MULTI-# UNKNOTTING OPERATIONS: A NEW FAMILY OF LOCAL MOVES ON A KNOT DIAGRAM AND RELATED INVARIANTS OF KNOTS

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Abstract. We define new families of (so called multi-#) local moves on knot projections (which contain the #-local move and the ordinary crossing change) and study some of their properties together with related knot invariants. We show that they define unknotting operations and hence resulting unknotting numbers. We use 4-manifold theory as a tool.

1. **Introduction**

In [M] H. Murakami defines a #-local move on knot diagrams and shows it to be an unknotting operation. The proof that it is indeed an unknotting operation is based on knot projections related to non-orientable spanning surfaces for the knot. Consequently he defines a metric $d_{\#}$ on the space $\mathcal{K}$ of all knots: There are finite length sequence of operations from any knot K_1 to any knot K_2. Take the minimum such length over all knot projections and sequences to be the distance between the K_i's. An #-unknotting number of knots is then defined as a special case (unknotting number of $K =$ the distance to the unknot). Murakami studies some of the properties of these numbers. In [A] I recovered slightly stronger versions of his estimates on these invariants by using input from four-manifold theory (P. Gilmer's thesis [G] is taylored ideally for this and it is the main estimating tool here). There are, besides the ordinary crossing change, a number of unknotting operations, defined and studied by various authors, all of which result in producing metrics on $\mathcal{K}$ and associated knot invariants: the unknotting numbers. In this paper we want to generalize the #-unknotting operation by producing a family of multi-# unknotting operations. These are nothing but local moves on a knot diagram, a finite sequence of which exists, for any given knot which will unknot it. In the next section we define the family of moves study some of their immediate properties and then proceed to establish that they are unknotting operations. The proofs in this section are entirely combinatorial and in fact pictorial. In the last section we define the relevant resulting invariants and study some of their properties. It is in this section that one can see that the definition of these operations is taylored in order to involve four-manifold theory in their study.

Key words and phrases. 4-manifolds, multi-# local moves, knots, unknotting numbers of knots, singularities.

2. The Multi-# local move

Define the multi-# local move of type (t, s, d) on a knot projection as in the figure below: The direction shown in the figure will be called a + to − multi-

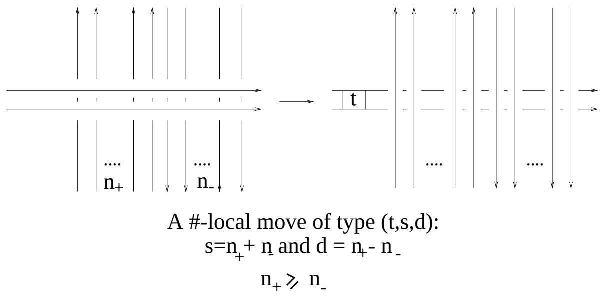

FIGURE 1

crossing change of type (t, s, d) and its inverse a − to + one. The boxed t in the figure is always even and indicates $|t|$ sign(t)-half twists (we will follow this convention every time we abbreviate a number of twists by boxing them). We will extract a family of unknotting operations out of these local moves. Consequently related metrics on the space of knots will be defined, as well as related unknotting numbers.

We begin with some preliminary lemmas which will be used later in the proofs. We will work with non-orientable spanning surfaces of knots as in [M]. It is immediate that a multi-# local move of type $(0, 1, 1)$ is nothing but Murakami's #-local move and that one of type $(2, 0, 0)$ is nothing but the ordinary unknotting operation. The moves of type $(0, s, 1)$ and $(2, s, 0)$ are interesting restricted generalizations of them respectively.

Lemma 2.1. *The effect of a $\pm$ to $\mp$, multi-# move of type (t, s, d) on two like-oriented parallel strings is $\pm(t - 4d)$ half twists.*

Proof Notice the equality in the figure below. Applying the move inside the dotted box finishes the proof in the case of a + to − move. The inverse is entirely similar.

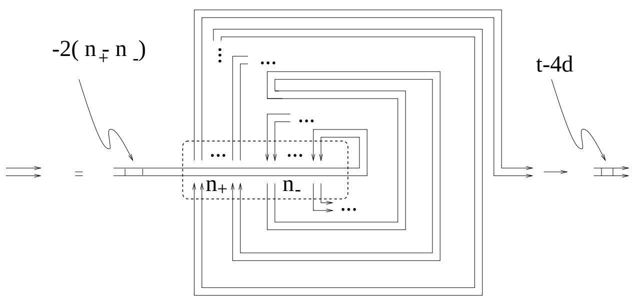

FIGURE 2

Lemma 2.2. *A + to − multi-# local move of type* (t, s, d) *can be locally expressed as one + to − multi-# local move of type* $(t \pm 4, s+1, d \mp 1)$.

Proof Appropriately apply lemma 2.1 above.

Now let D_m be a 2-disk bundle over a 2-sphere of Euler class m and let $2D_m$ be its double (the latter is therefore diffeomorphic to a punctured $S^2 \times S^2$ or $CP^2 \# \overline{CP}^2$ according as m is even or odd). The handlebody picture of D_m is then given in the figure below where we also introduce a homology basis and intersection number conventions:

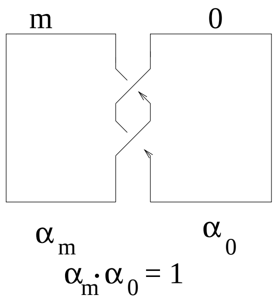

FIGURE 3

Lemma 2.3. *Let K be any knot and suppose that it can be unknotted as a result of applying a ± to ∓ multi-# local move of type* (t, s, d) *on one of its projections. Then there is a 2-sphere embedded in* $2D_{\mp t/2}$ *which is smooth everywhere except at one point where it is a cone on K and represents* $2(\alpha_{\mp t/2} \pm \frac{d}{2}\alpha_0)$ *on homology.*

Proof The idea is contained in the figure below. Just multiply the strands appropriately and include the t twists.

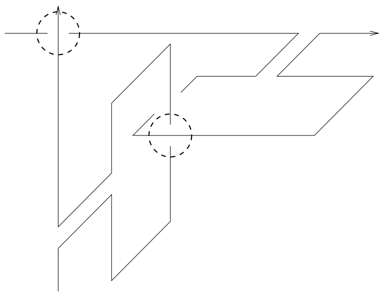

FIGURE 4

Remark 2.1. *It is an easy consequence of lemma 2.3 and theorem 3.2 that if K_1 and K_2 are two knots which differ by an odd (even) number of multi-# moves of type* (t, s, d) *with* $t \equiv 0 \bmod 4$ *and* $d \equiv 1 \bmod 2$ *then they have different (equal) Arf invariants. A move of type* $(0, s, d)$ *with* $d \equiv 0 \bmod 2$ *preserves the Arf invariant.*

The following lemma is a kind of inverse to lemma 2.2 above.

Lemma 2.4. *A multi-# local move of type (t, s, d) is locally expresible by 2s pass moves and $4s$ half twists.*

Proof The proof is contained in the figure below:

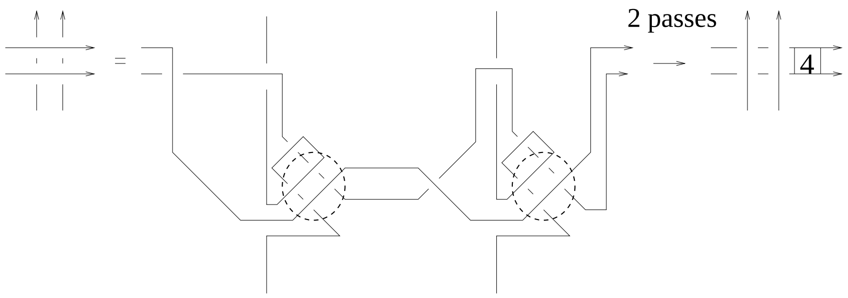

FIGURE 5

We now want to extract some unknotting operations out of these multi-# local moves. For this we need a definition:

Definition 2.1. *A multi-# local move of type (t, d) will be the union of all multi-# local moves of type (t, s, d). In other words we care only about the algebraic intersections of the two bundles of strings.*

We will then prove the main theorem of this section:

Theorem 2.1. *Let K be any knot and t and $d \geq 0$ any even integers. If $t \equiv 2 \bmod 4$ or $t \equiv 0 \bmod 4$ and $d \equiv 1 \bmod 2$ then a multi-# local move of type (t, d) is an unknotting operation. If $t \equiv 0 \bmod 4$ and $d \equiv 0 \bmod 2$ then a finite sequence of $\pm$ to $\mp$, multi-# local moves of type (t, d) leads to the unknot (trefoil) if $Arf(K) = 0$ ($Arf(K) = 1$).*

We will prove this theorem in the remainder of this section by using non-orientable Seifert surfaces for knots and the resulting knot projections as in the example below. We can obviously assume that the feet of each non orientable handle are next to each other and have no other feet in between and that each handle has an odd number of half twists.

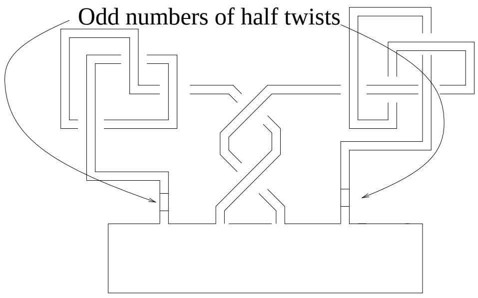

FIGURE 6

Lemma 2.5. *Every knot can be reduced to a connected sum* $\#_{i=1} sK(q_i, 2)$ *of torus knots where the* $q_i^{\prime}s$ *are defined only modulo* $t - 4d$.

Proof We will show how to, by applying our move once, split off a connected summand of the knot which is a torus knot of the desired type and the rest of the knot is again the boundary of a non-orientable surface as before. Then we can repeat the proccess as many times as we have handles and we will be done.

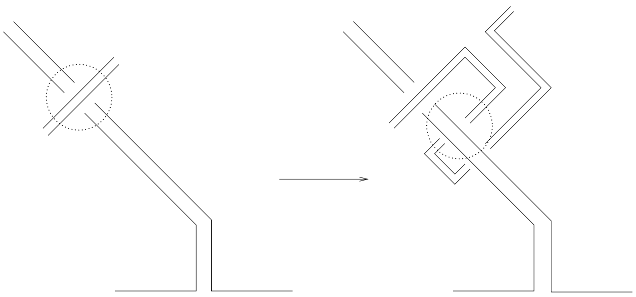

FIGURE 7

The idea is as follows; along each handle there is a number of (over or under) #-crossings which when changed all at once unknot the handle from all other handles as well as from itself so that we do get an unknotted handle with an odd number of half twists. Therefore all we have to do is show that we can bring all of these together. The first step is to turn all of them to (say) cross our handle from under. The picture above shows that we can do that as the pcitures are equal and changing the marked #-crossings in either has the same effect. Now just slide together all these undercrossings by following the handle as in the figure below and you are done. We should only point out that if say we are sliding all the crossings to a given foot of the handle then organize the sliding of the self #-crossings of the handle so that the furthest ones are slid first.

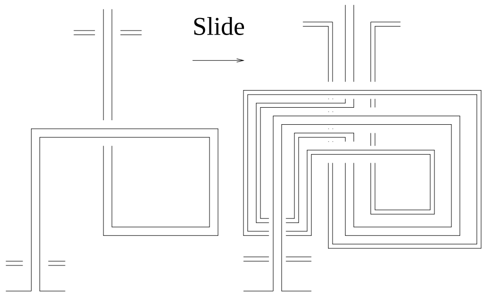

FIGURE 8

This establishes the reduction to a connected sum of torus knots of type $K(-, 2)$. That the first parameter can be changed modulo $t - 4d$ is a consequence of lemma 2.1. This finishes the proof of this lemma.

Now we have to deal with the case of a connected sum of $K(-, 2)$'s. For that we need some preliminary lemmas.

Lemma 2.6. *We can pass from (a) to any of (b) or (c) (resp. any of (d) or (e)) in the figure below by applying two multi-# moves of type* (t,d) *with* $t \equiv 0 \bmod 4$ *(resp.* $t \equiv 2 \bmod 4$*) for any* d.

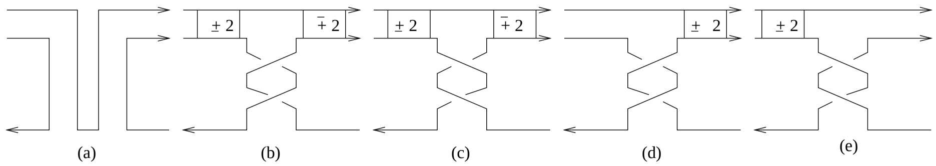

FIGURE 9

Proof The proof is based on observing the following equalities:

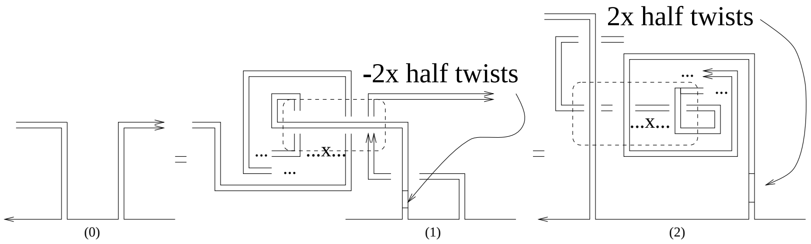

FIGURE 10

Applying a $+$ to $-$ multi-# move of type $(t, x+1)$ in the figure above inside the dotted box of (1) and a $-$ to $+$ multi-# move of type $(t, x+1)$ in (2), we get respectively the figure below. Applying our lemma 2.1 to each of these for $d = x+1$ finishes the proof.

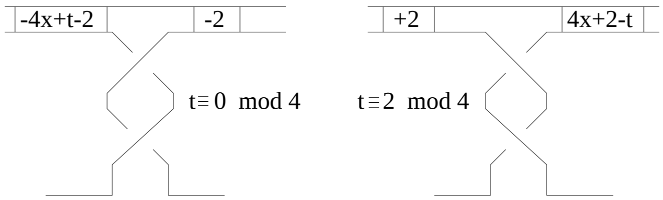

FIGURE 11

We need two further lemmas in the spirit of lemma 2.1 (which introduces twists on a pair of like-oriented parallel strings but the number of such twists depends on the parameters t, d). Our lemma 2.7 below introduces a number of twists which is independent of t, d but needs two summands when $t \equiv 0 \bmod 4$ and our lemma 2.8 introduces twists independently of t, d and with no need of extra summands in the knot. These two last pieces will allow us to state and prove the final piece of the proof of our main theorem 2.1 namely lemma 2.9. It is clear that lemmas 2.5 and 2.9 finish the proof of Theorem 2.1.

Lemma 2.7. *We can pass from $K(y,2)\#K(3,2)$ to $K(y+4,2)$ for any y achieved by one multi-# move of type $(0,d)$ for any d. Similarly from $K(y,2)$ to $K(y+4,2)$ with one move of type $(2,d)$.*

Proof The proof is given in the figure below where the arrows mean application of lemma 2.6.

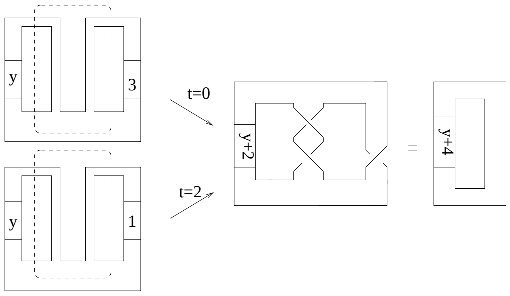

FIGURE 12

Lemma 2.8. *Any multi-# move of type $(0,d)$ can produce ± 8 half twists on two parallel like-oriented strings.*

Proof In the figure below arrow (1) is produced as in lemma 2.6 and arrow (2) is lemma 2.1.

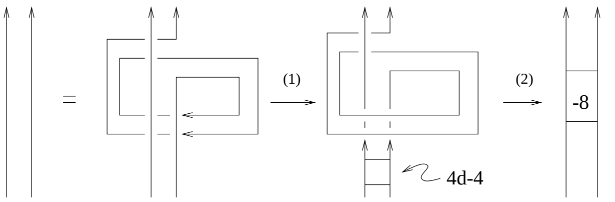

FIGURE 13

Now we are finally ready to prove:

Lemma 2.9. *Every connected sum $\#_{i=1}^{s} K(q_i,2)$ of torus knots can be unknotted via any multi-# move of type (t,d) if $t \equiv 2 \bmod 4$ or if $t \equiv 0 \bmod 4$ and $\frac{t}{4} - d \equiv 1 \bmod 2$ and can be reduced to either the unknot or the trefoil if $t \equiv 0 \bmod 4$ and $\frac{t}{4} - d \equiv 0 \bmod 2$.*

Proof By lemma 2.2 it suffices to show it for $t = 0, 2$. For $t = 2$ it is immediate from lemma 2.6. In case $t = 0$ apply lemma 2.7 (in reverse) to reduce any $\#_{i=1} sK(q_i, 2)$ to a connected sum of trefoil knots. Then every connected summand of the form $K(3,2)\#K(3,2)$ can be changed to $K(3,2)\#K(-5,2)$ by lemma 2.8 which is taken care off by 2.6. In case d is odd $K(3,2)$ can be changed, by lemma 2.1, to $K(4d+3,2) = K(4(d+1)-1,2)$ which in turn can be changed to the unknot $K(-1,2)$ by lemma 2.8. This finishes the proof of this lemma as well as that of Theorem 2.1.

3. Multi-# Knot Invariants, 4-manifolds and Estimates

We can now define knot invariants resulting from the multi-# unknotting operations we defined in the previous section. As in [M] we can define for every move of type (t, d) a metric $\mathrm{d}^{\#}_{(t,d)}$ on the space $\mathcal{K}$ of all knots, which we will some times abbreviate as $\mathrm{d}_{(t,d)}$ or d_d if t is fixed. For $i = 0, 1$, $\mathcal{K}_i$ will denote the subsets of $\mathcal{K}$ of Arf invariant equal to i.

Definition 3.1. *For any knots K_1, K_2 define their multi-# distance $\mathrm{d}^{\#}_{(t,d)}(K_1, K_2)$ to be the minimum, over all projections and multi-# sequences, number of multi-# moves necessary to go from K_1 to K_2. In case $t \equiv 0 \bmod 4$ and $d \equiv 0 \bmod 2$ these metrics are defined only on $\mathcal{K}_i$. The multi-# unknotting number $\mathrm{u}^{\#}_{(t,d)}$ of a knot is nothing but its distance from the unknot.*

We can also define the various unknotting sets:

Definition 3.2. *By $\mathcal{U}^{\#}_{(t,d)}(p, n)$ we will denote the set of all knots which can be unknotted via p + to - and n - to + multi-# moves of type (t, d). In other words these are nothing but the circles of radius $p + n$ in $(\mathcal{K}, \mathrm{d}^{\#}_{(t,d)})$.*

We make some further definitions which we find interesting:

Definition 3.3. *A local move will be called a* **slicing operation** *if every knot leads to a slice knot after a finite number of moves. We will say that the operation slices the knot. Every unknotting operation is a slicing operation.*

We find the concept an interesting one because of the following definition which can be defined for every unknotting operation:

Definition 3.4. *By $\mathcal{S}^{\#}_{(t,d)}(p, n)$ we will denote the set of all knots which can be sliced via p + to - and n - to + multi-# moves of type (t, d). In other words these are nothing but the subset of $(\mathcal{K}, \mathrm{d}^{\#}_{(t,d)})$ whose distance from the subset of slice knots is $p + n$. We can similarly to unknotting numbers define slicing numbers $s^{\#}_{(t,d)}$.*

Notice the inclusion $\mathcal{U}^{\#}_{(t,d)}(p, n) \subset \mathcal{S}^{\#}_{(t,d)}(p, n)$. We can also define such invariants as multi-# distances of knots from their mirror images, ribbon sets and "ribboning numbers" etc.

Question 3.1. *Do there exist local moves on knot diagrams which are slicing operations but are not unknotting operations?*

What is interesting here is first that slicing numbers seem to, in principle, be smaller than unknotting numbers and second the difference seems undetectable by the various methods available. The reader can see that definitions of slicing sets and numbers of knots are motivated by proposition 3.1 and the question that follows it.

We now quote the various 4-manifold theoretic results which will be used in giving estimates regarding the multi-# unknotting numbers. The following theorem is proven in [G] by P. Gilmer (from where we copy it here) and as mentioned there a special case of it (in the case $d = 2$ and one singular point) is also due to O. Ya. Viro. The version we quote here is only a special case of Gilmer's result (Theorem 4.1, Corollary 4.2).

Theorem 3.1. *Suppose $A \subset M^4$ is an embedded sphere in a simply connected 4-manifold which is smooth everywhere except at q points where it is the cone on knots $K_i,\ \ i = 1, \ldots, q$ and is such that on homology $[A] = \alpha = d\beta \in H_2(M^4)$, with d a power of a prime p. Then for every $0 < j < d$, $j \not\equiv 0 \bmod p$:*

$$b_2(M^4) + q - 1 \geq |\sigma(M^4) - \frac{2}{d^2} j(d-j)\alpha^2 - \sum_i \sigma_{j/d}(K_i)|$$

For the following theorem see p. 66 of [K] but beware because it is written down incorrectly there! See also theorem 1.2 on p. 31 of [K]:

Theorem 3.2. *Suppose M^4 is closed, smooth, simply connected and $\omega \in H_2(M^4)$ is characteristic. If ω is represented by an embedded sphere which is smooth everywhere except at q singular points where it is a cone on knots $K_i,\ \ i = 1, \ldots, q$ then it satisfies*

$$\frac{\sigma(M^4) - \omega \cdot \omega}{8} \equiv \sum_i Arf(K_i) \bmod 2$$

Such an ω is always representable by an embedded sphere which is smooth everywhere except at one singular point where it is a cone on a knot K.

I quote the following theorem from [K] where it is attributed to D. Ruberman.

Theorem 3.3. *The minimal genus of a surface in $S^2 \times S^2$ representing the class (a, b) with respect to basis $S^2 \times q$, $p \times S^2$ with $ab \neq 0$ is $(|a|-1)(|b|-1)$. The classes $(a, 0), (0, b)$ are represented by embedded spheres. The minimal genus of a surface in $CP^2 \# \overline{CP^2}$ which represents (a, b) in the obvious basis is*

$$\binom{|a|-1}{2} - \binom{|b|}{2}$$

if $|a| > |b|$. If $|a| < |b|$, the roles of a and b are reversed in the formula. If $|a| = |b|$ the class is represented by an embedded sphere.

These are the tools I know of that can be used to estimate the multi-# unknotting numbers defined above. Lemma 2.3 is the connecting tool. We have the following basic estimate:

Theorem 3.4. *If $K \in \mathcal{U}^{\#}_{(t,d)}(p, n)$ then:*

$$2(n-p)(2d - \frac{t}{2} + 1) - 4n \leq \sigma(k) \leq 2(n-p)(2d - \frac{t}{2} + 1) + 4p$$

Proof By a slight adaptation of lemma 2.3 if $K \in \mathcal{U}^{\#}_{(t,d)}(p, n)$ then there is a 2-sphere embedded in $\#_p(2D_{\frac{-t}{2}}) \#_n (2D_{\frac{t}{2}})$ smooth everywhere except at one point where it is the cone on K representing $\oplus_p(\alpha_{\frac{-t}{2}} + d\alpha_0) \oplus_n (\alpha_{\frac{t}{2}} - d\alpha_0)$ on homology. Now apply Theorem 3.1.

Remark 3.1. *It is clear that we need only assume that $K \in \mathcal{S}^{\#}_{(t,d)}(p, n)$ in the theorem above and we would get the same estimates. This method produces estimates on slicing numbers rather than unknotting ones.*

By observing then that $u^{\#}_{(t,d)}(K) = min\{p+n \colon K \in \mathcal{U}^{\#}_{(t,d)}(p, n)\}$ and $s^{\#}_{(t,d)}(K) = min\{p + n \colon K \in \mathcal{S}^{\#}_{(t,d)}(p, n)\}$ we get the following immediate:

Corollary 3.1. *If* $2d - \frac{t}{2} \geq 0$ *(resp.* < 0*) then* $|\sigma(K)| \leq 2u^{\#}_{(t,d)}(K)(2d - \frac{t}{2} + 1)$ *(resp.* $|\sigma(K)| \leq 2u^{\#}_{(t,d)}(K)(2d - \frac{t}{2} - 1)$*). Same for* $s^{\#}_{(t,d)}(K)$.

We can also, by applying Theorem 3.2 get:

Theorem 3.5. *If* $t \equiv 0 \bmod 4$ *and* $K \in \mathcal{U}^{\#}_{(t,d)}(p, n)$ *then*

$$(p + n)(\frac{t}{4} - d) \equiv Arf(K) \bmod 4.$$

In particular

$$u^{\#}_{(t,d)}(K)(\frac{t}{4} - d) \equiv Arf(K) \bmod 4.$$

Applying Theorem 3.3 in the obvious way we obtain:

Theorem 3.6. *If* $t \equiv 0 \bmod 4$ *then* $K \in \mathcal{U}^{\#}_{(t,d)}(1, 0)$ *or* $K \in \mathcal{U}^{\#}_{(t,d)}(0, 1)$ *imply*

$$g_{B^4}(K) \geq |2d - \frac{t}{2}| - 1.$$

Since any unknotting operation produces an immersed disk in B^4 we can get obstructions for a knot to have multi-# unknotting number equal to one in terms of unknotting numbers of other operations as in the following:

Corollary 3.2. *Suppose* $K \in \mathcal{U}^{\#}_{(t,d)}(p, n)$ *with* $p+n = 1$*. Then* $K \in \mathcal{U}(k, l)$ *implies that* $\max\{k, l\} \geq |2d - \frac{t}{2}| - 1$*. Similarly if for example* $K \in \mathcal{U}^{\#}_{(0,1)}(k, l)$ *then* $4 \max\{k, l\} \geq |2d - \frac{t}{2}| - 1$.

The proof of the following proposition can easily be extracted from the proof of lemma 2.5:

Proposition 3.1. *Any two-strand cable knot (or equivalently any knot which is the boundary of non-orientable Seifert surface with rank of first homology equal to one) has multi-# unknotting number equal to one for infinitely many unknotting operations of type* (t, d).

Any knot which is the boundary of non-orientable Seifert surface with rank of first homology equal to one has multi-# slicing number equal to one for infinitely many unknotting operations of type (t, d).

There are of course knots whose slicing (and hence also unknotting) numbers cannot be one. $K(7, 4)$ is one such. Its signature is $4 \mod 8$ and it cannot be sliced or unknotted at once by any multi-# unknotting operation!

We finish with some questions:

Question 3.2. *1. Can knots with multi-# unknotting number equal to one be characterized in some nice way? Corollary 3.2 provides characterizations of some kind. What if we ask to characterize "knots with multi-# unknotting number of type* (t, d) *equal to one for some* t, d*"?*

2. How can knots with slicing number smaller than unknotting number be detected?

3. For multi-# unknotting operations of type (t, d) *with* $4d - t =$*const. the estimates of their unknotting numbers coincide. Is there some way to tell them apart for some knots or are they the same?*

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