NORMAL AFFINE SURFACES WITH $\mathbb{C}^*$-ACTIONS

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Abstract. A classification of affine surfaces admitting a $\mathbb{C}^*$-action was given in the work of Białynicki-Birula, Fieseler and L. Kaup, Orlik and Wagreich, Rynes and others. We provide a simple alternative description of normal quasihomogeneous affine surfaces in terms of their graded rings as well as by defining equations. This is based on a generalization of the Dolgachev-Pinkham-Demazure construction.

Contents

Introduction

A classification of (normal) affine surfaces admitting a $\mathbb{C}^*$-action was given e.g., in [Bi, BiSo, OrWa, Pi, BasHa, Ry] and [FiKa$_1$]-[FiKa$_3$]. Here we obtain a simple alternative description of normal affine surfaces V with a $\mathbb{C}^*$-action in terms of their graded coordinate rings as well as by defining equations. Our approach is based on a generalization of the Dolgachev-Pinkham-Demazure construction [Do, Pi, De]. Recall (see [FiKa$_1$]-[FiKa$_3$]) that a $\mathbb{C}^*$-action on a normal affine surface V is called

elliptic if it has a unique fixed point which belongs to the closure of every 1-dimensional orbit,

parabolic if the set of its fixed points is 1-dimensional, and

hyperbolic if V has only a finite number of fixed points, and these fixed points are of hyperbolic type, that is each one of them belongs to the closure of exactly two 1-dimensional orbits.

In the elliptic case, the complement V^* of the unique fixed point in V is fibered by the 1-dimensional orbits over a projective curve C. In the other two cases V is fibered over an affine curve C, and this fibration is invariant under the $\mathbb{C}^*$-action.

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Vice versa, given a smooth curve C and a $\mathbb{Q}$-divisor D on C, the Dolgachev-Pinkham-Demazure construction provides a normal affine surface $V = V_{C,D}$ with a $\mathbb{C}^*$-action such that C is just the algebraic quotient of V^* or of V, respectively. This surface V is of elliptic type if C is projective and of parabolic type if C is affine.

We remind this construction in sections 1 and 2 below. In section 3 we use it to present any normal affine surface V with a parabolic $\mathbb{C}^*$-action as a normalization of the surface $x^d - P(z)y = 0$ in $\mathbb{C}^3$ for a certain $d \in \mathbb{N}$ and a certain polynomial $P \in \mathbb{C}[t]$ (see Theorem 3.11).

In section 4 we deal with the hyperbolic case. We generalize the Dolgachev-Pinkham-Demazure construction in order to make it work for any hyperbolic $\mathbb{C}^*$-surface. Instead of one $\mathbb{Q}$-divisor D on a smooth affine curve C as before, it involves now two $\mathbb{Q}$-divisors D_+ and D_- on C. By our result *isomorphism classes of normal affine hyperbolic $\mathbb{C}^*$-surfaces are in 1-1-correspondence to equivalence classes of triples (C, D_+, D_-), where C is a smooth affine curve and D_+, D_- is a pair of $\mathbb{Q}$-divisors on C with $D_+ + D_- \le 0$; two such triples (C, D_+, D_-) and (C, D_+, D_-) are considered to be equivalent if and only if $C \cong C'$ and $D_\pm = D'_\pm \pm D_0$ with a principal divisor D_0;* cf. Theorem 4.5. We also determine the type of singularities and of the special orbits in terms of the divisors $D_\pm$, see Theorems 4.16 and 4.19.

Using our description it is possible to represent any normal hyperbolic $\mathbb{C}^*$-surface fibered over $C = \mathbb{A}^1_{\mathbb{C}}$ as the normalization of a surface in $\mathbb{C}^4$ given by

$$x^d - P(t)y = 0, \quad x^e z - Q(t) = 0 \quad \text{and} \quad y^e z^d - R(t) = 0\,,$$

for certain polynomials $P, Q, R \in \mathbb{C}[t]$ satisfying the relation $P^e R = Q^d$. These polynomials can be easily computed in terms of the data $(D_+,\, D_-)$ (see Remark 4.10). For instance, if the divisor D_- is integral then this system reduces to one equation $x^e z - Q(t) = 0$ in $\mathbb{C}^3$, and vice versa. When $\gcd(e, d) = 1$ then it again reduces to one equation $y^e z^d - R(t) = 0$ in $\mathbb{C}^3$.

In Proposition 4.13 we show how the pair $(D_+,\, D_-)$ is transformed when passing to an equivariant cyclic cover of V. We deduce, in particular, a characterization of normal hyperbolic $\mathbb{C}^*$-surfaces over $C = \mathbb{A}^1_{\mathbb{C}}$ with the fractional part of D_- supported at one point, as normalized cyclic quotients of the surfaces $x^e z - Q(t) = 0$ in $\mathbb{C}^3$.

In the forthcoming paper [FlZa], which is actually Part II of the present one, we will apply these results to give a simple description of all normal affine $\mathbb{C}^*$-surfaces equipped in addition by a $\mathbb{C}^+$-action. In fact, this class consists of all normal affine surfaces which admit an algebraic group action with an open orbit.

We note that the results of this paper hold m.m. for graded 2-dimensional normal algebras of finite type over a Dedekind domain.

1. Generalities on graded rings

A $\mathbb{Z}$-graded ring $A = \bigoplus_{i\in\mathbb{Z}} A_i$ contains $A_{\ge 0} = \bigoplus_{i\ge 0} A_i$ and $A_{\le 0} = \bigoplus_{i\le 0} A_i$ as subrings. The following lemma is "well known"; in lack of a reference we provide a short argument.

Lemma 1.1. *If $A = \bigoplus_{i\in Z} A_i$ is a finitely generated A_0-algebra, then so are $A_{\ge 0}$ and $A_{\le 0}$. Moreover, A is normal if and only if so are both $A_{\ge 0}$ and $A_{\le 0}$.*

Proof. Reversing the grading interchanges the subrings $A_{\ge 0}$ and $A_{\le 0}$. Thus it is sufficient to prove the first part for $A_{\ge 0}$. If $a_{ij} \in A_i$ with $-n \le i \le n$, $j = 1, \dots, n_i$, is a system of homogeneous generators of A, then $A_{\geq 0}$ is generated (as a module over A_0) by the multiplicatively closed system of monomials

$$a^k := \prod_{i,j} a_{ij}^{k_{ij}}\,,$$

where $k := (k_{ij}) \in \mathbb{Z}^N$ satisfies the inequalities

$$k_{ij} \geq 0, \quad -n \leq i \leq n, \quad j = 1, \ldots, n_i, \qquad \sum_{i,j} ik_{ij} \geq 0\,. \tag{1}$$

By Gordan's Lemma (see [Od]) the rational polyhedral lattice cone $K \subseteq \mathbb{Z}^N$ defined by (1) is a finitely generated semigroup. Hence the algebra $A_{\geq 0}$ is generated by a finite system of monomials $a^k \in A_{\geq 0}$.

Next we show that the subalgebra $A_{\geq 0}$ (and then also $A_{\leq 0}$) is normal if so is A. Indeed, the integral closure $(A_{\geq 0})_{\mathrm{norm}} \subseteq A = A_{\mathrm{norm}}$ is graded. Take a homogeneous element $x \in (A_{\geq 0})_{\mathrm{norm}}$ of degree $d := \deg x$, and let

$$x^n + \sum_{i=1}^{n} b_i x^{n-i} = 0, \qquad \text{where} \quad b_i \in A_{\geq 0}\,, \tag{2}$$

be an equation of integral dependence. We may assume that b_i are also homogeneous, of degree $\deg b_i = di \geq 0$. Since $\deg b_i \geq 0$ we have $d \geq 0$, and so $x \in A_{\geq 0}$.

Conversely, suppose that both $A_{\geq 0}$ and $A_{\leq 0}$ are normal. The ring $A \otimes_{A_0} \mathrm{Frac}(A_0)$ is normal and so is equal to $\mathrm{Frac}(A_0)[u, u^{-1}]$ for a homogeneous element u of minimal degree > 0 in $A \otimes_{A_0} \mathrm{Frac}(A_0)$. Hence A_{norm} is contained in this subring of $\mathrm{Frac}\, A$. If $f \in A \otimes_{A_0} \mathrm{Frac}(A_0)$ belongs to the normalization A_{norm} of A then so does its top homogeneous component. Thus it is enough to deal with homogeneous elements. Let a be such an element satisfying an equation of integral dependence (2) over A. We may suppose as above that $b_i \in A_{di}$ $(i = 1, \ldots, n)$. Since di has the same sign as $d := \deg a$, we have $a \in (A_{\geq 0})_{\mathrm{norm}} = A_{\geq 0}$ if $d \geq 0$ and $a \in (A_{\leq 0})_{\mathrm{norm}} = A_{\leq 0}$ if $d \leq 0$, respectively. Anyhow, $a \in A$, whence A is normal, as stated. □

Notation 1.2. Let $V = \mathrm{Spec}\, A$ be a normal affine surface over $\mathbb{C}$ with an effective $\mathbb{C}^*$-action. The coordinate ring $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is then naturally graded so that A_i is the set of elements of A on which $t \in \mathbb{C}^*$ acts via $t.f = t^i f$. Thus, $A_0 = A^{\mathbb{C}^*}$ is the subalgebra of invariants, and A_i $(i \neq 0)$ consists of the quasi-invariants of weight i. Up to reversing the grading we may assume that $A_+ := \bigoplus_{i>0} A_i \neq 0$. The subsets A_+ and $A_- := \bigoplus_{i<0} A_i$ of A are ideals in $A_{\geq 0}$ and $A_{\leq 0}$, respectively.

The following lemma is well known (see e.g., [De], [FiKa$_1$, Lemma 1.5]).

Lemma 1.3. (a) *If $A_0 \neq \mathbb{C}$ then the set $M := \{i \in \mathbb{Z} \mid A_i \neq 0\}$ coincides either with $\mathbb{N}$ or with $\mathbb{Z}$, and A_i is a locally free A_0-module of rank 1 for all $i \in M$. Moreover, if $u \in \mathrm{Frac}(A_0) \cdot A_1$ is a non-zero element then*

$$A \subseteq \mathrm{Frac}(A_0)[u, u^{-1}]\,, \qquad \textit{and even} \qquad A \subseteq \mathrm{Frac}(A_0)[u] \quad \textit{if} \quad M = \mathbb{N}\,.$$

(b) *In particular, if $A_0 \cong \mathbb{C}[t]$ then A_i is a free A_0-module of rank 1 for all $i \in M$.*

Proof. (a) The $K_0 := \mathrm{Frac}(A_0)$-algebra $A \otimes_{A_0} K_0$ is a 1-dimensional normal graded domain over the field K_0. Hence it is isomorphic to the free polynomial ring $K_0[u]$ or the ring of Laurent polynomials $K_0[u, u^{-1}]$, where $u \in K_0 A_d$ and $d > 0$. As the $\mathbb{C}^*$-action is effective $d = 1$, and (a) follows.

(b) follows from [Bou, Ch. VII, §4, Corollary 2]. □

Lemma 1.3(a) does not hold in general without the assumption that $A_0 \neq \mathbb{C}$ as is seen by the Pham-Brieskorn surfaces $V_{p,q,r} := \{x^p + y^q + z^r = 0\} \subseteq \mathbb{C}^3$.

1.4. Usually (cf. [FiKa$_1$]) one distinguishes between the following three cases.

(i) *The elliptic case*: $A_- = 0, \quad A_0 = \mathbb{C}$.
(ii) *The parabolic case*: $A_- = 0, \quad A_0 \neq \mathbb{C}$.
(iii) *The hyperbolic case*: $A_- \neq 0$.

Below we provide more information in each of these cases.

2. The elliptic case

In the elliptic case the $\mathbb{C}^*$-action on V is good. In particular, its fixed point set $F := V^{\mathbb{C}^*}$ (which is the zero set of the augmentation ideal A_+ of A) consists of a unique point called *the vertex* of V, and the surface V is smooth outside the vertex. One considers the smooth projective curve $C := \mathrm{Proj}\, A \cong V^*/\mathbb{C}^*$, where $V^* := V \setminus F$, together with the orbit morphism $\pi : V^* \to C$ (the fibers of π are the orbits of the $\mathbb{C}^*$-action on V^*).

A useful class of examples of normal affine surfaces with a good $\mathbb{C}^*$-action is provided by the affine cones over projective curves. For an ample divisor D on a smooth projective curve C the ring

$$A_{C,D} := \bigoplus_{k \geq 0} H^0(C, \mathcal{O}_C(kD))) \cdot u^k \subseteq \mathrm{Frac}(C)[u]\,,$$

where u is an indeterminate, is the coordinate ring of a normal affine surface $V := \mathrm{Spec}\, A_{C,L}$ with a good $\mathbb{C}^*$-action. This surface V is a cone over C obtained by blowing down the zero section of the line bundle associated to $\mathcal{O}_C(-D)$.

Let furthermore a finite group G act on V freely off the vertex, and assume that this action commutes with the given good $\mathbb{C}^*$-action on V. Then the quotient V/G is again a normal affine surface with a good $\mathbb{C}^*$-action. Conversely, the following result is true.

Theorem 2.1. *(*[Do, Pi, De, Ru]*) Every normal affine surface with a good $\mathbb{C}^*$-action appears as the quotient of an affine cone over a smooth projective curve by a finite group acting freely off the vertex of the cone.*

Generalizing the construction above, for a smooth projective curve C and a $\mathbb{Q}$-divisor D on C one considers the graded ring

$$A_{C,D} := \bigoplus_{k \geq 0} H^0(C, \mathcal{O}(\lfloor kD \rfloor)) \cdot u^k\,,$$

where $\lfloor E \rfloor$ denotes the integral part of a $\mathbb{Q}$-divisor E. We have the following result.

Theorem 2.2. *(*[Pi], [De, Theorem 3.5]*) Given a normal affine surface $V = \mathrm{Spec}\, A$ with a good $\mathbb{C}^*$-action there exists a $\mathbb{Q}$-divisor D on the curve $C = \mathrm{Proj}\ A$ such that $A \cong A_{C,D}$.*

The affine toric surfaces provide an interesting family of elliptic $\mathbb{C}^*$-surfaces.

Example 2.3. ([Od, Co]) We remind that a normal affine toric surface $V = V_\sigma$ is associated to a strictly convex rational polyhedral cone $\sigma \subseteq \mathbb{R}^2$. If $\dim \sigma = 0$ or $= 1$ then $V_\sigma \cong \mathbb{C}^* \times \mathbb{C}^*$ or $V_\sigma \cong \mathbb{A}^1_\mathbb{C} \times \mathbb{C}^*$, respectively, and so $A^\times \neq \mathbb{C}^*$. Consequently, these two cannot be elliptic $\mathbb{C}^*$-surfaces. Otherwise, if $\dim \sigma = 2$ then choosing an appropriate base e_1, e_2 of the lattice one may suppose that σ is the cone $C(e_2, de_1 - ee_2)$, where $d \geq 1$, $0 \leq e < d$ and $\gcd(e, d) = 1$. We denote $V_{d,e} := V_\sigma$; then $V_{d,e} = \operatorname{Spec} A_{d,e}$, where

$$A_{d,e} := \bigoplus_{b \geq 0,\ ad - be \geq 0} \mathbb{C} \cdot x^a y^b \subseteq \mathbb{C}[x, x^{-1}, y, y^{-1}]$$

is the semigroup algebra of the dual cone $\sigma^\vee = C\,(e_1, ee_1 + de_2)$.

The 2-torus $\mathbb{T} = (\mathbb{C}^*)^2$ acts on $V_{d,e}$ with an open orbit $V^*_{d,e} := V_{d,e} \backslash \{\bar{0}\}$. Thus one can introduce on $V_{d,e}$ a number of elliptic, parabolic as well as hyperbolic $\mathbb{C}^*$-actions by choosing appropriate 1-parameter algebraic subgroups of the torus $\mathbb{T}$.

In [Ri, BeRi$_1$, BeRi$_2$, Co] one can find a description of minimal sets of generators of the algebras $A_{d,e}$ as above, as well as defining equations for the affine varieties $V_{d,e} = \operatorname{Spec} A_{d,e} \hookrightarrow \mathbb{C}^N$.

We would like to emphasize the well known relation between affine toric surfaces and cyclic quotient singularities [Od, Proposition 1.24].

Lemma 2.4. *If B is the normalization of $A := A_{d,e}$ in the field $L := \operatorname{Frac}(A)[u]$ with $u := \sqrt[d]{x}$, then B is the polynomial ring $B = \mathbb{C}[u, v]$ with $v := u^e y$. The Galois group $\langle \zeta \rangle \cong \mathbb{Z}_d$ of $L : \operatorname{Frac}(A)$ acts on B via*

$$\zeta.u = \zeta u, \qquad \zeta.v = \zeta^e v\,,$$

and $A = B^{\mathbb{Z}_d}$. Consequently, there is an isomorphism

$$V_{d,e} \cong \mathbb{A}^2_\mathbb{C}/G_{d,e} = \mathbb{A}^2_\mathbb{C}/\mathbb{Z}_d\,.$$

Proof. For the convenience of the reader we give a short argument. By definition, A is generated over $\mathbb{C}$ by the monomials

$$x^a y^b \qquad \text{with} \qquad b \geq 0,\ \ ad - be \geq 0\,.$$

As $x^a y^b = u^{ad-be} v^b$, this shows that A embeds naturally into $\mathbb{C}[u, v]$ and that even $A = \mathbb{C}[x, x^{-1}, y] \cap \mathbb{C}[u, v]$. In particular A is a normal domain. Because of $u^d = x \in A$ and $v^d = x^e y^d \in A$ the ring B is integral over A, whence it is the normalization of A.

The second part follows from the first one, since L is a cyclic extension of $\operatorname{Frac}(A)$ with Galois group $\mathbb{Z}_d$ acting via $\zeta.u = \zeta u$ and $\zeta.x = x$ for all $x \in A$. □

Remark 2.5. Assuming that $e > 0$ and letting $\xi := \zeta^e$ one obtains

$$(\zeta u, \zeta^e v) = (\xi^{e'} u, \xi v)\,,$$

where $0 \leq e' < d$ and $ee' \equiv 1 \mod d$ (note that for $d = 1$ this means $e' = 0$). Hence, with $\tau(u, v) := (v, u)$ the conjugate $\mathbb{Z}_d$-action $G'_{d,e'} := \tau^{-1} G_{d,e'} \tau$ on $\mathbb{A}^2_\mathbb{C}$

$$\xi.(u, v) = (\xi^{e'} u, \xi v)$$

has the same orbits as $G_{d,e}$ thus providing an isomorphism of toric surfaces

$$V_{d,e} \cong \mathbb{A}^2_\mathbb{C}/G_{d,e} \cong \mathbb{A}^2_\mathbb{C}/G'_{d,e'} \cong \mathbb{A}^2_\mathbb{C}/G_{d,e'} \cong V_{d,e'}\,.$$

Moreover, $V_{d,e} \cong V_{d',e'}$ if and only if $d = d'$ and either $e = e'$ or $ee' \equiv 1 \mod d$.

3. The parabolic case

In the parabolic case one considers a normal affine surface V with a $\mathbb{C}^*$-action such that the coordinate ring $A = \bigoplus_{i \geq 0} A_i$ is positively graded and A_0 is a 1-dimensional domain. Thus A_0 corresponds to a smooth affine curve $C = \operatorname{Spec} A_0$, which can be identified with the algebraic quotient $V//\mathbb{C}^*$ (indeed, $A_0 = A^{\mathbb{C}^*}$ is the ring of invariants of the $\mathbb{C}^*$-action on A). The embedding $A_0 \hookrightarrow A$ corresponds to the quotient morphism $\pi : V \to C$, and the projection $A \to A_0$ gives an embedding $\iota : C \hookrightarrow V$ which provides a retraction of π and whose image is the fixed point set. Every fiber of $\pi : V \to C$ is the closure of a non-trivial orbit; it contains a unique fixed point (a *source* of this orbit) [FiKa$_1$, Lemma 1.7].

A simple example of a parabolic $\mathbb{C}^*$-surface is the cylinder $C \times \mathbb{A}^1_{\mathbb{C}}$ over a smooth affine curve C, where $\mathbb{C}^*$ acts on the second factor. More examples can be produced by applying equivariant affine modifications to $C \times \mathbb{A}^1_{\mathbb{C}}$ (see [KaZa$_1$, Theorem 1.1]). Actually, one obtains in this way all normal affine surfaces with a parabolic $\mathbb{C}^*$-action.

3.1. The Dolgachev-Pinkham-Demazure construction (see Theorem 2.2) is available also in the parabolic case. Let $C = \operatorname{Spec} A_0$ be an affine curve over $\mathbb{C}$ with function field $K_0 := \operatorname{Frac}(A_0)$, and let D be a $\mathbb{Q}$-Cartier divisor on C. Similarly as in the elliptic case we can introduce the algebra

$$A_0[D] := A_{C,D} = \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C(\lfloor nD \rfloor)) \cdot u^n \subseteq K_0[u]\,.$$

More explicitly, if $f \in K_0$ then

$$fu^n \in A := A_0[D] \Leftrightarrow \operatorname{div} f + nD \geq 0. \tag{3}$$

We note that the algebra A is normal (see Corollary 3.8(b) below). Notice also that $u \in A_1$ if and only if $D \geq 0$.

The following theorem is well known (cf. [De, Theorem 3.5]); for the convenience of the reader we include a short proof.

Theorem 3.2. *Let $C = \operatorname{Spec} A_0$ be a normal affine algebraic curve with function field $K_0 := \operatorname{Frac}(A_0)$. If $A = \bigoplus_{i \geq 0} A_i$ is a normal finitely generated A_0-algebra of dimension 2 then the following hold.*

(a) *A is isomorphic to $A_0[D]$ for some $\mathbb{Q}$-divisor D on C. More precisely, if $u \in K_0 \cdot A_1$ is a non-zero element and if the divisor D is defined by the equality*

$$\pi^* D = \operatorname{div} u - \iota(C)\,,$$

then A and $A_0[D]$ are equal when considered as subrings of $K_0[u]$.

(b) *For two $\mathbb{Q}$-divisors D and D' on C, the rings $A = A_0[D]$ and $A' = A_0[D']$ are isomorphic as graded A_0-algebras if and only if D and D' are linearly equivalent.*

Proof. (a) Since $u \in K_0 \cdot A_1$ is homogeneous, the divisor $\operatorname{div} u$ on the normal surface $V = \operatorname{Spec} A$ is invariant under the induced $\mathbb{C}^*$-action on V, and so we have

$$\operatorname{div} u = \sum_{i=1}^{m} p_i F_i + \iota(C)$$

with $p_i \in \mathbb{Z}$, where $F_i = \pi^{-1}(x_i)$ are the fibers of π over distinct points $x_i \in C$, $i = 1, \ldots, m$. Letting $\pi^* x_i = q_i F_i$ with $q_i \in \mathbb{N}$ $(i = 1, \ldots, m)$, the $\mathbb{Q}$-divisor $D := \sum_{i=1}^m p_i/q_i x_i$ on V satisfies

$$\operatorname{div} u = \pi^*(D) + \iota(C)\,.$$

Since V is normal, for a rational function $\varphi \in K_0$ on C the following equivalences hold:

$$\varphi u^n \in A_n \Leftrightarrow \operatorname{div}(\varphi u^n) \geq 0 \Leftrightarrow \pi^* \operatorname{div} \varphi + n \operatorname{div} u \geq 0 \Leftrightarrow$$

$$\pi^* \operatorname{div} \varphi + n\pi^*(D) + n\iota(C) \geq 0 \Leftrightarrow \operatorname{div} \varphi + nD \geq 0 \Leftrightarrow \varphi \in H^0(C, \mathcal{O}_C(\lfloor nD \rfloor))\,.$$

Hence $A_n = H^0(C, \mathcal{O}_C(\lfloor nD \rfloor)) \cdot u^n$ for all $n \geq 0$, as desired.

(b) Any isomorphism of graded A_0-algebras

$$\varphi : A_0[D] = \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C(\lfloor nD \rfloor)) \cdot u^n \longrightarrow A_0[D'] = \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C(\lfloor nD' \rfloor)) \cdot u'^n\,,$$

extends to an isomorphism of graded K_0-algebras

$$\varphi_{K_0} : K_0[u] \to K_0[u']$$

and so has the form $u^n \mapsto f^n u'^n$, $n \geq 0$, for some non-zero $f \in K_0$. Conversely, such a morphism φ_{K_0} maps $A_0[D]$ isomorphically onto $A_0[D']$ if and only if

$$H^0(C, \mathcal{O}_C(\lfloor nD' \rfloor)) = f^n \cdot H^0(C, \mathcal{O}_C(\lfloor nD \rfloor)) \quad \forall n.$$

As

$$f^n \cdot H^0(C, \mathcal{O}_C(\lfloor nD \rfloor)) = H^0(C, \mathcal{O}_C(\lfloor nD - n\operatorname{div} f \rfloor))\,,$$

the existence of an isomorphism φ as above is equivalent to the existence of an element $f \in K_0$ with $D' = D - \operatorname{div} f$. □

3.3. We denote $\{D\} = D - \lfloor D \rfloor$ the fractional part of a $\mathbb{Q}$-divisor D. Since principal divisors are $\mathbb{Z}$-divisors, we have $\{D\} = \{D'\}$ as soon as $D \sim D'$.

If $C = \operatorname{Spec} \mathbb{C}[t] = \mathbb{A}^1_{\mathbb{C}}$ then the converse is also true. Indeed, any $\mathbb{Z}$-divisor on $\mathbb{A}^1_{\mathbb{C}}$ is principal, and so the linear equivalence class of a $\mathbb{Q}$-divisor D on $\mathbb{A}^1_{\mathbb{C}}$ is uniquely determined by the fractional part $\{D\}$ of D. Thus we obtain the following corollary.

Corollary 3.4. *For every normal parabolic $\mathbb{C}^*$-surface $V = \operatorname{Spec} A$ with $A = \bigoplus_{n \geq 0} A_n$ and $A_0 = \mathbb{C}[t]$, there is a unique isomorphism $A \cong A_0[D]$ of graded A_0-algebras, where $D = 0$ or $D = \sum_{i=1}^n \frac{p_i}{q_i} x_i$ with $0 < p_i < q_i$, $\gcd(p_i, q_i) = 1$ $\forall i = 1, \ldots, n$ and $x_i \in \mathbb{A}^1_{\mathbb{C}}$, $x_i \neq x_j$ for $i \neq j$.*

The next lemma is also well known; in lack of a reference we provide a short argument.

Lemma 3.5. *Let D be a $\mathbb{Q}$-divisor on a normal affine variety S and consider the graded ring $A := \bigoplus_{i \geq 0} A_i$, where $A_i := H^0(S, \mathcal{O}_S(\lfloor iD \rfloor)) \cdot u^i$. For $d \in \mathbb{N}$ the following conditions are equivalent.*

(i) *dD is integral.*

(ii) *$A_{d+m} = A_d A_m$ for all $m \geq 0$.*

(iii) *The d-th Veronese subring $A^{(d)} := \bigoplus_{m \geq 0} A_{md}$ is isomorphic to the symmetric algebra $S_{A_0}(A_d)$ i.e., $A_{kd} = S^k_{A_0} A_d$.*

Proof. Condition (ii) is equivalent to

$$\mathcal{O}_S(\lfloor (m+d)D \rfloor) \cong \mathcal{O}_S(\lfloor mD \rfloor) \otimes \mathcal{O}_S(\lfloor dD \rfloor) \qquad \forall m \geq 0\,,$$

and the latter condition is equivalent to

$$(ii') \qquad \lfloor (m+d)D \rfloor = \lfloor mD \rfloor + \lfloor dD \rfloor \qquad \forall m \geq 0\,.$$

Similarly, (iii) is equivalent to

$$(iii') \qquad \lfloor kdD \rfloor = k \lfloor dD \rfloor \qquad \forall k \geq 0\,.$$

The equivalence of (i), (ii') and (iii') now follows from the elementary fact that for a rational number $r = \frac{p}{q}$ the following conditions are equivalent:

(1) $dr \in \mathbb{Z}$ (2) $\lfloor mr + d \rfloor = \lfloor mr \rfloor + \lfloor d \rfloor$ $\forall m \geq 0$ (3) $\lfloor kdr \rfloor = k \lfloor dr \rfloor$ $\forall m \geq 0$.

□

Notation 3.6. We denote $d(A)$ the smallest positive integer d satisfying the equivalent conditions of Lemma 3.5.

Remark 3.7. In the situation of Theorem 3.2, one can recover D from the graded ring $A = A_0[D]$ more algebraically as follows. Consider $d \in \mathbb{N}$ with $A_d A_i = A_{d+i}$ for all $i \geq 0$ (or, equivalently, $A_{id} = S^i(A_d)$, see Lemma 3.5) and let v be a generator of A_d as A_0-module; this exists after a suitable localization of A_0. *If $u^d = fv$ with $f \in \operatorname{Frac} A_0$, then $D = \operatorname{div}(f)/d$.* In fact, the ideal vA is equal to $A_{\geq d}$ and so its set of zeros has no irreducible components in the fibres of π. Thus $\operatorname{div} v = d \cdot \iota(C)$ on V. Since

$$\pi^*(D) = \operatorname{div} u - \iota(C) \quad \text{and} \quad d \cdot \operatorname{div} u = \operatorname{div} v + \operatorname{div} f$$

as divisors on V, we obtain $D = \operatorname{div}(f)/d$.

A parabolic $\mathbb{C}^*$-surface $V = \operatorname{Spec} A_0[D]$ has at most cyclic quotient singularities, as follows from Miyanishi's Theorem (see [Miy, Lemma 1.4.4(1)]). In the next result (see [De, Section 5]) we describe their structure in terms of the divisor D.

Proposition 3.8. (a) *If $A_0 = \mathbb{C}[t]$ and if D is supported on the origin in $\operatorname{Spec} A_0 = \mathbb{A}^1_{\mathbb{C}}$ so that $D = -\frac{e}{d}[0]$ with $\gcd(e,d) = 1$, then $A := A_0[-\frac{e}{d}[0]]$ is naturally isomorphic to the semigroup algebra*

$$A = A_{d,e} = \bigoplus_{b \geq 0,\ ad-be \geq 0} \mathbb{C} \cdot t^a v^b$$

graded via $\deg t = 0$, $\deg v = 1$ (cf. Example 2.3). Consequently, $V := \operatorname{Spec} A$ is isomorphic to the toric surface $V_{d,e'} = \operatorname{Spec} A_{d,e'} \cong \mathbb{A}^2_{\mathbb{C}}/G_{d,e'}$, where $e' \equiv e \mod d$ and $0 \leq e' < d$.

(b) *If $C = \operatorname{Spec} A_0$ is any normal affine curve over $\mathbb{C}$ and D is a $\mathbb{Q}$-divisor on C, then the surface $V = \operatorname{Spec} A_0[D]$ is normal with at most cyclic quotient singularities. More precisely, if $D(a) = -e/d$ with $\gcd(e,d) = 1$ then V has a quotient singularity of type (d, e') at $\iota(a)$, where e' is as in (a).*

Proof. The first part of (a) follows immediately from (3) in 3.1, whereas the second one is a consequence of Lemma 2.4.

Tensoring the isomorphism in (a) with $-\otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]$ we obtain that (b) holds if $A_0 \cong \mathbb{C}[[t]]$. The general case follows from this by taking completions at the maximal ideals of A_0. □

The algebra $A_0[D]$ is finitely generated over A_0, so there exist $f_1, \ldots, f_n \in K_0$ and $m_1, \ldots, m_n \in \mathbb{N}$ such that

$$A = A_0[f_1 u^{m_1}, \ldots, f_n u^{m_n}] \subseteq K_0[u]\,.$$

In the next result we show how to compute D from such a representation.

Proposition 3.9. *Let $C = \operatorname{Spec} A_0$ be a smooth affine curve and $K_0 := \operatorname{Frac} A_0$. If a 2-dimensional subring B of the polynomial ring $K_0[u]$ is represented as*

$$B = A_0[f_1 u^{m_1}, \ldots, f_n u^{m_n}] \subseteq K_0[u]\,, \quad m_i > 0 \ \forall i$$

with $f_1, \ldots, f_n \in K_0$, then its normalization $A = B_{\text{norm}}$ coincides as an A_0-subalgebra of $K_0[u]$ with $A_0[D]$, where

$$D := -\min_{1 \le i \le n} \frac{\operatorname{div} f_i}{m_i}\,.$$

Proof. By definition of D we have $\operatorname{div} f_i + m_i D \ge 0$ so by (3) $f_i u^{m_i} \in A_0[D]$ and B is a subring of $A_0[D]$. As $A_0[D]$ is normal (see Proposition 3.8(b)), A is also contained in $A_0[D]$. Let us show that these subrings coincide.

According to Theorem 3.2, we can represent A as $A = A_0[D']$ with $\pi^*(D') = \operatorname{div} u - \iota(C)$. In particular $f_i u^{m_i} \in A = A_0[D']$, so again by (3) $\operatorname{div} f_i + m_i D' \ge 0$ or, equivalently, $D' \ge -\frac{1}{m_i}\operatorname{div} f_i$. Thus $D' \ge D$ and $A_0[D] \subseteq A_0[D'] = A$. As we have already shown the converse inclusion we obtain that $A = A_0[D]$, as desired. □

The following examples of parabolic $\mathbb{C}^*$-surfaces ruled over $\mathbb{A}^1_{\mathbb{C}}$ are basic (see Theorem 3.11 below).

Example 3.10. For a unitary polynomial $P \in \mathbb{C}[t]$ and for an integer $d \ge 1$ we let

$$B^+_{d,P} := \mathbb{C}[t, u, v]/(u^d - P(t)v) \cong \mathbb{C}\left[t, u, \frac{u^d}{P(t)}\right]$$

graded via

$$\deg t = 0, \qquad \deg u = 1, \qquad \deg v = d\,.$$

The normalization

$$A^+_{d,P} := (B^+_{d,P})_{\text{norm}}$$

is a positively graded finitely generated $\mathbb{C}$-algebra of dimension 2 with $A_0 = \mathbb{C}[t]$. By Proposition 3.9 and Corollary 3.4 we have

$$A^+_{d,P} \cong A_0[D] \cong A_0[\{D\}], \qquad \text{where} \qquad D = D(d, P) := \frac{\operatorname{div}(P)}{d}\,.$$

For $P(t) = \prod_{i=1}^n (t - x_i)^{r_i}$ (where $x_i \ne x_j$ if $i \ne j$) we obtain

$$D = \sum_{i=1}^n \frac{r_i}{d} x_i\,, \qquad \text{and} \qquad \{D\} = \sum_{i=1}^n \left\{\frac{r_i}{d}\right\} x_i\,,$$

whereas $D = 0$ if $P = 1$. Replacing D by $\{D\}$ we may suppose that

$(*)$ $\gcd(d, r_1, \ldots, r_n) = 1$, $0 < r_i < d \ \forall i = 1, \ldots, n$, if $d \ge 2$, and $P = 1$ if $d = 1$.

If two pairs (d, P) and $(\tilde{d}, \tilde{P})$ satisfy this condition $(*)$ and $A^+_{d,P} \cong A^+_{\tilde{d},\tilde{P}}$ as graded A_0-algebras then by Corollary 3.4 we have $\frac{\operatorname{div}(P)}{d} = \frac{\operatorname{div}(\tilde{P})}{\tilde{d}}$, and so $d = \tilde{d}$ and $P = \tilde{P}$.

Thus we obtain the following classification result.

Theorem 3.11. *For every normal affine surface $V = \operatorname{Spec} A$, where $A = \bigoplus_{i\geq 0} A_i$ with $A_0 = \mathbb{C}[t]$, there is a unique pair (d, P) satisfying condition $(*)$ and an equivariant isomorphism of A_0-schemes*

$$\varphi : V \longrightarrow V^+_{d,P} := \operatorname{Spec} A^+_{d,P} .$$

Remark 3.12. 1. In the situation of Theorem 3.11 above, the Veronese subring $A^{(d)}$ is equal to $A_0[v] = \mathbb{C}[t, v]$. The cyclic group $\mathbb{Z}_d$ acts on A via the $\mathbb{C}^*$-action and $A^{(d)}$ coincides with the ring of invariants $A^{\mathbb{Z}_d}$, whereas A is the normalization of $A^{(d)}$ in the fraction field $\operatorname{Frac}(A)$. Thus the morphism $V \to \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t, v]$ induced by the inclusion $\mathbb{C}[t, v] \subseteq A$ represents V as a cyclic covering of the plane branched along the curve $u = 0$, and V is the normalization of a surface $\{u^d - P(t)v = 0\}$ in $\mathbb{C}^3$.

2. More generally, let C be any affine smooth curve and let $A = \bigoplus_{i\geq 0} A_i$ be a normal 2-dimensional A_0-algebra of finite type. If $A_1 = u \cdot A_0$ and $A_d = v \cdot A_0$, $d := d(A_{\geq 0})$, for suitable elements $u \in A_1$ and $v \in A_d$ then A is the normalization of an algebra $A_0[u, v]/(u^d - P_+v)$ graded via $\deg u = 1$, $\deg v = d$, for a certain $d \in \mathbb{N}$ and a certain element $P_+ \in A_0$.

4. The hyperbolic case

Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be the coordinate ring of a normal affine surface $V = \operatorname{Spec} A$ with $\mathbb{C}^*$-action such that A_+, A_- are both non-zero. Here again there is a quotient morphism $\pi : V \to C = \operatorname{Spec} A_0$ induced by the inclusion $A_0 \hookrightarrow A$. Every fiber of π is either a non-trivial orbit or a union of two 1-dimensional orbits and a hyperbolic fixed point, which is a source for one of them and a sink for the other one [FiKa$_1$, Lemma 1.7]. Thus the fixed point set F is finite and contains $\operatorname{Sing} V$.

By Lemma 1.1 the proper subalgebras $A_{\geq 0}$ and $A_{\leq 0}$ of A are normal and finitely generated, and so $V_+ := \operatorname{Spec} A_{\geq 0}$ and $V_- := \operatorname{Spec} A_{\leq 0}$ are normal affine surfaces with a parabolic $\mathbb{C}^*$-action birationally dominated by V. The natural embeddings $A_0 \hookrightarrow A_{\geq 0} \hookrightarrow A$ and $A_0 \hookrightarrow A_{\leq 0} \hookrightarrow A$ yield the commutative diagram

$$\begin{tikzcd} V_+ & V \arrow[l, "\sigma_+"'] \arrow[r, "\sigma_-"] \arrow[d, "\pi"] & V_- \arrow[dl, "\pi_-"] \\ & C \arrow[from=ul, "\pi_+"'] & \end{tikzcd} \tag{4}$$

where $\sigma_\pm$ are equivariant birational morphisms. Hence $\sigma_\pm$ are equivariant affine modifications [KaZa$_1$, Theorem 1.1]. More precisely the following result holds.

Proposition 4.1. *V can be obtained from $V_\pm$ by blowing up a $\mathbb{C}^*$-invariant subscheme and deleting the proper transform of a $\mathbb{C}^*$-invariant divisor $D^\pm$ on $V_\pm$, which contains the fixed point curve $\iota_\pm(C) \subseteq V_\pm$.*

Proof. Let us show this for V_+, the proof for V_- being similar. Choose a system of homogeneous generators $a_1, \ldots, a_n$ of the finitely generated A_0-subalgebra $A_{\leq 0}$ and let $f_0 \in A_+$ be a non-zero element of degree $m = -\min_i \deg a_i$. Letting $f_i := a_i f_0$ for $i = 1, \ldots, n$ we obtain

$$A = A_{\geq 0}\left[\frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0}\right] = A_{\geq 0}[I/f_0] := \left\{\frac{x_k}{f_0^k} \mid x_k \in I^k,\ \ k \geq 0\right\} ,$$

where I is the graded ideal of $A_{\geq 0}$ generated by $f_0, \ldots, f_n$. Thus $V = \operatorname{Spec} A$ is obtained by blowing up $V_+ = \operatorname{Spec} A_{\geq 0}$ with center I and deleting the proper transform of the $\mathbb{C}^*$-invariant divisor $\operatorname{div} f_0$ on V_+. As this divisor contains $\iota_+(C)$, the result follows. $\square$

Remark 4.2. The exceptional divisor of $\sigma_\pm$ contains extra components of the reducible fibers of π. If the center $V(I_\pm)$ of $\sigma_\pm$ is nonempty then $\sigma_\pm$ contracts exactly one irreducible curve at each point of the center $V(I_\pm)$. It might happen that the center is empty, and $V = V_\pm \setminus \iota_\pm(C)$. But in general (in contrary to the situation considered in [De, 2.2]) a blowup is present, as the following simple example shows.

Example 4.3. If $A = \mathbb{C}[x, y]$ is graded via $\deg x = 1$, $\deg y = -1$ then $A_0 = \mathbb{C}[xy]$ and $A_{\geq 0} = \mathbb{C}[x, xy]$. Thus $V \cong \mathbb{A}^2_\mathbb{C}$ with the hyperbolic $\mathbb{C}^*$-action $t.(x, y) = (tx, t^{-1}y)$ and $V_+ \cong \mathbb{A}^2_\mathbb{C}$ with the parabolic $\mathbb{C}^*$-action $t.(u, v) = (tu, v)$. Furthermore, $C \cong \mathbb{A}^1_\mathbb{C}$, $\pi(x, y) = xy$, $\pi_+(u, v) = v$ and $\sigma(x, y) = (x, xy)$ is the standard affine modification of $\mathbb{A}^2_\mathbb{C}$ at the origin along a coordinate axes.

4.4. The Dolgachev-Pinkham-Demazure construction is still available in the hyperbolic case. In [De, Theorem 3.5] it is done under the additional assumption that $A_{-n} \otimes A_n \to A_0$ is an isomorphism for all n. Here we generalize the construction in order to make it work for any hyperbolic $\mathbb{C}^*$-surface.

Let D_+, D_- be $\mathbb{Q}$-divisors on the smooth affine curve $C := \operatorname{Spec} A_0$. For $n \geq 0$ we consider the A_0-submodules

$$A_{-n} := H^0(C, \mathcal{O}_C(\lfloor nD_- \rfloor)) \cdot u^{-n} \quad \text{and} \quad A_n := H^0(C, \mathcal{O}_C(\lfloor nD_+ \rfloor)) \cdot u^n$$

of $\operatorname{Frac}(A_0)[u, u^{-1}]$, where u is an indeterminate of degree 1. If $D_+ + D_- \leq 0$ then for $n \geq m \geq 0$ we have

$$\lfloor nD_+ \rfloor + \lfloor mD_- \rfloor \leq \lfloor (n-m)D_+ \rfloor,$$

whence $A_n \cdot A_{-m} \subseteq A_{n-m}$. Similarly, for $0 \leq n \leq m$ we have $A_n \cdot A_{-m} \subseteq A_{n-m}$. Thus

$$A := A_0[D_+, D_-] := \bigoplus_{n \in \mathbb{Z}} A_n$$

is a finitely generated A_0-subalgebra of $\operatorname{Frac}(A_0)[u, u^{-1}]$ with $A_{\geq 0} = A_0[D_+]$ and $A_{\leq 0} = A_0[D_-]$. The grading on A defines a natural hyperbolic $\mathbb{C}^*$-action on the surface $V := \operatorname{Spec} A$. The latter surface is normal as so are the algebras $A_0[D_+]$ and $A_0[D_-]$ (see Lemma 1.1 and Corollary 3.8(b)). Conversely, we have the following theorem.

Theorem 4.5. *If $C = \operatorname{Spec} A_0$ is a smooth affine curve and $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a normal graded finitely generated domain of dimension 2 with $A_\pm \neq 0$, then the following hold.*

(a) *A is isomorphic to $A_0[D_+, D_-]$, where D_+, D_- are $\mathbb{Q}$-divisors on C satisfying $D_+ + D_- \leq 0$. More precisely, if $u \in \operatorname{Frac}(A_0) \cdot A_1$ and if the divisors D_+, D_- on C are defined by*

$$\pi_+^*(D_+) = \operatorname{div}(u) - \iota_+(C) \quad \textit{and} \quad \pi_-^*(D_-) = \operatorname{div}(u^{-1}) - \iota_-(C), \tag{5}$$

where $\pi_\pm$ are as in diagram (4) above and $\iota_\pm : C \hookrightarrow V_\pm$ are the natural embeddings, then $D_+ + D_- \leq 0$ and $A \cong A[D_+, D_-]$.

(b) *$A_0[D_+, D_-] \cong A_0[D'_+, D'_-]$ as graded A_0-algebras if and only if, for a rational function $\varphi \in \operatorname{Frac}(A_0)$, one has*

$$D'_+ = D_+ + \operatorname{div} \varphi \qquad \textit{and} \qquad D'_- = D_- - \operatorname{div} \varphi\,.$$

Proof. (a) By Theorem 3.2 and its proof we have equalities

$$A_{\geq 0} = A_0[D_+] \quad \text{and} \quad A_{\leq 0} = A_0[D_-]$$

as subalgebras of $\mathrm{Frac}(A_0)[u, u^{-1}]$, whence $A = A_0[D_+, D_-]$. It remains to show that $D_+ + D_- \leq 0$. Applying in (5) the functors σ_+^* and σ_-^* respectively, we obtain

$$\pi^*(D_+) = \mathrm{div}(u) - \sigma_+^*\iota_+^*(C) \quad \text{and} \quad \pi^*(D_-) = \mathrm{div}(u^{-1}) - \sigma_-^*\iota_-^*(C).$$

Taking the sum of these equalities yields $\pi^*(D_+ + D_-) = -(\sigma_+^*\iota_+^*(C) + \sigma_-^*\iota_-^*(C))$, whence $D_+ + D_- \leq 0$, as required. Finally (b) follows from Theorem 3.2(b) and its proof. □

Consequently, if $A_0 = \mathbb{C}[t]$ then A admits a unique presentation $A = A_0[D_+, D_-]$ with $D_+ = \{D_+\}$ and $D_+ + D_- \leq 0$.

It follows from Theorem 4.5 that outside $|D_+| \cup |D_-|$, the map $\pi : V \to C$ is a locally trivial principal $\mathbb{C}^*$-bundle. More generally, the Dolgachev-Pinkham-Demazure construction shows the following result (cf. [BasHa], [FiKa$_1$, Proposition 1.11]).

Corollary 4.6. *In all three cases, outside of a finite subset of the curve C the projection $\pi : V^* \to C$ and $\pi : V \to C$, respectively, defines a locally trivial fiber bundle. This is a principal $\mathbb{C}^*$-bundle in the elliptic and hyperbolic cases, and a line bundle in the parabolic case.*

Note that if $u \in A_1 \cup A_{-1}$ is a non-zero element then its restriction to a general fiber of π gives a fiber coordinate and so a trivialization over a Zariski open subset of C.

Remarks 4.7. 1. If $v \in A$ is an invertible element of degree $d > 0$ then we can write

$$v = fu^d \in A_d \qquad \text{and} \qquad v^{-1} = f^{-1}u^{-d} \in A_{-d}\,,$$

where $f \in \mathrm{Frac}(A_0)$ satisfies

$$\mathrm{div}(f) + dD_+ \geq 0 \qquad \text{and} \qquad -\mathrm{div}(f) + dD_- \geq 0\,.$$

Thus $0 \geq D_+ + D_- \geq 0$, whence $D_- = -D_+$. Conversely, if $D_+ = -D_-$ and if $vA_0 = A_d$ with $d := d(A_{\geq 0}) = d(A_{\leq 0})$, then $v = fu^d$ with $\mathrm{div} f + dD_+ = 0$ by Remark 3.7. Hence also $\mathrm{div} f^{-1} - dD_- = 0$, so $f^{-1}u^{-d} \in A$ and fu^d is a unit in A.

2. Suppose now that $A_0 \cong \mathbb{C}[t]$ and D_+ is the $\mathbb{Q}$-divisor $-\frac{e}{d}[0]$, as in Lemma 2.4, where $0 \in \mathbb{A}^1_{\mathbb{C}}$ is the the origin. Consider the element

$$v := t^e u^d \in A := A_0[D_+].$$

As A has an isolated singularity, the localized ring A_v is smooth and graded of hyperbolic type. Since $v^{-1} = t^{-e}u^{-d} \in A_v$, the associated divisor D_- is just $D_- = \frac{e}{d}[0]$. Thus

$$A_v = A_0[D_+, -D_+].$$

Of course, the same result holds if $A_0 = \mathbb{C}[t]$ is replaced by the formal power series ring $A_0 = \mathbb{C}[[t]]$.

3. Similarly, if $C = \mathrm{Spec}\, A_0$ is any 1-dimensional smooth curve and if D_+ is a $\mathbb{Q}$-divisor on C, then $A_0[D_+, -D_+]$ is a smooth algebra. This is easily reduced to the case $A_0 = \mathbb{C}[[t]]$ in (2) using completions at the maximal ideals of A_0.

The following analogue of Proposition 3.8 holds with a similar proof.

Lemma 4.8. *Let $C = \operatorname{Spec} A_0$ be a smooth affine curve with function field $K_0 = \operatorname{Frac}(A_0)$. If a graded 2-dimensional domain $B \subseteq K_0[u, u^{-1}]$ is represented as*

$$B = A_0[h_1 u^{-n_1}, \dots, h_k u^{-n_k},\ f_1 u^{m_1}, \dots, f_n u^{m_n}] \qquad (\textit{where} \quad n_i,\ m_j > 0\ \forall i, j)$$

with $h_1, \dots, h_k,\ f_1, \dots, f_n \in K_0$ and $B_0 = A_0$, then its normalization $A = B_{\text{norm}}$ coincides (as a graded A_0-subalgebra of $K_0[u]$) with $A_0[D_+, D_-]$, where

$$D_- = -\min_{1\le i\le k} \frac{\operatorname{div} h_i}{n_i} \qquad \textit{and} \qquad D_+ = -\min_{1\le j\le n} \frac{\operatorname{div} f_j}{m_j}\,.$$

We notice that the assumption $A_0 = B_0$ amounts to the inequalities

$$\frac{\operatorname{div} h_i}{n_i} + \frac{\operatorname{div} f_j}{m_j} \ge 0 \quad \forall i, j\,,$$

which in turn are equivalent to $D_+ + D_- \le 0$.

The following lemma provides a useful additional information in the particular case where V is ruled over the affine line $\mathbb{A}^1_{\mathbb{C}}$.

Lemma 4.9. *Let $A = \bigoplus_{i\in\mathbb{Z}} A_i = A_0[D_+, D_-] \subseteq \operatorname{Frac}(A_0)[u, u^{-1}]$ with $A_0 \cong \mathbb{C}[t]$, and let $d_\pm = d_\pm(A)$ be the minimal positive integer such that the divisor $d_\pm D_\pm$ is integral. If $A_{\pm 1} = u_\pm \cdot A_0$, $A_{\pm d_\pm} = v_\pm \cdot A_0$ and*

$$u_+ u_- = Q(t), \qquad u_\pm^{d_\pm} = P_\pm(t) v_\pm$$

for some unitary polynomials Q, $P_\pm \in \mathbb{C}[t]$ then

$$D_+ = \frac{\operatorname{div} P_+}{d_+} + D_0 \qquad \textit{and} \qquad D_- = \frac{\operatorname{div} P_-}{d_-} - D_0 - \operatorname{div} Q\,, \tag{6}$$

where D_0 is the integral divisor $D_0 = \operatorname{div}(u/u_+)$ on $\mathbb{A}^1_{\mathbb{C}}$. Consequently,

$$\frac{\operatorname{div} P_+}{d_+} + \frac{\operatorname{div} P_-}{d_-} \le \operatorname{div} Q\,. \tag{7}$$

Furthermore, P_+ and P_- are uniquely determined by D_+ and D_- through

$$\{D_+\} = \frac{\operatorname{div} P_+}{d_+} \qquad \textit{and} \qquad \{D_-\} = \frac{\operatorname{div} P_-}{d_-}\,. \tag{8}$$

Proof. We have $u^{d_+} = P_+ \cdot (u/u_+)^{d_+} v_+$ and $u^{-d_-} = P_- \cdot (u/u_+)^{-d_-} Q^{-d_-} v_-$ and so by Remark 3.7

$$D_+ = \frac{\operatorname{div}(P_+ \cdot (u/u_+)^{d_+})}{d_+} = \frac{\operatorname{div} P_+}{d_+} + D_0$$

$$D_- = \frac{\operatorname{div}(P_- \cdot (u/u_+)^{-d_-} Q^{-d_-}}{d_-} = \frac{\operatorname{div} P_-}{d_-} - D_0 - \operatorname{div} Q\,.$$

Now (7) follows from the inequality $D_+ + D_- \le 0$. To show (8) we let $P_\pm = S_\pm^{d_\pm} T_\pm$, where $S_\pm, T_\pm \in \mathbb{C}[t]$ are polynomials with

$$\operatorname{div} S_\pm = \left\lfloor \frac{\operatorname{div} P_\pm}{d_\pm} \right\rfloor \qquad \text{and} \qquad \operatorname{div} T_\pm = \left\{ \frac{\operatorname{div} P_\pm}{d_\pm} \right\}$$

respectively. The relation $(u_\pm/S_\pm)^{d_\pm} = T_\pm v_\pm$ then shows that $u_\pm/S_\pm$ is integral over A and so by the normality of A is contained in $A_{\pm 1}$. As $u_\pm$ is a generator of $A_{\pm 1}$ this forces $S_\pm = 1$, proving (8). □

Remark 4.10. The relation $u_+^{d_+} = v_+P_+$ shows that u_+ is integral over $A_0[u_-, v_\pm] \subseteq A$. Moreover, A and its subring $A_0[u_-, v_\pm]$ have the same field of fractions. Therefore A is the normalization of $A_0[u_-, v_\pm]$. To find the relations between the generators of $A_0[u_-, v_\pm]$, write $d_\pm = d'_\pm k$, where $k := \gcd(d_+, d_-)$. From $v_\pm = u_\pm^{d_\pm}/P_\pm$ we get

$$v_+^{d'_-} v_-^{d'_+} = \frac{u_+^{d_+d'_-} u_-^{d'_+d_-}}{P_+^{d'_-} P_-^{d'_+}} = \frac{Q^{kd'_+d'_-}}{P_+^{d'_-} P_-^{d'_+}}$$

(in particular, $Q^{kd'_+d'_-}$ is divisible by $P_+^{d'_-} P_-^{d'_+}$; cf. (7)). Similarly,

$$v_+u_-^{d_+} = \frac{u_+^{d_+}u_-^{d_+}}{P_+} = \frac{Q^{d_+}}{P_+} \in A_0 \,.$$

Thus A as above is the normalization of the algebra

$$B := \mathbb{C}[t, u_-, v_+, v_-]/\left(u_-^{d_-} - v_-P_-,\ v_+^{d'_-} v_-^{d'_+} - \frac{Q^{kd'_+d'_-}}{P_+^{d'_-} P_-^{d'_+}},\, v_+u_-^{d_+} - \frac{Q^{d_+}}{P_+}\right) \tag{9}$$

graded via $\deg t = 0, \deg u_- = -1, \deg v_\pm = \pm d_\pm$. Indeed, $\operatorname{Spec} B$ is irreducible because the general fibers of the map $t : \operatorname{Spec} B \to \operatorname{Spec} A_0$ are irreducible, and every fiber is 1-dimensional and in the closure of the generic fiber.

Since u_- satisfies the relation $u_-^{d_-} = v_-P_-$, it is integral over the ring $A_0[v_\pm]$. Hence A is integral over $A_0[v_\pm]$. The normalization of $A_0[v_\pm]$ is just the Veronese subring $A^{(k)}$ of A, since the fraction fields of both rings coincide. In other words, $A^{(k)}$ is the normalization of the hypersurface algebra

$$\tilde{B} := \mathbb{C}[t, v_+, v_-]/\left(v_+^{d'_-} v_-^{d'_+} - P(t)\right), \quad \text{where} \quad P := \frac{Q^{kd'_+d'_-}}{P_+^{d'_-} P_-^{d'_+}} \,.$$

The cyclic group $\mathbb{Z}_d$ acts on A via the $\mathbb{C}^*$-action with invariant ring $A^{(k)}$ or, geometrically speaking, $V = \operatorname{Spec} A$ is a cyclic branched covering of degree k of the normalization of the hypersurface $\{v_+^{d'_-} v_-^{d'_+} = P(t)\}$ in $\mathbb{A}^3_C$.

In particular, in the case $\gcd(d_+, d_-) = 1$ the algebra A is the normalization of $A_0[v_+, v_-]/(v_+^{d_-} v_-^{d_+} - P(t))$. It is worthwhile to remark how to get a representation $A \cong A_0[D_+, D_-]$ in terms of P. Choose $p, q \in \mathbb{Z}$ with $\left|\begin{smallmatrix} d_+ & p \\ d_- & q \end{smallmatrix}\right| = 1$ so that $u' = v_+^p v_-^q$ has degree 1. By an easy calculation $u'^{d_+} = v_+P^p$ and $u'^{-d_-} = v_-/P^q$, whence by Remark 3.7 $A \cong A_0[D_+, D_-]$ with

$$D_+ = \frac{p}{d_+}\operatorname{div} P \quad \text{and} \quad D_- = -\frac{q}{d_-}\operatorname{div} P \,.$$

The special case $d_+ = 1$ leads to the following example.

Example 4.11. (Cf. [Be, Example 4.11]) For a unitary polynomial $P \in \mathbb{C}[t]$, we let $A = A_{d,P} = B_{\text{norm}}$ be the normalization of the $\mathbb{C}$-algebra

$$B = B_{d,P} := \mathbb{C}[t, u, v]/(u^d v - P(t))$$

graded via

$$\deg t = 0, \quad \deg u = 1, \quad \deg v = -d \,,$$

so that the normal affine surface $V := \operatorname{Spec} A$ is equipped with a hyperbolic $\mathbb{C}^*$-action. The regular function t on V is $\mathbb{C}^*$-invariant, so the $\mathbb{C}^*$-action preserves its fibers, and the general fibers of t are irreducible. Hence t gives the quotient morphism $\pi : V \to C \cong \mathbb{C} = \operatorname{Spec} \mathbb{C}[t]$. As $A \cong A_0[u, Pu^{-d}]$ we can write

$$A \cong A_0[D_+, D_-], \qquad \text{where} \qquad D_+ = 0 \quad \text{and} \quad D_- = -\operatorname{div}(P)/d$$

(see Lemma 4.8).

Moreover, we can recover $P_\pm$ and Q in the representation of Remark 4.10. Indeed, by the construction given there $P_+ = 1$ and by (8) $\{D_-\} = \operatorname{div}(P_-)/d_-$ (note that $d = d_-$). This gives

$$\operatorname{div} P_- = d\left\{-\frac{\operatorname{div} P}{d}\right\} \qquad \text{and} \qquad \operatorname{div} Q = \frac{\operatorname{div} P + \operatorname{div} P_-}{d} \tag{10}$$

(see (6)). In particular,

$$A_{\geq 0} \cong A_0[u] \cong \mathbb{C}[t, u] \qquad \text{and} \qquad A_{\leq 0} \cong A^+_{d,P_-}$$

as graded A_0-algebras, where for the second isomorphism we have to reverse the grading of one of the rings.

This discussion provides the following characterization of the algebras $A_{d,P}$.

Proposition 4.12. *If $A = A_0[D_+, D_-]$, where $A_0 \cong \mathbb{C}[t]$ and D_+, D_- are $\mathbb{Q}$-divisors on $\mathbb{A}^1_{\mathbb{C}}$ with $D_+ + D_- \leq 0$, then the following conditions are equivalent.*

(i) *D_+ is integral i.e., $\{D_+\} = 0$.*
(ii) *$A_{\geq 0} \cong A_0[u]$ as graded A_0-algebras, where* $\deg u = 1$.
(iii) *$A \cong A_{d,P}$ as graded A_0-algebras, where $D_+ + D_- = -\frac{\operatorname{div} P}{d}$.*

Next we study the effect of a base change to the Dolgachev-Pinkham-Demazure representation.

Proposition 4.13. *Let $C = \operatorname{Spec} A_0$ be an affine curve with function field $K_0 =$ $\operatorname{Frac}(A_0)$ and let*

$$A := A_0[D_+, D_-] \subseteq K_0[u, u^{-1}]\,,$$

where $D_\pm$ are $\mathbb{Q}$-divisors on C satisfying $D_+ + D_- \leq 0$. Let L be the field $L :=$ $\operatorname{Frac}(A)[\sqrt[d]{tu^b}]$, where $t \in K_0$ and $b, d \in \mathbb{N}$. If A' is the normalization of A in L then the following hold.

1. *A'_0 is the normalization of A_0 in $K_0[s]$ with $s := \sqrt[k]{t}$, where $k := \gcd(b, d)$.*
2. *$A' \cong A'_0[D'_+, D'_-]$ with*

$$D'_\pm := \frac{k}{d}\left(p^*(D_\pm) \pm \beta \operatorname{div} s\right)\,,$$

where $p : C' := \operatorname{Spec} A'_0 \to C$ is the projection and β is defined by $\beta b \equiv k \mod d$.

Proof. We let $b = b'k$ and $d = d'k$. The normalization A' admits a natural $\frac{1}{d}$-grading, and the element $u^* := \sqrt[d]{tu^b}$ is of degree $b/d = b'/d'$. If we write $k = \beta b + \delta d$, then the element $u' := u^{*\beta} u^\delta \in \operatorname{Frac}(A')$ has minimal possible positive degree $1/d'$. Thus

$$A' \subseteq \operatorname{Frac}(A'_0)[u', u'^{-1}].$$

To compute A'_0, we note that $u^{*n} u^{-m}$ with n, $m \in \mathbb{N}$ has degree 0 if and only if $nb'/d' = m$. In particular, $n = n'd'$ is an integer multiple of d'. Thus $K'_0 := \operatorname{Frac} A'_0$ is generated over K_0 by $u^{*d'} u^{-b'} = t^{d'/k}$ (i.e., $n' = 1$). As d' and k are coprime, it follows that $s=\sqrt[k]{t}$ also belongs to K_0' and that this field is actually generated by s over K_0, proving (1).

After localizing A_0 we may assume that there is an element $v_+\in A$ of degree $d_+=d(A_{\ge 0})$ with $A_{d_+}=v_+A_0$ (see 3.6). We claim that then $A'_{sd_+}=v_+^sA_0'$ for all $s\ge 0$. If not, then for some $s>0$ and some non-unit $x\in A_0'$ the element v_+^s/x belongs to A', so it is integral over A and there is an equation

$$\frac{v_+^{sm}}{x^m}+a_1\frac{v_+^{s(m-1)}}{x^{m-1}}+\cdots+a_m=0\,,$$

where $m\ge 0$ and $a_i\in A_{id_+}$. Thus $a_i=v_+^{si}q_i$ for some elements $q_i\in A_0$, whence dividing the equation above by v_+^{sm} we obtain that

$$\frac{1}{x^m}+q_1\frac{1}{x^{m-1}}+\cdots+q_m=0\,.$$

As A_0' is integrally closed this is only possible if $x\in A_0'$ contradicting the choice of x.

Thus $v=v_+$ is an element satisfying the assumptions of Remark 3.7, and we compute with it the divisor D'_+ as follows (the calculation for D'_- is analogous). If we consider the new grading of A' by assigning to u' the degree 1, then v_+^k becomes an element of degree dd_+. Moreover, if $u^{d_+}=P_+v_+$ with $P_+\in K_0$ then by Remark 3.7 $D_+=\frac{\mathrm{div}(P_+)}{d_+}$. Since

$$\begin{aligned} u'^{dd_+} &= (u^{*\beta}u^{\delta})^{dd_+}=(tu^b)^{\beta d_+}u^{\delta dd_+}\\ &= t^{\beta d_+}u^{d_+(\beta b+\delta d)}=t^{\beta d_+}u^{d_+k}\\ &= t^{\beta d_+}P_+^kv_+^k \end{aligned}$$

we obtain again by Remark 3.7 that on C'

$$D'_+=\frac{\mathrm{div}(t^{\beta d_+}P_+^k)}{dd_+}=\frac{\beta}{d}\mathrm{div}(t)+\frac{k}{d}p^*(D_+)$$

and (2) follows. □

Let us consider the following important example.

Example 4.14. With $A_0:=\mathbb{C}[t]$, suppose that $D_+=-\frac{e}{d}[0]$ and that D_- is any $\mathbb{Q}$-divisor on $\mathbb{A}^1_{\mathbb{C}}=\operatorname{Spec}A_0$. Applying Proposition 4.13 to $s:=\sqrt[d]{t}$ (i.e. $b=0$) we get that the normalization of $A:=A_0[D_+,D_-]$ in the field $L:=\mathrm{Frac}(A)[s]$ is given by

$$A'=A_0'[-e[0],D'_-]\subseteq\mathbb{C}(s)[u,u^{-1}]\,,$$

where $A_0'=\mathbb{C}[s]$ and $D'_-=p^*(D_-)$ (as before, $p:\operatorname{Spec}\mathbb{C}[s]\to\operatorname{Spec}\mathbb{C}[t]$ denotes the projection $s\mapsto s^d$). The divisor $D'_+=-e[0]$ being integral we have

$$A'\cong A_0'[0,D'_++D'_-]\subseteq\mathbb{C}(s)[\tilde u,\tilde u^{-1}]\,,$$

where $\tilde u:=s^eu$.

More concretely, if $k:=d_-(A)$ and if we choose a unitary polynomial $Q\in\mathbb{C}[t]$ with $D_-=-\frac{\mathrm{div}(Q)}{k}$ then $D'_++D'_-=-\frac{\mathrm{div}(Q(s^d)s^{ke})}{k}$. By Example 4.11 $A'\cong A_{k,P}$ is the normalization of

$$B_{k,P}=\mathbb{C}[s,\tilde u,v]/\left(\tilde u^kv-P(s)\right)\,,\quad\text{where}\quad P(s):=Q(s^d)s^{ke}\,. \tag{11}$$

The field extension $\mathrm{Frac}(A)\subseteq\mathrm{Frac}(A)[s]$ is Galois with Galois group $\mathbb{Z}_d=\langle\zeta\rangle$, where $\zeta.s=\zeta s$. Thus

$$A\cong(A_{k,P})^{\mathbb{Z}_d}\,,$$

and the action of ζ on $\tilde{u} = s^e u$ is given by $\zeta.\tilde{u} = \zeta^e \tilde{u}$. Therefore, the group $\mathbb{Z}_d$ acts on $A_{k,P}$ via

$$\zeta.s = \zeta s, \quad \zeta.\tilde{u} = \zeta^e \tilde{u} \quad \text{and} \quad \zeta.v = v\,. \tag{12}$$

Thus we obtain the following characterization.

Proposition 4.15. *For an algebra $A = A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[s]$ the following conditions are equivalent.*

(i) $\{-D_+\} = \frac{e}{d}[0]$*, where* $0 \le e < d$ *and* $\gcd(e, d) = 1$.
(ii) $A \cong (A_{k,P})^{\mathbb{Z}_d}$*, where $A_{k,P}$ is the normalization of $B_{k,P}$ in (11) and where* $\mathbb{Z}_d = \langle\zeta\rangle$ *acts via the formulas in (12).*

Like in the parabolic case V may possess at most cyclic quotient singularities, as follows from Miyanishi's Theorem (see [Miy, Lemma 1.4.4(1)]). The type of quotient singularities is determined from the divisors D_+, D_- by the following result. As before, $C = \operatorname{Spec} A_0$ is a smooth affine curve with function field $K_0 = \operatorname{Frac}(A_0)$ and $A := A_0[D_+, D_-]$ with $\mathbb{Q}$-divisors D_+ and D_- on C. Denote $\pi : V = \operatorname{Spec} A \to C$ the canonical projection.

Theorem 4.16. (a) *The set of singular points $\operatorname{Sing} V$ is contained in the fixed point set F which is the zero locus $F = V(I)$ of the ideal $I := A_+A + A_-A$ of A.*
(b) *The map $\pi|F : F \to C$ is injective, and $\pi(F) = \{a \in C | D_+(a) + D_-(a) < 0\}$.*
(c) *For a point $a' \in F$ with image $a := \pi(a') \in C$ we write*

$$D_+(a) = -\frac{e_+}{m_+} \qquad \textit{and} \qquad D_-(a) = \frac{e_-}{m_-}$$

with the convention that

$$m_+ > 0, \quad m_- < 0, \qquad \gcd(e_+, m_+) = \gcd(e_-, m_-) = 1 \qquad \textit{and}$$

$$m_+ = 1 \quad \textit{if} \quad D_+(a) = 0, \qquad m_- = -1 \quad \textit{if} \quad D_-(a) = 0\,.$$

Let p, $q \in \mathbb{Z}$ with $\left|\begin{smallmatrix} p & e_+ \\ q & m_+ \end{smallmatrix}\right| = 1$. Then $a' \in F$ is a quotient singularity of type

$$(\Delta(a), e), \quad \textit{where} \quad \Delta(a) := -\begin{vmatrix} e_+ & e_- \\ m_+ & m_- \end{vmatrix} \quad \textit{and} \quad e \equiv \begin{vmatrix} p & e_- \\ q & m_- \end{vmatrix} \mod \Delta(a)\,.$$

In particular, $a' \in \operatorname{Sing} V$ if and only if $\Delta(a) \neq 1$.

Proof. As in the proof of Proposition 3.8(b) we can reduce the statement to the case that $A_0 = \mathbb{C}[t]$ and $|D_+| \cup |D_-|$ is contained in the origin, so that $D_\pm = \mp e_\pm/m_\pm[0]$.

(a) If $v \in A$, $v \neq 0$, is a homogeneous element of non-zero degree then the localization A_v is smooth (see Remark 4.7(2)). Thus $\operatorname{Sing} V \subseteq V(I)$, as claimed.

(b) The map $A_0 \to A/I$ is obviously surjective. Thus

$$\pi|F : F = \operatorname{Spec}(A/I) \to C$$

is a closed embedding. Moreover, $F = \emptyset$ if and only if $1 \in I$ if and only if $1 = a_+a_-$ for some homogeneous elements of A of opposite degrees, and the latter happens if and only if $D_+ + D_- = 0$ by Remark 4.7(1).

(c) Notice first that the elements

$$t^{e_+}u^{m_+}, \quad t^{e_-}u^{m_-} \in K_0[u, u^{-1}]$$

belong to A. By definition, the ideal $I+tA$ of A (this is just the maximal ideal of the point $a'\in F$) is generated by the monomials $t^e u^m$ with $(e,m)\in\mathbb{Z}\times\mathbb{Z}$, where $(e,m)\neq(0,0)$ and

$$e+mD_+(0)\geq 0\quad\text{if}\quad m\geq 0,\qquad e-mD_-(0)\geq 0\quad\text{if}\quad m\leq 0\,.$$

In other words, (e,m) is an element of the cone $C:=\mathrm{C}\left((e_+,m_+),\,(e_-,m_-)\right)$ generated by the vectors $(e_\pm,m_\pm)$ in the plane. Hence A is a toric algebra generated by the semigroup $C\cap\mathbb{Z}^2$ and so is a quotient $A_{d,e}$ for some $d,\ e\geq 0$ (see Lemma 2.4). To determine d,e, we must find a basis of $\mathbb{Z}^2$ such that (e_+,m_+) is one of the basis vectors. This is done as follows.

If we choose $p,q\in\mathbb{Z}$ with $\left|\begin{smallmatrix}p & e_+\\ q & m_+\end{smallmatrix}\right|=1$, then the vectors $\tilde e_1:=(e_+,m_+)$ and $\tilde e_2:=(p,q)$ form a basis of $\mathbb{Z}^2$, and

$$(e_-,m_-)=\Delta'\tilde e_1+\Delta\tilde e_2,\quad\text{where}\quad \Delta':=\begin{vmatrix}p & e_-\\ q & m_-\end{vmatrix}\quad\text{and}\quad \Delta:=\Delta(0)\,.$$

As $\tilde e_1$ and (e_-,m_-) form a basis of the cone C, it follows from Lemma 2.4 that A has a quotient singularity of type (Δ,e), where $0\leq e<d$ and $e\equiv\left|\begin{smallmatrix}p & e_-\\ q & m_-\end{smallmatrix}\right|\mod\Delta$. Note that Δ and Δ' are coprime since so are e_- and m_-.

The determinant Δ has always positive sign as

$$D_+(0)+D_-(0)=\frac{\Delta}{m_+m_-}\leq 0\qquad\text{and}\qquad m_+>0,\ \ m_-<0\,,$$

and so (c) follows. ☐

Corollary 4.17. *If $A_{d,P}$ is the normalization of the algebra*

$$B_{d,P}=\mathbb{C}[t,u,v]/(u^dv-P(t))\,,$$

where $P(t)=\prod_{i=1}^k(t-a_i)^{r_i}$ with $a_i\neq a_j$ for $i\neq j$ (see Example 4.11) then the singular points of the surface $V_{d,P}=\operatorname{Spec}A_{d,P}$ are the points $a_i'\in V_{d,P}$ $(1\leq i\leq k)$, where $t=a_i,u=v=0$ and $r_i\nmid d$.

Proof. It was shown in Example 4.11 that $D_-=0$ and $D_+(a_i)=-\frac{r_i}{d}$. Therefore, $\Delta(a_i)=e_+>1$ if and only if $r_i\nmid d$, which implies our assertion. ☐

In the sequel we use the following notation.

Definition 4.18. Let $O=\mathbb{C}^*z$ be the orbit through a point $z\in V\backslash F$. Following [FiKa$_1$] we say that O is of type (d,q) if d is the order of the stabilizer

$$\mathrm{Stab}_z=\ker(\mathbb{C}^*\to\operatorname{Aut}O)\subseteq\mathbb{C}^*,\qquad\text{so that}\quad \mathrm{Stab}_z=\langle\zeta\rangle\cong\mathbb{Z}_d\,,$$

and q $(0\leq q<d)$ is determined from the tangent representation of Stab_z on the tangent plane T_zV via pseudo-reflections

$$\mathrm{Stab}_z\ni\zeta\longmapsto\begin{pmatrix}1 & 0\\ 0 & \zeta^q\end{pmatrix}\,.$$

The orbit O is called *principal* if $d=1$ and *exceptional* otherwise (see [FiKa$_1$]-[FiKa$_3$] for a detailed description of the structure of V near the exceptional orbits).

In the next result we will characterize the orbit types of the surface $V = \operatorname{Spec} A$ with $A := A_0[D_+, D_-]$, where D_+ and D_- are $\mathbb{Q}$-divisors on the smooth affine curve $C = \operatorname{Spec} A_0$. Let $\pi : V \to C$ denote the projection. To examine the orbits over a point $a \in C$, we write

$$D_+(a) = -e_+/m_+ \quad \text{and} \quad D_-(a) = e_-/m_-$$

with the conventions as in Theorem 4.16(c). Let q_+ be defined by $0 \le q_+ < m_+$ and $q_+e_+ \equiv -1 \mod m_+$, and similarly q_- by $0 \le q_- < -m_-$ and $q_-e_- \equiv 1 \mod m_-$. With these notations the following result holds.

Theorem 4.19. *The exceptional orbits of V are located over $|D_+| \cup |D_-|$. The orbits over a given point $a \in |D_+| \cup |D_-|$ are as follows.*

(a) If $D_+(a) + D_-(a) = 0$ then $\pi^{-1}(a)$ consists of one orbit of type (m_+, q_+).

(b) If $D_+(a) + D_-(a) < 0$ then $\pi^{-1}(a)$ contains two orbits O^+ and O^- of types (m_+, q_+) and $(-m_-, q_-)$, respectively. Moreover, their closures $\bar{O}^\pm$ intersect in the unique fixed point, say a', of the fiber, and $\pi^{-1}(a) = O^+ \cup \{a'\} \cup O^-$.

Proof. With the same reasoning as in the proof of Proposition 3.8(b), it is sufficient to treat the case that $A_0 = \mathbb{C}[t]$ and that D is supported on $a = 0 \in \mathbb{A}^1_{\mathbb{C}}$, i.e. $D_\pm = \mp e_\pm/m_\pm[0]$. Note that in this case $m_+ = d(A_{\ge 0})$ and $m_- = -d(A_{\le 0})$.

Let us first assume that $D_+ + D_- = 0$, so that $e := e_+ = -e_-$ and $m := m_+ = -m_-$. The algebra A is then the semigroup algebra $\mathbb{C}[C \cap \mathbb{Z}^2]$ of the real cone C generated by the vectors $\pm(e, m)$ and $(1, 0)$. If we choose $p, q \in \mathbb{Z}$ with $\left|\begin{smallmatrix} p & e \\ q & m \end{smallmatrix}\right| = 1$, then

$$C \cap \mathbb{Z}^2 = \{(a, b) | (a, b) \in \mathbb{Z}(e, m) + \mathbb{N}(p, q)\}.$$

Hence A is the algebra of Laurent polynomials $A = A_0[x, x^{-1}, y]$, where

$$x := t^e u^m \in A_m \quad \text{and} \quad y := t^p u^q \in A_q.$$

Clearly $t = x^{-q}y^m$. The action of $\mathbb{C}^*$ is given by $\lambda.x = \lambda^m x$ and $\lambda.y = \lambda^q y$, whence there is only one orbit O over $t = 0$ and it is given by the equation $y = 0$. The stabilizer of any point of O is the group $E_m \subseteq \mathbb{C}^*$ of m-th roots of unity, and the type of the orbit is $(m, q) = (m_+, e_+)$, as required.

Let us now treat the general case. Consider a generator $v_+ = fu^{m_+}$ of A_{m_+} as A_0-module so that $D_+ = \operatorname{div}(f)/m_+$. The localization $A_{v_+} = A[f^{-1}u^{-m_+}]$ is the subring $A_0[D_+, -D'_-]$ of $\operatorname{Frac}(A_0)[u, u^{-1}]$ with $D'_- := \max(D_-, -D_+)$. (cf. Lemma 4.8). As $D_+ + D_- \le 0$ we have $D'_- = -D_+$, so by the first case the open subset $\operatorname{Spec} A_{v_+}$ of V contains an orbit O^+ of type (m_+, q_+). Similarly, if $v_- = gu^{m_-} \in A_{m_-}$ is a generator, then $\operatorname{Spec} A_{v_-}$ contains an orbit O^- of type $(-m_-, q_-)$. As the whole fiber is given by $v_+ \cdot v_- = 0$, the functions v_+, v_- must vanish on $\bar{O}^-$ and $\bar{O}^+$, respectively. The intersection $\bar{O}^+ \cap \bar{O}^-$ is given by $v_+ = v_- = 0$ and so is the unique fixed point of the fiber. ☐

Example 4.20. In the example of the algebra $A = A_{d,P}$ treated in Corollary 4.17 we have $D_+ = 0$ and $D_- = -\operatorname{div}(P)/d = \sum_i -\frac{r_i}{d}[a_i]$ (see Example 4.11). The exceptional orbits are located over the points $a_i \in \mathbb{A}^1_{\mathbb{C}}$, and $\pi^{-1}(a_i) = O_i^+ \cup \{z_i\} \cup O_i^-$, where z_i is the unique fixed point of the fiber (located over the point $(0, 0, a_i)$ of $\operatorname{Spec} B \subseteq \mathbb{C}^3$). Applying Theorem 4.19, the orbit O_i^+ is principal, and if we write $r_i/d = e_i/m_i$ with

$\gcd(e_i, m_i) = 1$ then O_i^- is of type (m_i, q_i), where

$$q_i e_i \equiv -1 \mod d \quad \text{with} \quad 0 \le q_i < m_i\,.$$

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