CONGRUENCES OF ALTERNATING MULTIPLE HARMONIC SUMS

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Abstract. In this sequel to [10], we continue to study the congruence properties of alternating multiple harmonic sums (MHS). As a contrast to the study of MHS where Bernoulli numbers and Bernoulli polynomials play the key roles, in the alternating setting the Euler numbers and the Euler polynomials are also essential.

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1. Introduction

In proving a Van Hamme type congruence the first author was led to consider some congruences involving alternating multiple harmonic sums (AMHS for short) which are defined as follows. Let $d > 0$ and let $\vec{s} := (s_1, \dots, s_d) \in (\mathbb{Z}^*)^d$. We define the *alternating multiple harmonic sum* as

$$H(\vec{s}; n) := \sum_{1 \le k_1 < k_2 < \cdots < k_d \le n} \prod_{i=1}^{d} \frac{\operatorname{sgn}(s_i)^{k_i}}{k_i^{|s_i|}}.$$

By convention we set $H(\vec{s}; n) = 0$ any $n < d$. We call $\ell(\vec{s}) := d$ and $\operatorname{wt}(\vec{s}) := \sum_{i=1}^{d} |s_i|$ its *depth* and *weight*, respectively. We point out that $\ell(\vec{s})$ is sometimes called length in the literature. When every s_i is positive we recover the multiple harmonic sums (MHS for short) whose congruence properties are studied in [5, 6, 11, 12]. There is another “non-strict” version of the AMHS defined as follows:

$$S(\vec{s}; n) := \sum_{1 \le k_1 \le k_2 \le \cdots \le k_d \le n} \prod_{i=1}^{d} \frac{\operatorname{sgn}(s_i)^{k_i}}{k_i^{|s_i|}}.$$

By Inclusion and Exclusion Principle it is easy to see that

$$S(\vec{s}; n) = \sum_{\vec{r} \preceq \vec{s}} H(\vec{r}; n), \tag{1}$$

$$H(\vec{s}; n) = \sum_{\vec{r} \preceq \vec{s}} (-1)^{\ell(\vec{s}) - \ell(\vec{r})} S(\vec{r}; n) \tag{2}$$

where $\vec{r} \prec \vec{s}$ means $\vec{r}$ can be obtained from $\vec{s}$ by combining some of its parts.

The feature of this paper is a systematic treatment of the congruence property of $H(\vec{s}; p-1)$ for primes $p > |\vec{s}| + 2$ by using intimate relations between Bernoulli polynomials, Bernoulli numbers, Euler polynomials, and Euler numbers.

We now sketch the outline of the paper. We start §2 by recalling some important relations among AMHS such as the stuffle and reversal relations. Then we present some basic properties of Euler polynomials which is one of the fundamental tools for us in the alternating setting. Then we describe two reduction procedures in §2.3 for general $\vec{s}$, which are used to derive congruences in the depth two and depth three cases in §3 and §4, respectively.

Theorem 1. *Let $a, \ell \in \mathbb{N}$, $\vec{s} = (s_1, \dots, s_\ell) \in (\mathbb{Z}^*)^\ell$ and $\vec{s}' = (s_2, \dots, s_\ell)$. Then for every prime $p \ge a+2$ we have the reduction formulae*

$$\begin{aligned} H(a, \vec{s}) \equiv & \frac{-1}{a} H((a-1) \oplus s_1, \vec{s}') - \frac{1}{2} H(a \oplus s_1, \vec{s}') \\ & + \sum_{k=2}^{p-1-a} \binom{p-a}{k} \frac{B_k}{p-a} H((k+a-1) \oplus s_1, \vec{s}'), \\ H(-a, \vec{s}) \equiv & \frac{(1-2^{p-a}) B_{p-a}}{p-a} \Big(H(\vec{s}) - H(-s_1, \vec{s}') \Big) \\ & - \sum_{k=0}^{p-2-a} \binom{p-1-a}{k} \frac{E_k(0)}{2} H\big((k+a) \oplus (-s_1), \vec{s}'\big), \end{aligned}$$

where $E_k(0) = 2(1-2^{k+1}) B_{k+1}/(k+1)$.

In §5 we deal with the homogeneous AMHS of arbitrary depth and provide an explicit formula using the relation between the power sum and elementary symmetric functions and the partition functions. The last section is devoted to a comprehensive study of the weight four AMHS in which identities involving Bernoulli numbers such as those proved in [11] play the leading roles. For example, by writing $H(-) = H(-; p-1)$ we find the following interesting relations (see Prop. 16, Prop. 17 and Prop. 18):

$$\begin{aligned} H(1,-3) \equiv \frac{1}{2} H(-2,2) \equiv & \sum_{k=0}^{p-3} 2^k B_k B_{p-3-k} && \pmod p, \\ H(-1,3; p-1) \equiv & -\frac{1}{2} q_p B_{p-3} && \pmod p, \\ H(1,-2,-1) \equiv & H(1,-3) - \frac{5}{4} q_p B_{p-3} && \pmod p, \\ H(-1,1,-1,1) \equiv H(1,-1,1,-1) \equiv & -\frac{1}{12} \Big(q_p B_{p-3} + 2 q_p^4 \Big) && \pmod p, \\ H(-1,1,1,-1) \equiv & \frac{1}{12} \Big(6H(1,-3) + 7 q_p B_{p-3} + 2 q_p^4 \Big) && \pmod p. \end{aligned}$$

for all prime $p \ge 7$, where $q_p = (2^{p-1}-1)/p$ is the Fermat's quotient. None of the above congruences can be obtained simply by the stuffle and reversal relations.

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2. Properties of AMHS

2.1. **Stuffle relation.** The most important relation between AMHS is the so called stuffle relation. It is possible to formalize this using words as in [13, §2] or [8, §2.2] which is a generalization of the MHS case (see [5, §2]). Unfortunately, for AMHS we don't have the integral representations which provide another product structure for the alternating multiple zeta values.

Fix a positive integer n. Let $\mathfrak{A}$ be the algebra generated by letters y_s for $s \in \mathbb{Z}^*$. Define a multiplication $*$ on $\mathfrak{A}$ by requiring that $*$ distribute over addition, that $\mathbf{1} * w = w * \mathbf{1} = w$ for the empty word $\mathbf{1}$ and any word w, and that, for any two words w_1, w_2 and two letters y_s, y_t $(s, t \in \mathbb{Z}^*)$

$$y_s w_1 * y_t w_2 = y_s(w_1 * y_t w_2) + y_t(y_s w_1 * w_2) + y_{s\oplus t}(w_1 * w_2) \tag{3}$$

where $s \oplus t = \operatorname{sgn}(st)(|s| + |t|)$. Then we get an algebra homomorphism

$$\begin{aligned} H: \quad (\mathfrak{A}, *) &\longrightarrow \{H(\vec{s}; n) : \vec{s} \in \mathbb{Z}^r, r \in \mathbb{N}\} \\ \mathbf{1} &\longmapsto 1 \\ y_{s_1} \dots y_{s_r} &\longmapsto H(s_1, \dots, s_r; n). \end{aligned}$$

For example,

$$\begin{aligned} H(-2; n)H(-3, 2; n) = H(-2, -3, 2; n) &+ H(-3, -2, 2; n) + H(-3, 2, -2; n) \\ &+ H(5, 2; n) + H(-3, 4; n). \end{aligned}$$

There is another kind of relation caused by the reversal of the arguments which we call the *reversal relations.* They have the form

$$\begin{aligned} H(s_1, \dots, s_r; p-1) &\equiv \operatorname{sgn}\Big(\prod_{j=1}^r s_j\Big)(-1)^r H(s_r, \dots, s_1; p-1) \pmod p, \\ S(s_1, \dots, s_r; p-1) &\equiv \operatorname{sgn}\Big(\prod_{j=1}^r s_j\Big)(-1)^r S(s_r, \dots, s_1; p-1) \pmod p, \end{aligned} \tag{4}$$

for any odd prime $p > \operatorname{wt}(\vec{s})$.

2.2. **Euler polynomials.** In the study of congruences of MHS [5, 11, 12] we have seen that Bernoulli numbers play the key roles by virtue of the following identity: ([1, p. 804, 23.1.4-7])

$$\sum_{j=1}^{n-1} j^d = \sum_{r=0}^{d} \binom{d+1}{r} \frac{B_r}{d+1} n^{d+1-r}, \quad \forall n, d \geq 1. \tag{5}$$

In the case of AMHS, however, the Euler polynomials and the Euler numbers are indispensable, too. Recall that the Euler polynomials $E_n(x)$ are defined by the generating function

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

Lemma 2. *Let $n \in \mathbb{Z}_{\geq 0}$. Then we have*

$$\sum_{i=1}^{d-1} (-1)^i i^n = \frac{1}{2}\Big((-1)^{d-1} E_n(d) + E_n(0)\Big) = \sum_{a=0}^{n} \binom{n}{a} F_{n,d,a} d^{n-a} \tag{6}$$

where

$$F_{n,d,a} = \begin{cases} (-1)^{d-1} E_a(0)/2, & \text{if } a < n; \\ (1 - (-1)^d) E_n(0)/2, & \text{if } a = n > 0; \\ -(1 + (-1)^d)/2, & \text{if } a = n = 0. \end{cases}$$

Moreover, $E_0(0) = 1$ and for all $a \in \mathbb{N}$

$$E_a(0) = \frac{2^{a+1}}{a+1}\Big(B_{a+1}\Big(\frac{1}{2}\Big) - B_{a+1}\Big) = \frac{2}{a+1}(1 - 2^{a+1}) B_{a+1} \tag{7}$$

Proof. Consider the generating function

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{i=1}^{d-1}(-1)^i i^n\right)\frac{t^n}{n!} =&\sum_{i=1}^{d-1}(-1)^i e^{ti}\\ =&\frac{(-e^t)^d-1}{-e^t-1}-1\\ =&\frac{(-1)^{d-1}e^{dt}+1}{e^t+1}-1\\ =&\frac{1}{2}\sum_{n=0}^{\infty}\Big((-1)^{d-1}E_n(d)+E_n(0)\Big)\frac{t^n}{n!}-1 \end{aligned}\tag{8}$$

Now (6) follows from the notorious equation (see for e.g., [1, p. 805, 23.1.7])

$$E_n(x)=\sum_{a=0}^{n}\binom{n}{a}E_a(0)x^{n-a}\tag{9}$$

for all $n>0$. Equation (7) is also well-known (see for e.g., item 23.1.20 on p. 805 of loc. cit.). □

Remark 3. The classical Euler numbers E_k is defined by

$$\frac{2}{e^t+e^{-t}}=\sum_{k=0}^{\infty}E_k\frac{t^k}{k!}.$$

They are related to $E_k(0)$ by the formula (see [1, p. 805, 23.1.7])

$$E_m(0)=\sum_{k=0}^{m}\binom{m}{k}\frac{E_k}{2^k}\left(-\frac{1}{2}\right)^{m-k}.$$

Corollary 4. *Let $a\in\mathbb{Z}_{\geq 0}$ and p be a prime such that $p\geq a+2$. Then*

$$H(-a;p-1)\equiv\begin{cases}-\dfrac{2(1-2^{p-a})}{a}B_{p-a} & \pmod p, \quad \text{if } a \text{ is odd;}\\ \dfrac{a(1-2^{p-1-a})}{a+1}pB_{p-1-a} & \pmod{p^2}, \quad \text{if } a \text{ is even.}\end{cases}\tag{10}$$

Proof. Taking $d=p$ and $n=p(p-1)-a$ in the Lemma we see that

$$\begin{aligned}H(-a;p-1)\equiv&F_{p(p-1)-a,p,p(p-1)}+p(p(p-1)-a)F_{p(p-1)-a,p,p(p-1)-1-a}\\ \equiv&E_{p(p-1)-a}(0)-\frac{1}{2}paE_{p(p-1)-1-a}(0) \pmod{p^2},\end{aligned}$$

since all the coefficients in (6) are p-integral by (7) and the property of Bernoulli numbers: B_m is not p-integral if and only if $p-1|m>0$. Then the corollary directly follows from (7) and Kummer congruences

$$\begin{aligned}\frac{B_{p(p-1)-a}}{p(p-1)-a}\equiv&\frac{B_{p-1-a}}{p-1-a} \pmod p,\\ \frac{B_{p(p-1)-a+1}}{p(p-1)-a+1}\equiv&\frac{B_{p-a}}{p-a} \pmod p.\end{aligned}$$

□

Remark 5. The corollary can be obtained also from [10, Thm. 2.1] combined with [9, Thm. 5.2]. Notice that when a is even both terms in [10, Thm. 2.1] contribute nontrivially since the modulus is p^2.

2.3. **Two reduction formulae.** We now prove two reduction formulae of $H(\vec{s})$ for arbitrary composition $\vec{s}$, corresponding to the two cases where $\vec{s}$ begins with a positive or a negative number.

Theorem 6. *Let $a, \ell \in \mathbb{N}$, $\vec{s} = (s_1, \dots, s_\ell) \in (\mathbb{Z}^*)^\ell$ and $\vec{s}' = (s_2, \dots, s_\ell)$. Then for any prime $p \geq a+2$*

$$H(a,\vec{s}) \equiv \frac{-1}{a} H((a-1)\oplus s_1, \vec{s}') - \frac{1}{2} H(a \oplus s_1, \vec{s}') + \sum_{k=2}^{p-1-a} \binom{p-a}{k} \frac{B_k}{p-a} H((k+a-1)\oplus s_1, \vec{s}'). \tag{11}$$

Proof. By definition

$$\begin{aligned} H(a,\vec{s}) &\equiv \sum_{j_\ell=1}^{p-1} \frac{\operatorname{sgn}(s_\ell)^{j_\ell}}{j_\ell^{|s_\ell|}} \sum_{j_{\ell-1}=1}^{j_\ell-1} \frac{\operatorname{sgn}(s_{\ell-1})^{j_{\ell-1}}}{j_{\ell-1}^{|s_{\ell-1}|}} \cdots \sum_{j_1=1}^{j_2-1} \frac{\operatorname{sgn}(s_1)^{j_1}}{j_1^{|s_1|}} \sum_{j=1}^{j_1-1} j^{p-1-a} \\ &\equiv \sum_{k=0}^{p-1-a} \binom{p-a}{k} \frac{B_k}{p-a} \sum_{j_\ell=1}^{p-1} \frac{\operatorname{sgn}(s_\ell)^{j_\ell}}{j_\ell^{|s_\ell|}} \sum_{j_{\ell-1}=1}^{j_\ell-1} \frac{\operatorname{sgn}(s_{\ell-1})^{j_{\ell-1}}}{j_{\ell-1}^{|s_{\ell-1}|}} \cdots \sum_{j_1=1}^{j_2-1} \frac{\operatorname{sgn}(s_1)^{j_1}}{j_1^{|s_1|}} j_1^{p-a-k} \end{aligned}$$

which is exactly the right hand side of (11). □

Theorem 7. *Let $a, \ell \in \mathbb{N}$, $\vec{s} = (s_1, \dots, s_\ell) \in (\mathbb{Z}^*)^\ell$ and $\vec{s}' = (s_2, \dots, s_\ell)$. Then for any prime $p \geq a+2$*

$$\begin{aligned} H(-a,\vec{s}) \equiv & \frac{(1-2^{p-a})B_{p-a}}{p-a} \Big(H(\vec{s}) - H(-s_1, \vec{s}') \Big) \\ & - \sum_{k=0}^{p-2-a} \binom{p-1-a}{k} \frac{(1-2^{k+1})B_{k+1}}{k+1} H\big((k+a)\oplus(-s_1), \vec{s}'\big). \end{aligned}$$

Proof. By definition and Lemma 2

$$\begin{aligned} H(a,\vec{s}) &\equiv \sum_{j_\ell=1}^{p-1} \frac{\operatorname{sgn}(s_\ell)^{j_\ell}}{j_\ell^{|s_\ell|}} \sum_{j_{\ell-1}=1}^{j_\ell-1} \frac{\operatorname{sgn}(s_{\ell-1})^{j_{\ell-1}}}{j_{\ell-1}^{|s_{\ell-1}|}} \cdots \sum_{j_1=1}^{j_2-1} \frac{\operatorname{sgn}(s_1)^{j_1}}{j_1^{|s_1|}} \sum_{j=1}^{j_1-1} (-1)^j j^{p-1-a} \\ &\equiv \sum_{k=0}^{p-2-a} \binom{p-1-a}{k} \frac{E_k(0)}{2} \sum_{j_\ell=1}^{p-1} \frac{\operatorname{sgn}(s_\ell)^{j_\ell}}{j_\ell^{|s_\ell|}} \sum_{j_{\ell-1}=1}^{j_\ell-1} \frac{\operatorname{sgn}(s_{\ell-1})^{j_{\ell-1}}}{j_{\ell-1}^{|s_{\ell-1}|}} \cdots \sum_{j_1=1}^{j_2-1} \frac{\operatorname{sgn}(s_1)^{j_1}}{j_1^{|s_1|+k+a}} (-1)^{j_1-1} \\ &\quad + \frac{E_{p-1-a}(0)}{2} \sum_{j_\ell=1}^{p-1} \frac{\operatorname{sgn}(s_\ell)^{j_\ell}}{j_\ell^{|s_\ell|}} \sum_{j_{\ell-1}=1}^{j_\ell-1} \frac{\operatorname{sgn}(s_{\ell-1})^{j_{\ell-1}}}{j_{\ell-1}^{|s_{\ell-1}|}} \cdots \sum_{j_1=1}^{j_2-1} \frac{\operatorname{sgn}(s_1)^{j_1}}{j_1^{|s_1|}} \Big(1-(-1)^{j_1}\Big) \\ &\equiv \frac{E_{p-1-a}(0)}{2} \Big(H(\vec{s}) - H(-s_1, \vec{s}') \Big) - \sum_{k=0}^{p-2-a} \binom{p-1-a}{k} \frac{E_k(0)}{2} H\big((k+a)\oplus(-s_1), \vec{s}'\big). \end{aligned}$$

The theorem follows from (7) easily. □

3. AMHS of depth two

In this section we will provide congruence formulae for all depth two AMHS. All but one case are given by very concise values involving Bernoulli numbers or Euler numbers (which are closely related by the identity (7)). Throughout the rest of the paper we often use the shorthand $H(-) = H(-; p-1)$.

Theorem 8. *Let $a, b \in \mathbb{N}$ and p be a prime such that $p \geq a+b+2$. If $a+b$ is odd then*

$$H(a,b) \equiv \frac{(-1)^b}{a+b}\binom{a+b}{a} B_{p-a-b} \pmod p, \tag{12}$$

$$H(-a,-b) \equiv (2^{p-a-b}-1)H(a,b) \pmod p, \tag{13}$$

$$H(-a,b) \equiv H(a,-b) \equiv \frac{1-2^{p-a-b}}{a+b} B_{p-a-b} \pmod p. \tag{14}$$

If $a+b$ is even then we have

$$H(a,b) \equiv 0 \pmod p, \tag{15}$$

$$H(-a,-b) \equiv \frac{(2^a-2)(2^b-2)}{2^{a+b-1}ab} B_{p-a}B_{p-b} \pmod p. \tag{16}$$

Proof. Congruences (12) and (15) follow from [11, Thm. 3.1] (see [5, Thm. 6.1] for a different proof). Congruences (13) and (16) are given by [10, Thm. 2.1] combined with [9, Thm. 5.2]. Let's consider the two cases $H(a,-b), H(-a,b)$ when $a+b$ is odd. By stuffle relation (see §2.1) it's easy to see that for $(\alpha,\beta) = (a,-b)$ or $(-a,b)$ we have

$$\begin{aligned} 0 \equiv H(\alpha)H(\beta) &= H(\alpha,\beta) + H(\beta,\alpha) + H(\alpha \oplus \beta) && \pmod p \\ &\equiv H(\alpha,\beta) + \operatorname{sgn}(\alpha\beta)(-1)^{a+b}H(\alpha,\beta) + H(\alpha\oplus\beta) && \pmod p \\ &\equiv 2H(\alpha,\beta) + H(\alpha\oplus\beta) && \pmod p. \end{aligned}$$

Thus

$$H(-a,b) \equiv H(a,-b) \equiv -\frac{1}{2}H(-a-b) \equiv \frac{1-2^{p-a-b}}{a+b}B_{p-a-b} \pmod p$$

from Cor. 4, as desired. □

Even though we don't have compact congruence formulae for $H(-a,b)$ and $H(a,-b)$ when $a+b$ is even we can prove two general statements using the two reduction procedures provided by Thm. 7 and Thm. 6.

Proposition 9. *Let $a, b \in \mathbb{N}$ and p a prime such that $p \geq a+b+2$. If $a+b$ is even then*

$$H(a,-b) \equiv -H(-b,a) \equiv \sum_{k=0}^{p-a-1} \binom{p-a}{k} \frac{2(1-2^{2p-a-b-k})B_k B_{2p-a-b-k}}{(p-a)(2p-a-b-k)} \pmod p. \tag{17}$$

Proof. By the theorem modulo p we have

$$H(a,-b) \equiv \sum_{k=0}^{p-a-1} \binom{p-a}{k} \frac{B_k}{p-a} \cdot H(-(a+b+k-1)).$$

To use Cor. 4 we need to break the sum into two parts, i.e., when $a+b+k < p$ and when $a+b+k \geq p$. In the first case we can replace k by $k+p-1$ and then to get the correct term in (17) we only need to use Fermat's Little Theorem $2^{p+1-a-b-k} \equiv 2^{2p-a-b-k} \pmod p$ and Kummer congruence $B_{p+1-a-b-k}/(p+1-a-b-k) \equiv B_{2p-a-b-k}/(2p-a-b-k) \pmod p$. This finishes the proof of the proposition. □

The second reduction procedure, Theorem 7, provides us another useful result on AMHS of even weight and depth two with two arguments having opposite signs.

Proposition 10. *Let $a,b\in\mathbb{N}$ and p a prime such that $p\ge a+b+2$. If $a+b$ is even then we have*

$$H(-a,b)\equiv\sum_{k=1}^{p-2-a-b}\binom{p-1-a}{k}\frac{2(1-2^{k+1})(1-2^{p-a-b-k})B_{k+1}B_{p-a-b-k}}{(k+1)(a+b+k)}$$
$$+\sum_{k=p-1-a-b}^{p-1-a}\binom{p-1-a}{k}\frac{2(1-2^{k+1})(1-2^{2p-1-a-b-k})B_{k+1}B_{2p-1-a-b-k}}{(k+1)(1+a+b+k)}\pmod p.$$

Proof. By Theorem 7 we have modulo p

$$H(-a,b)\equiv\sum_{k=0}^{p-1-a}\binom{p-1-a}{k}\frac{(1-2^{k+1})B_{k+1}}{k+1}H\big(-(k+a+b)\big)$$

since $H(b)\equiv 0$. The rest follows from Cor. 4 similar to the proof of Prop. 9. □

The two propositions above will be used in §6 to compute some AMHS congruences explicitly.

4. AMHS of depth three

All the congruences in this section are modulo a prime p. Recall that for MHS we have the following statement (see [11, Thm. 3.5] or [5, Thm. 6.2]).

Theorem 11. *Let p be a prime and $a,b,c\in\mathbb{N}$ such that $p>w:=a+b+c$ is odd. Then*

$$H(a,b,c;p-1)\equiv(-1)^c\left[\binom{w}{a}-\binom{w}{c}\right]\frac{B_{p-w}}{2w}\pmod p.$$

When w is even and $p\ge w+3$ [11, (3.13)] yields

$$\begin{aligned}H(a,b,c;p-1)\equiv&-\sum_{k=0}^{p+1-w}(-1)^c\binom{a+k}{k}\frac{B_k}{a+k}\binom{w+k-1}{c}\frac{B_{p+1-w-k}}{w+k-1}\\&-\sum_{k=p+1-a-b}^{p-1-a}(-1)^c\binom{a+k}{k}\frac{B_k}{a+k}\binom{w+k}{c}\frac{B_{2p-w-k}}{w+k}.\end{aligned}\tag{18}$$

For AMHS, we first observe that for any $\alpha,\beta,\gamma\in\mathbb{Z}$ we have

$$\begin{aligned}H(\alpha,\beta,\gamma)=&H(\alpha)H(\beta)H(\gamma)-H(\gamma)H(\beta,\alpha)-H(\gamma)H(\beta\oplus\alpha)\\&-H(\gamma,\beta)H(\alpha)-H(\gamma\oplus\beta)H(\alpha)+H(\gamma,\beta,\alpha)\\&+H(\gamma\oplus\beta,\alpha)+H(\gamma,\beta\oplus\alpha)+H(\gamma\oplus\beta\oplus\alpha).\end{aligned}$$

This can be easily checked by stuffle relations but the idea is hidden in the general framework set up by Hoffman [4]. Combining with the reversal relations we can obtained the following results without much difficulty. We leave the proof to the interested reader.

Theorem 12. *Let p be a prime and $a,b,c\in\mathbb{N}$ such that $p>w$ where $w:=a+b+c$.*

(1). If w is even then

$$2H(a,-b,c)\equiv H(-c-b,a)+H(c,-b-a),\tag{19}$$

$$2H(a,b,-c)\equiv-H(-c)H(b,a)+H(-c-b,a)+H(-c,b+a),\tag{20}$$

$$2H(-a,-b,-c)\equiv-H(-c)H(-b,-a)-H(-c,-b)H(-a)+H(c+b,-a)+H(-c,a+b).\tag{21}$$

(2). If w is odd then

$$2H(a,-b,-c)\equiv H(c+b,a)+H(-c,-b-a)-H(-c)H(-b,a), \tag{22}$$

$$2H(-a,b,-c)\equiv -H(-c)H(b,-a)-H(-c,b)H(-a)+H(-c-b,-a)+H(-c,-b-a). \tag{23}$$

Because of the reversal relations when the weight is even there remains essentially only one more case to consider in depth three. This is given by the next result which will be used in §6.

Theorem 13. *Let a,b,c be positive integers such that $w:=a+b+c$ is even. Then for any prime $p\ge w+3$ we have*

$$\begin{aligned}H(a,-b,-c)\equiv &-\sum_{k=2}^{p-w+1}\binom{p-a}{p-w-k+1}\binom{k+c-1}{c}\frac{(1-2^k)B_kB_{p-w-k+1}}{ak}\\ &-\sum_{k=p+1-b-c}^{p-c}\binom{p-a}{2p-w-k}\binom{k+c-1}{c}\frac{(1-2^k)B_kB_{2p-w-k}}{ak}\\ &-\frac{(1-2^{p-c})(1-2^{p-a-b})B_{p-a-b}B_{p-c}}{(a+b)c}.\end{aligned}$$

Remark 14. The condition $p\ge w+3$ can not be weakened since

$$H(1,-2,-3)\equiv \text{RHS}+5\not\equiv \text{RHS} \pmod 7.$$

Proof. The proof is essentially a repeated application of Thm. 6. But we spell out all the details below because there are some subtle details that we need to attend to.

By (5) and Fermat's Little Theorem we have modulo p

$$\begin{aligned}H(a,-b,-c)&\equiv\sum_{i=1}^{p-1}\frac{(-1)^i}{i^c}\sum_{j=1}^{i-1}(-1)^j j^{p-1-b}\sum_{k=1}^{j-1}k^{p-1-a}\\ &\equiv\sum_{i=1}^{p-1}\frac{(-1)^i}{i^c}\sum_{k=0}^{p-1-a}\binom{p-a}{k}\frac{B_k}{p-a}\sum_{j=1}^{i-1}(-1)^j j^{\rho(k)+p-a-b-k},\end{aligned}$$

where $\rho(k)=0$ if $k<p-a-b$ and $\rho(k)=p-1$ if $k\ge p-a-b$ (to make sure all exponents are positive in the sum of the second line above). By Lemma 2

$$\begin{aligned}H(a,-b,-c)&\equiv\sum_{k=0}^{p-1-a}\binom{p-a}{k}\frac{B_k}{p-a}\sum_{r=0}^{n}\binom{n}{r}\sum_{i=1}^{p-1}(-1)^i i^{n-r-c}F_{n,i,r}\\ &\equiv-\sum_{k=0}^{p-1-a}\binom{p-a}{k}\sum_{r=1}^{n-1}\binom{n}{r}\frac{(1-2^{r+1})B_kB_{r+1}}{(p-a)(r+1)}\sum_{i=1}^{p-1}i^{n-c-r}\\ &\quad+\sum_{k=0}^{p-1-a}\binom{p-a}{k}\frac{(1-2^{n+1})B_kB_{n+1}}{(p-a)(n+1)}\sum_{i=1}^{p-1}((-1)^i-1)i^{-c},\end{aligned}$$

where $n=\rho(k)+p-a-b-k$. Here we have used the fact that when $r=0$ we have $F_{n,i,r}=(-1)^{i-1}$ and thus the inner sum is $\sum_{i=1}^{p-1}i^{n-c}\equiv 0 \pmod p$ except when $p-1|n-c$, i.e., except when $n = c$ and $k = p - w$. But then $B_k = 0$ since w is even by assumption. Thus

$$\begin{aligned} H(a,-b,-c) \equiv & \sum_{\substack{0\le k<p-a\\ c<n}} \binom{p-a}{k}\binom{n}{n-c}\frac{(1-2^{n-c+1})B_kB_{n-c+1}}{(p-a)(n-c+1)} \\ & + H(-c)\sum_{k=0}^{p-1-a}\binom{p-a}{k}\frac{(1-2^{n+1})B_kB_{n+1}}{(p-a)(n+1)}. \end{aligned}$$

Now if c is even then $H(-c) \equiv 0 \pmod p$. So we may assume c is odd in the last line above. Then $k+n+1$ is always odd so that $B_kB_{n+1} \ne 0$ if and only if $k = 1$ and $n = p-a-b-1$. Hence

$$\begin{aligned} H(a,-b,-c) \equiv & \sum_{k=0}^{p-w-1}\binom{p-a}{k}\binom{p-a-b-k}{p-w-k}\frac{(1-2^{p-w-k+1})B_kB_{p-w-k+1}}{(p-a)(p-w-k+1)} \\ & + \sum_{k=p-a-b}^{p-1-a}\binom{p-a}{k}\binom{2p-1-a-b-k}{2p-1-w-k}\frac{(1-2^{2p-w-k})B_kB_{2p-w-k}}{(p-a)(2p-w-k)} \\ & - \frac{(1-2^{p-c})(1-2^{p-a-b})B_{p-a-b}B_{p-c}}{(a+b)c}. \end{aligned}$$

After substitutions $k \to p-w+1-k$ in the first sum and $k \to 2p-w-k$ in the second sum the theorem follows immediately from Cor. 4, □

5. AMHS of arbitrary depth

In this section we provide some general results on AMHS without restrictions on the depth. We first consider the homogeneous case for which the key idea comes from [11, Lemma 2.12] and [5, Thm. 2.3].

Let $p_i = \sum_{j\ge1} x_j^i$ be the power-sum symmetric functions and $e_i = \sum_{j_1<\cdots<j_i} x_{j_i}\cdots x_{j_i}$ be the elementary symmetric functions of degree i. Let $P(\ell)$ be the set of unordered partitions of ℓ. For $\lambda = (\lambda_1,\dots,\lambda_r) \in P(\ell)$ we set $p_\lambda = \prod_{i=1}^r p_{\lambda_i}$. Recall that the expression of e_l in terms of p_i is given by the following formula (see [7, p.28]):

$$\ell! e_\ell = \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ & & & \ddots & 0 \\ p_{\ell-1} & p_{\ell-2} & p_{\ell-3} & \cdots & \ell-1 \\ p_\ell & p_{\ell-1} & p_{\ell-2} & \cdots & p_1 \end{vmatrix} = \sum_{\lambda\in P(\ell)} c_\lambda p_\lambda. \tag{24}$$

Denote by $O(\ell) \subset P(\ell)$ the subset of odd partitions $\lambda = (\lambda_1,\dots,\lambda_r)$ (i.e., λ_i is odd for every part).

Lemma 15. *Let $a, \ell \in \mathbb{N}$ and p a prime such that $a\ell < p-1$. Set $H(-) = H(-;p-1)$. For an odd partition $\lambda = (\lambda_1,\dots,\lambda_r) \in O(\ell)$ we put $H_\lambda(-a) = \prod_{i=1}^r H(-\lambda_i a)$. Then*

$$\ell! H\big(\{-a\}^\ell\big) \equiv \sum_{\lambda\in O(\ell)} c_\lambda H_\lambda(-a) \pmod p \tag{25}$$

where c_λ are given by (24). In particular, if a is even then (25) $\equiv 0 \pmod p$. If $a > 1$ is odd then (25) is congruent to a $\mathbb{Q}$-linear combination of $B_{p-\lambda_1 a}\cdots B_{p-\lambda_r a}$ for odd partitions $\lambda = (\lambda_1,\dots,\lambda_r)$. If $a = 1$ then (25) is congruent to a $\mathbb{Q}$-linear combination of $q_p^s B_{p-\lambda_{s+1}}\cdots B_{p-\lambda_r}$

for odd partitions $\lambda = (\{1\}^s, \lambda_{s+1}, \dots, \lambda_r)$ *($\lambda_i > 1$ for all $i > s$), where* $q_p = (2^{p-1} - 1)/p$ *is the Fermat's quotient.*

Proof. (25) follows from [11, Lemma 2.12] and [5, Thm. 2.3]. The last part follows from [10, Thm. 2.1] and [9, Thm. 5.2] □

For example, by [10, Thm. 2.1] and [9, Thm. 5.2(c)] we have

$$H(-1) \equiv H(1;(p-1)/2) \equiv -2q_p. \tag{26}$$

we have $O(2) = \{(1,1)\}, O(3) = \{(1,1,1),(3)\}, O(4) = \{(1,1,1,1),(1,3)\}$. It is obvious that $c_{(1,\cdots,1)} = 1$, $c_{(3)} = 2$, $c_{(1,3)} = 8$, $c_{(1,1,3)} = 20$, and $c_{(1,1,1,3)} = c_{(3,3)} = 40$, which implies that

$$2H(\{-1\}^2) \equiv 4q_p^2, \tag{27}$$

$$6H(\{-1\}^3) \equiv -8q_p^3 + 2H(-3), \qquad \text{so } H(\{-1\}^3) \equiv -\frac{4}{3}q_p^3 - \frac{1}{6}B_{p-3} \tag{28}$$

$$24H(\{-1\}^4) \equiv 16q_p^4 + 8H(-1)H(-3), \quad \text{so } H(\{-1\}^4) \equiv \frac{2}{3}q_p^4 + \frac{1}{3}q_pB_{p-3}, \tag{29}$$

$$5!H(\{-1\}^5) \equiv -32q_p^5 + 20H(-1)^2H(-3), \quad \text{so } H(\{-1\}^5) \equiv -\frac{4}{15}q_p^5 - \frac{1}{3}q_p^2B_{p-3} \tag{30}$$

and

$$\begin{aligned} 6!H(\{-1\}^6) &\equiv 64q_p^6 + 40H(-1)^3H(-3) + 40H(-3)^2, \\ H(\{-1\}^6) &\equiv \frac{4}{45}q_p^6 + \frac{2}{9}q_p^3B_{p-3} + \frac{1}{72}B_{p-3}^2. \end{aligned} \tag{31}$$

6. AMHS of weight four

In [10] the first author studied the congruence properties of AMHS of weight less than four. In this section, applying the results obtained in the previous sections we can analyze the weight four AMHS in some detail. First we treat some special congruences which can not be obtained by just using the stuffle relations and the reversal relations. Let $p \geq 7$ be a prime and set $A = A_p, \cdots, K = K_p$ as follows:

$$A := \sum_{k=2}^{p-3} B_kB_{p-3-k}, \quad B := \sum_{k=2}^{p-3} 2^kB_kB_{p-3-k}, \quad C := \sum_{k=2}^{p-3} 2^{p-3-k}B_kB_{p-3-k},$$

$$D := \sum_{k=2}^{p-3} \frac{B_kB_{p-3-k}}{k}, \quad E := \sum_{k=2}^{p-3} \frac{2^kB_kB_{p-3-k}}{k}, \quad F := \sum_{k=2}^{p-3} \frac{2^{p-3-k}B_kB_{p-3-k}}{k},$$

$$G := \sum_{k=2}^{p-3} kB_kB_{p-3-k}, \quad J := \sum_{k=2}^{p-3} 2^kkB_kB_{p-3-k}, \quad K := \sum_{k=2}^{p-3} 2^{p-3-k}kB_kB_{p-3-k}.$$

Then by [11, Cor. 3.6] and simple computation

$$A \equiv -B_{p-3}, \quad G \equiv 0, \quad C \equiv B - \frac{3}{4}A, \quad K \equiv -3B - J + 3A \pmod p. \tag{32}$$

Proposition 16. *Let $p \geq 7$ be a prime. Then we have the following depth two congruences*

$$H(1,-3) \equiv \frac{1}{2}H(-2,2) \equiv B - A \equiv \sum_{k=0}^{p-3} 2^kB_kB_{p-3-k} \pmod p, \tag{33}$$

$$H(-1,3) \equiv -\frac{1}{2}q_pB_{p-3} \pmod p. \tag{34}$$

Proof. We take congruence modulo p throughout this proof. By Prop. 10 we have

$$\begin{aligned} H(-3,1) \equiv & -\frac{1}{3}\sum_{k=1}^{p-6}(k+2)(k+3)(1-2^{k+1})(1-2^{p-4-k})\frac{B_{k+1}B_{p-4-k}}{k+4} - \frac{1}{2}q_pB_{p-3} \\ \equiv & \frac{1}{3}\sum_{k=2}^{p-3}(k+1)(k+2)(1-2^{k})(1-2^{p-3-k})\frac{B_kB_{p-3-k}}{k} - \frac{1}{2}q_pB_{p-3}. \end{aligned}$$

by the substitution $k \to p-4-k$. Similarly we can get

$$\begin{aligned} H(-2,2) \equiv & -\sum_{k=2}^{p-3}(k+2)(1-2^{k})(1-2^{p-3-k})\frac{B_kB_{p-3-k}}{k} + \frac{3}{2}q_pB_{p-3} \\ H(-1,3) \equiv & 2\sum_{k=2}^{p-3}(1-2^{k})(1-2^{p-3-k})\frac{B_kB_{p-3-k}}{k} - 2q_pB_{p-3} \end{aligned}$$

Using (32) we reduce the above to

$$3H(-3,1) \equiv 3A-3B+\frac{5}{2}D-2E-2F-\frac{3}{2}q_pB_{p-3} \tag{35}$$

$$H(-2,2) \equiv 2B-2A-\frac{5}{2}D+2E+2F+\frac{3}{2}q_pB_{p-3} \tag{36}$$

$$H(-1,3) \equiv \qquad +\frac{5}{2}D-2E-2F-2q_pB_{p-3} \tag{37}$$

On the other hand, by Prop. 9 we get

$$\begin{aligned} H(1,-3) \equiv & \frac{2}{p-1}\sum_{k=0}^{p-2}\binom{p-1}{k}\frac{1-2^{2p-4-k}}{2p-4-k}B_kB_{2p-4-k} \\ \equiv & -2\sum_{k=0}^{p-5}\frac{1-2^{p-3-k}}{p-3-k}B_kB_{p-3-k} + 2q_pB_{p-3} \\ \equiv & -2\sum_{k=2}^{p-3}\frac{1-2^{k}}{k}B_kB_{p-3-k} + 2q_pB_{p-3} \\ \equiv & 2E-2D+2q_pB_{p-3}. \end{aligned}$$

by the substitution $b \to p-3-b$. Thus by the reversal relation

$$H(-3,1) \equiv -H(1,-3) \equiv 2D-2E-2q_pB_{p-3}. \tag{38}$$

Similarly we find

$$H(-2,2) \equiv -H(2,-2) \equiv B-A+2E-2D+2q_pB_{p-3}, \tag{39}$$

$$H(-1,3) \equiv -H(3,-1) \equiv \frac{1}{3}\Big(-J+3A-3B+2D-2E\Big). \tag{40}$$

Then by adding (35), (36), (38) and (39) altogether we have

$$4H(-3,1)+2H(-2,2) \equiv 0$$

which implies the first congruence in (33). Now adding (38) and (39) yields

$$-H(-3,1) \equiv H(-3,1)+H(-2,2) \equiv B-A \tag{41}$$

which is the second congruence in (33). Plugging this into (35) we see that

$$\frac{5}{2}D-2E-2F-\frac{3}{2}q_pB_{p-3} \equiv 0$$

which combined with (37) produces (34). This finishes the proof of the proposition. □

Proposition 17. *Let $p \geq 7$ be a prime. Then we have the following depth three congruences*

$$H(1,-1,-2) \equiv \frac{1}{2}H(1,-3) + \frac{1}{2}J \pmod p, \tag{42}$$

$$H(1,-2,-1) \equiv H(1,-3) - \frac{5}{4}q_pB_{p-3} \pmod p, \tag{43}$$

$$H(2,-1,-1) \equiv -H(1,-3) - \frac{1}{2}J + \frac{3}{4}q_pB_{p-3} \pmod p. \tag{44}$$

Proof. By Thm. 13, Prop. 16 and (32) we get

$$2H(1,-1,-2) \equiv -\sum_{k=2}^{p-3}(k+1)(1-2^k)B_kB_{p-3-k} \equiv B - A + J - G = H(1,-3) + J \tag{45}$$

$$H(1,-2,-1) \equiv -\sum_{k=2}^{p-3}(1-2^k)B_kB_{p-3-k} - \frac{5}{4}q_pB_{p-3} \equiv H(1,-3) - \frac{5}{4}q_pB_{p-3}$$

$$H(2,-1,-1) \equiv \sum_{k=2}^{p-3}\frac{(k+2)(1-2^k)B_kB_{p-3-k}}{-2} + \frac{3}{4}q_pB_{p-3} \equiv -H(1,-3) - \frac{1}{2}J + \frac{3}{4}q_pB_{p-3},$$

as claimed. □

By (20) it is readily seen that

$$2H(1,1,-2) \equiv H(-3,1) + H(-2,2) \equiv -H(-3,1)$$

from (41). In fact, by stuffle and reversal relations we can now find congruences for all weight four AMHS of depth up to three. By reversal relations we only need to list about half of the values. Write $H_{211} := H(-2,1,1)$. Then

$$H(4) \equiv H(-4) \equiv H(2,2) \equiv H(-2,-2) \equiv H(1,3) \equiv H(1,-2,1) \equiv H(-1,-2,-1) \equiv 0,$$

$$\begin{aligned}
&H(1,-3) \equiv -2H_{211}, \quad H(2,-2) \equiv 4H_{211}, && H(1,-1,2) \equiv 3H_{211},\\
&H(-1,-3) \equiv \tfrac{1}{2}q_pB_{p-3}, && H(3,-1) \equiv \tfrac{1}{2}q_pB_{p-3},\\
&H(1,-1,-2) \equiv -H_{211} + \tfrac{1}{2}J, && H(-2,-1,-1) \equiv 2H_{211} - q_pB_{p-3},\\
&H(-1,2,1) \equiv -H_{211} + \tfrac{5}{4}q_pB_{p-3}, && H(-1,1,2) \equiv 2H_{211} - \tfrac{3}{4}q_pB_{p-3},\\
&H(-2,1,-1) \equiv 3H_{211} - \tfrac{1}{2}J + \tfrac{3}{4}q_pB_{p-3}, && H(1,-2,-1) \equiv -2H_{211} - \tfrac{5}{4}q_pB_{p-3},\\
&H(-1,2,-1) \equiv -4H_{211} + J - \tfrac{5}{2}q_pB_{p-3}, && H(2,-1,-1) \equiv 2H_{211} - \tfrac{1}{2}J + \tfrac{3}{4}q_pB_{p-3}.
\end{aligned}$$

We now turn to the depth four cases.

Proposition 18. *Let $p \geq 7$ be a prime. Then we have the following depth four congruences*

$$\begin{aligned}
H(1,-1,-1,1) &\equiv -\tfrac{1}{2}\Big(H(1,-3) + J + q_p^4\Big) \pmod p,\\
H(-1,-1,1,1) \equiv H(1,1,-1,-1) &\equiv \tfrac{1}{24}\Big(6J + 7q_pB_{p-3} + 8q_p^4\Big) \pmod p,\\
H(-1,1,-1,1) \equiv H(1,-1,1,-1) &\equiv -\tfrac{1}{12}\Big(q_pB_{p-3} + 2q_p^4\Big) \pmod p,\\
H(-1,1,1,-1) &\equiv \tfrac{1}{12}\Big(6H(1,-3) + 7q_pB_{p-3} + 2q_p^4\Big) \pmod p.
\end{aligned}$$

Proof. By Thm. 6

$$H(1,-1,-1,1) \equiv -H(-1,-1,1) - \frac{1}{2}H(-2,-1,1) - \sum_{k=2}^{p-3} B_k H(-(k+1),-1,1),$$
$$H(1,1,-1,-1) \equiv -H(1,-1,-1) - \frac{1}{2}H(2,-1,-1) - \sum_{k=2}^{p-3} B_k H(k+1,-1,-1).$$

Using reversal relations and (22) we see that

$$H(1,-1,-1,1) \equiv -H(-1,-1,1) - \frac{1}{2}H(-2,-1,1) + \frac{1}{2}\sum_{k=2}^{p-3} B_k\Big(H(k+2,1) + H(-(k+1),-2) - H(-(k+1))H(-1,1)\Big), \tag{46}$$

$$H(1,1,-1,-1) \equiv -H(1,-1,-1) - \frac{1}{2}H(2,-1,-1) - \frac{1}{2}\sum_{k=2}^{p-3} B_k\Big(H(2,k+1) + H(-1,-(k+2)) - H(-1)H(-1,k+1)\Big). \tag{47}$$

Note that by Thm. 6

$$-H(-1,1) \equiv H(1,-1) \equiv -\sum_{k=0}^{p-3}(-1)^k B_k H(-(k+1)),$$
$$H(1,1,-1) \equiv -H(1,-1) - \frac{1}{2}H(2,-1) + \sum_{k=2}^{p-3} B_k H(-1,k+1).$$

We may use (12), and (13) to simplify (46) and (47) further. For all $k = 2, \dots, p-5$ we have

$$H(k+2,1) \equiv -B_{p-3-k}, \qquad H(-(k+1),-2) \equiv \frac{1}{2}(2^{p-3-k}-1)(k+2)B_{p-3-k}, \tag{48}$$

$$H(2,k+1) \equiv -\frac{(k+2)B_{p-3-k}}{2}, \qquad H(-1,-(k+2)) \equiv (2^{p-3-k}-1)B_{p-3-k}. \tag{49}$$

However, one has to be very careful in applying these formulae because the formulae might fail when $k = p-3$. We need to compute these separately as follows:

$$H(p-1,1) = \sum_{i=1}^{p-1} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j^{p-1}} \equiv \sum_{i=1}^{p-1} \frac{i-1}{i} \equiv -1 \equiv -B_0,$$

$$H(-(p-2),-2) = \sum_{i=1}^{p-1} \frac{(-1)^i}{i^2} \sum_{j=1}^{i-1} \frac{(-1)^j}{j^{p-2}} \equiv \sum_{i=1}^{p-1} \frac{(-1)^i}{i^2} \left((-1)^i \frac{1-2i}{4} - \frac{1}{4} \right) \equiv 0,$$

$$H(2,p-2) \equiv H(p-2,2) = -\sum_{i=1}^{p-1} \frac{1}{i^2} \sum_{j=1}^{i-1} \frac{1}{j^{p-2}} \equiv -\sum_{i=1}^{p-1} \frac{i-1}{2i} \equiv \frac{1}{2} \equiv 0,$$

$$H(-1,-(p-1)) \equiv -H(-(p-1),-1) = -\sum_{i=1}^{p-1} \frac{(-1)^i}{i} \sum_{j=1}^{i-1} \frac{(-1)^j}{j^{p-1}} \equiv \sum_{i=1}^{p-1} \frac{1+(-1)^i}{2i} \equiv \frac{1}{2} H(-1) \equiv -q_p$$

by (26). We see that only $H(-1,-(p-1))$ fails the formula in (49) and therefore we get

$$H(1,-1,-1,1) \equiv -H(-1,-1,1) - \frac{1}{2}H(-2,-1,1) + \frac{1}{2}H(-1)H(-1,1) - \frac{1}{2}H(-1,1)^2 + \frac{1}{2}\sum_{k=2}^{p-3} B_k \left(-B_{p-3-k} + \frac{1}{2}(2^{p-3-k}-1)(k+2)B_{p-3-k} \right).$$

and

$$H(1,1,-1,-1) \equiv H(-1,-1,1) - \frac{1}{2}H(2,-1,-1) + \frac{1}{2}H(-1)\Big(H(1,1,-1) + H(1,-1) + \frac{1}{2}H(2,-1)\Big) - \frac{1}{2}\sum_{k=2}^{p-3} B_k \left(-\frac{(k+2)B_{p-3-k}}{2} + (2^{p-3-k}-1)B_{p-3-k} \right) + \frac{1}{2} q_p B_{p-3}.$$

Now by [10, Cor. 2.4, Cor. 2.5] we know $H(-1,1) \equiv -q_p^2$, $H(-1,-1,1) \equiv q_p^3 + \frac{7}{8}B_{p-3}$, and $H(1,1,-1) \equiv -\frac{1}{3}q_p^3 - \frac{7}{24}B_{p-3}$. Together with (26) these yield

$$H(1,-1,-1,1) \equiv -\frac{1}{2}B_{p-3} - \frac{1}{2}H(-2,-1,1) - \frac{1}{2}q_p^4 - \frac{1}{2}A + \frac{1}{4}\sum_{k=0}^{p-3}(2^{p-3-k}-1)(k+2)B_k B_{p-3-k},$$

$$H(1,1,-1,-1) \equiv B_{p-3} - \frac{1}{2}H(2,-1,-1) + \frac{1}{3}q_p^4 + \frac{2}{3}q_p B_{p-3} + A + \frac{1}{4}G - \frac{1}{2}\sum_{k=0}^{p-3} 2^{p-3-k} B_k B_{p-3-k}.$$

It now follows from (32) and the substitution $k \to p-3-k$ that

$$H(1,-1,-1,1) \equiv -\frac{1}{2}H(-2,-1,1) - \frac{1}{2}q_p^4 + \frac{1}{4}\sum_{k=0}^{p-3}(1-2^k)(k+1)B_k B_{p-3-k},$$

$$H(1,1,-1,-1) \equiv -\frac{1}{2}H(2,-1,-1) - \frac{1}{2}H(1,-3) + \frac{2}{3}q_p B_{p-3-k} + \frac{1}{3}q_p^4.$$

Hence (18) and (18) quickly follow from (45) and (44).

Finally, (18) follows from stuffle relations applied to $H(-1)H(1,-1,1)$ and then (18) from stuffle relations applied to $H(-1)H(1,1,-1)$. This finishes the proof of the proposition. □

For other depth four and weight four AMHS we have the following relations derived from the stuffle relations and the congruences obtained above:

$$\begin{aligned} H(1,1,-1,1) \equiv& 2H_{211} + 3H(-1,1,1,1) + \frac{1}{2}q_p B_{p-3}, \\ H(-1,-1,1,-1) \equiv& 6H_{211} + 3H(1,-1,-1,-1) - 4q_p B_{p-3} - 2q_p^4, \\ H(-1,-1,-1,-1) \equiv& \qquad \frac{1}{3}q_p B_{p-3} + \frac{2}{3}q_p^4. \end{aligned}$$

On the other hand, we can only deduce from Thm. 6 and Thm. 7 that

$$\begin{aligned} H(1,1,1,-1) \equiv& -H(1,1,-1) - \frac{1}{2}H(2,1,-1) - \sum_{k=2}^{p-3} B_k H(k+1,1,-1), \\ H(-1,-1,-1,1) \equiv& -q\Big(H(-1,-1,1) - H(1,-1,1)\Big) - \frac{1}{2}H(2,-1,1) \\ &+ \sum_{k=2}^{p-3}(1-2^k)B_k H(k+1,-1,1), \end{aligned}$$

where, by the reduction theorems again,

$$\begin{aligned} H(k+1,1,-1) \equiv& \frac{-1}{k+1}H(k+1,-1) - \frac{1}{2}H(k+2,-1) \\ &+ \sum_{j=2}^{p-k-2} \binom{p-k-1}{j} \frac{B_j}{p-k-1} H(j+k+1,-1), \\ H(k+1,-1,1) \equiv& \frac{-1}{k+1}H(-(k+1),1) - \frac{1}{2}H(-(k+2),1) \\ &+ \sum_{j=2}^{p-k-2} \binom{p-k-1}{j} \frac{B_j}{p-k-1} H(-(j+k+1),1). \end{aligned}$$

Observe that the indices k and j in the above sums can be both taken to be even numbers. Thus by Prop. 9 and Prop. 10

$$\begin{aligned}
H(j+k+1,-1) &\equiv \sum_{i=0}^{p-j-k-2} \binom{p-j-k-1}{i} \frac{2(1-2^{2p-i-j-k-2})B_i B_{2p-i-j-k-2}}{(p-j-k-1)(2p-i-j-k-2)}, \\
&\equiv \sum_{i=0}^{p-j-k-2} \binom{p-j-k-1}{i} \frac{2(1-2^{p-i-j-k-1})B_i B_{p-i-j-k-1}}{(j+k+1)(i+j+k+1)}, \\
H(-(j+k+1),1) &\equiv \sum_{i=1}^{p-j-k-4} \binom{p-2-j-k}{i} \frac{2(1-2^{i+1})(1-2^{p-i-j-k-2})B_{i+1} B_{p-i-j-k-2}}{(i+1)(i+j+k+2)} \\
&\quad + \frac{2(1-2^{p-j-k-1})(1-2^{p-1})B_{p-j-k-1}B_{p-1}}{p-j-k-1} \\
&\equiv \sum_{i=2}^{p-j-k-3} \binom{p-2-j-k}{i} \frac{2(1-2^{i})(1-2^{p-i-j-k-1})B_{i} B_{p-i-j-k-1}}{i(i+j+k+1)} \\
&\quad - q_p \frac{2(1-2^{p-j-k-1})B_{p-j-k-1}}{j+k+1}.
\end{aligned}$$

Consequently, both $H(1,1,1,-1)$ and $H(-1,-1,-1,1)$ can be written as a triple sum with most of the terms given involving products $B_iB_jB_kB_{p-i-j-k-2}$. It is very likely that modulo p we cannot reduce $H(1,1,1,-1)$ and $H(-1,-1,-1,1)$ to a linear combination of AMHS of depths up to three. At least in theory one possible way to check this hypothesis is to find six infinite sets of primes $S_1 = \{p_1^{(k)} : k \geq 1\}, \dots, S_6 = \{p_6^{(k)} : k \geq 1\}$ for each of the following six elements:

$$\begin{aligned}
b_1(p) &= J_p, \quad b_2(p) = H(1,-3), \quad b_3(p) = H(1,-1,-1,-1), \\
b_4(p) &= q_p^4, \quad b_5(p) = q_p B_{p-3}, \quad b_6(p) = H(-1,1,1,1),
\end{aligned}$$

such that for each choice $(p_1^{(k)}, \dots, p_6^{(k)})$ we always have $b_j(p_i^{(k)}) \equiv 0 \pmod{p_i^{(k)}}$ for all $i \neq j$ and $b_j(p_j^{(k)}) \not\equiv 0 \pmod{p_j^{(k)}}$ for all $j = 1, \dots, 6$. In practice this is extremely difficult to carry out. For example, if $b_4(p) \equiv 0 \pmod p$ then the prime p is called a Wieferich prime. The only known Wieferich primes are 1093 and 3511 and if any other Wieferich primes exist, they must be greater than 6.7×10^{15} according to [2]. It turns out that

$$\begin{aligned}
[J_p, H(1,-3), H(1,-1,-1,-1), H(-1,1,1,1)] &\equiv [1023, 529, 670, 952] \pmod{1093}, \\
[J_p, H(1,-3), H(1,-1,-1,-1), H(-1,1,1,1)] &\equiv [1618, 2160, 1620, 540] \pmod{3511}.
\end{aligned}$$

In order to understand the general mod p structure of AMHS we need to consider some infinite algebras similar to the adeles (see [14]).

References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, (1972) Dover, New York.

[2] F. Dorais aAnd D. Klyve, Near Wieferich primes up to 6.7×10^{15}, preprint.

[3] A. Gardiner, Four problems on prime power divisibility, *Amer. Math. Monthly* **95** (1988), 926–931.

[4] M.E. Hoffman, The Hopf algebra structure of multiple harmonic sums, *Nuclear Phys. B (Proc. Suppl.)* **135** (2004), 214-219; arXiv: QA/0406589.

[5] M.E. Hoffman, Quasi-symmetric functions and mod p multiple harmonic sums, arXiv: NT/0401319

[6] M.E. Hoffman, Algebraic aspects of multiple zeta values, in: Zeta Functions, Topology and Quantum Physics, Developments in Mathematics 14, T. Aoki et. al. (eds.), Springer, New York, 2005, pp. 51–74; preprint QA/0309425.

[7] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Oxford University Press, New York, 1995.

[8] G. Racinet, Doubles mélanges des polylogarithmes multiples aux racines de l'unité, *Publ. Math. IHES* **95** (2002), 185–231.

[9] Z.-H. Sun, Congruences concerning Bernoulli numbers and Bernoulli polynomials. *Discrete Appl. Math.* **105** (1-3)(2000), 193–223.

[10] R. Tauraso, Congruences involving alternating multiple harmonic sum, arXiv:0906.5150.

[11] J. Zhao, Wolstenholme Type Theorem for multiple harmonic sums, *Intl. J. of Num. Thy.* **4**(1) (2008) 73–106.

[12] J. Zhao, Multiple harmonic sums II: finiteness of p-divisible sets, math.NT/0303043

[13] J. Zhao, On a conjecture of Borwein, Bradley and Broadhurst, in press: *J. reine angew. Math.*, arxiv: 0705.2267.

[14] J. Zhao, Mod p structure of alternating and non-alternating multiple harmonic sums, in preparation.