Coxeter groups and interpolation of operators

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ABSTRACT

Let V be a finite dimensional real Euclidean space and let G be a finite irreducible group generated by orthogonal reflections across hyperplanes in V. We study interpolation of operators in G-invariant norms on V. A collection of G-invariant norms is called G-sufficient if any G-invariant norm is a strict interpolation norm for this collection. Using the general theory of sufficient collections we calculate explicitly two remarkable minimal sufficient collections and study their extremal properties.

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To our teacher, Professor Selim Krein,
on his 75^{th} birthday

Introduction.

Here we present a detailed exposition of our results on interpolation of operators in finite dimensional spaces with norms invariant under the action of a Coxeter group.

The first result of this sort was a finite dimensional version of the well-known theorem due to B. Mityagin [5] and A.P. Calderon [3], asserting that every B_n-invariant norm on $\mathbb{R}^n$ is a strict interpolation norm between the l_1^n - and l_∞^n - norms (B_n is the group generated by all permutations and all changes of signs of canonical coordinates in $\mathbb{R}^n$).

It follows from our results that, say, l_∞^n - norm is **not** a strict interpolation norm for any finite collection of B_n- invariant norms, all different from the l_∞^n - norm. So, the above two norms are , in this sense, extremal B_n- invariant norms . What is the reason for such an extremality ? What are the analogs of these norms if we consider other groups (or semigroups) ? These questions were studied for general groups [13,14] and, further on, for general semigroups [10,11], a full exposition of the general theory is contained in [12].

The case of Coxeter groups is especially interesting because it turned out that it is possible to give final answers to almost all natural questions.

The first results were obtained in [8,9], but at that time we had no general theory and the results were very far from being final. It is interesting that a bit earlier M.L. Eaton and M.D. Perlman in their investigation of analogs of Schur majorization motivated by problems of statistics [4], came to a necessity to study geometry of convex hulls of orbits of vectors under the action of Coxeter groups. This was a crucial point of our research and there are some intersections in their and our results in this theme. A new approach was proposed in [13,14] and it gave a possibility to understand the problem deeper. An intense research was undertaken in 1979-1989 and we have obtained final results, which were partially announced in [11], the proofs were very complicated and depended heavily on the classification of Coxeter groups. Recently we found new ideas which permitted us to give new and simpler proofs.

The paper is organized as follows. In § 1 we briefly describe the general theory of sufficient collections, introduce main notions and formulate main results. § 2 is devoted to a short introduction into the theory of Coxeter groups, adjusted to our needs. § 3 contains some additional material on Coxeter groups (maybe, it is known to specialists, but we could not find it in literature). § 4 is devoted to a realization of the general construction of sufficient collections in the specific situation of Coxeter groups. In § 5 we study deeper extremal properties of the canonical collections which are not covered by the general theory. § 6 contains explicit formulas for standard collections for some Coxeter groups. § 7 is devoted to some remarks on connections between the results of the paper and other problems of the Interpolation theory.

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§ 1. A review of the general theory.

PROBLEM. Let V be a real finite dimensional linear space. Let G be a group of linear operators acting on V. A closed convex G-invariant set U is called <u>G-symmetric</u> . A collection of G-symmetric sets $\{ U_\alpha \}_{\alpha\in A}$ is called <u>sufficient</u> (or, better, G-sufficient) if for any linear operator $L: V \to V$ the inclusions $LU_\alpha \subset U_\alpha$ $(\forall\ \alpha \in A)$ imply the inclusions $LU \subset U$ for <u>any</u> G-symmetric set U.

The problem is to describe all sufficient collections, to construct certain canonical collections and to investigate them. This was done in [14]. The papers [8,10,13,14] are mostly short announcements. The most complete exposition of the theory for general semigroups of operators is contained in [12], a brief survey of the theory is contained in [11].

The main goal of this paper is to give a complete account of the results concerning the realization of the general theory in the case when G is a finite irreducible Coxeter group.

NOTIONS. Our approach is based on a systematic exploitation of the canonical duality between the space End V of linear operators on V and the tensor product space $V \otimes V'$.

Sufficient collections are described in geometric terms connected with certain sets : $\mathfrak{A}$, $K(\mathfrak{A})$, Extr $K(\mathfrak{A})$, U° , $S(U)$, defined below :

$$\mathfrak{A} = \{ a\otimes f \in V\otimes V' : \sup_{g\in G} \langle ga, f\rangle \le 1 \}$$

(For a uniformly bounded group G the set $\mathfrak{A}$ is compact if and only if G acts irreducibly).

$K(\mathfrak{A}) = \operatorname{conv} \mathfrak{A}$ - the closed convex hull of the set $\mathfrak{A}$.

Extr $K(\mathfrak{A})$ - the set of extreme points of $K(\mathfrak{A})$.

$U^{o} = \{ f \in V' : \langle x,f \rangle \le 1, \ \forall x \in U \}$ - the polar set of U.

$S(U) = \{ x \otimes f \in V \otimes V' : x \in U, \ f \in U^{o} \}$.

$\mathrm{Co}_G x = \operatorname{conv} \{gx : g \in G\}$ - the closed convex hull of the G-orbit of x.

The following Theorem 1.1 gives a description of all sufficient collections.

THEOREM 1.1. *A collection* $\{ U_\alpha \}_{\alpha\in A}$ *of G-symmetric sets is sufficient if and only if*

$$K(\mathfrak{A}) = \operatorname{conv} \bigcup_{\alpha\in A} S(U_\alpha)$$

CANONICAL COLLECTIONS .

Collections, consisting of G-symmetric sets of the form $\mathrm{Co}_G x$ are called simple collections .

Collections consisting of G-symmetric sets of the form $(\mathrm{Co}_{G^*} f)^{o}$ are called dual-simple collections.

There are two canonical collections constructed with the help of the following sets $\mathfrak{N}$ and $\mathfrak{N}'$:

$$\mathfrak{N} = \{ a \in V : \exists f_a \in V', a \otimes f_a \in \operatorname{Extr} K(\mathfrak{A}) \}$$

$$\mathfrak{N}' = \{ f \in V' : \exists a_f \in V , a_f \otimes f \in \operatorname{Extr} K(\mathfrak{A}) \}$$

The collection $\{ Co_G a \}_{a \in \mathfrak{N}}$ is called the simple canonical collection; the collection $\{ (Co_{G^*} f)^{o} \}_{f \in \mathfrak{N}'}$ is called the dual-simple canonical collection.

THEOREM 1.2. *The canonical collections are sufficient.*

EQUIVALENCE OF SUFFICIENT COLLECTIONS. Consider the set of all compact convex subsets in V. This set of subsets is equipped with the so called Hausdorff topology. The Hausdorff distance between two sets is defined as follows :

$$d_H (U_1, U_2) = \inf \{ \lambda : U_1 \subset U_2 + \lambda B, U_2 \subset U_1 + \lambda B \},$$

where B is a fixed neighborhood of the origin in V. It is clear that the Hausdorff topology does not depend on the choice of B.

Let $\overline{}^{H}$, $\xrightarrow{H}$, $\lim_H$ denote, respectively, the closure, the convergence and the limit in the Hausdorff topology.

Let $\{ U_\alpha \}_{\alpha \in A}$ be a sufficient collection of bounded G-symmetric sets . It is clear that any collection of the type $\{ \lambda_\alpha U_\alpha \}_{\alpha \in A}$ is sufficient and any collection $\{W_\beta\}_{\beta \in B}$ such that

$$\forall \alpha \in A , \exists \{\beta_i\} \subset B \quad U_\alpha = \lim_{i}{}_H W_{\beta_i},$$

is also sufficient. This remark implies the following

DEFINITION 1.1. Sufficient collections $\{ U_\alpha \}_{\alpha \in A}$ and $\{ W_\beta \}_{\beta \in B}$ of G-symmetric bounded sets are called equivalent if

$$\overline{\{ \lambda U_\alpha \}_{\lambda\in\mathbb{R},\alpha\in A}}^{H} = \overline{\{ \mu W_\beta \}_{\mu\in\mathbb{R},\beta\in B}}^{H}$$

If

$$\overline{\{ \lambda U_\alpha \}_{\lambda\in\mathbb{R},\alpha\in A}}^{H} \subseteq \overline{\{ \mu W_\beta \}_{\mu\in\mathbb{R},\beta\in B}}^{H}$$

then the collection $\{ U_\alpha \}_{\alpha\in A}$ is said to be <u>smaller</u> than the collection $\{ W_\beta \}_{\beta\in B}$.

THEOREM 1.3. *The simple canonical collection* $\{ Co_G a \}_{a\in\Pi}$ *is the smallest simple sufficient collection. The dual-simple canonical collection* $\{ (Co_{G^*} f)^\circ \}_{f\in\Pi'}$, *is the smallest dual-simple sufficient collection.*

THEOREM 1.4. *The smallest sufficient collection exists if and only if the two canonical collections coincide.*

§ 2. A survey of the theory of Coxeter groups

A Coxeter group G is a group of linear operators in a real Euclidean finite dimensional space V, which can be described as follows: fix a finite number of hyperplanes in V containing the origin, then the group G is generated by orthogonal reflections across these hyperplanes. Let n be a unit vector orthogonal to a hyperplane across which a reflection g acts. Then the reflection g is defined by the formula

$$gx = x - 2n \langle x, n\rangle$$

We consider finite Coxeter groups. The finiteness condition on the group G implies severe restrictions upon positions of the hyperplanes. If reflections across two hyperplanes belong to a finite group, then the angle between these two hyperplanes must be $\frac{\pi}{m}$, $m \in N$, $m \geq 2$.

All hyperplanes, such that the reflections across them belong to the Coxeter group, split the space into connected components - interiors of polyhedral cones; these cones are called Weil chambers.

A hyperplane containing a (dim V - 1) - dimensional face of a Weil chamber is called a wall. The group G is generated by reflections across the walls of any Weil chamber ([1], ch.V, 3.1, Lemma 2). Any Weil chamber is a fundamental domain for the group G ([1], ch.V, 3.3, Th.2), this means that the G-orbit of any x has exactly one point in common with any Weil chamber. The group G acts transitively on the set of Weil chambers - for any two Weil chambers there exists exactly one element of the group mapping the first chamber onto the second one. ([1], ch.V, 3.1, Lemma 2, and ch.V, 3.3, Prop. 1). Weil chambers of any Coxeter group such that the origin is its only fixed point are simplicial cones, this means that every extreme ray of the chamber does not belong to exactly one wall of the chamber ([1], ch V, 3.9, Prop.7).

A Coxeter group G is usually described with the help of its Coxeter graph $\Gamma(G)$. The vertices of the graph are in a one-to-one correspondence with the walls of a Weil chamber (or with the extreme rays of the chamber - an extreme ray corresponds to that unique wall of the chamber which does not contain it); two vertices are connected by a bond if and only if the angle between the corresponding walls is $\frac{\pi}{m}$, $m \geq 3$. This number m is attributed to the bond of the graph.

For a Coxeter group without nontrivial fixed points the irreducibility is equivalent to the connectedness of its Coxeter graph ([1], ch.V, 3.7, Corollary).

If G is a finite Coxeter group then its Coxeter graph has no cycles ([1], ch.V, 4.8, Prop.8). A vertex of a graph is called an end vertex if it is connected with exactly one other vertex. A vertex is called a branching vertex if it is connected with at least three other vertices.

The Coxeter graph completely describes the Weil chamber and the Coxeter group as well. All Coxeter graphs are classified and, hence, all finite irreducible Coxeter groups are classified too (see [1]).

§ 3. Coxeter groups : stabilizers and supports

Let C be a Weil chamber. Let $W(i)$ denote the wall of C corresponding the vertex $\pi(i) \in \Gamma(G)$. Let $\mathbf{n}(i)$ denote the unit vector of the inner (with respect to C) normal to $W(i)$. For every i there exists exactly one extreme ray of C not contained in $W(i)$. We let $\omega(i)$ denote the vector situated on this extreme ray, normalized by the condition $\langle \mathbf{n}(i), \omega(i) \rangle = 1$. So, we obtain

$$\langle \mathbf{n}(i), \omega(j) \rangle = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases}$$

In the theory of Coxeter groups the vectors $\lambda_i \mathbf{n}(i)$ (with special λ's) are called roots, and the vectors $\frac{1}{\lambda_i}\omega(i)$ are called fundamental weights.

Let $g(i)$ denote the orthogonal reflection across the wall $W(i)$

$$g(i)x = x - 2\langle \mathbf{n}(i), x \rangle \, \mathbf{n}(i)$$

Let a^* denote the unique element of $\mathrm{orb}_G a$, belonging to C.

DEFINITION 3.1.

$$\mathrm{supp}_G a = \{ \pi(i) \in \Gamma(G) : \langle \mathbf{n}(i), a^* \rangle > 0 \} =$$
$$= \{ \pi(i) \in \Gamma(G) : a^* \notin W(i) \}$$

One can easily see that if we decompose $a^* = \sum \lambda_i \omega(i)$ ($\lambda_i \geq 0$), then $\mathrm{supp}_G a = \{ \pi(i) : \lambda_i > 0 \}$. One can easily prove that $\mathrm{supp}_G a$ does not depend upon the choice of the Weil chamber C.

Let $J_1, \dots, J_s$ be the sets of vertices of connected components of the Coxeter graph $\Gamma(G)$. Let G_p denote the subgroup of G generated by reflections across the walls $W(i)$, $\pi(i) \in J_p$. Let

$$V_p = \{ x \in V : gx = x \quad \forall g \in G_p \}^{\perp}$$

It is clear that G_p are normal subgroups in G , V_p are invariant under G. It is known ([1], ch.V, 3.7, Prop.5), that

$$G = G_1 \times G_2 \times \dots \times G_s \quad , \quad V = V_1 \oplus V_2 \oplus \dots \oplus V_s ,$$

the actions of G_p in V_p are irreducible. It follows from the above that a vector x from V belongs to a proper G-invariant subspace if and only if supp x intersects with every J_p , $1 \leq p \leq s$.

Let $a \in C$, consider

$$C(a) = \Big(\bigcap_{\pi(i) \notin \mathrm{supp}_G a} W(i) \Big) \cap C = \Big(\bigcap_{i : a \in W(i)} W(i) \Big) \cap C$$

C(a) is called the cell of a.

Let $K \subset V$, let $\mathrm{Stab}_G K$ denote the stabilizer of K, i.e.,

$$\mathrm{Stab}_G K = \{ g \in G : gx = x \; , \; \forall x \in K \}$$

It is known that -

(i) $\mathrm{Stab}_G a = \mathrm{Stab}_G C(a)$ and $\bigcap_{\pi(i) \notin \mathrm{supp}_G a} W(i)$ is the set of fixed points of $\mathrm{Stab}_G a$ ([1], ch.V, 3.3, Th.2);

(ii) $\mathrm{Stab}_G a$ is a Coxeter group, it is generated by reflections across the walls of the chamber C, containing a, i.e., it is generated by the reflections $g(i)$ ($\pi(i) \notin \mathrm{supp}_G a$) ([1], Ch.V, 3.3, Prop.1). We consider the action of $\mathrm{Stab}_G a$ on $C(a)^{\perp}$ to avoid nontrivial fixed points,

$$C(a)^{\perp} = [(\bigcap_{\pi(i) \notin \mathrm{supp}_G a} W(i)) \cap C]^{\perp} = \mathrm{span} \{ n(i) : \pi(i) \notin \mathrm{supp}_G a \}$$

One can easily see that

$$\dim C(a)^{\perp} = \dim V - \mathrm{card}\ \mathrm{supp}_G a \qquad (*)$$

If S is a subset of vertices of the graph $\Gamma(G)$, then let $\Gamma(G)\backslash S$ denote the graph obtained from $\Gamma(G)$ by erasing all the vertices belonging to S together with all bonds incident to these vertices.

PROPOSITION. 3.1. $\Gamma(\mathrm{Stab}_G a|_{C(a)^{\perp}}) = \Gamma(G)\backslash \mathrm{supp}_G a.$

PROOF. Consider the traces of the hyperplanes $W(i)$ on $C(a)^{\perp}$, i.e., consider the hypersubspaces $W(i) \cap C(a)^{\perp}$ in the space $C(a)^{\perp}$. The angles between the walls $W(i)$ coincide with the angles between their traces. These traces obviously form a Weil chamber for the group $\mathrm{Stab}_G a|_{C(a)^{\perp}}$, so the Coxeter graph $\Gamma(\mathrm{Stab}_G a|_{C(a)^{\perp}})$ is completely defined by the angles between the walls $W(i)$, $\pi(i) \notin \mathrm{supp}_G a$, i.e.,

$$\Gamma(\mathrm{Stab}_G a|_{C(a)^{\perp}}) = \Gamma(G)\backslash \mathrm{supp}_G a. \qquad \blacksquare$$

Let $pr_a b$ denote the orthogonal projection of the vector b on the subspace $C(a)^{\perp}$.

PROPOSITION 3.2. $\mathrm{supp}_{\mathrm{Stab}_G a}(pr_a b) = \mathrm{supp}_G b \backslash \mathrm{supp}_G a.$

PROOF. $b = c + pr_a b$, $c \in \bigcap_{i : a \in W(i)} W(i)$. Note that $b \in W(i)$ ($\pi(i) \notin \mathrm{supp}_G a$) if and only if $pr_a b \in W(i) \cap C(a)^{\perp}$

$$\mathrm{supp}_{\mathrm{Stab}_G a}(\mathrm{pr}_a b) = \{\ \pi(i) \in \Gamma(\mathrm{Stab}_G a\big|_{C(a)^\perp}) : \mathrm{pr}_a b \notin C(a)^\perp \cap W(i)\ \} =$$

$$= \{\ \pi(i) \in \Gamma(G) : \pi(i) \notin \mathrm{supp}_G a,\ \mathrm{pr}_a b \notin W(i) \cap C(a)^\perp\ \} =$$

$$= \{\ \pi(i) \in \Gamma(G) : a \in W(i),\ \mathrm{pr}_a b \notin W(i)\ \} =$$

$$= \{\ \pi(i) \in \Gamma(G) : a \in W(i),\ b \notin W(i)\ \} =$$

$$= \mathrm{supp}_G b \setminus \mathrm{supp}_G a. \qquad \blacksquare$$

PROPOSITION 3.3. *Let G be irreducible. $\mathrm{Stab}_G a\big|_{a^\perp}$ is irreducible if and only if $\mathrm{supp}_G a$ consists of an end vertex.*

PROOF. If the group $\mathrm{Stab}_G a\big|_{a^\perp}$ is irreducible then $C(a)$ is 1-dimensional (or else there exist nontrivial fixed points for the action of $\mathrm{Stab}_G a$ on $a^\perp$). Applying the equality (*) above, we obtain

$$\mathrm{Card}\ \mathrm{supp}_G a = \dim V - \dim C(a)^\perp = \dim V - (\dim V - 1) = 1$$

So, $\mathrm{supp}_G a$ consists of one vertex.

Let us show that this vertex from $\mathrm{supp}_G a$ is an end vertex. As $C(a)^\perp = a^\perp$ then $\mathrm{Stab}_G a\big|_{a^\perp} = \mathrm{Stab}_G a\big|_{C(a)^\perp}$ and the action is irreducible if and only if $\Gamma(\mathrm{Stab}_G a\big|_{C(a)^\perp})$ is connected ($\mathrm{Stab}_G a\big|_{C(a)^\perp}$ acts without nontrivial fixed points, see (i) above) or, equivalently, $\Gamma(G)\setminus\mathrm{supp}_G a$ is connected. But $\mathrm{supp}_G a$ consists of one vertex, so this vertex must be an end vertex, because $\Gamma(G)$ is connected (G is irreducible) and $\Gamma(G)$ does not contain cycles (see § 2). ■

§ 4. Canonical collections for Coxeter groups

Let G be a finite irreducible Coxeter group acting in a finite dimensional real Euclidean space V. We consider $V = V'$ and the duality is given by the G-invariant scalar product $< , >$. These agreements and the orthogonality of operators from G imply the coincidence of the sets $\mathfrak{N}$ and $\mathfrak{N}'$.

LEMMA 4.1.

(i) $\sup_{g\in G} \langle gx,f\rangle = \langle x,f\rangle$ *if and only if x and f belong to the same Weil chamber;*

(ii) if $\sup_{h\in G} \langle hx,f\rangle = \langle x,f\rangle = \langle gx,f\rangle$ *then there exists* $w \in G$ *such that* $gx = wx$ *and* $wf = f$.

PROOF.

(i) Let f belong to a Weil chamber C_0, take any $h \in \mathrm{Stab}_G f$ and consider another Weil chamber hC_0. Obviously, f belongs to the chamber hC_0. Conversely, if C_0 and C_1 are Weil chambers and $f \in C_0 \cap C_1$, then take the element $h \in G$ such that $hC_0 = C_1$ (it exists because of the transitivity of the action of G on the set of Weil chambers - see § 2) and notice that f and hf belong to the same Weil chamber C_1, so $f = hf$, and $h \in \mathrm{Stab}_G f$. So, we have proved that the elements of $\mathrm{Stab}_G f$ and the Weil chambers containing f are in a one-to one correspondence. So, if x and f do not belong to the same Weil chamber then $hx \notin C_0$ for any $h \in \mathrm{Stab}_G f$. Consider the walls $W(i)$ of C_0, such that $f \in W(i)$. The subgroup $\mathrm{Stab}_G f$ is generated by the reflections across these walls, so it is possible to find $h \in \mathrm{Stab}_G f$ such that $\langle hx, n(i)\rangle \geq 0$ for all i such that $f \in W(i)$. But if $hx \notin C_0$, then there exists i_0 such that $\langle hx,n(i_0)\rangle < 0$, so $f \notin W(i_0)$ and $\langle f,n(i_0)\rangle > 0$. Then

$$\langle g(i_0)hx,f\rangle = \langle hx - 2n(i_0)\langle hx,n(i_0)\rangle,f\rangle =$$

$$= \langle hx,f\rangle - 2\langle hx,n(i_0)\rangle\langle n(i_0),f\rangle > \langle hx,f\rangle = \langle x,h^{-1}f\rangle = \langle x,f\rangle$$

and therefore if x and f do not belong to one Weil chamber then $\langle x,f\rangle <$ $< \max \{ \langle gx,f\rangle : g \in G \}$. If there exist two Weil chambers C_0 and C_1 such that $x,f \in C_0$ and $gx,f \in C_1$ then there exists $h \in \mathrm{Stab}_G f$ such that $hC_0 =$ $= C_1$ and therefore hx, $gx \in C_1$, so $hx = gx$, and $\langle x,f\rangle = \langle hx,hf\rangle =$ $= \langle gx,f\rangle$. The assertion (i) is proved.

(ii) If $\sup_{h\in G} \langle hx,f\rangle = \langle x,f\rangle = \langle gx,f\rangle$ then by (i) there exist Weil chambers C_1 and C_2 such that $f,x \in C_1$ and $f,gx \in C_2$. As the group G acts transitively on the set of Weil chambers (see § 2), there exists an element $w \in G$ such that $wC_1 = C_2$, so gx, $wx \in C_2$ and f, $wf \in C_2$, therefore $gx = wx$ and $fx = x$. ∎

REMARK 4.1. The assertion (ii) of Lemma 1 may be reformulated as follows:

Let x,y belong to one G-orbit and let

$$\langle x,f\rangle = \langle y,f\rangle = \max \{ \langle t,f\rangle : t \in orb_G x = orb_G y \}.$$

Then x,y belong to one $Stab_G f$-orbit.

THEOREM 4.1. $z \in Extr(Co_G a)^\circ$ *if and only if* $supp_G z$ *consists of one vertex and for every connected component U of* $\Gamma(G)\backslash supp_G s$ *the intersection* $supp_G a \cap U$ *is nonempty.*

PROOF. Let $z \in \mathrm{Extr}(\mathrm{Co}_G a)^\circ$. We may assume that z and a belong to the same Weil chamber C. As $z \in \mathrm{Extr}(\mathrm{Co}_G a)^\circ$ there exists a $(\dim V - 1)$ - dimensional face of $\mathrm{Co}_G a$ such that z is orthogonal to this face, i.e., the system

$$S = \{ x \in orb_G a : 1 = \langle x,z\rangle = \max (\langle h,z\rangle : h \in orb_G a) \}$$

is complete in V. Note that the vector a belongs to this system. Due to the assertion (ii) of Lemma 4.1 the system S coincides with the $Stab_G z$-orbit of a.

Decompose $a = a_1 + a_2$, $a_1 \in C(z)$, $a_2 \in C(z)^{\perp}$ $a_2 = pr_{C(z)^{\perp}} a$. Then for every $x \in S$, $x = a_1 + y$, $y \in orb_{Stab_G z} a_2$, as $C(z)$ belongs to the set of fixed vectors of $Stab_G z$. So, $S \subset C(z)^{\perp} + \{ \nu a_1 : \nu \in \mathbb{R} \}$, and if the system S is complete then $\dim C(z)^{\perp} + 1 = \dim V$.

As $\text{card } supp_G z = \dim V - \dim C(z)^{\perp}$ we obtain that $\text{card } supp_G z = 1$. Moreover as the system $S = a_1 + orb_{Stab_G z} a_2$ is complete in V, then a_2 cannot belong to a proper $Stab_G z$-invariant subspace in $C(z)^{\perp}$, therefore

$supp_{Stab_G z|_{C(z)^{\perp}}} a_2$ must intersect with every connected component of $\Gamma(Stab_G z|_{C(z)^{\perp}})$ or, due to the Propositions 3.1, 3.2, $supp_G a \setminus supp_G z$ must intersect with every connected component of $\Gamma(G) \setminus supp_G z$.

These arguments may be obviously reverted. ∎

THEOREM 4.2. *$a \in \mathfrak{M}$ if and only if $supp_G a$ consists of exactly one end vertex of $\Gamma(G)$.*

PROOF. Let $a \in \mathfrak{M}$, then there exists $f \in V$ such that $a \otimes f \in \text{Extr } K(\mathfrak{A})$, hence $a \in (Co_G f)^{\circ}$, $f \in (Co_G a)^{\circ}$, so the supports of a and f consist of one vertex each and $supp_G f$ intersects with every component of $\Gamma(G) \setminus supp_G a$ and $supp_G a$ intersects with every connected component of

$\Gamma(G)\backslash \mathrm{supp}_G f$, hence $\mathrm{supp}_G a$ consists of an end vertex.

Conversely, suppose that $a \in C$ and $\mathrm{supp}_G a$ consists of an end vertex of $\Gamma(G)$. Take any $f \in C \cap \mathrm{Extr}(\mathrm{Co}_G a)^\circ$. Then $\mathrm{supp}_G f$ also consists of an end vertex of $\Gamma(G)$ and surely $\mathrm{supp}_G a \neq \mathrm{supp}_G f$ (Th. 4.1). Note that $a \in \mathrm{Extr}(\mathrm{Co}_G f)^\circ$ (Th. 4.1), and that $\mathrm{Stab}_G a$ acts irreducibly on $a^\perp$ (Prop. 3.3).

Let us show that $a \otimes f \in \mathrm{Extr}\, K(\mathfrak{A})$. Consider a decomposition

$$a\otimes f = \sum \lambda_i a_i \otimes f_i \ , \quad \lambda_i \geq 0,\ \sum_i \lambda_i = 1,\ a_i\otimes f_i \in \mathfrak{A}.$$

We show that $a_i \otimes f_i = a\otimes f$. Apply the operators $g\otimes \mathbb{I}$ ($g \in \mathrm{Stab}_G a$) to the equality and sum up the results. We obtain

$$(\ \mathrm{Card\ Stab}_G a\ )\ a\otimes f = \sum_i \lambda_i \Big(\sum_{g\in \mathrm{Stab}_G a} g a_i \Big)\otimes f_i$$

Decompose $a_i = a_{i1} + a_{i2}$, $a_{i1} = \nu_i a$, $a_{i2} \in a^\perp$. Then

$$\sum_{g\in \mathrm{Stab}_G a} g a_i = \sum_{g\in \mathrm{Stab}_G a} g(a_{i1} + a_{i2}) = (\ \mathrm{Card\ Stab}_G a\ ) a_{i1} + \sum_{g\in \mathrm{Stab}_G a} g a_{i2}$$

The second term vanishes because of the irreducibility of $\mathrm{Stab}_G a$ on $a^\perp$ (this term belongs to $a^\perp$ and it is $\mathrm{Stab}_G a$-invariant). So, we obtain

$$a\otimes f = \sum_i \lambda_i a_{i1} \otimes f_i = \sum_i \lambda_i \nu_i (a\otimes f_i)$$

therefore $f = \sum \lambda_i \nu_i f_i$.

Let us prove that $\nu_i f_i \in (\mathrm{Co}_G a)^\circ$. Really

$$\nu_i a = a_{i1} = \frac{1}{\mathrm{Card\ Stab}_G a} \sum_{g\in \mathrm{Stab}_G a} g a_i \ .$$

Therefore

$$\nu_i a\otimes f_i = \frac{1}{\mathrm{Card\ Stab}_G a} \sum_{g\in \mathrm{Stab}_G a} g a_i \otimes f_i \ .$$

Since $a_i \otimes f_i \in \mathfrak{A}$ we conclude that for any $h \in G$

$$\langle h\nu_i a\ ,\ f_i\rangle = \frac{1}{\mathrm{Card\ Stab}_G a} \sum_{g\in \mathrm{Stab}_G a} \langle hga_i, f_i\rangle \leq 1$$

So , $\nu_i f_i \in (Co_G a)^\circ$. But $f \in Extr(Co_G a)^\circ$ and $f = \sum \lambda_i (\nu_i f_i)$, $\nu_i f_i \in (Co_G a)^\circ$. Therefore $\nu_i f_i = f$ and as

$$a \otimes f = \sum \lambda_i a_i \otimes f_i = \sum \lambda_i a_i \otimes \frac{1}{\nu_i} f = \sum \lambda_i (\frac{1}{\nu_i} a_i) \otimes f$$

we obtain that $a = \sum \lambda_i (\frac{1}{\nu_i} a_i)$.

As $\frac{1}{\nu_i} a_i \otimes f = a_i \otimes f_i \in \mathfrak{A}$ then $\frac{1}{\nu_i} a_i \in (Co_G f)^\circ$. But $a \in Extr\ (Co_G f)^\circ$ and $a = \sum \lambda_i (\frac{1}{\nu_i} a_i)$, therefore $a = \frac{1}{\nu_i} a_i$. So, $a_i \otimes f_i = \nu_i a \otimes \frac{1}{\nu_i} f = a \otimes f$. ■

THEOREM 4.3. *Let G be a finite irreducible Coxeter group. There exists the smallest sufficient collection (= the canonical collections coincide) if and only if the Coxeter graph has no branching vertices.*

PROOF. The canonical collections coincide

$$\{ Co_G a \}_{a \in \mathfrak{N}} = \{ (Co_G f)^\circ \}_{f \in \mathfrak{N}}$$

if and only if for any $a \in \mathfrak{N}$ there exists $f \in \mathfrak{N}$ such that $Co_G a = (Co_G f)^\circ$, or Extr $(Co_G a)^\circ = orb_G f$, or, equivalently, for any $a \in \mathfrak{N}$ all vectors from $Extr(Co_G a)^\circ$ have the same support, but every end vertex, different from $supp_G a$, is supporting a vector from $(Co_G a)^\circ$ - see Th. 4.1. So, there is only one end vertex in $\Gamma(G)$, different from $supp_G a$, so there are only two end vertices in $\Gamma(G)$ and this happens if and only if there is no branching vertices in $\Gamma(G)$ (because $\Gamma(G)$ contains no cycles - see § 2). ■

§ 5. Standard collections

We want to investigate the structure of the canonical collections in more details. One can easily see that there are "surplus" sets in the canonical collections: if a set U belongs to a canonical collection then any set kU also belongs to it ($k \in \mathbb{R}$). We want to eliminate such surplus sets. Consider the group $\hat{G}$ consisting of operators kg, $k \in \mathbb{R}\setminus\{0\}$, $g \in G$. Obviously $\hat{G}\,\mathfrak{N} = \mathfrak{N}$, i.e. $\hat{G}$ transforms $\mathfrak{N}$ into $\mathfrak{N}$ and $\mathfrak{N}$ is fibered into nonintersecting $\hat{G}$-orbits).

Choose a representative b in every $\hat{G}$-orbit in $\mathfrak{N}$ and let $\hat{\mathfrak{N}}$ denote the set of such representatives.

DEFINITION 5.1. Every collection $\{ Co_G b \}_{b \in \hat{\mathfrak{N}}}$ is called a <u>standard simple collection</u>. Every collection $\{ (Co_G b)^{o} \}_{b \in \hat{\mathfrak{N}}}$ is called a <u>standard dual-simple collection</u>.

Obviously, standard collections are equivalent to the corresponding canonical ones and therefore inherit many of their properties.

It is possible to calculate the number of elements in standard collections and to prove that this number is the smallest possible among all sufficient collections.

I. **COXETER GROUPS AND SPECIAL PERMUTATIONS OF COXETER GRAPHS.** Consider a Weil chamber C. Obviously -C is also a Weil chamber. As the group G acts transitively on the set of Weil chambers, then there exists exactly one element $w_0 \in G$ such that $w_0(-C) = C$. If $-\mathbb{I} \in G$ then $w_0 = -\mathbb{I}$. Consider the operator $w_0(-\mathbb{I})$. It maps C onto C, it is an orthogonal operator, therefore it preserves angles between the walls of C and it maps extreme rays of C to extreme rays of C. Therefore it gives rise to a special permutation π of vertices of the Coxeter graph $\Gamma(G)$. π is trivial if $-\mathbb{I} \in G$. This permutation π preserves bonds and their multiplicities because the operator $w_0(-\mathbb{I})$ preserves angles between walls. So, the permutation π maps end vertices to end vertices, giving rise to a permutation $\hat{\pi}$ of the set of end vertices, the permutation π is completely defined by the permutation $\hat{\pi}$. The permutation π contains cycles of length at most two, because $[w_0(-\mathbb{I})]^2 = \mathbb{I}$ (really, $[w_0(-\mathbb{I})]^2 = w_0{}^2 \in G$, $w_0{}^2$ maps C to C, therefore $w_0{}^2 = \mathbb{I}$).

LEMMA 5.1. *Let $supp_G x$ and $supp_G y$ consist of one vertex each. Then x is $\hat{G}$-equivalent to y if and only if $\pi\, supp_G x = supp_G y$ or $supp_G x = supp_G y$.*

PROOF.

$$\begin{aligned} x \text{ is } \hat{G}\text{-equivalent to } y &\Leftrightarrow \exists\, g \in G,\ k \in \mathbb{R}\setminus\{0\},\ x = kgy \\ &\Leftrightarrow \exists\, k \in \mathbb{R}\setminus\{0\}\ \operatorname{supp}_G \tfrac{1}{k}x = \operatorname{supp}_G y \\ &\Leftrightarrow \operatorname{supp}_G x = \operatorname{supp}_G y \text{ or } \operatorname{supp}_G(-x) = \operatorname{supp}_G y \\ &\Leftrightarrow \operatorname{supp}_G x = \operatorname{supp}_G y \text{ or } \operatorname{supp}_G w_0(-x) = \operatorname{supp}_G y \\ &\Leftrightarrow \operatorname{supp}_G x = \operatorname{supp}_G y \text{ or } \pi\, \operatorname{supp}_G x = \operatorname{supp}_G y \end{aligned}$$ ∎

COROLLARY 5.1.

If $-\mathbb{I} \in G$ then x is $\hat{G}$-equivalent to y if and only if $supp_G x = supp_G y$.
If $-\mathbb{I} \notin G$ then x is $\hat{G}$-equivalent to y if and only if $supp_G x = supp_G y$ or $\pi\, supp_G x = supp_G y$.

II. SUFFICIENT COLLECTIONS CONTAINING THE SMALLEST NUMBER OF SETS.

THEOREM 5.1. *Let G be a finite irreducible Coxeter group. The number of elements in a standard collection equals*

(i) the number of end vertices of the Coxeter graph provided the operator $-\mathbb{I} \in G$.

(ii) the number of end vertices of the Coxeter graph minus 1 provided the operator $-\mathbb{I} \notin G$.

PROOF. It is known from the classification of connected Coxeter graphs ([1], ch. VI, 4.1, Th.1) that there may be two or three end vertices in a Coxeter graph. So, if $-\mathbb{I} \notin G$ then,

(i) in the case of two end vertices, $\hat{\pi}$ changes places of these vertices and therefore vectors supported at these vertices are $\hat{G}$-equivalent.

(ii) in the case of three end vertices $\hat{\pi}$ changes places of two of them and leaves the third end vertex fixed (because it contains cycles of length at most two), therefore vectors supported at the first two end vertices are $\hat{G}$-equivalent and as for vectors supported at the third vertex they are $\hat{G}$-unequivalent to the previous vectors.

The number of sets in a standard collection is equal to the number of vectors in $\hat{\mathfrak{N}}$, or, equivalently, to the number of pairwise $\hat{G}$ - nonequivalent elements in $\mathfrak{N}$, or, equivalently, to the number of pairwise $\hat{\pi}$-nonequivalent end vertices of $\Gamma(G)$.

So, if $-\mathbb{I} \in G$ this number equals the number of end vertices.

If $-\mathbb{I} \notin G$ then there is exactly one pair of $\hat{\pi}$-equivalent end vertices so the number of elements of $\hat{\mathfrak{N}}$ equals the number of end vertices minus 1. ∎

Now we want to study general sufficient collections containing the minimal possible number of sets. We shall prove that in the most cases these sufficient collections are the standard ones.

Let $\{ U_i \}$ be a finite sufficient collection. Its sufficiency is equivalent to the inclusion

$$\operatorname{Extr} K(\mathfrak{A}) \subset \bigcup_i S(U_i)$$

But we know all vectors from $\operatorname{Extr} K(\mathfrak{A})$: consider the set $\{ \pi(\alpha) \}$ of end vertices of the Coxeter graph $\Gamma(G)$ and let $\omega(\alpha) \in C$, $\operatorname{supp}_G \omega(\alpha) = \pi(\alpha)$.

One can easily see that

$$\text{Extr } K(\mathfrak{A}) = \{ k(\alpha,\beta) g\omega(\alpha)\otimes h\omega(\beta) : g,h \in G,\ \alpha \neq \beta,\ \pi(\alpha),\ \pi(\beta) \text{ are end vertices of } \Gamma(G),\ k(\alpha,\beta) = \frac{1}{\langle\omega(\alpha),\omega(\beta)\rangle} \}$$

Note that if $k(\alpha,\beta)\omega(\alpha)\otimes\omega(\beta) \in S(U)$ and U is G-symmetric then $k(\alpha,\beta)g\omega(\alpha)\otimes h\omega(\beta) \in S(U)$ for all $g,h \in G$. So, the collection $\{ U_i \}$ is sufficient if and only if we can distribute all elements of the type $k(\alpha,\beta)\omega(\alpha)\otimes\omega(\beta)$ among the sets $S(U_i)$. We must know which of the elements $h(\alpha,\beta)\omega(\alpha)\otimes\omega(\beta)$ are "compatible", i.e., can belong to one set S(U), and which are not.

Let G be a finite irreducible Coxeter group.

LEMMA 5.2. *Let* $e_1\otimes e_2 \in \text{Extr } K(\mathfrak{A})$. *Let U be a G-symmetric closed set such that* $e_1\otimes e_2 \in S(U)$, *and* $\nu e_1 \in \text{Extr } U$, $\frac{1}{\nu} e_2 \in \text{Extr } U^\circ$ ($\nu > 0$). *Then* $\mu e_1 \notin \text{Extr } U^O$ *for any* $\mu > 0$.

PROOF. We may consider that e_1, e_2 belong to the same Weil chamber, Let $\hat{g} \in \text{Stab}_G e_1$. Then $\langle \hat{g} e_2, e_1 \rangle = \langle e_2, e_1 \rangle = 1$

Represent the element $\frac{1}{\nu} e_2$:

$$\frac{1}{\nu} e_2 = \alpha\, \nu e_1 + a, \qquad a \in e_1^{\perp}$$

It follows from the irreducibility of the group $\text{Stab}_G e_1$ on $e_1^{\perp}$ that

$$\sum_{\hat{g}\in \text{Stab}_G e_1} \hat{g}\, \frac{1}{\nu} e_2 = \alpha\, \nu e_1 \,\text{Card}(\text{Stab}_G e_1)$$

Hence

$$\frac{1}{\text{Card}(\text{Stab}_G e_1)} \sum_{\hat{g}} \hat{g}\, \frac{1}{\nu} e_2 = \alpha\, \nu e_1.$$

But the left part of this equality is a convex combination of elements from U°. Hence $\alpha\,\nu e_1 \in U^\circ$, and

$$< \alpha\nu e_1, \nu e_1 > = < \frac{1}{\mathrm{Card}(\mathrm{Stab}_G e_1)} \sum_{\hat{g}\in \mathrm{Stab}_G e_1} < \hat{g}\,\frac{1}{\nu}\,e_2\,, \nu e_1 > = 1$$

Now assume that $\mu e_1 \in \mathrm{Extr}\ U^\circ$ for some $\mu > 0$. As $\alpha\nu > 0$ ($\alpha = \dfrac{<e_2, e_1>}{\nu^2 <e_1, e_1>}$) and $\alpha\nu e_1 \in U^\circ$, then $\mu \geq \alpha\nu$. But if $\mu > \alpha\nu$ then

$$1 = <\alpha\nu e_1, \nu e_1> < <\mu e_1, \nu e_1> \leq 1,$$

since $\mu e_1 \in U^\circ$, $\nu e_1 \in U$. Thus, $\mu = \alpha\nu$ and therefore μe_1 ($= \alpha\nu e_1$) is a convex combination of the elements $\hat{g}\,\frac{1}{\nu}\,e_2 \in U^\circ$ ($\hat{g} \in \mathrm{Stab}_G e_1$), which differ from μe_1. But this contradicts to our assumption, that $\mu e_1 \in \mathrm{Extr}\ U^\circ$. ∎

LEMMA 5.3. *Let* $-\mathbb{I} \in G$. *Let* $e_1 \otimes e_2 \in \mathit{Extr}\ K(\mathfrak{A})$. *Let U be a closed G-symmetric set and* $e_1 \otimes e_2 \in S(U)$. *Consider any* $x \otimes e_1 \in \mathit{Extr}\ K(\mathfrak{A})$. *Then* $x \otimes e_1 \notin S(U)$.

PROOF. It follows from the inclusion $e_1 \otimes e_2 \in S(U) \cap \mathit{Extr}\ K(\mathfrak{A})$ that for some γ $\gamma e_1 \in \mathit{Extr}\ U$ and $\frac{1}{\gamma}\,e_2 \in \mathit{Extr}\ U^\circ$. As $-\mathbb{I} \in G$ we may consider $\gamma > 0$.

Analogously , if $x \otimes e_1 \in S(U) \cap \mathrm{Extr}\ K(\mathfrak{A})$, then for some $\delta > 0$ $x \in \mathrm{Extr}\ U$, and $\frac{1}{\delta}\,e_1 \in \mathrm{Extr}\ U^\circ$, but this contradicts to the assertion of Lemma 5.2. ∎

It follows from the Classification of Coxeter graphs that if $\Gamma(G)$ has a branching vertex then it has three end vertices.

LEMMA 5.4. *Let $-\mathbb{I} \in G$ and let $\Gamma(G)$ have a branching vertex. Let $\pi(1)$, $\pi(2)$, $\pi(3)$ denote the three end vertices of $\Gamma(G)$. Let e_i be supported at $\pi(i)$, $e_i \in C$ ($i = 1, 2, 3$).*

(i) Let $e_1 \otimes e_2$, $e_1 \otimes e_3 \in \mathrm{Extr}\, K(\mathfrak{A})$. If $e_1 \otimes e_2$, $e_1 \otimes e_3 \in S(U)$ then $U = \lambda\, Co_G e_1$.

(ii) Let $e_2 \otimes e_1$, $e_3 \otimes e_1 \in \mathrm{Extr}\, K(\mathfrak{A})$. If $e_2 \otimes e_1$, $e_3 \otimes e_1 \in S(U)$ then $U = \mu\, (Co_G e_1)^{\circ}$.

PROOF.

(i) As $e_1 \otimes e_2, e_1 \otimes e_3 \in S(U) \cap \mathrm{Extr}\, K(\mathfrak{A})$ then there exist λ, μ such that $\lambda e_1, \mu e_1 \in \mathrm{Extr}\, U$, and $\frac{1}{\lambda} e_2$, $\frac{1}{\mu} e_3 \in \mathrm{Extr}\, U^{\circ}$. As $-\mathbb{I} \in G$ we may assume that $\lambda, \mu > 0$ and therefore $\lambda = \mu$. So, $\lambda e_1 \in \mathrm{Extr}\, U$, $\frac{1}{\lambda} e_2$, $\frac{1}{\lambda} e_3 \in \mathrm{Extr}\, U^{\circ}$. Then

$$\lambda\, Co_G e_1 \subset U \subset (Co_G \tfrac{1}{\lambda} e_2)^{\circ} \cap (Co_G \tfrac{1}{\lambda} e_3)^{\circ} = \lambda \{ \mathrm{conv}\, [\, (Co_G e_2) \cup (Co_G e_3)\,]\, \}^{\circ}$$

It follows from Theorem 4.1 that $\mathrm{Extr}\, (Co_G e_1)^{\circ} = (orb_G e_2) \cup (orb_G e_3)$. Therefore $(Co_G e_1)^{\circ} = \mathrm{conv}\, \mathrm{Extr}\, (Co_G e_1)^{\circ} = \mathrm{conv}\, [\, (orb_G e_2) \cup (orb_G e_3)\,] =$ $= \mathrm{conv}\, [\, (Co_G e_2) \cup (Co_G e_3)\,]$, therefore

$$\lambda\, Co_G e_1 = \lambda\, \{\mathrm{conv}\, [\, (Co_G e_2) \cup (Co_G e_3)\,]\}^{\circ} = U$$

The assertion (ii) is proved similarly. ∎

REMARK 5.1. Note that the condition $-\mathbb{I} \in G$ is very substantial. If $-\mathbb{I} \notin G$ then we may only assert that if $e_1 \otimes e_2$, $e_1 \otimes e_3 \in S(U) \cap \mathrm{Extr}\, K(\mathfrak{A})$

then there exist λ, $\mu \in \mathbb{R}$ such that $\lambda e_1 \in \text{Extr } U$, $\mu e_1 \in \text{Extr } U$, $\frac{1}{\lambda} e_2 \in \text{Extr } U^\circ$ and $\frac{1}{\mu} e_3 \in \text{Extr } U^\circ$. Certainly, if λ and μ are of the same sign then we can assert that $\lambda = \mu$ and repeat the arguments of the previous proof. But if $-\mathbb{I} \notin G$ then it may happen that λ and μ are of the different signs and, for example $\lambda = -\mu$, then we only know that $\frac{1}{\lambda} e_2$ and $-\frac{1}{\lambda} e_3$ belong to $\text{Extr } U^\circ$. It may happen (and it really happens) that $-e_3 \in \text{orb}_G e_2$ and therefore we only know that $e_1 \otimes e_2 \in S(U)$, this is certainly not sufficient for the validity of the assertion that $U = \lambda \text{Co}_G e_1$.

THEOREM 5.2. *Let G be a finite irreducible Coxeter group .*

(i) The number of sets in <u>*any*</u> *sufficient collection is not smaller then the number of sets in a standard collection.*

(ii) If the Coxeter graph has no branching vertices or the operator $-\mathbb{I} \in G$ then any sufficient collection consisting of the same number of elements as a standard one - is a standard collection itself.

(iii) If the Coxeter graph has a branching vertex and the operator $-\mathbb{I} \notin G$ then there exist non-standard sufficient collections consisting of the same number of elements as the standard ones. They may be described as follows :

let $\hat{\Pi} = \{ e_1, e_2 \}$, every sufficient collection $\{ U_1, U_2 \}$ is of the form :

$$U_1 = \alpha\, (Co_G e_2)^\circ \quad , \quad \beta\, Co_G e_2 \subset U_2 \subset \beta\, \langle e_2, e_1 \rangle\, (Co_G e_1)^\circ,$$

$$U_2 = \alpha\, Co_G e_2 \quad , \quad \beta\, Co_G e_1 \subset U_1 \subset \beta\, \langle e_1, e_2 \rangle\, (Co_G e_2)^\circ.$$

PROOF.

(i) If the Coxeter graph has no branching vertices then by Theorem 4.3 the two canonical collections coincide and they form the smallest sufficient collection. A standard collection (which is equivalent to the respective canonical one) is also the smallest collection. Hence there is a standard subcollection in any finite sufficient collection. Therefore it remains to prove the assertion only for groups with branching graphs.

Let $\{ U_1, \dots ,U_m \}$ be a sufficient collection.

As we know from the classification of Coxeter graph ([1], ch.VI, 4.1, Th.1), the branching Coxeter graphs have three end vertices. Therefore standard collections consist of three sets if $-I \in G$ and of two sets if $-I \notin G$ (see Theorem 5.1). Consider these two cases.

I. $-1 \in G$, $\hat{\Pi} = \{ e_1, e_2, e_3 \}$.

One may consider that all e_i's belong to the same Weil chamber . The points $\nu_{ij} e_i \otimes e_j$, $i \neq j$, $i, j = 1,2,3$, are extreme points of $K(\mathfrak{A})$, here $\nu_{ij} = \dfrac{1}{\langle e_i, e_j \rangle}$. Consider the points

$$\nu_{12} e_1 \otimes e_2 , \ \nu_{23} e_2 \otimes e_3 , \ \nu_{31} e_3 \otimes e_1 \in \mathrm{Extr}\, K(\mathfrak{A}).$$

Assume that $\nu_{12} e_1 \otimes e_2 \in S(U_1)$ then by Lemma 5.3 $\nu_{23} e_2 \otimes e_3 \notin S(U_1)$ and $\nu_{31} e_3 \otimes e_1 \notin S(U_1)$. Let $\nu_{23} e_2 \otimes e_3 \in S(U_2)$ then $\nu_{31} e_3 \otimes e_1 \notin S(U_2)$. Hence $\nu_{31} e_3 \otimes e_1$ must belong to the third set $S(U_3)$. This means that the collection $\{ U_1, \dots ,U_m \}$ consists of no less than three sets.

II. Let $-I \notin G$. Consider the permutation $\hat{\pi}$ of the end vertices of the Coxeter graph, which was defined in part I of § 5. The permutation $\hat{\pi}$ transposes two end vertices and leaves the third one fixed.

Let $\hat{\Pi} = \{e_1, e_2\}$ and let e_1 be the vector corresponding to the $\hat{\pi}$-fixed vertex of the Coxeter graph. (One may consider that e_1 and e_2 belong to the same Weil chamber).

So, $\nu_{12} e_1 \otimes e_2 \in \operatorname{Extr} K(\mathfrak{A})$, $\nu_{21} e_2 \otimes e_1 \in \operatorname{Extr} K(\mathfrak{A})$ $(\nu_{ij} > 0)$.

Assume that $\nu_{12} e_1 \otimes e_2 \in S(U)$ and $\nu_{21} e_2 \otimes e_1 \in S(U)$. (We cannot use Lemma 5.3, since $-\mathbb{I} \notin G$). Then for some γ and δ

$$\gamma e_1 \in \operatorname{Extr} U \ , \ \frac{\nu_{12}}{\gamma} e_2 \in \operatorname{Extr} U^{o}$$

and

$$\delta e_2 \in \operatorname{Extr} U \ , \ \frac{\nu_{21}}{\delta} e_1 \in \operatorname{Extr} U^{o}.$$

Remark that in this case the numbers γ and δ can be of arbitrary signs.

But by the choice of e_1 : $-e_1 = w_o e_1$ (the operator w_o was defined in part I of § 5). As U is a G-symmetric set then $w_o U = U$ and therefore $\gamma w_o e_1$ $(=-\gamma e_1) \in \operatorname{Extr} U$. Analogously, $\frac{\nu_{21}}{\delta} w_o e_1$ $(=- \frac{\nu_{21}}{\delta} e_1) \in \operatorname{Extr} U^{o}$. Then $|\gamma|\, e_1 \in \operatorname{Extr} U$ and $\frac{\nu_{21}}{|\delta|} e_1 \in \operatorname{Extr} U^{o}$, but this is impossible by Lemma 5.2.

Thus , $\nu_{12} e_1 \otimes e_2$ and $\nu_{21} e_2 \otimes e_1$ cannot belong to $S(U)$ simultaneously. Hence the collection $\{ U_i \}$ consists of no less then two sets. (i) is proved.

(ii) Let $\{ U_i \}$ be a finite sufficient collection consisting of the same number of sets as a standard one. We show that $\{ U_i \}$ is a standard collection itself.

If the Coxeter graph $\Gamma(G)$ has no branching vertices then the canonical collections coincide and there exists the smallest sufficient collection (see Theorem 4.3). Then there exists a standard subcollection of the collection $\{ U_i \}$. So, the assertion (ii) is obvious.

Let the Coxeter graph have a branching vertex , and $-I \in G$. Then every standard collection consists of three sets . Let $\hat{\Pi} = \{ e_1, e_2, e \}$ (we consider e_1, e_2, e_3 belonging to the same Weil chamber).

Every extreme point $\nu_{ij} e_i \otimes e_j$ ($\nu_{ij} > 0$, $i,j = 1,2,3$) of $K(\mathfrak{A})$ must belong to one of the sets $S(U_1)$, $S(U_2)$, $S(U_3)$.

Assume that $\nu_{12} e_1 \otimes e_2 \in S(U_1)$, then by Lemma 5.2

$$\nu_{23} e_2 \otimes e_3 \notin S(U_1)$$

$$\nu_{21} e_2 \otimes e_1 \notin S(U_1)$$

$$\nu_{31} e_3 \otimes e_1 \notin S(U_1)$$

Let $\nu_{23} e_2 \otimes e_3 \in S(U_2)$, then by Lemma 5.2

$$\nu_{32} e_3 \otimes e_2 \notin S(U_2)$$

$$\nu_{31} e_3 \otimes e_1 \notin S(U_2)$$

$$\nu_{12} e_1 \otimes e_2 \notin S(U_2)$$

As $\nu_{31} e_3 \otimes e_1 \notin S(U_1)$, $S(U_2)$ then $\nu_{31} e_3 \otimes e_1 \in S(U_3)$, and then $\nu_{13} e_1 \otimes e_3 \notin S(U_3)$. Consider $\nu_{13} e_1 \otimes e_3$. As it does not belong to $S(U_3)$ then it must belong to $S(U_1)$ or to $S(U_2)$. We consider both cases.

<u>The first case</u>. Let $\nu_{13} e_1 \otimes e_3 \in S(U_1)$ then $\nu_{32} e_3 \otimes e_2 \notin S(U_1)$. As $\nu_{32} e_3 \otimes e_2 \notin S(U_2)$ then $\nu_{32} e_3 \otimes e_2 \notin S(U_3)$. Then immediately $\nu_{21} e_2 \otimes e_1 \notin$ $\notin S(U_3)$ (by Lemma 5.3).

So, $\nu_{21} e_2 \otimes e_1 \in S(U_2)$, and we obtain the following :

$$\nu_{12} e_1 \otimes e_2 \, , \nu_{13} e_1 \otimes e_3 \in S(U_1)$$

$$\nu_{23} e_2 \otimes e_3 \, , \nu_{21} e_2 \otimes e_1 \in S(U_2)$$

$$\nu_{31}e_3\otimes e_1 \ ,\nu_{32}e_3\otimes e_2 \in S(U_3)$$

By Lemma 5.4 these inclusions imply the following equalities:

$$U_1 = \lambda_1 Co_G e_1,\ U_2 = \lambda_2 Co_G e_2,\ U_3 = \lambda_3 Co_G e_3.$$

<u>The second case</u>. Let $\nu_{13}e_1\otimes e_3 \in S(U_2)$ then $\nu_{21}e_2\otimes e_1 \notin S(U_2)$ (and earlier we had $\nu_{31}e_3\otimes e_1,\ \nu_{32}e_3\otimes e_2 \notin S(U_2)$). As $\nu_{21}e_2\otimes e_1 \notin S(U_1),\ \notin S(U_2)$ then $\nu_{21}e_2\otimes e_1 \in S(U_3)$. Hence

$$\nu_{13}e_1\otimes e_3 \notin S(U_3),\ \nu_{12}e_1\otimes e_2 \notin S(U_3)\ ,\ \nu_{32}e_3\otimes e_2 \notin S(U_3).$$

Now, as $\nu_{32}e_3\otimes e_2 \notin S(U_2),\ S(U_3)$ then $\nu_{32}e_3\otimes e_2 \in S(U_1)$. Thus we obtain the following

$$\nu_{12}e_1\otimes e_2\ ,\ \nu_{32}e_3\otimes e_2 \in S(U_1)$$

$$\nu_{13}e_1\otimes e_3\ ,\ \nu_{23}e_2\otimes e_3 \in S(U_2)$$

$$\nu_{21}e_2\otimes e_1\ ,\ \nu_{31}e_3\otimes e_1 \in S(U_3)$$

By Lemma 5.4 these inclusions imply the following equalities :

$$U_1 = \mu_1(Co_G e_2)^o\ ,\ U_2 = \mu_2(Co_G e_3)^o\ ,\ U_3 = \mu_3(Co_G e_1)^o.$$

(ii) is proved.

(iii) Consider a sufficient collection $\{U_1,U_2\}$. Consider (as above) that e_1 and e_2 belong to the same Weil chamber. There are three end vertices of the Coxeter graph $\Gamma(G)$ corresponding to the vectors e_1, e_2, e_3. Let $w_o e_1 = -e_1$, $w_o e_2 = -e_3$, $w_o e_3 = -e_2$. We consider elements $\nu_{ij}e_i\otimes e_j$ ($\nu_{ij} = \dfrac{1}{\langle e_i, e_j\rangle}$), they all belong to Extr $K(\mathfrak{A})$, so we must distribute the elements $\nu_{ij}e_i\otimes e_j$ among the sets $S(U_1)$ and $S(U_2)$.

Remark that $\nu_{12} = \nu_{13}$ and $\nu_{21} = \nu_{31}$. Indeed :

$$\nu_{12} = \frac{1}{\langle e_1, e_2\rangle} = \frac{1}{\langle w_o e_1, w_o e_2\rangle} = \frac{1}{\langle -e_1, -e_3\rangle} = \frac{1}{\langle e_1, e_3\rangle} = \nu_{13}\ ,$$

and the second equality may be proved similarly.

Let $\nu_{12}e_1\otimes e_2 \in S(U_1)$ then $\nu_{13}e_1\otimes e_3 \in S(U_1)$. (Indeed , $\nu_{13}e_1\otimes e_3 =$

$= \nu_{12}e_1\otimes e_3 = \nu_{12}(-e_1)\otimes(-e_3) = \nu_{12}(w_o e_1)\otimes(w_o e_2) \in S(U_1))$

By the same reason as in the proof of (ii) we obtain : as $v_{12}e_1\otimes e_2 \in S(U_1)$ then $\nu_{21}e_2\otimes e_1 \in S(U_2)$.

As above, one can show that the inclusion $\nu_{21}e_2\otimes e_1 \in S(U_2)$ implies the inclusion $\nu_{31}e_3\otimes e_1 \in S(U_2)$. So, we have got that

$$\nu_{12}e_1\otimes e_2 \text{ , } \nu_{13}e_1\otimes e_3 \in S(U_1) \text{ and } \nu_{21}e_2\otimes e_1 \text{ , } \nu_{31}e_3\otimes e_1 \in S(U_2).$$

We have distributed almost all elements of Extr $K(\mathfrak{A})$ between $S(U_1)$ and $S(U_2)$ with the only exceptions of $\nu_{23}e_2\otimes e_3$ and $\nu_{32}e_3\otimes e_2$. Let us see what is the situation with them.

Obviously $\nu_{23} = \nu_{32}$ and if $\nu_{23}e_2\otimes e_3 \in S(U_i)$, then $\nu_{32}e_3\otimes e_2 \in S(U_i)$.

Let $i = 1$, i.e., $\nu_{23}e_2\otimes e_3 \in S(U_1)$. Then $S(U_1) \ni \nu_{12}e_1\otimes e_2$, $\nu_{23}e_2\otimes e_3$.

It follows from the first inclusion that $\lambda e_1 \in$ Extr U_1 and $\frac{\nu_{12}}{\lambda}\, e_2 \in$ Extr U_1^o for some λ. From the second inclusion we have : $\mu e_2 \in$ Extr U_1 and $\frac{\nu_{23}}{\mu}\, e_3 \in$ Extr U_1^o. The signs of λ and μ must be different (really, if $\lambda\mu > 0$ then we consider vectors

$$\mu e_2 \in \text{Extr } U_1 \text{ and } \frac{\nu_{12}}{\lambda\mu}\,(\mu e_2) \in \text{Extr } U_1^o$$

and we obtain a contradiction with Lemma 5.2).

Without loss of generality we may assume that $\lambda > 0$, $\mu < 0$. Then $\mu e_2 = |\mu|\, w_o e_3$ and the element $|\mu| w_o e_3$ is an extreme point of U_1 and, hence $|\mu| e_3$ is also an extreme point of U_1. Then by the Remark 5.1, using the fact that λ and $|\mu|$ are both positive we conclude that

$U_1 = (\frac{\nu_{12}}{\lambda} Co_G e_2)^{\circ}$. Now the set U_2 is submit to the only restriction : $\nu_{21} e_2 \otimes e_1 \in S(U_2)$. For example, the set $Co_G e_2$ satisfies this condition.

Thus a non-standard collection $\{(\frac{\nu_{12}}{\lambda} Co_G e_2)^{O},\ Co_G e_2\}$ is sufficient.

Note that one can take U_2 to be any set such that

$$\beta\ Co_G e_2 \subset U_2 \subset \beta \frac{1}{\nu_{21}} (Co_G e_1)^{\circ}.$$

Let $i = 2$: $\nu_{23} e_2 \otimes e_3 \in S(U_2)$. Then $\nu_{21} e_2 \otimes e_1$, $\nu_{23} e_2 \otimes e_3 \in S(U_2)$

Then for some λ , μ

$$\lambda e_2 \in \text{Extr } U_2 , \ \frac{\nu_{21}}{\lambda} e_1 \in \text{Extr } U_2^{\circ}$$

$$\mu e_2 \in \text{Extr } U_2 , \ \frac{\nu_{23}}{\mu} e_3 \in \text{Extr } U_2^{\circ}$$

Signs of $\frac{-\nu_{23}}{\mu}$ and λ must be different (by Lemma 5.2) , since $\frac{-\nu_{23}}{\mu} e_2 = \frac{\nu_{23}}{\mu} w_o e_3$ is an extreme point of U_2°. Hence the signs of λ and μ must be the same, therefore $\lambda = \mu$. Consider $\lambda, \mu > 0$.

Then $U_2 = \lambda\ Co_G e_2$. And U_1 is submit to the only one restriction : $\nu_{12} e_1 \otimes e_2 \in S(U_1)$. It follows that U_1 may be any closed G-symmetric set satisfying the condition

$$\beta\ Co_G e_1 \subset U_1 \subset \frac{\beta}{\nu_{12}} (Co_G e_2)^{O}. \qquad \blacksquare$$

§ 6. Explicit formulas for standard collections.

All finite irreducible Coxeter groups are classified, they are divided into 4 countable families : A_k, B_k, D_k, $J_2(p)$, and 6 exceptional groups : E_6, E_7, E_8, F_4, H_3, H_4. Their Coxeter graphs are classified

(see [1], ch.VI, 4.1, Th.1).

As it was mentioned above, in the theory of Coxeter groups special vectors on extreme rays of Weil chambers are called <u>fundamental</u> <u>weights</u> and they are calculated explicitly (see [1]). There are also descriptions of the action of the operator π on the vertices of Coxeter graphs. In some cases there are explicit descriptions of Weil chambers and the related operators $a \longmapsto a^*$.

We calculate standard collections for Coxeter groups such that there exist simple descriptions of their actions. Slightly different formulas were given in the survey [11] but we prefer to give them here for the sake of completeness.

Group A_k ($k \geq 2$) .

V is a hyperplane of the space R^{k+1}

$$V = \{ x = (x_1, \ldots x_{k+1}) : \sum_{i=1}^{k+1} x_i = 0 \}$$

ε_i are the vectors of canonical basis :

$$\varepsilon_1 = (1,0,\ldots 0),\ \varepsilon_2 = (0,1,\ldots,0),\ \ldots\ ,\ \varepsilon_{k+1} = (0,0,\ldots 0,1).$$

The action of the group A_k: permutations of coordinates of a vector in the canonical basis (the trivial case is not under consideration).

The Coxeter graph is : o—o— ... —o—o . As the Coxeter graph has no branching vertices then there exists the smallest sufficient collection (Theorem 4.3).

A Weil chamber:

$$C = \{ x = (x_1,\ldots,x_{k+1}):\ x_1 \geq x_2 \geq \ldots \geq x_k \geq x_{k+1},\ \sum_{i=1}^{k+1} x_i = 0 \}$$

Let x^* be the only vector of $\mathrm{orb}_G x$ in the Weil chamber C, i.e., the operation $x \longmapsto x^*$ is the permutation of coordinates in the nonincreasing order. The fundamental weights are the following

$$\omega_i = (\varepsilon_1 + \dots + \varepsilon_i) - \frac{i}{k+1} \sum_{j=1}^{k+1} \varepsilon_j$$

The fundamental weights corresponding to the end vertices of the Coxeter graph (i.e., the set $\mathfrak{R}$) are

$$\omega_1 = \varepsilon_1 - \frac{1}{k+1} \sum_{j=1}^{k+1} \varepsilon_j = \frac{1}{k+1} (k, -1, -1, \dots, -1)$$

...

$$\omega_k = (\varepsilon_1 + \dots + \varepsilon_k) - \frac{k}{k+1} \sum_{j=1}^{k+1} \varepsilon_j = \frac{1}{k+1} (1, 1, \dots 1, -k)$$

Remark that $-\mathbb{I} \notin A_k$. Then by Theorem 5.1 standard collections consist of one set (the number of end vertices minus 1). The permutation π transposes the end vertices and therefore $\hat{\mathfrak{R}} = \{ \omega_1 \}$. A standard collection $\{ \mathrm{Co}_{A_k} \omega_1 \}$.

$$\mathrm{Co}_{A_k} \omega_1 = \{ x : \langle x, g\hat{\omega}_k \rangle \le \sup_{g \in A_k} \langle g\omega_1, g\hat{\omega}_k \rangle \} =$$

$$= \{ x : \langle x, g\hat{\omega}_k \rangle \le \langle x^*, \omega_k \rangle \le \langle \omega_1, \omega_k \rangle \} =$$

$$= \{ x : \frac{1}{k+1} (\sum_{i=1}^{k} x_i^* - x_{k+1}^* \cdot k) \le \frac{1}{(k+1)} \} =$$

$$= \{ x : \sum_{i=1}^{k} x_i^* - x_{k+1}^* \cdot k \le 1 \} = \{ x : - x_{k+1}^* - x_{k+1}^* \cdot k \le 1 \}$$

$$= \{ x : - x_{k+1}^* \cdot (k+1) \le 1 \} = \{ x : (\min_i x_i)(k+1) \ge -1 \}.$$

Taking $\hat{\omega}_1 = \dfrac{\omega}{k+1}$, we obtain

$$\mathrm{Co}_{A_k} \hat{\omega}_1 = \{ x : \min_i x_i \ge -1 \}$$

Group B_k ($k \ge 2$).

$V = \mathbb{R}^k$. The action of the group : permutations and sign changes of coordinates in the canonical basis.

The Coxeter graph : $\underset{1}{o}$—$\underset{2}{o}$— ... —o$\overset{4}{—}\underset{k}{o}$. As graph has no branching vertices, then the canonical collections coincide and this is the smallest sufficient collection.

The fundamental weights :

$$\omega_i = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_i = (1,1, \ \ldots \ 1,0, \ \ldots \ ,0) \ (i \ \text{units})$$

$$\omega_k = \frac{1}{2} (\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_k)$$

A Weil chamber

$$C = \{ x = (x_1,\ldots,x_k) : x_1 \ge x_2 \ge \ldots \ge x_k \ge 0 \}$$

x^* denotes, as above, the image of the vector x in the Weil chamber, i.e., the operation $x \to x^*$ is a non-increasing permutation of the vector $(|x_1|, |x_2|, \ \ldots \ , |x_k|)$.

The fundamental weights corresponding to the end vertices of the Coxeter graph (i.e., the set $\mathfrak{N} = \{ \omega_1, \omega_k \}$) are :

$$\omega_1 = \varepsilon_1 = (1,0, \ \ldots \ ,0)$$

$$\omega_k = \frac{1}{2} (1,1, \ \ldots \ ,1)$$

The operator $-\mathbb{I} \in B_k$. It follows that standard collections consist of two sets : $\{ \mathrm{Co}_{B_k}\omega_1 \ , \ \mathrm{Co}_{B_k}\omega_k \}$.

$$\mathrm{Co}_{B_k}\omega_1 = \{ x: < x^*, \omega_k > \le <\omega_1, \omega_k> \} =$$

$$= \{ x: \frac{1}{2} \sum_{i=1}^{k} x_i^* \le \frac{1}{2} \} =$$

$$= \{\ x:\ \sum_{i=1}^{k} x_i^* \le 1\ \} =$$

$$= \{\ x:\ \sum_{i=k}^{1} |x_i| \le 1\ \}$$

This is the unit ball of the space l_1^k.

$$Co_{B_k}\omega_k = \{\ x:\ \langle x^*, \omega_1 \rangle \le \langle \omega_k, \omega_1 \rangle\ \} =$$

$$= \{\ x:\ x_1^* \le \frac{1}{2}\ \} = \{\ x:\ \max |x_i| \le \frac{1}{2}\ \}$$

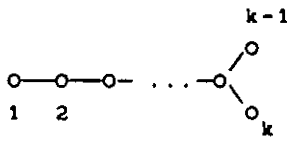
This is a ball of the space l_∞^k.

The assertion about the sufficiency of the collection $\{Co_{B_k}\omega_1\ ,\ Co_{B_k}\omega_k\}$ is the Mityagin-Calderon theorem ([3,5]).

It follows from our general theory that this collection is the smallest one. This means that the norms of l_1^k and l_∞^k are not strict interpolation norms for any finite collection of B_k-symmetric norms.

Group D_k ($k \ge 3$)

$V = \mathbb{R}^k$. The action of the group : permutations and sign changes of even numbers of coordinates .

The Coxeter graph :  .

The fundamental weights :

$$\omega_i = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_i = (\ 1,1,\ \ldots 1,0,\ \ldots,\ 0\ ),$$

$$i \text{ units},\ 1 \le i \le k-2;$$

$$\omega_{k-1} = \frac{1}{2}\ (\ \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{k-2} + \varepsilon_{k-1} - \varepsilon_k\ ) =$$

$$= \frac{1}{2}\ (\ 1,1,\ \ldots\ 1,-1\ )$$

$$\omega_k = \frac{1}{2} (\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{k-2} + \varepsilon_{k-1} + \varepsilon_k) = \frac{1}{2} (1,1, \dots ,1).$$

A Weil chamber

$$C = \{x = (x_1, \dots ,x_k): x_1 \ge x_2 \ge \dots \ge x_{k-1} \ge x_k, |x_k| \le x_{k-1}\}$$

The fundamental weights of the Weil chamber C corresponding to the end vertices (i.e., elements of the set Π) are

$$\omega_1 = \varepsilon_1 = (1,0, \dots ,0)$$

$$\omega_{k-1} = \frac{1}{2} (1,1, \dots 1,-1)$$

$$\omega_k = \frac{1}{2} (1,1, \dots 1)$$

We omit the factor $\frac{1}{2}$ in the formulas for ω_{k-1}, ω_k.

The graph has a branching vertex therefore the canonical collections do not coincide and the smallest collection does not exist.

The vector x^* may be obtained from x as follows : the coordinates of x are rearranged in the modulus nonincreasing order. If the number of negative coordinates of x is even then all coordinates of x^* get the signs "+". If the number of negative coordinates of x is odd then the last coordinate of x^* gets "-" and the other coordinates get "+".

To describe the canonical collections we need to consider two cases.

I. Let k be an odd number. Then the operator $-\mathbb{I} \notin D_k$ and therefore (Theorem 5.1) that standard collections consist of two sets. The action of the operator $w_o \cdot (-1)$ on the fundamental weights of Weil chamber is the following :

$$w_o \cdot (-1) : \omega_1 \to \omega_1 , \omega_{k-1} \to \omega_k , \omega_k \to \omega_{k-1} .$$

A standard simple collection : $\{ Co_{D_k} \omega_1 , Co_{D_k} \omega_{k-1} \}$.

$$Co_{D_k} \omega_1 = \{ x: \langle x^*, \omega_{k-1} \rangle \le \langle \omega_1, \omega_{k-1} \rangle , \langle x^*, \omega_k \rangle \le \langle \omega_1, \omega_k \rangle \} =$$

$$= \{ x : \sum_{i=1}^{k-1} (x^*)_i - (x^*)_k \le 1,\ \sum_{i=1}^{k-1} (x^*)_i + (x^*)_k \le 1 \} =$$

$$= \{ x : \sum_{i=1}^{k-1} (x^*)_i + |(x^*)_k| \le 1 \}$$

$$= \{ x : \sum_{i=1}^{k} |x_i| \le 1 \}$$

$$\mathrm{Co}_{D_k}\omega_{k-1} = \{ x : \langle x^*,\omega_1\rangle \le \langle \omega_{k-1},\omega_1\rangle ,\ \langle x^*,\omega_k\rangle \le \langle \omega_{k-1},\omega_k\rangle \} =$$

$$= \{ x : (x^*)_1 \le 1,\ \sum_{i=1}^{k} (x^*)_i \le k-2 \} =$$

$$= \{ x : \max_i |x_i| \le 1,\ \sum_{i=1}^{k} |x_i| + (x^*)_k - \min_i |x_i| \le k-2 \} =$$

$$= \{ x : \max_i |x_i| \le 1,\ \sum_{i=1}^{k} |x_i| + \min_i |x_i| \ ((-1)^{N(x)} - 1) \le k-2 \}$$

Here $N(x)$ is the number of <u>negative</u> components in the vector x.

A standard dual simple collection : $\{ (\mathrm{Co}_{D_k}\omega_1)^{\circ} ,\ (\mathrm{Co}_{D_k}\omega_k)^{\circ} \}$.

$$(\mathrm{Co}_{D_k}\omega_1)^{\circ} = \{ x : \langle x^*,\omega_1\rangle \le 1 \} =$$

$$= \{ x : x^*_1 \le 1 \} = \{ x : \max_{1\le i\le k} |x_i| \le 1 \}$$

$$(\mathrm{Co}_{D_k}\omega_k)^{\circ} = \{ x : \langle x^*,\omega_k\rangle \le 1 \} =$$

$$= \{ x : \sum_{i=1}^{1-1} (x^*)_i \le 1 \} =$$

$$= \{ x : \sum_{i=1}^{k} |x_i| + (x^*)_k - \min_i |x_i| \le 1 \} =$$

$$= \{ x : \sum_{i=1}^{k} |x_i| + \min_i |x_i| \ ((-1)^{N(x)} - 1) \le 1 \}$$

Let k be an even number, then $-1 \in D_k$. A standard simple collection

$\{ \mathrm{Co}_{D_k}\omega_1 ,\ \mathrm{Co}_{D_k}\omega_{k-1} ,\ \mathrm{Co}_{D_k}\omega_k \}$ is the following

$$\mathrm{Co}_{D_k}\omega_1 = \{ x : \sum_{i=1}^{k} |x_i| \le 1 \}$$

$$Co_{D_k}\omega_{k-1} = \{x : \max |x_i| \le 1, \ \sum_{i=1}^{k} |x_i| + \min_i |x_i| \ ((-1)^{N(x)} - 1) \le k-2 \}$$

$$Co_{D_k}\omega_k = \{ x : \max_i |x_i| \le 1, \ \sum_{i=1}^{k} |x_i| + \min_i |x_i| \ (-(-1)^{N(x)} - 1) \le k-2 \}$$

A standard dual simple collection

$$\{ (Co_{D_k}\omega_1)^\circ, \ (Co_{D_k}\omega_{k-1})^\circ, \ (Co_{D_k}\omega_k)^\circ \}$$

$$(Co_{D_k}\omega_1)^\circ = \{ x : \max_{1 \le i \le k-1} |x_i| \le 1 \}$$

$$(Co_{D_k}\omega_{k-1})^\circ = \{ x : \sum_{i=1}^{k} |x_i| + \min_i |x_i| \ (-(-1)^{N(x)} - 1) \le 1 \}$$

$$(Co_{D_k}\omega_k)^\circ = \{ x : \sum_{i=1}^{k} |x_i| + \min_i |x_i| \ ((-1)^{N(x)} - 1) \le 1 \}$$

Group $I_2(p)$

The group $I_2(p)$ acts in $\mathbb{R}^2$ as follows : fix two straight lines containing the origin with the angle $\frac{\pi}{p}$ between them. The group $I_2(p)$ is generated by orthogonal reflections across these lines. In particular, $I_2(3) = A_3$, $I_2(4) = B_2$, $I_2(6) = G_2$. The corner bounded by two rays, situated on these lines (the angle between them equals $\frac{\pi}{p}$) is a Weil chamber. The Weil chamber C has two fundamental weights ω_1 and ω_2. The Coxeter graph is : $\circ\overset{p}{\text{——}}\circ$. The Coxeter graph has no branching vertices, so the canonical collections coincide and the smallest sufficient collection exists.

If p is an even number then $-1 \in I_2(p)$, therefore standard collections consist of two sets : regular p-polygons, one with the vertex on the ray, containing ω_1, the other -- on the ray, containing ω_2.

If p is odd then $-1 \notin I_2(p)$ therefore standard collections consist of one set : regular p-polygon with a vertex on one of the fixed lines, i.e. on one of the rays ω_i ($i = 1,2$).

Any sufficient collection consists of no less then two (p even) or one (p odd) sets and any sufficient collection with such number of elements is a standard one.

Groups E_6 , E_7 , E_8 , F_4 , H_3 , H_4 .

As for the rest Coxeter groups mentioned above we don't know simple descriptions of their actions, so we don't know simple descriptions of the operations $x \longmapsto x^*$ and this is why formulas for standard collections in terms of this operation are noneffective. As Coxeter graphs of these groups are well known ([1]) we can only answer the question on the the existence of the smallest sufficient collection and the number of sets in standard collections.

Group E_6. The Coxeter graph is :

```
o—o—o—o—o
1 3 |4 5 6
    o
    2
```

. As the graph has a branching vertex then the canonical collections do not coincide, so the smallest sufficient collection does not exist. The operator $w_o \cdot (-1)$ acts as follows

$$w_o \cdot (-1) \; : \; \omega_1 \to \omega_6 \; , \; \omega_2 \to \omega_2 \; , \; \omega_6 \to \omega_1 \; .$$

Every standard collection consists of two sets.

Group E_7 . The Coxeter graph is :

```
o—o—o—o—o—o
1 3 |4 5 6 7
    o
    2
```

.The canonical

collections do not coincide, so the smallest sufficient collection does not exist. $-1 \in E_7$, every standard collection consists of three sets.

Group E_8 . The Coxeter graph is :

```
o—o—o—o—o—o—o .
1 3 |4 5 6 7 8
    o
    2
```

The canonical collections do not coincide, so the smallest sufficient collection does not exist. $-1 \in E_8$, so every standard collection consists of three sets .

Group F_4 . The Coxeter graph is :

```
     4
o—o——o—o .
1 2  3 4
```

There exists a description of the action of F_4 but it is rather complicated. The canonical collections coincide, forming the smallest sufficient collection, since there is no branching vertices in the graph . $-1 \in F_4$, hence standard collections consist of two sets.

Group H_3 . The Coxeter graph is :

```
  5
o——o—o .
```

The canonical collections coincide and form the smallest sufficient collection. $-1 \in H_3$, so standard collections consist of two sets .

Group H_4 . The Coxeter graph is :

```
  5
o——o—o—o .
```

The canonical collections coincide and form the smallest sufficient collection. $-1 \in$
$\in H_4$, so standard collections consist of two sets .

§ 7. Some remarks.

K - MONOTONICITY. Consider a collection of pseudonorms $\{ \|\cdot\|_\alpha \}$. One may construct the following K-functional (see, e.g. [2]) for $x \in V$, $t_\alpha \geq 0$

$$K\,(x;\ t_\alpha;\ \|\cdot\|_\alpha) = \inf\,\{\ \sum_\alpha t_\alpha \|\ x_\alpha\ \|_\alpha\ :\ x = \sum_\alpha x_\alpha\ \}$$

The pseudonorm $\|\cdot\|$ is said to be <u>K-monotone</u> (with respect to the pseudonorms $\|\cdot\|_\alpha$) if the following implication holds :

$$\text{if}\quad K\,(x;\ t_\alpha;\ \|\cdot\|_\alpha) \leq K\,(y;\ t_\alpha;\ \|\cdot\|_\alpha) \text{ for all } t_\alpha \geq 0$$

$$\text{then } \|\ x\ \| \leq \|\ y\ \|$$

One can easily prove that if a pseudonorm is K-monotone with respect to a collection pseudonorms, then it is a strict interpolation norm for this collection. In [3] it was shown that every B_n-invariant norm is K-monotone with respect to the norms (l_1^n , l_∞^n).

It was observed in [6] that a pseudonorm is K-monotone with respect to a collection of pseudonorms { $\|\cdot\|_\alpha$ } if and only if for every $x \in V$, $f \in V'$

$$\|\ x\ \|\ \|\ f\ \|' \geq \inf\,\{\ \sum_{\alpha,k} \|\ z_{\alpha,k}\ \|_\alpha\ \|\ \varphi_k\ \|^\alpha\ :\quad f = \sum_k \varphi_k\ ,\ x = \sum_\alpha z_{\alpha,k}\ \}$$

(Here $\|\cdot\|'$, (and, respectively, $\|\cdot\|^\alpha$) denotes the pseudonorm on V', conjugate to the pseudonorm $\|\cdot\|$ (respectively, $\|\cdot\|_\alpha$).

PROPOSITION 7.1. *Let G be a finite irreducible Coxeter group. Any G-invariant pseudonorm is K-monotone with respect to any standard simple collection.*

PROOF. Take any G-invariant pseudonorm, then its unit ball U is a G-symmetric set. U° is the unit ball of the conjugate pseudonorm. Take $x \in \|\ x\ \|\ U$, $f \in \|\ f\ \|'\ U^\circ$. Then $f \in \|\ x\ \|\ \|\ f\ \|'(Co_G x)^\circ$. Decompose f: $f = \sum \nu_k f_k$, where $f_k \in \mathrm{Extr}\ \|\ x\ \|\ \|\ f\ \|'(Co_G x)^\circ$, $\nu_k \geq 0$, $\sum \nu_k = 1$. So, Card $\mathrm{supp}_G f_k = 1$ (see Th. 4.1). Then $x \otimes f = \sum_k \nu_k x \otimes f_k$. Obviously

$x \in \| x \| \, \| f \|' (Co_G f_k)^{\circ}$, so we may decompose x:

$$x = \sum_{\alpha, k} \lambda_{\alpha k} x_{\alpha k} \ ,$$

$$x_{\alpha k} \in \mathrm{Extr} \| x \| \, \| f \|' (Co_G f_k)^{\circ} \quad , \quad \lambda_{\alpha k} \geq 0 \quad , \quad \sum_{\alpha} \lambda_{\alpha k} = 1.$$

So, $\mathrm{supp}_G x_{\alpha k}$ consists of an end vertex of $\Gamma(G)$ for every α, k. Decompose $x \otimes f$:

$$x \otimes f = \sum_{\alpha} \sum_{k} \nu_k \lambda_{\alpha k} x_{\alpha k} \otimes f_k$$

Obviously $f_k \in \| x \| \, \| f \|' (Co_G x_{\alpha k})^{\circ}$. Let $\mathrm{supp}_G x_{\alpha k} = \pi(s(\alpha, k))$, where $\pi(s(\alpha,k))$ is an end vertex of $\Gamma(G)$, so $Co_G x_{\alpha k}$ belongs to the simple canonical collection.

Let $\hat{\Pi} = \{ \omega(s) : s \in S \}$ and let $\| \cdot \|_s$ denote the pseudonorm whose unit ball is $Co_G \omega(s)$, $\pi(s)$ is an end vertex of $\Gamma(G)$.

We obtain

$$\sum_{\alpha, k} | \nu_k \lambda_{\alpha k} | \, \| x_{\alpha k} \|_{s(\alpha, k)} \| f_k \|^{s(\alpha, k)} =$$

$$= \sum_{\alpha, k} | \nu_k \lambda_{\alpha k} | \, \| x_{\alpha k} \|_{s(\alpha, k)} \left\| \frac{f_k}{\|x\| \, \|f\|'} \right\|^{s(\alpha, k)} \| x \| \, \| f \|'$$

As $\dfrac{f_k}{\|x\| \, \|f\|'} \in (Co_G x_{\alpha k})^{\circ}$, then $\| x_{\alpha k} \|_{s(\alpha, k)} \left\| \dfrac{f_k}{\|x\| \, \|f\|'} \right\|^{s(\alpha, k)} \leq 1,$

so

$$\sum_{\alpha, k} \| \lambda_{\alpha k} x_{\alpha k} \|_{s(\alpha, k)} \, \| \nu_k f_k \|^{s(\alpha, k)} \leq \sum_{\alpha, k} \nu_k \lambda_{\alpha k} \| x \| \, \| f \|' =$$
$$= \| x \| \, \| f \|' \ .$$

and $f = \sum_{k} \nu_k f_k$, $x = \sum_{\alpha} \lambda_{\alpha k} x_{\alpha k}$ ∎

RECONSTRUCTING COLLECTIONS OF NORMS FROM THE SET OF INTERPOLATION NORMS.

In the survey [2] the following question was asked : is it possible to reconstruct two norms, defined on the space V, knowing the set of all strict interpolation norms for this couple of norms ?

Recently O. Tikhonov and L. Veselova have shown that the answer is "yes" (private communication). The answer to the above question is "no", if to consider not two but three initial norms on V - one may consider two different standard collections - a simple one and a dual simple one - for the group D_n, n even (the set of all strict interpolation norms here is exactly the set of all D_n-invariant norms). If we replace the word "norm" by the word "pseudonorm" in the above question, then again the answer is "no" - a counterexample is given by two different standard collections for the group D_n, n odd. These collections consist of two sets each, they are certainly nonequivalent and have the same set of strict interpolation norms.

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