+\kappa}}\right)\times$$
$$\left(\prod_{\kappa=k+1}^{c_1(\nu)+k}\binom{n}{\lambda_{c_2(\nu)-k+\kappa}}\left(\frac{-d_{1,c_2(\nu),c_5(\nu)}d_{c_2(\nu),c_1(\nu),c_5(\nu)}}{\kappa}\right)^{\lambda_{c_2(\nu)-k+\kappa}}\right)\times$$
$$\left(d_{c_2(\nu),c_1(\nu),c_5(\nu)}(\nu)\right)^{\lambda_{c_5(\nu)+1}},$$

where

$$\lambda_{c_5(\nu)+1} = min(m-\lambda,nc_1(\nu)) - \sum_{\kappa=k+1}^{c_1(\nu)+k}\lambda_{c_2(\nu)-k+\kappa}$$

depends, of course, on λ and k and, in view of (48), $\lambda_{c_5(\nu)+1}\geq 0$. Further, $c_2(\nu) < c_5(\nu)$; moreover, if $1\leq\kappa\leq k$, then κ and therefore any prime dividing κ do not exceed $c_2(\nu)$. Consequently,

$$v_p\left(\prod_{\kappa=1}^{k}\binom{m+\lambda_\kappa-1}{\lambda_\kappa}\left(\frac{d_{1,c_2(\nu),c_5(\nu)}}{\kappa}\right)^{\lambda_\kappa}\right)\geq 0$$

for every prime number p,

$$v_p\left(\prod_{\kappa=1}^{c_2(\nu)-k}\binom{m+\lambda_{k+\kappa}-1}{\lambda_{k+\kappa}}\left(\frac{-d_{1,c_1(\nu),c_5(\nu)}}{\kappa}\right)^{\lambda_{k+\kappa}}\right)\geq 0$$

for every prime number p, and

$$v_p\left(\prod_{\kappa=k+1}^{c_1(\nu)+k}\binom{n}{\lambda_{c_2(\nu)-k+\kappa}}\left(\frac{-d_{1,c_2(\nu),c_5(\nu)}d_{c_2(\nu),c_1(\nu),c_5(\nu)}}{\kappa}\right)^{\lambda_{c_2(\nu)-k+\kappa}}\right)\geq 0$$

for every prime number p not exceeding $c_4(\nu)$. Therefore

$$v_p\left(\alpha^{\sim}_{k,\lambda}(d_{2,c_1(\nu),c_5(\nu)})^{m-\lambda}(d_{c_2(\nu),c_1(\nu),c_5(\nu)})^{min(m-\lambda,nc_1(\nu))}\right)\geq 0$$

for every prime number p not exceeding $c_4(\nu)$. It follows now from Lemma 3 that

$$v_p\left(\alpha^{\sim}_{k,\lambda}(d_{1,c_1(\nu),c_5(\nu)})^{m-\lambda}(d_{c_2(\nu),c_1(\nu),c_5(\nu)})^{min(m-\lambda,nc_1(\nu))}\right)\geq 0 \tag{62}$$

for every prime number p. The lemma is proved.

§8. The first progress in our attempts to satisfy the conditions of Lemma 1; the plan of the further proceedings.

Since

$$\sum_{t=1}^{\infty}z^{-t}/(t+k)^{\lambda+\mu}=z^kL_{\lambda+\mu}(z^{-1})-\sum_{l=1}^{k}z^{k-l}/l^{\lambda+\mu}$$

with $\lambda=1,\ldots,m$, $\mu\in\mathbb{N}-1$, it follows from (33) that

$$\psi_\mu(z;\nu)=-P_{m+1+\mu}(z;\nu)+\sum_{\lambda=1}^{m}P_\lambda(z;\nu)\binom{\mu+\lambda-1}{\mu}L_{\lambda+\mu}(z^{-1}), \tag{64}$$

where

$$P_\lambda(z;\nu)=\sum_{k=0}^{c_2(\nu)}\alpha^{\sim}_{k,\lambda}z^k$$

(65)

$$P_{m+1+\mu}(z;\nu)=\sum_{l=0}^{c_2(\nu)-1} z^l \sum_{k=l+1}^{c_2(\nu)}\sum_{\lambda=1}^{m}\binom{\mu+\lambda-1}{\mu}\frac{\alpha^{\sim}_{k,\lambda}}{(k-l)^{\lambda+\mu}}$$

for $\lambda=1\,,\dots,m,\ \mu\in\mathbb{N}-1$.

Lemma 6. *Let $a\in\mathbb{Z}$, $b\in\mathbb{N}$, $z=\frac{a}{b}$, $s\in\mathbb{N}$, $s\leq n$, $\nu\in\mathbb{N}$, and*

(66) $$d_s^{\sim}(\nu)=(d_{1,c_2(\nu),c_5(\nu)})^{m+s-1}(d_{c_2(\nu),c_1(\nu),c_5(\nu)})^{min(m-1,nc_1(\nu))}.$$

Then

$$P^*_\lambda(z;\nu)=b^{c_2(\nu)}d_s^{\sim}(\nu)P_\lambda(z;\nu)\in\mathbb{Z}, P^*_{m+1+\mu}=b^{c_2(\nu)}d_s^{\sim}(\nu)P_{m+1+\mu}(z;\nu)\in\mathbb{Z},$$

(67)

$$P^*_{m+1+\mu}=b^{c_2(\nu)}d_s^{\sim}(\nu)P_{m+1+\mu}(z;\nu)\in\mathbb{Z}$$

where $\lambda=1\,,\dots,m$ and $\mu=0\,,\dots,s-1$.

Proof. The assertion of the lemma is an immediate consequence of the equalities (65) and Lemma 5.

Lemma 7. *For every $\varepsilon>0$, there exists $c(\varepsilon)>1$ such that*

$$\left(b^{c_2(\nu)}d_s^{\sim}(\nu)\right)^{1/(nd^{\wedge}\nu)}\leq$$

(68)

$$c(\varepsilon)\left(\exp(s/n+(\alpha-1/n)\max(\delta,1))b^{1/n}\right)^{1+\varepsilon},$$

where $\alpha=m/n$ and $\delta=d^{\vee}/d^{\wedge}$.

Proof. Since $c_5(\nu)/c_2(\nu)<1+\delta$, we may apply Lemma 4 with $\alpha=c_2(\nu)$, $\beta=c_1(\nu)$, $\gamma=c_5(\nu)$, and $\gamma_0=1+\delta_0$. By Lemma 4, for every $\varepsilon>0$, there exists $c_0(\varepsilon)>1$ such that

$$(d_{1,c_2(\nu),c_5(\nu)})^{m+s-1}\leq(c_0(\varepsilon))^{m+s-1}\exp(d^{\wedge}\nu(1+1/d^{\wedge}\nu)(1+\varepsilon/2)(m+s-1))$$

and

$$d_{c_2,c_1(\nu),c_5(\nu)})^{m-1}\leq(c_0(\varepsilon))^{m-1}\exp(d^{\wedge}\nu(1+1/d^{\wedge}\nu)max(\delta-1,0)(1+\varepsilon/2)(m-1))$$

since $c_1(\nu)/c_2(\nu)<\delta_0$. Therefore it follows from (66) that

$$d_s^{\sim}(\nu)=(d_{1,c_2(\nu),c_5(\nu)})^{m+s-1}(d_{c_2(\nu),c_1(\nu),c_5(\nu)})^{m-1}\leq$$

$$(d_{1,c_2(\nu),c_5(\nu)})^s(d_{1,c_2(\nu),c_5(\nu)}d_{c_2(\nu),c_1(\nu),c_5(\nu)})^{m-1}\leq$$
$$(c_0(\varepsilon))^{2m-2+s}\exp(d^\wedge\nu(1+1/d^\wedge\nu)(1+\varepsilon/2)s)\times$$
$$\exp(d^\wedge\nu(1+1/d^\wedge\nu)(1+\max(\delta-1,0))(1+\varepsilon/2)(m-1))=$$
$$(c_0(\varepsilon)^{2m-2+s}\exp(d^\wedge\nu(1+1/d^\wedge\nu)(1+\varepsilon/2)s)\times$$
$$\exp(d^\wedge\nu(1+1/d^\wedge\nu)\max(\delta,1)(1+\varepsilon/2)(m-1)).$$

Consequently,

$$\left(b^{c_2(\nu)}d_s^\sim(\nu)\right)^{1/(nd^\wedge\nu)}\leq$$
$$b^{(1+1/d^\wedge\nu)/n}(c_0(\varepsilon))^{(2m-2+s)/(nd^\wedge\nu)}\exp(d^\wedge\nu(1+1/d^\wedge\nu)(1+\varepsilon/2)s/n)\times$$
$$\exp(d^\wedge\nu(1+1/d^\wedge\nu)\max(\delta,1)(1+\varepsilon/2)(\alpha-1/n)).$$

The last inequality implies the assertion of the lemma. The lemma is proved.

Making use of (64), we wish to apply Lemma 1 to the matrix $\Lambda_{m,s}\left(\frac{b}{a}\right)$ of Theorem 1, taking m in (67) as "m" in (3), s in (67) as "n" in (3), λ in (67) as "j" in (3), $P^*_\lambda(z;\nu)=b^{c_2(\nu)}d_s^\sim(\nu)P_\lambda(z;\nu)$ in (67) as "$\alpha_j(\nu)$" in (3), $\mu=0,\ldots,s-1$ in (67) as "$k-1$" in (3), and $P^*_{m+1+\mu}(z;\nu)=b^{c_2(\nu)}d_s^\sim(\nu)P_{m+1+\mu}(z;\nu)$ in (67) as "$\alpha_{m+k}(\nu)$" in (3). To conclude the proof of Theorem 1, we have to prove that the relation

$$\lim_{\nu\to\infty}\psi^*_\mu(z;\nu)=0,$$

for $\psi^*_\mu(z;\nu)=b^{c_2(\nu)}d_s^\sim(\nu)\psi_\mu(z;\nu)$ with $\psi_\mu(z;\nu)$ from (64), follows from (26) and that for every $\kappa\in\mathbb{N}$ the $m+s$ sequences

$$\{b^{c_2(\nu)}d_s^\sim(\kappa+\omega)P_l(z;\nu):\omega\in\mathbb{N}\},$$

where $l=1,\ldots,m+s$, are linearly independent over $\mathbb{R}$.

§9. Linear independence over $\mathbb{R}$ of certain sequences in $\mathbb{Q}^{\mathbb{N}}$.

For $d\in\mathbb{N}-1$ and $\sigma=0,\ldots,d$, let $\nu_\sigma\in\mathbb{Z}$. Let further

$$\Pi^*_{i,\lambda}(d;\nu_0,\ldots,\nu_d)=\binom{m+\nu_0-1}{i\lambda+\nu_0-i}\prod_{\kappa=1}^{d}\binom{n}{\nu_\kappa}\kappa^{-\nu_\kappa}\tag{69}$$

and

$$h_{i,\lambda}(d)=\sum_{\substack{0\leq\nu_1\leq n,\ldots,0\leq\nu_d\leq n\\ \nu_0+\ldots+\nu_d=m-\lambda}}\Pi^*_{i,\lambda}(d;\nu_0,\ldots,\nu_d),\tag{70}$$

where $\lambda = 1, \dots, m$ and $i = 0, 1$. It follows from (69) that

$$h_{i,\lambda}(d) = \sum_{\substack{i - i\lambda \le \nu_0, 0 \le \nu_1 \le n, \dots, 0 \le \nu_d \le n \\ \nu_0 + \dots + \nu_d = m - \lambda}} \Pi^*_{i,\lambda}(d; \nu_0, \nu_1, \dots, \nu_d). \tag{71}$$

where, as above, $\lambda = 1, \dots, m$ and $i = 0, 1$.

Lemma 8. *We have*

$$h_{1,\lambda}(0) = \binom{2m - \lambda - 1}{m - \lambda} = h_{0,\lambda}(0) \tag{72}$$

for $\lambda = 1, \dots, m$. Moreover, if $2 \le \lambda \le m, 0 < d < m/n$ and $\varepsilon^3 = \varepsilon$, then

$$h_{1,\lambda}(d) + \varepsilon h_{0,\lambda}(d) > 0, \tag{73}$$

and

$$h_{1,1}(d) = h_{0,1}(d) > 0. \tag{74}$$

Let $\lambda = 1, \dots, m$; the coefficient of $z^{m-\lambda}$ in the Taylor's series expansion of the function

$$S_0(z) = (1 - z)^{-m} \prod_{\kappa=1}^{d} (1 + z/\kappa)^n \tag{75}$$

is equal to $h_{0,\lambda}(d)$ and, if $0 \le d < m/n$, then the coefficient of $z^{m-\lambda}$ in the Taylor's series expansion of the function

$$S_1(z) = (1 - z)^{-m} (d!)^{-n} \prod_{\kappa=1}^{d} (\kappa + 1 - z)^n \tag{76}$$

is equal to $h_{1,\lambda}(d)$.

Proof. Equalities (72) are a direct consequence of (70). It follows from (72) that $h_{1,1}(0) = h_{0,1}(0) > 0$. On rewriting (71), one obtains the following equations

$$h_{0,\lambda}(d) = \sum_{\substack{0 \le \nu_0, 0 \le \nu_1 \le n, \dots, 0 \le \nu_d \le n \\ \nu_0 + \dots + \nu_d = m - \lambda}} \Pi^*_{i,\lambda}(d; \nu_0 \le n, \dots, \nu_d), \tag{77}$$

$$h_{1,\lambda}(d) = \sum_{\substack{1 - \lambda \le \nu_0, 0 \le \nu_1 \le n, \dots, 0 \le \nu_d \le n \\ \nu_0 + \dots + \nu_d = m - \lambda}} \Pi^*_{i,\lambda}(d; \nu_0 \le n, \dots, \nu_d). \tag{78}$$

for $\lambda = 1, \dots, m$. It follows from (77) that $0 \le \nu_0 \le \lambda + \nu_0 - 1 \le m + \nu_0 - 1$ and $\nu_0 \le m - \lambda$. Therefore

$$\binom{m+\nu_0-1}{m-\lambda} = \binom{m+\nu_0-1}{\lambda+\nu_0-1} \ge 1 \tag{79}$$

and

$$\binom{m+\nu_0-1}{\nu_0} \le \binom{m+\nu_0-1}{\lambda+\nu_0-1}. \tag{80}$$

Equality in (80) holds only in one of the following cases:

(i) if $\lambda = 1$,

or

(ii) if $2 \le \lambda \le m$ and $\nu_\kappa = 0$ for all $\kappa = 1, \dots, m$,

or

(iii) if $m - 1 - \sum\limits_{\kappa=1}^{d} \nu_\kappa = \nu_0 + \lambda - 1 < 0$.

If $\lambda = 1$, then $\nu_0 \ge 0$; therefore relation (74), with $d > 0$, follows from the relations (77) - (79). Further, let $2 \le \lambda \le m$ and $0 < d < m/n$, then condition (iii) does not hold and therefore relation (80) is a strict inequality for $\nu_1 = n, \dots, \nu_d = n$. Thus the inequality (73) follows from the relations (77) - (80). The right hand side of the equation (70) for $i = 0$ clearly coincides with the coefficient of $z^{m-\lambda}$ in the Taylor's series expansion of the function (75). The function (76) is equal to

$$\sum_{0 \le \nu_1 \le n, \dots, 0 \le \nu_d \le n} \Pi_d^{\sim}(z; \nu_1, \dots, \nu_d),$$

where

$$\Pi_d^{\sim}(z; \nu_1, \dots, \nu_d) = (1-z)^{(\sum\limits_{\kappa=1}^{d} \nu_\kappa) - m} \prod_{\kappa=1}^{d} \binom{n}{\nu_\kappa} \kappa^{-\nu_\kappa};$$

if $d < m/n$, then $m - \sum\limits_{\kappa=1}^{d} \nu_\kappa > 0$ in the last expression, and therefore the coefficient of $z^{m-\lambda}$ in the Taylor's series expansion of the function $\Pi_d^{\sim}(z; \nu_1, \dots, \nu_d)$ is equal to

$$\binom{m+\nu_0-1}{\lambda+\nu_0-1} \prod_{\kappa=1}^{d} \binom{n}{\nu_\kappa} \kappa^{-\nu_\kappa}$$

with

$$\nu_0 = m - \lambda - \sum_{\kappa=1}^{d} \nu_\kappa.$$

Thus the right hand side of the equation (70) for $i = 1$ coincides with the coefficient of $z^{m-\lambda}$ in the Taylor's series expansion of the function (76). This proves the lemma.

Corollary. *If* $0 \leq d < m/n$ *and* $|z| > 1$, *then*

$$\pm z h_{1,\lambda}(d) + h_{0,\lambda}(d) \neq 0 \tag{81}$$

for $\lambda = 1, \ldots, m$.

Lemma 9. *Let* p *be a prime; suppose that*

$$d \in \mathbb{N} - 1, r \in \mathbb{N} - 1, d_1 \in \mathbb{N} - 1, d_2 \in \mathbb{N} - 1, r_1 \in \mathbb{N} - 1, r_2 \in \mathbb{N} - 1,$$

and

$$max(r_1, r_2) < p. \tag{82}$$

Then

$$p^{-d}(dp + r)! \in (-1)^d d! r! + p\mathbb{Z} \tag{83}$$

and

$$\binom{(d_1 + d_2)p + r_1 + r_2}{d_1 p + r_1} \in \binom{d_1 + d_2}{d_1}\binom{r_1 + r_2}{r_1} + p\mathbb{Z}. \tag{84}$$

Proof. Since

$$p^{-d}(dp + r)! = \left(\prod_{k=1}^{d} k \prod_{l=1}^{p-1}((k-1)p + l)\right) \prod_{j=1}^{r}(dp + j), \tag{85}$$

relation (83) follows from Wilson's theorem. Since

$$v_p\left(\left(\prod_{j=1}^{r_i}(d_i p + j)\right) - r_i!\right) \geq 1,$$

for $i = 1, 2$ and, in view of (82), $v_p(r_i!) = 0$ for $i = 1, 2$, it follows from (85) and Wilson's theorem that

$$v_p\left(\binom{(d_1 + d_2)p + r_1 + r_2}{d_1 p + r_1} - \binom{d_1 + d_2}{d_1}\binom{r_1 + r_2}{r_1}\right) > 0. \tag{86}$$

This proves the lemma.

Lemma 10. *Let p be a prime; suppose that*

$$d \in \mathbb{N}, r \in \mathbb{N}, r < p, d^{\sim} \in \mathbb{N} - 1,$$

and $d^{\sim} < d$. Then

$$\binom{dp}{d^{\sim}p + r} \in d\binom{d-1}{d^{\sim}}\binom{p}{r} + p^2\mathbb{Z}. \tag{87}$$

Proof. Since

$$\binom{dp}{d^{\sim}p + r} = \binom{(d-1)p + p - 1}{d^{\sim}p + r - 1}\frac{dp}{d^{\sim}p + r},$$

$$v_p(d^{\sim}p + r) = v_p(r) = 0, v_p((d^{\sim}p + r)^{-1} - r^{-}) > 0,$$

$$\binom{(d-1)p + p - 1}{d^{\sim}p + r - 1} \in \binom{d-1}{d^{\sim}}\binom{p-1}{r-1} + p\mathbb{Z}$$

by Lemma 9, it follows that

$$v_p\left(\binom{(d-1)p + p - 1}{d^{\sim}p + r - 1}\frac{d}{d^{\sim}p + r} - \binom{d-1}{d^{\sim}}\binom{p-1}{r-1}\frac{d}{r}\right) > 0,$$

$$v_p\left(\binom{(d-1)p + p - 1}{d^{\sim}p + r - 1}\frac{dp}{d^{\sim}p + r} - \binom{d-1}{d^{\sim}}\binom{p-1}{r-1}\frac{dp}{r}\right) > 1,$$

and

$$\binom{d-1}{d^{\sim}}\binom{p-1}{r-1}\frac{dp}{r} = d\binom{d-1}{d^{\sim}}\binom{p}{r}.$$

This proves the lemma.

Lemma 11. *Given a prime number p, let*

$$r \in \mathbb{N} - 1, r < p, d \in \mathbb{N} - 1,$$

let

$$c_2(\nu) = p, c_1(\nu) = dp + r, c_1(\nu) + c_2(\nu) < p^2 \tag{88}$$

for some $\nu \in \mathbb{N}$, and let $\alpha^{\sim}_{k,\lambda}(\nu)$ be defined by the equations (48), (49). Then the following inequalities hold true:

$$v_p(\alpha^{\sim}_{k,\lambda}(\nu)) \geq \lambda, \tag{89}$$

where $k = 1, \ldots, p-1$ and $\lambda = 1, \ldots, m$,

$$v_p(\alpha^{\sim}_{sp,\lambda}(\nu)) \geq \lambda - m, \tag{90}$$

where $s = 0, 1$ *and* $\lambda = 1, \ldots, m$,

$$v_p\left(\alpha^{\sim}_{sp,\lambda}(\nu)(-1)^{spm+nc_1(\nu)+(1-s)(m-\lambda)} - p^{\lambda-m}h_{s,\lambda}(d)\right) \geq \lambda - m + 1, \tag{91}$$

where $s = 0, 1$ *and* $\lambda = 1, \ldots, m$.

Proof. The inequalities (89) and (90) follow directly from the equalities (48), (49). Let us prove (91). Since the parameters in the equations (48), (49) satisfy the following inequalities: $0 \leq k \leq c_2(\nu)$ and $1 \leq \kappa \leq max(c_1(\nu)+k, c_2(\nu)-k) \leq c_1(\nu)+c_2(\nu) < p^2$, it follows that

$$(v_p(\kappa))^2 = v_p(\kappa),$$

and that

$$v_p\left(\Pi_{sp,\lambda}(\nu;\lambda_1, \ldots, \lambda_{c_1(\nu)+c_2(\nu)})\right) =$$

$$\left(-\sum_{\sigma=1}^{s}\lambda_{p\sigma}\right) - \left(\sum_{\sigma=1}^{1-s}\lambda_{p(s+\sigma)}\right) - \left(\sum_{\sigma=s+1}^{d^{\sim}+s}\lambda_{p(1-s+\sigma)}\right) = \lambda - m +$$

$$\left(\sum_{\substack{1\leq\kappa\leq^{s}p\\(\kappa,p)=1}}\lambda_\kappa\right) + \left(\sum_{\substack{1\leq\kappa\leq c_2(\nu)-sp\\(\kappa,p)=1}}\lambda_{sp+\kappa}\right) + \left(\sum_{\substack{ps+1\leq\kappa\leq c_1(\nu)+sp\\(k,p)=1}}\lambda_{c_2(\nu)-k+\kappa}\right).$$

Therefore

$$v_p\left(\Pi_{sp,\lambda}(\nu;\lambda_1, \ldots, \lambda_{u(\nu)})\right) \geq \lambda - m + 1$$

if at least one of the last three sums strictly positive. Thus

$$(-1)^{nc_1(\nu)+msp}\alpha^{\sim}_{sp,\lambda} \in p^{\lambda-m+1}\mathbb{Z} + \binom{c_1(\nu)+sp}{sp}^n\binom{c_2(\nu)}{sp}^m \times$$

$$\sum_{\substack{\lambda_1+\ldots+\lambda_{u(\nu)}=m-\lambda\\ \lambda_1\in\mathbb{N}-1, \ldots, \lambda_{c_5(\nu)}\in\mathbb{N}-1\\ \sum_{\sigma=1}^{1+d}\lambda_{p\sigma}=m-\lambda}}\Pi_{k,\lambda}(\nu;\lambda_1, \ldots, \lambda_{u(\nu)}).$$

It follows from (49) that if the integers $\lambda_1, \ldots, \lambda_{u(\nu)}$ lie in $\mathbb{N}-1$ and satisfy the relations

$$\lambda_1 + \ldots + \lambda_{u(\nu)} = m - \lambda,$$

$$\lambda_1 \in \mathbb{N}-1, \ldots, \lambda_{c_5(\nu)} \in \mathbb{N}-1,$$

and

$$\sum_{\sigma=1}^{1+d} \lambda_{p\sigma} = m - \lambda,$$

then

$$\Pi_{sp,\lambda}(\nu; \lambda_1, \dots, \lambda_{u(\nu)}) = \tag{92}$$

$$\left(\prod_{\sigma=1}^{s} \binom{m + \lambda_{p\sigma} - 1}{\lambda_{p\sigma}} (p\sigma)^{-\lambda_{p\sigma}}\right) \times$$

$$\left(\prod_{\sigma=1}^{1-s} \binom{m + \lambda_{p(s+\sigma)} - 1}{\lambda_{p(s+\sigma)}} (-p\sigma)^{-\lambda_{p(s+\sigma)}}\right) \times$$

$$\left(\prod_{\sigma=s+1}^{d+s} \binom{n}{\lambda_{p(1-s+\sigma)}} (-p\sigma)^{-\lambda_{p(1-s+\sigma)}}\right) =$$

$$p^{\lambda-m} \left(\prod_{\sigma=1}^{s} \binom{m + \lambda_{p\sigma} - 1}{\lambda_{p\sigma}} (\sigma)^{-\lambda_{p\sigma}}\right) \times$$

$$\left(\prod_{\sigma=1}^{1-s} \binom{m + \lambda_{p(s+\sigma)} - 1}{\lambda_{p(s+\sigma)}} (-\sigma)^{-\lambda_{p(s+\sigma)}}\right) \times$$

$$\left(\prod_{\sigma=s+1}^{d+s} \binom{n}{\lambda_{p(1-s+\sigma)}} (-\sigma)^{-\lambda_{p(1-s+\sigma)}}\right)$$

for $s = 0, 1$ and $\lambda = 1, \dots, m$. For $s = 0$, the last product (92) is equal to

$$p^{\lambda-m} \binom{m + \lambda_p - 1}{\lambda_p} (-1)^{-\lambda_p} \left(\prod_{\sigma=1}^{d} \binom{n}{\lambda_{p(1+\sigma)}} (-\sigma)^{-\lambda_{p(1+\sigma)}}\right). \tag{93}$$

For $\nu_0 = \lambda_p$ and $\nu_\sigma = \lambda_{p(1+\sigma)}$, with $\sigma = 1, ..., d^\sim$, the product (93) is equal to $(-p)^{\lambda-m} \Pi^*_{0,\lambda}(d; \nu_0, \dots, \nu_{d^\sim})$. This proves relation (91) for $s = 0$. For $s = 1$, the last product (92) is equal to

$$p^{\lambda-m} \binom{m + \lambda_p - 1}{\lambda_p} \left(\prod_{\sigma=2}^{d+1} \binom{n}{\lambda_{p\sigma}} (-\sigma)^{-\lambda_{p(1-s+\sigma)}}\right). \tag{94}$$

For $\sigma = \kappa+1, \nu_\kappa = \lambda_{p(\kappa+1)}$, with $\kappa = 1, \ldots, d$, the product (94) is equal to $p^{\lambda-m}\Pi^*_{2,\lambda}(d;\nu_0, \ldots, \nu_d)$, where

$$\Pi^*_{2,\lambda}(d;\nu_0, \ldots, \nu_d) = \binom{m+\nu_0-1}{\nu_0}\left(\prod_{\kappa=1}^{d}\binom{n}{\nu_\kappa}(-(\kappa+1))^{-\nu_\kappa}\right).$$

It follows from Lemma 9 that

$$\binom{c_1(\nu)+p}{p}^n\binom{c_2(\nu)}{p}^m =$$

$$\binom{(d+1)p+r}{p+0}^n\binom{p}{p}^m \in \binom{d+1}{1}^n\binom{r}{0}^n + p\mathbb{Z}.$$

Therefore

$$(-1)^{nc_1(\nu)+mp}\alpha^{\sim}_{p,\lambda} \in p^{\lambda-m+1}\mathbb{Z} + h_{2,\lambda}(d),$$

where

$$h_{2,\lambda}(d) = (d+1)^n \sum_{\substack{0\le\nu_1\le n, \ldots, 0\le\nu_d\le n \\ \nu_0+\ldots+\nu_d=m-\lambda}} \Pi^*_{2,\lambda}(d;\nu_0, \ldots, \nu_d).$$

On rewriting (76) as follows:

$$S_1(z) = (1-z)^{-m}(d+1)^n\prod_{\kappa=1}^{d}(1-z/(\kappa+1))^n, \tag{95}$$

one concludes that

$$h_{2,\lambda}(d) = h_{1,\lambda}(d),$$

with $d\in\mathbb{Z}$, $0\le d< m/n$, $\lambda = 1, \ldots, m$. This proves the lemma.

Lemma 12. *Suppose the assumptions of Lemma 11 be satisfied. Let $z\in\mathbb{Q}$, $\lambda\in\mathbb{N}$, $\lambda\le m$ and suppose that*

$$v_p(z) = v_p(\sum_{s=0}^{1}(-1)^s(h_{s,\lambda}(d)z^s)^2) = 0.$$

Then the values of the polynomial $P_\lambda(z;\nu)$ defined by the equations (65) satisfy the following relation:

$$v_p(P_\lambda(z;\nu)) = \lambda - m.$$

Proof. This lemma follows directly from (65),Lemma 11 and small Fermat theorem.

Lemma 13. *Suppose the assumptions of Lemma 11 be satisfied. Let* $z \in \mathbb{Q}$, $\mu \in \mathbb{N} - 1$, $\mu \leq n - 1$ *and suppose that*

$$v_p(z) = v_p\left(\sum_{\lambda=1}^{m} h_{1,\lambda}(d)\binom{\lambda+\mu-1}{\mu}\right) = 0.$$

Then the values of the polynomial $P_{m+\mu+1}(z;\nu)$ *defined by the equations (65) satisfy the following relation:*

$$v_p(P_{m+\mu+1}(z;\nu)) = -\mu - m.$$

Proof. It follows from Lemma 11 that the terms of the second sum in (65) with

$$l^2 + (c_2(\nu) - k)^2 > 0$$

lie in $p^{1-m}\mathbb{Z}$, while the sum of all the other terms lies in

$$p^{1-m-\mu}\mathbb{Z} + (-1)^{pm+nc_1(\nu)} p^{-m-\mu} \sum_{\lambda=1}^{m} h_{1,\lambda}(d)\binom{\lambda+\mu-1}{\mu}.$$

This proves the lemma.

Lemma 14.*Let* $z \in \mathbb{Q}$, $|z| > 1$, *then for every* $\kappa \in \mathbb{N}$ *the* $m+n$ *sequences*

$$\{P_i(z, \kappa+\omega) : \omega \in \mathbb{N}\},$$

with $i = 1, \ldots, m+n$, *are linearly independent over* $\mathbb{R}$.

Proof. Let

$$T_i(z,\nu) = P_{m+1-i}(z,\nu)$$

for $i = 1, \ldots, m$, and let

$$T_i(z,\nu) = P_i(z,\nu)$$

for $i = m+1, \ldots, m+n$. Let us prove first that, for each $\kappa \in \mathbb{N}$, the $m+n$ sequences

$$\{T_i(z, \kappa+\omega) : \omega \in \mathbb{N}\},$$

with $i = 1, \ldots, m+n$, are linearly independent over $\mathbb{Q}$. Suppose it is not the case. Then there exist $\kappa \in \mathbb{N}$ and $a_i \in \mathbb{Q}$, with $i = 1, \ldots, m+n$, satisfying the conditions

$$\sum_{i=1}^{m+n} |a_i| > 0$$

and

$$\sum_{i=1}^{m+n} a_i T_i(z;\nu) = 0 \tag{96}$$

for $\nu \in \kappa + \mathbb{N}$. Let c be equal to the product of all the non-zero numbers a_i for $i \in [1, m+n] \cap \mathbb{N}$. Let $z = a/b$, where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $|a| > b$. Finally, let d denote the biggest integer which is strictly smaller than $\delta = d^{\vee}/d^{\wedge}$, and let

$$F = \left(abc \prod_{\lambda=1}^{m} ((h_{0,\lambda}(d))^2 - (zh_{1,\lambda}(d))^2) \right) \prod_{\mu=0}^{n-1} \left(\sum_{\lambda=1}^{m} h_{1,\lambda}(d) \binom{\lambda+\mu-1}{\mu} \right).$$

Since $c_1(\nu)/c_2(\nu) < d^{\vee}/d^{\wedge}$ and $\lim\limits_{\nu \to +\infty} c_1(\nu)/c_2(\nu) = d^{\vee}/d^{\wedge} = \delta$, it follows that

$$\lim_{\nu \to +\infty} [c_1(\nu)/c_2(\nu)] = d,$$

where $[y]$ stands for the integral part of $y \in \mathbb{R}$ as usual. Therefore there is $\kappa_1 \in \mathbb{N}$ such that

$$[c_1(\nu)/c_2(\nu)] = d$$

for $\nu \in \mathbb{N} + \kappa_1$. Since

$$d < d^{\vee}/d^{\wedge} = \delta \leq \alpha = m/n,$$

it follows from Lemma 8 and the Corollary to this lemma that $F \neq 0$. Let $\nu \in (\mathbb{N} + \kappa_1) \cap (\mathbb{N} + \kappa)$, and suppose that $c_2(\nu) = d^{\wedge}\nu + 1$ is equal to a prime p with $v_p(F) = 0$. Then $c_1(\nu) = dp + r$, with $r \in \mathbb{N} - 1$, $r < p$, $d < \delta \leq \alpha = m/n$. Therefore the conditions of Lemmata 12 and 13 are satisfied and consequently

$$v_p(T_i(z; \nu)) = -i + 1$$

for $i = 1, \ldots, m+n$. Let $i_0 = max\{i : 1 \leq i \leq m+n,\ a_i \neq 0\}$, then

$$v_p\left(\sum_{i=1}^{m+n} a_i T_i(z; \nu) \right) = -i_0 + 1.$$

The last equation, however, contradicts (96). Therefore there is an invertible $m \times m-$ submatrix of the $m \times \mathbb{N}$ - matrix

$$\mathfrak{T} = (T_i(z, \kappa + \omega))_{1 \leq i \leq m+n,\, \omega \in \mathbb{N}}.$$

The lemma is proved.

Corollary. *Let $z \in \mathbb{Q}, |z| > 1$. Then, for every $\kappa \in \mathbb{N}$, the $m+n$ sequences*

$$\{P_i^*(z, \kappa + \omega) : \omega \in \mathbb{N}\},$$

where $i = 1, \ldots, m+n$, are linearly independent over $\mathbb{R}$.

Proof. Assuming the assertion does not hold, one can find $\kappa \in \mathbb{N}$ and $a_i \in \mathbb{R}$, with $i = 1, \dots, m+n$, satifying the conditions

$$\sum_{i=1}^{m+n} |a_i| > 0$$

and

$$\sum_{i=1}^{m+n} a_i P_i^*(z;\nu) = 0$$

for all $\nu \in \mathbb{N} + \kappa$. Then

$$b^{c_2(\nu)} d_n^{\sim}(\nu) \sum_{i=1}^{m+n} a_i P_i(z;\nu) = \sum_{i=1}^{m+n} a_i P_i^*(z;\nu) = 0$$

and, since $b^{c_2(\nu)} d_n^{\sim}(\nu) > 0$,

$$\sum_{i=1}^{m+n} a_i P_i(z;\nu) = 0$$

for all $\nu \in \mathbb{N} + \kappa$, in cotradiction with Lemma 14. This proves the corollary.

§10. Theorem 1 (the end of the proof).

Suppose the inequality (26) be satisfied. Then

$$\tilde{\rho}(\alpha, \delta, \eta_1(\alpha, \delta, D))(e^{s/n+(\alpha-1/n)max(\delta,1)} b^{1/n})^{1+\varepsilon} < 1$$

for a small enough $\varepsilon > 0$. Let

$$\psi_{j-1}^*(z;\nu) = -P_{m+j}^*(z;\nu) + \sum_{i=1}^{m} P_i^*(z;\nu) \binom{i+j-2}{i-1} L_{i+j-1}(z^{-1}), \tag{97}$$

where $j = 1, ..., s$ with $s \in \mathbb{N}$, $s \le n$. It now follows from the relations (45),(64),(67), and (68) that

$$\lim_{\nu \to +\infty} [\psi_{j-1}^*(z;\nu)] = 0$$

for each j. In view of Lemmata 6 and 14, $P_l^*(z, \nu) \in \mathbb{Z}$ for $l = 1, \dots, m+s$ and, for each $\kappa \in \mathbb{N}$, the $m+s$ sequences

$$\{P_i^*(z, \kappa + \omega) : \omega \in \mathbb{N}\},$$

where $i = 1, \dots, m+s$, are linearly independent over $\mathbb{R}$. Therefore the matrix $\Lambda_{m,s}(b/a)$ satisfies condition (ii) of Lemma 1. This completes the proof of Theorem 1.

References

[1] G.V. Chudnovsky, *Pade approximations to the generalized hyper-geometric functions I.*, J.Math.Pures Appl. **58** (1979), 445 – 476..

[2] N.I. Fel'dman, *Estimation of the absolute value of a linear form from logarithms of certain algebraic numbers (Russian)*, Mat. Zametki **2** (1967), 245 – 256; English transl. in Math.Notes 2 (1967), 634-640..

[3] N.I. Fel'dman, *Estimate for a linear form of logarithms of algebraic numbers (Russian)*, Mat.Sb. (N.S.) 76 (118) (1968), 304-319; English transl. in Math. USSR Sb. 5 (1968), 291-307..

[4] N.I. Fel'dman, *An estimate of an incomplete linear form in several algebraic numbers (Russian)*, Mat.Zametki, 7 (1970), 569-580; English transl. in Math.Notes 7 (1970), 343-349..

[5] N.I. Fel'dman, *Effective bounds for the number of solutions of certain diophantine equations (Russian)*, Mat.Zametki, 8 (1970), 361-371; English transl. in Math.Notes 8 (1970), 674-679..

[6] A.I. Galoškin, *Estimates from below of polynomials in the values of analytic functions of a certain class (Russian)*, Mat.Sb. 95 (137) (1974), 396-417; English transl. in Math. USSR Sb. 24 (1974), 385-407. Mat.Zametki, 8 (1970), 361-371; English transl. in Math.Notes 8 (1970), 674-679..

[7] V.L. Gončarow, *Theory of interpolation and the approximation of functions*, ONTI, Moscow,, 1934, 2nd ed. GITTL., Moscow, 1954..

[8] L.A. Gutnik, *Linear independence over Q of dilogarithms at rational points (Russian)*, Uspekhi Mat.Nauk 37 (1982), 179-180; English transl. in Russ.Math. surveys 37 (1982), 176-177..

[9] W. Maier, *Potenzreihen irrationalen Grenzwertes*, J.reine angew. Math. **156** (1927), 93 – 148.

[10] E.M. Nikišin,, *On irrationality of the values of the functions F(x,s) (Russian)*, Mat.Sb. 109 (1969), 410-417; English transl. in Math. USSR Sb. 37 (1980), 381-388..

[11] S.Eckmann, *Über die lineare Unaqbhangigkeit der Werte gewisser Reihen*, Results in Mathematics, **11** (1987), 7 – 43.

[12] A.O. Gelfond., *Izbrannye trudy (Russian) (Selected works)*, NAUKA, Moscow, 1973.

[13] Hata M., *Legendre type polynomials and irrationality measures*, J. Reine Angew. Math. **407** (1990), 99 – 105.

[14] Gutnik L., *On a measure of the irrationality of dilogarithms at rational points,* **4345-84** (1984), VINITI, 1 – 74.

[15] ——, *On the rank over* $\mathbb{Q}$ *of some real matrices,* **5736-84** (1984), VINITI, 1 – 29.

[16] ——, *To the question of the smallness of some linear forms,* **2413-B93** (1993), VINITI, 1 – 94.

[17] ——, *About linear forms, whose coefficients are* $\mathbb{Q}$*-proportional to the number* $\log 2$, *and the values of* $\zeta(s)$ *for integer* s, **3258-B96** (1996), VINITI, 1 – 70.

[18] ——, *The lower estimate for some linear forms with coefficients* $\mathbb{N}\,\zeta(1+\mathbb{N})$, **3072-B97** (1997), VINITI, 1 – 77.

[19] ——, *About the behaviour of solutions of difference equations of Poincaré type,* **3384-B97** (1997), VINITI, 1 – 41.

[20] ——, *On some linear forms with coefficients in* $\mathbb{N}\,\zeta(1+\mathbb{N})$, **3** (2000), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 1 – 13.

[21] ——, *On some linear forms with coefficients in* $\mathbb{N}\,\zeta(1+\mathbb{N})$ *(the detailed version, part 1),* **15** (2001), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 1 – 20.

[22] ——, *On some linear forms with coefficients in* $\mathbb{N}\,\zeta(1+\mathbb{N})$ *(the detailed version, part 2),* **104** (2001), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 1 – 36.

[23] Bateman H. and Erdélyi A., *Higher transcendental functions*, New-York – Toronto – London, Mc. Grow-Hill Book Company, Inc., 1953.
[24] Titchmarsh E. C., *The Theory of Functions*, Oxford University Press, 1939.
[25] Whittaker E.T. and Watson G.N., *A course of modern analysis*, Cambridge University Press, 1927.

Moscow State Institute
of Electronics and Mathematics
Bolshoi Trekhsvyatitelskij pereulok, 1-3/12, str.8
Moscow 109028
Russian Federation
E-mail: gutnik@gutnik.mccme.ru